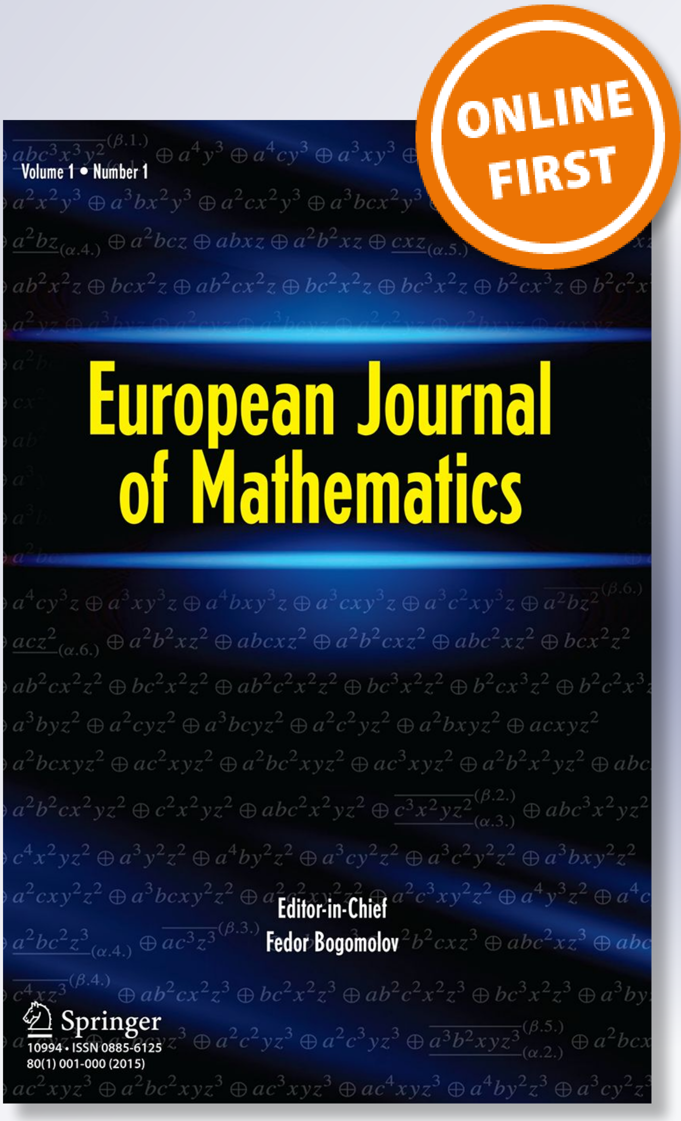
Symmetric polynomials on

Taras Vasylyshyn

European Journal of Mathematics

ISSN 2199-675X

European Journal of Mathematics
DOI 10.1007/s40879-018-0268-3



Springer

European Journal of Mathematics
https://doi.org/10.1007/s40879-018-0268-3

RESEARCH ARTICLE

Symmetric polynomials on $(L_p)^n$

Taras Vasylyshyn[1]

Received: 23 February 2018 / Revised: 19 May 2018 / Accepted: 23 June 2018

Abstract
We describe an algebraic basis of the algebra of symmetric continuous polynomials on the nth Cartesian power of the complex Banach space $L_p = L_p([0, 1])$, where $1 \leqslant p < +\infty$.

Keywords Polynomial · Symmetric polynomial · Block-symmetric polynomial · Algebraic basis

Mathematics Subject Classification 46G25 · 46E15 · 46E25 · 46G20

1 Introduction

Polynomials and analytic functions on a Banach space X invariant (symmetric) with respect to a group of operators acting on X were studied by a number of authors [1,2,4–12,14–19] (see also a survey [3]). For example, if X is a rearrangement-invariant sequence space, it is natural to consider the group of permutations of the natural basis of X isomorphic to the group of all bijections on the set of all positive integers. If X is a rearrangement-invariant function space of measurable functions on some measurable space Ω, it is natural to consider the group of operators acting as a composition of a function with some measure-preserving bijection on Ω. For the investigation of algebras of symmetric polynomials and symmetric analytic functions it is handy to know an algebraic basis (a set of elements of the algebra such that every element of the algebra can be uniquely represented as an algebraic combination of elements from this set) of the algebra (if such a basis exists), because every homomorphism on the algebra with an algebraic basis is completely determined by its values on the elements of the basis. Symmetric analytic functions and symmetric polynomials on ℓ_p and L_p were first studied by Nemirovski and Semenov in [14]. In particular, in [14]

✉ Taras Vasylyshyn
taras.v.vasylyshyn@gmail.com

1 Vasyl Stefanyk Precarpathian National University, 57 Shevchenka Str., Ivano-Frankivsk 76018, Ukraine

Published online: 30 July 2018

a description of algebraic bases of algebras of symmetric polynomials on ℓ_p and L_p was provided. In [9] González et al. generalized results of [14] and described algebraic bases of some algebras of symmetric polynomials on rearrangement-invariant separable Banach spaces. Algebras of symmetric polynomials and symmetric analytic functions on the complex Banach space $L_\infty([0,1])$ of essentially bounded measurable functions on $[0,1]$ were studied in [7,15,17,18]. In particular, in [7] the author together with Galindo and Zagorodnyuk described an algebraic basis of the algebra of symmetric continuous polynomials on $L_\infty([0,1])$. This result was applied for the description of the spectrum of the algebra of symmetric entire functions of bounded type on $L_\infty([0,1])$ in works [7,17,18]. Symmetric polynomials on the Cartesian powers of Banach spaces (which are also called "block-symmetric") were studied in [10–12,16].

