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SYMMETRIC POLYNOMIALS ON THE SPACE OF BOUNDED INTEGRABLE FUNCTIONS ON THE SEMI-AXIS

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Abstract: We describe an algebraic basis of the algebra of continuous symmetric polynomials on the complex Banach space of all essentially bounded Lebesgue integrable functions on the semi-axis.

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1. Introduction

Algebras of analytic functions which are invariant (symmetric) with respect to a group or semigroup of linear operators were investigated by many authors [1], [2], [4], [5], [6], [8], [9], [10], [11], [13], [14], [15] (see also a survey [7]).

In [11] Nemirovski and Semenov described algebraic bases of algebras of continuous symmetric polynomials on real spaces $L_p[0,1]$ and $L_p[0,+\infty)$ with respect to a natural group of operators, where $1 \leq p < +\infty$. Some of their results were generalized by González *et al.* [9] to real separable rearrangement-invariant function spaces.

Note that the non-separable case is much more complicated than the separable case. Symmetric continuous polynomials on the complex $L_\infty[0,1]$ have been studied in [8] and [13]. In this paper we consider the algebra of symmetric continuous polynomials on $L_1[0,+\infty) \cap L_\infty[0,+\infty)$ and describe its algebraic basis.

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2. Preliminaries

Let us denote $\mathbb{Z}_+$ the set of non-negative integers and $\mathbb{N}$ the set of positive integers. Let Ω be a Lebesgue measurable subset of $\mathbb{R}$. Let Ξ_Ω be the set of all measurable bijections of Ω that preserve the measure. For a given rearrangement-invariant complex Banach space $X(\Omega)$ of Lebesgue measurable functions $x : \Omega \to \mathbb{C}$, function $F : X(\Omega) \to \mathbb{C}$ is called *symmetric* if $F(x \circ \sigma) = F(x)$ for every $x \in X(\Omega)$ and $\sigma \in \Xi_\Omega$.

Let Y be a complex Banach space. A mapping $P : Y \to \mathbb{C}$ is called an n-homogeneous polynomial if there exists an n-linear mapping $A_P : Y^n \to \mathbb{C}$ such that $P(x) = A_P(x, \overset{n}{\ldots}, x)$ for every $x \in Y$. A mapping $P = P^{(0)} + P^{(1)} + \ldots + P^{(m)}$, where $P^{(0)} \in \mathbb{C}$ and $P^{(j)}$ is a j-homogeneous polynomial for every $j \in \{1, \ldots, m\}$, is called a polynomial (of degree at most m).

Let us denote $\mathcal{P}_s(X(\Omega))$ the algebra of all continuous symmetric polynomials $P : X(\Omega) \to \mathbb{C}$.

Let $L_\infty[0,1]$ be the complex Banach space of all Lebesgue measurable essentially bounded complex-valued functions x on $[0,1]$ with norm $\|x\|_\infty = \operatorname{ess\,sup}_{t\in[0,1]}|x(t)|$.

Theorem 1 ([8], Theorem 4.3)**.** *Every symmetric continuous n-homogeneous polynomial $P : L_\infty[0,1] \to \mathbb{C}$ can be uniquely represented as*

$$P(x) = \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1,\ldots,k_n} \tilde{R}_1^{k_1}(x) \cdots \tilde{R}_n^{k_n}(x),$$

where $k_1, \ldots, k_n \in \mathbb{Z}_+$, $\alpha_{k_1,\ldots,k_n} \in \mathbb{C}$ and $\tilde{R}_j(x) = \int_{[0,1]} (x(t))^j\, dt$.

In other words, $\{\tilde{R}_n\}$ forms an algebraic basis in the algebra $\mathcal{P}_s(L_\infty[0,1])$.

3. The Main Result

Let $L_1[0,+\infty)$ be the complex Banach space of all Lebesgue integrable functions $x : [0,+\infty) \to \mathbb{C}$ with norm $\|x\|_1 = \int_{[0,+\infty)} |x(t)|\, dt$ and let $L_\infty[0,+\infty)$ be the complex Banach space of all Lebesgue measurable essentially bounded functions $x : [0,+\infty) \to \mathbb{C}$ with norm $\|x\|_\infty = \operatorname{ess\,sup}_{t\in[0,+\infty)}|x(t)|$. Let us consider the space $L_1 \cap L_\infty := L_1[0,+\infty) \cap L_\infty[0,+\infty)$ with norm $\|x\| = \max\{\|x\|_1, \|x\|_\infty\}$. By [3, Theorem 1.3, p. 97], $L_1 \cap L_\infty$ is a Banach space.

For every $E \subset [0,+\infty)$ let

$$1_E(t) = \begin{cases} 1, & \text{if } t \in E \\ 0, & \text{otherwise.} \end{cases}$$

Note that a family of functions $\{1_{[0,a]} : a > 0\}$ is uncountable and a distance between any two different functions is not less than 1. Therefore $L_1 \cap L_\infty$ is non-separable.

