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# ON ALGEBRAIC BASES OF ALGEBRAS OF BLOCK-SYMMETRIC POLYNOMIALS ON BANACH SPACES

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The paper contains a description of algebraic basis of algebra of block-symmetric polynomials on the $\ell_1$-sum of the copies of $\ell_1$.

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В работе описан алгебраический базис алгебры блочно-симметрических полиномов на $\ell_1$-сумме копий пространства $\ell_1$.

In resent years there is increasing interest to investigations of invariants of permutation group $S_\infty$ of integer numbers. This group can be represented on a Banach space $X$ with a symmetric basis as a group of operators of perturbation of basis vectors. The action of this group has a natural extension to the action on the algebra $H_b(X)$ of analytic functions of bounded type on X. Invariants of this representation of $S_\infty$ are so called symmetric analytic functions of bounded type on X. The algebra of symmetric analytic functions $H_{bs}(X)$ were investigated by many authors ([3], [4], [6]). In particular, it is known that $H_{bs}(\ell_p)$ admits an algebraic basis for $1 \le p < \infty$.

On the other hand, there are more representations of $S_\infty$ in Banach spaces. For example, if $\mathcal{X}$ is a direct sum of infinitely many "blocks" which are copies of a Banach space X, then $S_\infty$ acts permutating the "blocks" (see for the definition below). For this case we have invariants — the algebra of block-symmetric analytic functions. Note that this algebra is much more complicated and in the general case has no algebraic basis (see e. g. [1, 2]). Note that if $\dim \mathcal{X} < \infty$, then block-invariant polynomials are investigated in the classical theory of invariants [5, 7].

Let
$$\mathcal{X} = \Big(\sum X\Big)_{\ell_1} = \bigoplus_{\ell_1} X$$
be a finite $\bigoplus_{\ell_1}^m X$ or an infinite $\bigoplus_{\ell_1} X$ $\ell_1$-sum of copies of Banach space $X$. So any element $\overline{x} \in \mathcal{X}$ can be represented as a sequence $\overline{x} = (x_1, \ldots, x_n, \ldots)$, where $x_n \in X$, with the norm $\|\overline{x}\| = \sum\limits_{k=1}^{\infty} \|x_k\|$.

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A polynomial $P$ on the space $\mathcal{X}^s_m = \bigoplus^m_{\ell_1} \mathbb{C}^s$ is called block-symmetric (or vector-symmetric) if:

$$P\left(\begin{pmatrix} u_1 \\ v_1 \\ \vdots \\ w_1 \end{pmatrix}_1, \ldots, \begin{pmatrix} u_m \\ v_m \\ \vdots \\ w_m \end{pmatrix}_m\right) = P\left(\begin{pmatrix} u_1 \\ v_1 \\ \vdots \\ w_1 \end{pmatrix}_{\sigma(1)}, \ldots, \begin{pmatrix} u_m \\ v_m \\ \vdots \\ w_m \end{pmatrix}_{\sigma(m)}\right),$$

for every permutation $\sigma$ on the set $\{1, 2, \ldots, m\}$, where $\begin{pmatrix} u_i \\ v_i \\ \vdots \\ w_i \end{pmatrix} \in \mathbb{C}^s$. Let us denote by $\mathcal{P}_{vs}(\mathcal{X})$ the algebra of block-symmetric polynomials on $\mathcal{X}$.

In paper [2] it was shown that the following vectors are generating elements of $\mathcal{P}_{vs}(\mathcal{X}^s_\infty)$ :

$$H^{k_1,k_2,\ldots,k_s}_n(x^1, x^2, \ldots, x^s) = \sum_{i=1}^{\infty} (x^1_i)^{k_1}(x^2_i)^{k_2} \ldots (x^s_i)^{k_s}, \quad k_1 + k_2 + \ldots + k_s = n, \tag{1}$$

where $x_i = (x^1_i, x^2_i, \ldots, x^s_i) \in \mathbb{C}^s$, $i \geq 1$.

The aim of this paper is to describe an algebraic basis of the block-symmetric polynomial algebra on the space $\mathcal{X}^\infty_\infty = \bigoplus_{\ell_1} \ell_1$.

**Lemma.** *Let $P_1, P_2, \ldots, P_n$ be algebraically independent polynomials on $\mathbb{C}^m$. Then $\{P_1(x), P_2(x), \ldots, P_n(x) \colon x \in \mathbb{C}^m\}$ is a dense subset of $\mathbb{C}^m$.*

*Proof.* We know from the algebraic geometry that the closure of the range of polynomial map $x \mapsto (P_1(x), P_2(x), \ldots, P_n(x))$ is an algebraic variety. So there exists a polynomial $Q$ on $\mathbb{C}^m$, such that $Q(P_1(x), P_2(x), \ldots, P_n(x)) = 0$ for any $x \in \mathbb{C}^m$. Since $P_1, P_2, \ldots, P_n$ are algebraically independent, $Q \equiv 0$. Hence, $\{P_1(x), P_2(x), \ldots, P_n(x) \colon x \in \mathbb{C}^m\}$ is a dense set in $\ker Q = \mathbb{C}^m$. □

Let us denote by $\mathcal{P}^{n+k}_{vs}(\mathcal{X}^s_\infty)$ the subalgebra of $\mathcal{P}_{vs}(\mathcal{X}^s_\infty)$ which is generated by the polynomials

$$H^{1,0,\ldots,0}_1(x^1, x^2, \ldots, x^s), \ldots, H^{k_1,k_2,\ldots,k_s}_m(x^1, x^2, \ldots, x^s), \tag{2}$$

where $k_1 + k_2 + \ldots + k_s = m$ and the number of these elements is equal to $n + k$.