In this work an algebraic basis of the algebra of all continuous symmetric polynomials on $(L_p)^n$ is described.

We denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{Z}_+$ the set of all nonnegative integers. Let X and Y be complex Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively. A mapping $A\colon X^m \to Y$, where $m \in \mathbb{N}$, is called an m-linear mapping if it is linear with respect to every of its m arguments separately. A mapping $P\colon X \to Y$ is called an m-homogeneous polynomial if there exists an m-linear mapping $A_P\colon X^m \to Y$ such that its restriction to the diagonal is equal to P, that is,

$$P(x) = A_P(\underbrace{x, \ldots, x}_{m}) \quad \text{for every} \quad x \in X.$$

The mapping A_P is called the m-linear mapping associated with P. Note that $P(\lambda x) = \lambda^m P(x)$ for every $\lambda \in \mathbb{C}$ and $x \in X$.

It is known that an m-homogeneous polynomial $P\colon X \to Y$ is continuous if and only if

$$\|P\| = \sup_{\|x\|_X \leqslant 1} \|P(x)\|_Y < +\infty.$$

Similarly, an m-linear mapping $A\colon X^m \to Y$ is continuous if and only if

$$\|A\| = \sup_{\|x_1\|_X \leqslant 1, \ldots, \|x_m\|_X \leqslant 1} \|A(x_1, \ldots, x_m)\|_Y < +\infty.$$

By [13, Theorem 2.2, p. 12], an m-homogeneous polynomial P is continuous if and only if the symmetric (with respect to the permutations of its arguments) m-linear mapping A_P associated with P is continuous.

A mapping $P = P_0 + P_1 + \cdots + P_N$, where $P_0 \in Y$ and P_j is a j-homogeneous polynomial for every $j \in \{1, \ldots, N\}$, is called a polynomial of degree at most N.

Let $p \in [1, +\infty)$ and $n \in \mathbb{N}$. Let $L_p = L_p([0,1])$ be the complex Banach space of functions $y\colon [0,1] \to \mathbb{C}$ for which the pth power of the absolute value is Lebesgue integrable, with norm

$$\|y\|_p = \left(\int_{[0,1]} |y(t)|^p \, dt\right)^{1/p}.$$

Let $(L_p)^n$ be the nth Cartesian power of L_p with norm

$$\|y\|_{p,n} = \left(\sum_{s=1}^{n} \int_{[0,1]} |y_s(t)|^p \, dt \right)^{1/p},$$

where $y = (y_1, \ldots, y_n) \in (L_p)^n$.

Let Ξ be the set of all bijections $\sigma\colon [0,1] \to [0,1]$ such that both σ and σ^{-1} are measurable and preserve the Lebesgue measure. A function $f\colon (L_p)^n \to \mathbb{C}$ is called symmetric if

$$f((y_1 \circ \sigma, \ldots, y_n \circ \sigma)) = f((y_1, \ldots, y_n))$$

for every $(y_1, \ldots, y_n) \in (L_p)^n$ and for every $\sigma \in \Xi$. Let us denote by $\mathcal{P}_s((L_p)^n)$ the algebra of all symmetric continuous complex-valued polynomials on $(L_p)^n$.

2 The main result

For every multi-index $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$ such that $1 \leqslant |k| \leqslant \lfloor p \rfloor$, where $|k| = k_1 + \cdots + k_n$ and $\lfloor p \rfloor$ is the integral part of p, let us define a mapping $R_k\colon (L_p)^n \to \mathbb{C}$ by

$$R_k(y) = \int_{[0,1]} \prod_{\substack{s=1 \\ k_s > 0}}^{n} (y_s(t))^{k_s} \, dt, \tag{1}$$

where $y = (y_1, \ldots, y_n) \in (L_p)^n$. Also we set $R_{(0,\ldots,0)}(y) \equiv 1$.

Theorem 2.1 *For $k \in \mathbb{Z}_+^n$ such that $1 \leqslant |k| \leqslant \lfloor p \rfloor$, the mapping R_k defined by (1) is a well-defined continuous symmetric $|k|$-homogeneous polynomial on $(L_p)^n$.*

Proof Clearly, R_k is symmetric. Let $B\colon (L_p)^{|k|} \to \mathbb{C}$ be defined by

$$B(x_1, \ldots, x_{|k|}) = \int_{[0,1]} x_1(t) \cdots x_{|k|}(t) \, dt.$$

Note that B is a $|k|$-linear mapping. Let $Q\colon L_p \to \mathbb{C}$ be the restriction of B to the diagonal. Since

$$|Q(x)| = \left| \int_{[0,1]} (x(t))^{|k|} dt \right| \leqslant \int_{[0,1]} |x(t)|^{|k|} dt = \|x\|_{|k|}^{|k|} \leqslant \|x\|_p^{|k|}$$

for every $x \in L_p$, it follows that

$$\|Q\| = \sup_{\|x\|_p \leqslant 1} |Q(x)| \leqslant \sup_{\|x\|_p \leqslant 1} \|x\|_p^{|k|} = 1.$$

Therefore, Q is continuous, hence B as well.