For $n \in \mathbb{N}$ let $R_n : L_1 \cap L_\infty \to \mathbb{C}$, $R_n(x) = \int_{[0,+\infty)} (x(t))^n\, dt$. Note that R_n is a symmetric n-homogeneous polynomial. Let us show that $\|R_n\| = 1$. Let $x \in L_1 \cap L_\infty$ be such that $\|x\| \le 1$. Then $\|x\|_1 \le 1$ and $\|x\|_\infty \le 1$. Since $\|x\|_\infty \le 1$, it follows that $|x(t)|^n \le |x(t)|$ for almost all $t \in [0, +\infty)$. Therefore,

$$|R_n(x)| \le \int_{[0,+\infty)} |x(t)|^n\, dt \le \int_{[0,+\infty)} |x(t)|\, dt = \|x\|_1 \le 1.$$

Hence, $\|R_n\| = \sup_{\|x\|\le 1} |R_n(x)| \le 1$. On the other hand, $\|1_{[0,1]}\| = 1$ and $R_n(1_{[0,1]}) = 1$. Therefore, $\|R_n\| = 1$ and, consequently, R_n is continuous.

Theorem 2. *Every symmetric continuous n-homogeneous polynomial $P : L_1[0, +\infty) \cap L_\infty[0, +\infty) \to \mathbb{C}$ can be uniquely represented as*

$$P(x) = \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1,\ldots,k_n} R_1^{k_1}(x) \cdots R_n^{k_n}(x),$$

where $k_1, \ldots, k_n \in \mathbb{Z}_+$ and $\alpha_{k_1,\ldots,k_n} \in \mathbb{C}$.

Proof. Let $P : L_1 \cap L_\infty \to \mathbb{C}$ be a continuous symmetric n-homogeneous polynomial. For $x \in L_1 \cap L_\infty$ let $\operatorname{supp} x = \{t \in [0, +\infty) : x(t) \ne 0\}$. For $a > 0$ let us denote X_a the subspace of all functions $x \in L_1 \cap L_\infty$ such that $\operatorname{supp} x \subset [0, a]$. Let us denote P_a the restriction of P to X_a. Note that X_1 is isometrically isomorphic to $L_\infty[0, 1]$. Therefore, by Theorem 1, there exist unique coefficients $\alpha_{k_1,\ldots,k_n} \in \mathbb{C}$ such that

$$P_1(x) = \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1,\ldots,k_n} \tilde{R}_1^{k_1}(x) \cdots \tilde{R}_n^{k_n}(x)$$

for every $x \in X_1$. For $a > 1$ let us define a mapping $J_a : X_1 \to X_a$ by $J_a(x)(t) = x(t/a)$. Evidently, J_a is a linear bijection. Note that $\|J_a(x)\|_1 = a\|x\|_1$ and $\|J_a(x)\|_\infty = \|x\|_\infty$, therefore $\|x\| \le \|J_a(x)\| \le a\|x\|$. Hence, J_a is an isomorphism. Let $G_a = P_a \circ J_a$. Note that G_a is a continuous symmetric n-homogeneous polynomial on X_1. Therefore, by Theorem 1, there exist coefficients $\alpha^{(a)}_{k_1,\ldots,k_n} \in \mathbb{C}$ such that

$$G_a(x) = \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha^{(a)}_{k_1,\ldots,k_n} \tilde{R}_1^{k_1}(x) \cdots \tilde{R}_n^{k_n}(x)$$

for every $x \in X_1$. Let $x = J_a^{-1}(y)$ for $y \in X_a$. Note that $G_a(x) = G_a(J_a^{-1}(y)) = P_a(J_a(J_a^{-1}(y))) = P_a(y)$ and

$$\tilde{R}_j(x) = \tilde{R}_j(J_a^{-1}(y)) = \int_{[0,1]} (y(at))^j \, dt = \frac{1}{a} \int_{[0,a]} (y(t))^j \, dt = \frac{1}{a} R_j(y).$$

Therefore

$$P_a(y) = \sum_{k_1+2k_2+\ldots+nk_n=n} \frac{\alpha^{(a)}_{k_1,\ldots,k_n}}{a^{k_1+\ldots+k_n}} R_1^{k_1}(y) \cdots R_n^{k_n}(y).$$

Note that the restriction of P_a to X_1 coincides with P_1. On the other hand, the restriction of R_j to X_1 coincides with $\tilde{R}_j$. Therefore, by the uniqueness of $\alpha_{k_1,\ldots,k_n}$, we have that $\frac{\alpha^{(a)}_{k_1,\ldots,k_n}}{a^{k_1+\ldots+k_n}} = \alpha_{k_1,\ldots,k_n}$. Hence, for every $a \geq 1$ and for every $y \in X_a$,

$$P_a(y) = \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1,\ldots,k_n} R_1^{k_1}(y) \cdots R_n^{k_n}(y). \tag{1}$$