**Theorem 1.** *The generating elements (2) are algebraically independent.*

*Proof.* Let $\{v_1, \ldots, v_k, \xi_1, \ldots, \xi_n\}$ be the same subset of generating elements system of algebra $\mathcal{P}^{n+k}_{vs}(\mathcal{X}^\infty_s)$, where $v_1, \ldots, v_k$ are symmetric polynomials, $\xi_1, \ldots, \xi_n$ are not symmetric. We will show that this system will be algebraically independent. The proof will be developed by the method of the mathematical induction. If $n = 0$ this result is obvious.

Let $\{v_1, \ldots, v_k, \xi_{j_1}, \ldots, \xi_{j_{n-1}}\}$ be algebraically independent for all $j_1, \ldots, j_{n-1} \in \{1, \ldots, n\}$. According to the lemma the set of ranges is dense in the prime $V_i = \{z \colon z_i = 0\}$, $i = 1, \ldots, n$.

If $\{v_1, \ldots, v_k, \xi_1, \ldots, \xi_n\}$ is an algebraically depending set, then there exists $Q$ from $\mathbb{C}^{k+n}$, such that $Q(v_1, \ldots, v_k, \xi_1, \ldots, \xi_n) = 0$ in the space of ranges. Without loss of the generality, we can suppose that $Q = \operatorname{rad} Q$. Then $\ker Q \supset \bigcup V_i$, $i = 1, \ldots, n$ and so $\ker Q \supset \ker z_1 \ldots z_n$. Thus by the Hilbert Nullstellensatz, $Q = Q_1 z_1 \ldots z_n$, where $Q_1$ is a constant. Hence $\xi_1(x^1, x^2, \ldots, x^s) \ldots \xi_n(x^1, x^2, \ldots, x^s) \equiv 0$, what is impossible. □

Since every polynomial from the algebra $\mathcal{P}_{vs}(\mathcal{X}_\infty^s)$ is uniquely representable as an algebraic combination of generating elements (1) Theorem 1 implies the following corollary.

**Corollary.** *Algebra $\mathcal{P}_{vs}(\mathcal{X}_\infty^s)$ has an algebraic basis which consists of polynomials (1).*

Now we consider the algebra of block-symmetric polynomials $\mathcal{P}_{vs}(\mathcal{X}_\infty^\infty)$ on the space $\mathcal{X}_\infty^\infty = \bigoplus_{\ell_1} \ell_1$.

**Theorem 2.** *The algebraic basis of algebra $\mathcal{P}_{vs}(\mathcal{X}_\infty^s)$ consists of polynomials*

$$H_n^{k_1,\dots,k_m,\dots}(x^1,x^2,\dots,x^m,\dots) = \sum_{i=1}^{\infty}\prod_{j=1}^{\infty}(x_i^j)^{k_j}, \quad \sum_{j=1}^{\infty} k_j = n, \quad n = 1,2,\dots. \tag{3}$$

*Proof.* Let $P(x^1,x^2,\dots,x^s,\dots)$ be a block-symmetric $m$ degree polynomial on $\mathcal{X}_\infty^\infty$. At first we are going to prove that the norm of polynomials (3) is finite, that is the series

$$\sum_{i=1}^{\infty}\prod_{j=1}^{\infty}(x_i^j)^{k_j}, \quad \sum_{j=1}^{\infty} k_j = n, \; n = 1,2,\dots$$

are convergent on the space $\mathcal{X}_\infty^\infty$ with norm $\|\overline{x}\| = \sum_{k,i=1}^{\infty} |x_k^i|$, where the vector $\overline{x} \in \mathcal{X}_\infty^\infty$ and $x_i = (x_i^1, x_i^2, \dots, x_i^s, \dots) \in \ell_1$. Indeed,

$$\left|\sum_{i=1}^{\infty}\prod_{j=1}^{\infty}(x_i^j)^{k_j}\right| = \lim_{m\to\infty}\left|\sum_{i=1}^{\infty}\prod_{j=1}^{m}(x_i^j)^{k_j}\right| \le \lim_{m\to\infty}\sum_{i=1}^{\infty}\prod_{j=1}^{m}|x_i^j|^{k_j} \le \lim_{m\to\infty}\prod_{j=1}^{m}\left(\sum_{i=1}^{\infty}|x_i^j|\right)^{k_j}$$
$$\le \lim_{m\to\infty}\prod_{j=1}^{m}\left(1+\sum_{i=1}^{\infty}|x_i^j|\right)^{k_j} \le \lim_{m\to\infty}\prod_{j=1}^{m}\left(1+\sum_{i=1}^{\infty}|x_i^j|\right)^{n} = \left(\prod_{j=1}^{\infty}\left(1+\sum_{i=1}^{\infty}|x_i^j|\right)\right)^{n}.$$