Note that

$$R_k(y) = B\big(\underbrace{y_1, \ldots, y_1}_{k_1}, \ldots, \underbrace{y_n, \ldots, y_n}_{k_n}\big), \tag{2}$$

where $y = (y_1, \ldots, y_n) \in (L_p)^n$. So R_k is well defined.

Note that for fixed $x, y \in (L_p)^n$ the function $\lambda \in \mathbb{C} \mapsto R_k(x + \lambda y) \in \mathbb{C}$ is a polynomial of degree at most $|k|$. Therefore, by [13, Theorem 3.6, p. 22], R_k is a polynomial of degree at most $|k|$. Also note that $R_k(\lambda y) = \lambda^{|k|} R_k(y)$ for every $\lambda \in \mathbb{C}$ and $y \in (L_p)^n$. Therefore, by [13, Exercise 2.C, p. 16], R_k is a $|k|$-homogeneous polynomial.

By (2),

$$|R_k(y)| \leqslant \|B\| \|y_1\|_p^{k_1} \cdots \|y_n\|_p^{k_n}$$

for every $y \in (L_p)^n$. Therefore, for every $y \in (L_p)^n$ such that $\|y\|_{p,n} \leqslant 1$, we have $|R_k(y)| \leqslant \|B\|$. Hence, $\|R_k\| \leqslant \|B\|$. Consequently, R_k is continuous. □

Lemma 2.2 *Let $P = P_0 + P_1 + \cdots + P_N$ be a symmetric continuous complex-valued polynomial on $(L_p)^n$, where $P_0 \in \mathbb{C}$ and P_j is a j-homogeneous polynomial for $j \in \{1, \ldots, N\}$. Then every P_j is symmetric and continuous, where $j \in \{0, \ldots, N\}$.*

Proof This is immediate from the Cauchy Integral Formula (see [13, Corollary 7.3, p. 47]) since $P_0 + P_1 + \cdots + P_N$ is the Taylor series of P at 0. □

Let M be a finite nonempty subset of $\mathbb{Z}_+^n$. Let $\mathbb{C}^M$ be the vector space of all mappings from M to $\mathbb{C}$. Note that every element $\xi \in \mathbb{C}^M$ can be considered as an $|M|$-dimensional complex vector $(\xi_k)_{k \in M}$, where $\xi_k = \xi(k)$ for $k \in M$ and $|M|$ is the cardinality of M. Therefore, $\mathbb{C}^M$ is isomorphic to $\mathbb{C}^{|M|}$. We endow the space $\mathbb{C}^M$ with norm

$$\|\xi\|_\infty = \max_{k \in M} |\xi_k|.$$

For $m \in \mathbb{N}$, let $c_{00}^{(m)}(\mathbb{C}^n)$ be the space of all sequences $x = (x_1, \ldots, x_m, 0, \ldots)$, where $x_j = \big(x_j^{(1)}, \ldots, x_j^{(n)}\big) \in \mathbb{C}^n$ for $j \in \{1, \ldots, m\}$, and $0 = (0, \ldots, 0) \in \mathbb{C}^n$. We endow the space $c_{00}^{(m)}(\mathbb{C}^n)$ with norm

$$\|x\|_{\ell_p} = \left(\sum_{j=1}^m \sum_{s=1}^n \left| x_j^{(s)} \right|^p \right)^{1/p}.$$

Note that $c_{00}^{(m)}(\mathbb{C}^n) \subset c_{00}^{(m+1)}(\mathbb{C}^n)$ for every $m \in \mathbb{N}$.

A function $f : c_{00}^{(m)}(\mathbb{C}^n) \to \mathbb{C}$ is called symmetric if

$$f((x_1, \ldots, x_m, 0, \ldots)) = f((x_{\tau(1)}, \ldots, x_{\tau(m)}, 0, \ldots))$$

for every $(x_1, \dots, x_m, 0, \dots) \in c_{00}^{(m)}(\mathbb{C}^n)$ and for every permutation $\tau\colon \{1, \dots, m\} \to \{1, \dots, m\}$.

For every $k \in \mathbb{Z}_+^n$, let $H_k^{(m)}\colon c_{00}^{(m)}(\mathbb{C}^n) \to \mathbb{C}$ be defined by

$$H_k^{(m)}(x) = \sum_{j=1}^{m} \prod_{\substack{s=1\\ k_s>0}}^{n} \big(x_j^{(s)}\big)^{k_s}.$$

Note that $H_k^{(m)}$ is a symmetric $|k|$-homogeneous polynomial. For an arbitrary nonempty finite set $M \subset \mathbb{Z}_+^n$ such that $|k| \geqslant 1$ for every $k \in M$, let us define a mapping $\pi_M^{(m)}\colon c_{00}^{(m)}(\mathbb{C}^n) \to \mathbb{C}^M$ by

$$\pi_M^{(m)}(x) = \big(H_k^{(m)}(x)\big)_{k\in M}$$

We will use following results, proved in [12].