Let E be a Lebesgue measurable subset of $[0, +\infty)$ such that $\mu(E) < +\infty$, where μ is the Lebesgue measure. For every $j \in \mathbb{N}$ let $E_j = [j-1, j) \cap E$ and $F_j = \tau_j(E_J)$, where $\tau_j(t) = t - (j-1)$. By [12, §2, No. 1–4], every measurable subset $F \subset [0,1]$ is isomorphic modulo zero to an interval of the length $\mu(F)$. Therefore, for every $j \in \mathbb{N}$ there exists $\sigma_j \in \Xi_{[0,1]}$ such that $\sigma_j(F_j) \stackrel{a.e.}{=} [0, \mu(F_j)]$ and $\sigma_j([0,1] \backslash F_j) \stackrel{a.e.}{=} [\mu(F_j), 1]$. Let us define a mapping $\sigma_E : [0, +\infty) \to [0, +\infty)$ by the following way: for $t \in [0, +\infty)$ such that $m - 1 \leq t < m$, where $m \in \mathbb{N}$, we set

$$\sigma_E(t) = \begin{cases} \sum_{k=1}^{m-1} \mu(E_k) + \sigma_m(\tau_m(t)), & \text{if } t \in E \\ \mu(E) + \sum_{k=1}^{m-1} (1 - \mu(E_k)) + \sigma_m(\tau_m(t)) - \mu(E_m), & \text{otherwise.} \end{cases}$$

It can be checked that $\sigma_E \in \Xi_{[0,+\infty)}$, $\sigma_E(E) \stackrel{a.e.}{=} [0, \mu(E)]$ and $\sigma_E([0,+\infty) \backslash E) \stackrel{a.e.}{=} [\mu(E), +\infty)$.

Let $y \in L_1 \cap L_\infty$ be such that $\mu(\text{supp } y) < +\infty$. Since P is symmetric, it follows that $P(y) = P(y \circ \sigma^{-1}_{\text{supp } y})$. Note that $y \circ \sigma^{-1}_{\text{supp } y} \in X_{\mu(\text{supp } y)} \subset X_{\max\{1, \mu(\text{supp } y)\}}$. Therefore, by (1),

$$\begin{aligned} P(y) &= \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1,\ldots,k_n} R_1^{k_1}(y \circ \sigma^{-1}_{\text{supp } y}) \cdots R_n^{k_n}(y \circ \sigma^{-1}_{\text{supp } y}) \\ &= \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1,\ldots,k_n} R_1^{k_1}(y) \cdots R_n^{k_n}(y). \end{aligned} \tag{2}$$

Let $x \in L_1 \cap L_\infty$. For every $m \in \mathbb{N}$ let

$$x_m(t) = \begin{cases} x(t), & \text{if } |x(t)| > \frac{1}{m}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $A_0 = \{t \in [0,+\infty) : |x(t)| > 1\}$ and $A_m = \{t \in [0,+\infty) : \frac{1}{m+1} < |x(t)| \leq \frac{1}{m}\}$ for $m \in \mathbb{N}$. Since $x \in L_1[0,+\infty)$, it follows that $\mu(A_m) < +\infty$ for every $m \in \mathbb{Z}_+$. Since the series

$$\|x\|_1 = \sum_{m=0}^{\infty} \int_{A_m} |x(t)|\, dt$$

is convergent, it follows that $\|x - x_j\|_1 = \sum_{m=j}^{\infty} \int_{A_m} |x(t)|\, dt \to 0$ as $j \to +\infty$. Note that $\|x - x_j\|_\infty \leq \frac{1}{j} \to 0$ as $j \to +\infty$. Hence, $\|x - x_j\| \to 0$, i. e. $x_j \to x$. Note that supp $x_j = \bigcup_{m=0}^{j-1} A_m$. Since $\mu(A_m) < +\infty$ for every $m \in \mathbb{Z}_+$, it follows that $\mu(\operatorname{supp} x_j) < +\infty$. Therefore, by (2),

$$P(x_j) = \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1,\ldots,k_n} R_1^{k_1}(x_j) \cdots R_n^{k_n}(x_j).$$

By the continuity of $R_1, \ldots, R_n$ and P,

$$P(x) = \lim_{j\to+\infty} P(x_j) = \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1,\ldots,k_n} R_1^{k_1}(x) \cdots R_n^{k_n}(x).$$

□

Let $R_0 : L_1 \cap L_\infty \to \mathbb{C}$, $R_0(x) = 1$.

Corollary 3. *$\{R_n\}_{n\in\mathbb{Z}_+}$ forms an algebraic basis in the algebra*

$$\mathcal{P}_s(L_1[0,+\infty) \cap L_\infty[0,+\infty)).$$

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