Note that the absolute convergence of the last product follows from the convergence of the series

$$\sum_{j=1}^{\infty}\left|\sum_{i=1}^{\infty}|x_i^j|\right| = \sum_{i,j=1}^{\infty}|x_i^j|.$$

Let $P^s(x^1,x^2,\dots,x^s)$ and $P^{s+l}(x^1,x^2,\dots,x^s)$ be the restriction of $m$ degree polynomial $P(x^1,x^2,\dots,x^s,\dots)$ to the spaces $\mathcal{X}_\infty^s$ and $\mathcal{X}_\infty^{s+l}$ respectively. According to the corollary of Theorem 1 we have that there exists a polynomials $Q_s$ and $Q_{s+l}$ on these spaces respectively such that

$$P^s(x^1,x^2,\dots,x^s) = Q_s(H_1^{1,\dots,0,\dots}(x^1,x^2,\dots,x^s),\dots,H_m^{k_1,\dots,k_m,\dots}(x^1,x^2,\dots,x^s)),$$
$$P^{s+l}(x^1,x^2,\dots,x^s,\dots,x^{s+l}) =$$
$$= Q_{s+l}(H_1^{1,\dots,0,\dots}(x^1,x^2,\dots,x^s,\dots,x^{s+l}),\dots,H_m^{k_1,\dots,k_m,\dots}(x^1,x^2,\dots,x^s,\dots,x^{s+l})),$$

where $\sum\limits_{j=1}^{\infty} k_j = m$. We remark that $H_m^{k_1,\dots,k_m,\dots}(x^1,x^2,\dots,x^s) = H_m^{k_1,\dots,k_m,\dots}(x^1,x^2,\dots,x^s,\dots)$ on the space $\mathcal{X}_\infty^s$. Let us show that $Q_{s+l} = Q_s$ for all $l = 0,1,\dots$.

Since $\overline{x} = (x^1,x^2,\dots,x^s)$, we have

$$P^{s+l}(x^1,x^2,\dots,x^s,\dots,x^{s+l}) = P^s(x^1,x^2,\dots,x^s) =$$
$$= Q_{s+l}(H_1^{1,\dots,0,\dots}(x^1,x^2,\dots,x^s),\dots,H_m^{k_1,\dots,k_m,\dots}(x^1,x^2,\dots,x^s)),$$
$$P^s(x^1,x^2,\dots,x^s) = Q_s(H_1^{1,\dots,0,\dots}(x^1,x^2,\dots,x^s),\dots,H_m^{k_1,\dots,k_m,\dots}(x^1,x^2,\dots,x^s)).$$

Since the polynomials (2) are algebraically independent on the space $\mathcal{X}_\infty^s$, it follows that $Q_{s+l} = Q_s$. This gives the equality

$$P(x^1, x^2, \ldots, x^s, \ldots) = Q_m(H_1^{1,\ldots,0,\ldots}(x^1, x^2, \ldots, x^s, \ldots), \ldots, H_m^{k_1,\ldots,k_m,\ldots}(x^1, x^2, \ldots, x^s, \ldots)),$$

where $\sum\limits_{j=1}^{\infty} k_j = m$ on the space $\mathcal{X}_\infty^\infty$ and this representing is unique. $\square$

## REFERENCES

1. A.V. Zagorodnyuk, V.V. Kravtsiv, *Spectra of algebra of block-symmetrical analitical functions*, Applied Problem of Mekh.-Mat., (2011) (in press) (in Ukrainian)
2. V. Kravtsiv, *Algebra of block-symmetric polynomials: generating elements and the transletion operator*, Matemetychnyi visnyk NTSh, **8** (2011), 107–121. (in Ukrainian)
3. I.V. Chernega, *Symmetric polynomials on Banach spaces*, Carpath. Mathem. Publications, **1** (2009), №2, 105–125. (in Ukrainian)
4. R. Alencar, R. Aron, P. Galindo, A. Zagorodnyuk, *Algebra of symmetric holomorphic functions on* $\ell_p$, Bull. Lond. Math. Soc., **35** (2003), 55–64.
5. J. Dieudone, J. Carrell, D. Mumford, Geometric invariant theory, Mir, Moskow, 1974.
6. A. Nemirovskii, S. Semenov, *On polynomial approximation of functions on Hilbert space*, Mat. USSR Sbornik, **21** (1973), 255–277.
7. H. Weyl, The classical group: their invariants and representations, Princeton university press, Princenton, New Jersey, 1973.

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