Theorem 2.3 ([12, Theorem 6]) *Let M be a finite nonempty subset of $\mathbb{Z}_+^n$ such that $|k| \geqslant 1$ for every $k \in M$. Then*

(i) *there exists $m \in \mathbb{N}$ such that for every $\xi = (\xi_k)_{k\in M} \in \mathbb{C}^M$ there exists $x_\xi \in c_{00}^{(m)}(\mathbb{C}^n)$ with $\pi_M^{(m)}(x_\xi) = \xi$;*

(ii) *there exists a constant $\rho > 0$ such that if $\|\xi\|_\infty < 1$, then $\|x_\xi\|_{\ell_p} < \rho$ for every $p \in [1, +\infty)$.*

Theorem 2.4 ([12, Theorem 8]) *Every symmetric N-homogeneous polynomial $P\colon c_{00}^{(m)}(\mathbb{C}^n) \to \mathbb{C}$, where m is an arbitrary positive integer, can be represented as an algebraic combination of polynomials $H_k^{(m)}$, where $k \in \mathbb{Z}_+^n$ is such that $1 \leqslant |k| \leqslant N$.*

Lemma 2.5 ([12, Lemma 11]) *For $m \in \mathbb{N}$, let $K \subset \mathbb{C}^m$ and $\varkappa\colon K \to \mathbb{C}^{m-1}$ be the orthogonal projection: $\varkappa((x_1, x_2, \dots, x_m)) = (x_2, \dots, x_m)$. Let $K_1 = \varkappa(K)$. Suppose* $\operatorname{int} K_1 \neq \varnothing$ *and that for every open set $U \subset K_1$ the set $\varkappa^{-1}(U)$ is unbounded. If the polynomial $Q(x_1, \dots, x_m)$ is bounded on K, then Q does not depend on x_1.*

For every $E \subset [0, 1]$, let

$$1_E(t) = \begin{cases} 1, & \text{if } t \in E,\\ 0, & \text{otherwise.}\end{cases}$$

For $m \in \mathbb{N}$, let $J_m\colon c_{00}^{(m)}(\mathbb{C}^n) \to (L_p)^n$ be defined by

$$J_m(x) = \left(\sum_{j=1}^{m} x_j^{(1)} 1_{[(j-1)/m,\, j/m]}, \dots, \sum_{j=1}^{m} x_j^{(n)} 1_{[(j-1)/m,\, j/m]}\right)$$

for $x = (x_1, \dots, x_m, 0, \dots) \in c_{00}^{(m)}(\mathbb{C}^n)$. Note that J_m is a linear operator. Let us show that J_m is continuous.

Lemma 2.6 *For every $x \in c_{00}^{(m)}(\mathbb{C}^n)$,*

$$\|J_m(x)\|_{p,n} = \frac{1}{m^{1/p}}\,\|x\|_{\ell_p}.$$

Proof For $x = (x_1, \ldots, x_m, 0, \ldots) \in c_{00}^{(m)}(\mathbb{C}^n)$,

$$\begin{aligned}\|J_m(x)\|_{p,n}^p &= \sum_{s=1}^n \int_{[0,1]} \left|\sum_{j=1}^m x_j^{(s)} 1_{[(j-1)/m,\, j/m]}(t)\right|^p dt\\ &= \sum_{s=1}^n \int_{[0,1]} \sum_{j=1}^m \left|x_j^{(s)}\right|^p 1_{[(j-1)/m,\, j/m]}(t)\, dt\\ &= \sum_{s=1}^n \sum_{j=1}^m \left|x_j^{(s)}\right|^p \int_{[0,1]} 1_{[(j-1)/m,\, j/m]}(t)\, dt\\ &= \frac{1}{m}\sum_{s=1}^n \sum_{j=1}^m \left|x_j^{(s)}\right|^p = \frac{1}{m}\,\|x\|_{\ell_p}^p.\end{aligned}$$

Therefore, $\|J_m(x)\|_{p,n} = \|x\|_{\ell_p}/m^{1/p}$. □

Consequently, J_m is continuous. For $l \in \mathbb{N}$, let

$$D_l = J_{2^l}\big(c_{00}^{(2^l)}(\mathbb{C}^n)\big).$$

Note that J_{2^l} is a bijection between $c_{00}^{(2^l)}(\mathbb{C}^n)$ and D_l. Since J_{2^l} is linear and continuous, it follows that J_{2^l} is a continuous isomorphism between $c_{00}^{(2^l)}(\mathbb{C}^n)$ and D_l. Let

$$D = \bigcup_{l=1}^{\infty} D_l.$$

Note that D is dense in $(L_p)^n$. For every $k \in \mathbb{Z}_+^n$ such that $|k| \geqslant 1$, let us define a mapping $\widetilde{R}_k \colon D \to \mathbb{C}$ analogously to (1):

$$\widetilde{R}_k(y) = \int_{[0,1]} \prod_{\substack{s=1\\ k_s>0}}^n (y_s(t))^{k_s}\, dt,$$

where $y = (y_1, \ldots, y_n) \in D$.

Lemma 2.7 *For every $k \in \mathbb{Z}_+^n$ such that $|k| \geqslant 1$ and $l \in \mathbb{N}$,*

$$\widetilde{R}_k(J_{2^l}(x)) = \frac{1}{2^l}\, H_k^{(2^l)}(x) \quad \text{for every} \quad x \in c_{00}^{(2^l)}(\mathbb{C}^n).$$

Proof For $x = (x_1, \ldots, x_{2^l}, 0, \ldots) \in c_{00}^{(2^l)}(\mathbb{C}^n)$,

$$\begin{aligned}
\widetilde{R}_k(J_{2^l}(x)) &= \int_{[0,1]} \prod_{\substack{s=1\\k_s>0}}^{n} \left(\sum_{j=1}^{2^l} x_j^{(s)} 1_{[(j-1)/2^l, j/2^l]}(t) \right)^{k_s} dt \\
&= \int_{[0,1]} \prod_{\substack{s=1\\k_s>0}}^{n} \sum_{j=1}^{2^l} \left(x_j^{(s)}\right)^{k_s} 1_{[(j-1)/2^l, j/2^l]}(t)\, dt \\
&= \int_{[0,1]} \sum_{j=1}^{2^l} \prod_{\substack{s=1\\k_s>0}}^{n} \left(x_j^{(s)}\right)^{k_s} 1_{[(j-1)/2^l, j/2^l]}(t)\, dt \\
&= \sum_{j=1}^{2^l} \prod_{\substack{s=1\\k_s>0}}^{n} \left(x_j^{(s)}\right)^{k_s} \int_{[0,1]} 1_{[(j-1)2^l, j/2^l]}(t)\, dt \\
&= \frac{1}{2^l} \sum_{j=1}^{2^l} \prod_{\substack{s=1\\k_s>0}}^{n} \left(x_j^{(s)}\right)^{k_s} = \frac{1}{2^l} H_k^{(2^l)}(x).
\end{aligned}$$

□

Let $a, b \in [0, 1]$ be such that $a < b$ and that there exist $r_1, r_2, s \in \mathbb{Z}_+$ such that $a = r_1/2^s$ and $b = r_2/2^s$. Let us define a mapping $S_{[a,b]} \colon D \to D$ in the following way. For $y \in D$, we set

$$(S_{[a,b]}(y))(t) = \begin{cases} y\left(\dfrac{t-a}{b-a}\right), & \text{if } \ t \in [a, b], \\ (0, \ldots, 0), & \text{if } \ t \in [0, 1] \setminus [a, b]. \end{cases}$$

Lemma 2.8 *For every $y \in D$,*

$$\|S_{[a,b]}(y)\|_{p,n} = (b-a)^{1/p} \|y\|_{p,n}.$$

Proof For $y = (y_1, \ldots, y_n) \in D$,

$$\begin{aligned}
\|S_{[a,b]}(y)\|_{p,n}^p &= \sum_{s=1}^{n} \int_{[a,b]} \left| y_s\left(\frac{t-a}{b-a}\right) \right|^p dt \\
&= (b-a) \sum_{s=1}^{n} \int_{[0,1]} |y_s(\theta)|^p \, d\theta = (b-a) \|y\|_{p,n}^p.
\end{aligned}$$

Therefore, $\|S_{[a,b]}(y)\|_{p,n} = (b-a)^{1/p} \|y\|_{p,n}$. □

Lemma 2.9 *For every $y \in D$ and $k \in \mathbb{Z}_+^n$ such that $|k| \geqslant 1$,*

$$\widetilde{R}_k(S_{[a,b]}(y)) = (b-a)\,\widetilde{R}_k(y).$$

Proof For $y = (y_1, \ldots, y_n) \in D$,

$$\begin{aligned}\widetilde{R}_k(S_{[a,b]}(y)) &= \int_{[a,b]} \prod_{\substack{s=1\\k_s>0}}^{n} \left(y_s\left(\frac{t-a}{b-a}\right)\right)^{k_s} dt\\ &= (b-a)\int_{[0,1]} \prod_{\substack{s=1\\k_s>0}}^{n} (y_s(\theta))^{k_s}\, d\theta = (b-a)\,\widetilde{R}_k(y).\end{aligned}$$

□

Theorem 2.10 *Every N-homogeneous symmetric continuous polynomial $P\colon (L_p)^n \to \mathbb{C}$ can be uniquely represented as an algebraic combination of polynomials R_k, where $k \in \mathbb{Z}_+^n$ is such that $1 \leqslant |k| \leqslant \min\{\lfloor p \rfloor, N\}$.*

Proof Let Q be the restriction of P to D. Let

$$M_N = \{k \in \mathbb{Z}_+^n : 1 \leqslant |k| \leqslant N\}.$$

By Theorem 2.3, where we set $M = M_N$, there exist $m \in \mathbb{N}$ and $\rho > 0$ such that for every $\xi \in \mathbb{C}^{M_N}$ there exists $x_\xi \in c_{00}^{(m)}(\mathbb{C}^n)$ with $\pi_{M_N}^{(m)}(x_\xi) = \xi$ and if $\|\xi\|_\infty < 1$, then $\|x_\xi\|_{\ell_p} < \rho$. Hence, $\pi_{M_N}^{(m)}$ is surjective. For $m' \geqslant m$, the restriction of $H_k^{(m')}$ to $c_{00}^{(m)}(\mathbb{C}^n)$ is equal to $H_k^{(m)}$ for every $k \in \mathbb{Z}_+^n$ such that $|k| \geqslant 1$. Consequently, the restriction of $\pi_{M_N}^{(m')}$ to $c_{00}^{(m)}(\mathbb{C}^n)$ is equal to $\pi_{M_N}^{(m)}$. Since $\pi_{M_N}^{(m)}$ is surjective, it follows that $\pi_{M_N}^{(m')}$ is surjective too.

Let

$$l_0 = \lfloor \log_2 m \rfloor + 1. \tag{3}$$

Then $2^{l_0} \geqslant m$. Let $l \geqslant l_0$. Since J_{2^l} is continuous and linear, it follows that the mapping $Q \circ J_{2^l}$ is an N-homogeneous continuous polynomial on $c_{00}^{(2^l)}(\mathbb{C}^n)$. Also note that $Q \circ J_{2^l}$ is symmetric. Therefore, by Theorem 2.4, $Q \circ J_{2^l}$ can be represented as an algebraic combination of polynomials $H_k^{(2^l)}$, where $k \in M_N$. In other words, there exists a polynomial $q_l\colon \mathbb{C}^{M_N} \to \mathbb{C}$ such that

$$(Q \circ J_{2^l})(x) = q_l\big(\pi_{M_N}^{(2^l)}(x)\big)$$

for every $x \in c_{00}^{(2^l)}(\mathbb{C}^n)$. Since $\pi_{M_N}^{(2^l)}$ is surjective, it follows that such a polynomial q_l is unique.

For $y \in D_l$, let $x = J_{2^l}^{-1}(y)$. Then

$$(Q \circ J_{2^l})\big(J_{2^l}^{-1}(y)\big) = q_l\big(\pi_{M_N}^{(2^l)}\big(J_{2^l}^{-1}(y)\big)\big).$$

By Lemma 2.7, $\widetilde{R}_k(J_{2^l}(x)) = H_k^{(2^l)}(x)/2^l$. Therefore, $H_k^{(2^l)}\big(J_{2^l}^{-1}(y)\big) = 2^l \widetilde{R}_k(y)$. Consequently,

$$\pi_{M_N}^{(2^l)}\big(J_{2^l}^{-1}(y)\big) = \big(2^l \widetilde{R}_k(y)\big)_{k \in M_N}.$$

Thus, for every $y \in D_l$,

$$Q(y) = q_l\big(\big(2^l \widetilde{R}_k(y)\big)_{k \in M_N}\big).$$

For $a \in \mathbb{C}$, let $\gamma_a : \mathbb{C}^{M_N} \to \mathbb{C}^{M_N}$ be defined by

$$\gamma_a\left((\xi_k)_{k \in M_N}\right) = (a\xi_k)_{k \in M_N},$$

where $(\xi_k)_{k \in M_N} \in \mathbb{C}^{M_N}$. Let $\widetilde{q}_l = q_l \circ \gamma_{2^l}$. Then $\widetilde{q}_l$ is a polynomial on $\mathbb{C}^{M_N}$ and

$$Q(y) = \widetilde{q}_l\left((\widetilde{R}_k(y))_{k \in M_N}\right) \tag{4}$$

for every $y \in D_l$.

Let us show that $\widetilde{q}_{l_0} \equiv \widetilde{q}_{l_0+1} \equiv \cdots$. For $l > l_0$ we have $D_{l_0} \subset D_l$. Therefore, by (4),

$$\widetilde{q}_l\left((\widetilde{R}_k(y))_{k \in M_N}\right) = Q(y) = \widetilde{q}_{l_0}\left((\widetilde{R}_k(y))_{k \in M_N}\right) \tag{5}$$

for every $y \in D_{l_0}$. Let $\eta = (\eta_k)_{k \in M_N}$ be an arbitrary element of $\mathbb{C}^{M_N}$. For $\xi = \gamma_{2^{l_0}}(\eta)$ there exists $x_\xi \in c_{00}^{(m)}(\mathbb{C}^n)$ such that $\pi_{M_N}^{(m)}(x_\xi) = \xi$. Since $2^{l_0} \geqslant m$, it follows that $\pi_{M_N}^{(2^{l_0})}(x_\xi) = \xi$, that is, $H_k^{(2^{l_0})}(x_\xi) = \xi_k$ for every $k \in M_N$. Let $y_\xi = J_{2^{l_0}}(x_\xi)$. By Lemma 2.7,

$$\widetilde{R}_k(y_\xi) = \frac{1}{2^{l_0}} H_k^{(2^{l_0})}(x_\xi) = \frac{1}{2^{l_0}} \xi_k = \eta_k$$

for every $k \in M_N$. Hence, $(\widetilde{R}_k(y_\xi))_{k \in M_N} = \eta$. By (5), where we set $y = y_\xi$, we have $\widetilde{q}_l(\eta) = \widetilde{q}_{l_0}(\eta)$. Since this equality holds for every $\eta \in \mathbb{C}^{M_N}$, it follows that $\widetilde{q}_l \equiv \widetilde{q}_{l_0}$. Thus, $\widetilde{q}_{l_0} \equiv \widetilde{q}_{l_0+1} \equiv \cdots$. Let $q = \widetilde{q}_{l_0}$. By (4),

$$Q(y) = q\left((\widetilde{R}_k(y))_{k \in M_N}\right) \tag{6}$$

for every $y \in D$.

Consider the case $N > p$. Let $k_0 \in \mathbb{Z}_+^n$ be such that $\lfloor p \rfloor < |k_0| \leqslant N$. Let us show that the polynomial $q = q((\xi_k)_{k \in M_N})$ does not depend on ξ_{k_0}. We will use Lemma 2.5. Let

$$V = \left\{ y \in D : \|y\|_{p,n} < \frac{\rho}{2^{l_0/p}} \right\}$$

and

$$K = \nu(V),$$

where $\nu\colon D \to \mathbb{C}^{M_N}$ is defined by

$$\nu(y) = (\widetilde{R}_k(y))_{k \in M_N}$$

for $y \in D$. Let $\varkappa\colon \mathbb{C}^{M_N} \to \mathbb{C}^{M_N \setminus \{k_0\}}$ be the orthogonal projection, defined by

$$\varkappa\left((\xi_k)_{k \in M_N}\right) = (\xi_k)_{k \in M_N \setminus \{k_0\}}$$

for $(\xi_k)_{k \in M_N} \in \mathbb{C}^{M_N}$. Let $K_1 = \varkappa(K)$. Let us show that int $K_1 \neq \varnothing$. For an arbitrary $\xi = (\xi_k)_{k \in M_N} \in \mathbb{C}^{M_N}$ such that $\|\xi\|_\infty < 1$ there exists $x_\xi \in c_{00}^{(m)}(\mathbb{C}^n)$ such that $\|x_\xi\|_{\ell_p} < \rho$ and $\pi_{M_N}^{(m)}(x_\xi) = \xi$. Let $y_\xi = J_{2^{l_0}}(x_\xi)$, where l_0 is defined by (3). Then, by Lemma 2.6,

$$\|y_\xi\|_{p,n} = \frac{1}{2^{l_0/p}} \|x_\xi\|_{\ell_p} < \frac{\rho}{2^{l_0/p}}.$$

Thus, $y_\xi \in V$. By Lemma 2.7, taking into account that $\pi_{M_N}^{(m)}$ is the restriction of $\pi_{M_N}^{(2^{l_0})}$,

$$\nu(y_\xi) = \left(\widetilde{R}_k(y_\xi)\right)_{k \in M_N} = \left(\frac{1}{2^{l_0}} H_k^{(2^{l_0})}(x_\xi)\right)_{k \in M_N}$$
$$= \frac{1}{2^{l_0}} \pi_{M_N}^{(2^{l_0})}(x_\xi) = \frac{1}{2^{l_0}} \pi_{M_N}^{(m)}(x_\xi) = \frac{1}{2^{l_0}} \xi.$$

Since $y_\xi \in V$ and $K = \nu(V)$, it follows that $\xi/2^{l_0} \in K$. Thus, K contains the open ball

$$F = \left\{ \eta \in \mathbb{C}^{M_N} : \|\eta\|_\infty < \frac{1}{2^{l_0}} \right\}.$$

Since $\|\varkappa(\eta)\|_\infty \leqslant \|\eta\|_\infty$, it follows that $\varkappa(F)$ contains the open ball

$$F' = \left\{ \eta' \in \mathbb{C}^{M_N \setminus \{k_0\}} : \|\eta'\|_\infty < \frac{1}{2^{l_0}} \right\}.$$

Therefore, K_1 contains F'. Thus, int $K_1 \neq \varnothing$.

Let U be a nonempty open subset of K_1. Let us show that $\varkappa^{-1}(U)$ is unbounded. Since U is open and nonempty, there exist $\zeta' \in U$ and $\varepsilon > 0$ such that the open ball

$$B(\zeta', \varepsilon) = \left\{\eta' \in \mathbb{C}^{M_N \setminus \{k_0\}} : \|\eta' - \zeta'\|_\infty < \varepsilon\right\}$$

is contained in U. Since $\zeta' \in U \subset K_1$ and $K_1 = \varkappa(K)$, there exists $\zeta \in K$ such that $\varkappa(\zeta) = \zeta'$. Since $K = \nu(V)$, there exists $y_\zeta \in V$ such that $\nu(y_\zeta) = \zeta$.

Let $\delta = (\delta_k)_{k \in M_N} \in \mathbb{C}^{M_N}$ be defined by

$$\delta_k = \begin{cases} \frac{1}{2}, & \text{if } k = k_0, \\ 0, & \text{if } k \in M_N \setminus \{k_0\}. \end{cases}$$

Since $\|\delta\|_\infty = 1/2 < 1$, there exists $x_\delta \in c_{00}^{(m)}(\mathbb{C}^n)$ such that $\|x_\delta\|_{\ell_p} < \rho$ and $\pi_{M_N}^{(m)}(x_\delta) = \delta$, that is,

$$H_k^{(m)}(x_\delta) = \begin{cases} \frac{1}{2}, & \text{if } k = k_0, \\ 0, & \text{if } k \in M_N \setminus \{k_0\}. \end{cases}$$

Let $l \geqslant l_0$ be such that

$$2^l > \frac{1}{\varepsilon} \max_{k \in M_N \setminus k_0} |\zeta_k|. \tag{7}$$

Let $y_\delta = J_{2^l}(x_\delta)$. Then, by Lemma 2.7, $\widetilde{R}_k(y_\delta) = H_k^{(2^l)}(x_\delta)/2^l$. Since $2^l \geqslant m$ and $x_\delta \in c_{00}^{(m)}(\mathbb{C}^n)$, it follows that $H_k^{(2^l)}(x_\delta) = H_k^{(m)}(x_\delta)$. Therefore,

$$\widetilde{R}_k(y_\delta) = \begin{cases} \frac{1}{2^{l+1}}, & \text{if } k = k_0, \\ 0, & \text{if } k \in M_N \setminus \{k_0\} \end{cases} = \frac{1}{2^l} \delta_k.$$

Since $\widetilde{R}_{k_0}(y_\delta) \neq 0$, it follows that $\|y_\delta\|_{p,n} > 0$. Since $|k_0| > \lfloor p \rfloor$ and both $|k_0|$ and $\lfloor p \rfloor$ are integers, it follows that $|k_0| \geqslant \lfloor p \rfloor + 1$. Therefore, $|k_0| > p$. Consequently, $p/|k_0| < 1$. Therefore, there exists β such that

$$0 < \beta < \frac{1}{\|y_\delta\|_{p,n}} \frac{\rho}{2^{l_0/p}} \left(1 - \left(\frac{1}{2}\right)^{1 - p/|k_0|}\right)^{1/p}.$$

For $j \in \mathbb{N}$, let

$$z_j = S_{[1/2^l, 1]}(y_\zeta) + \sum_{s=1}^{j} \alpha_s S_{[1/2^{l+s}, 1/2^{l+s-1}]}(y_\delta),$$

where

$$\alpha_s = \beta \left(\frac{2^s}{s} \right)^{1/|k_0|}.$$

Let us show that $z_j \in V$. By Lemma 2.8,

$$\begin{aligned}\|z_j\|_{p,n}^p &= \|S_{[1/2^l,1]}(y_\zeta)\|_{p,n}^p + \sum_{s=1}^{j} \alpha_s^p \left\| S_{[1/2^{l+s},1/2^{l+s-1}]}(y_\delta) \right\|_{p,n}^p \\ &= \left(1 - \frac{1}{2^l}\right) \|y_\zeta\|_{p,n}^p + \sum_{s=1}^{j} \alpha_s^p \left(\frac{1}{2^{l+s-1}} - \frac{1}{2^{l+s}} \right) \|(y_\delta)\|_{p,n}^p.\end{aligned}$$

Note that $\|y_\zeta\|_{p,n} < \rho/2^{l_0/p}$, since $y_\zeta \in V$. Also note that

$$\begin{aligned}\sum_{s=1}^{j} \alpha_s^p \left(\frac{1}{2^{l+s-1}} - \frac{1}{2^{l+s}} \right) \|(y_\delta)\|_{p,n}^p &= \frac{1}{2^l} \|y_\delta\|_{p,n}^p \sum_{s=1}^{j} \frac{\alpha_s^p}{2^s} \\ &= \frac{1}{2^l} \|y_\delta\|_{p,n}^p \sum_{s=1}^{j} \frac{\beta^p (2^s/s)^{p/|k_0|}}{2^s} \\ &= \frac{1}{2^l} \beta^p \|y_\delta\|_{p,n}^p \sum_{s=1}^{j} \left(\left(\frac{1}{2}\right)^{1-p/|k_0|} \right)^s \frac{1}{s^{p/|k_0|}} \\ &\leqslant \frac{1}{2^l} \beta^p \|y_\delta\|_{p,n}^p \sum_{s=1}^{j} \left(\left(\frac{1}{2}\right)^{1-p/|k_0|} \right)^s \\ &< \frac{1}{2^l} \beta^p \|y_\delta\|_{p,n}^p \sum_{s=1}^{\infty} \left(\left(\frac{1}{2}\right)^{1-p/|k_0|} \right)^s \\ &= \frac{1}{2^l} \beta^p \|y_\delta\|_{p,n}^p \frac{1}{1-(1/2)^{1-p/|k_0|}} \\ &< \frac{1}{\|y_\delta\|_{p,n}^p} \frac{\rho^p}{2^{l_0}} \left(1 - \left(\frac{1}{2}\right)^{1-p/|k_0|}\right) \\ &\quad \cdot \frac{1}{2^l} \|y_\delta\|_{p,n}^p \frac{1}{1-(1/2)^{1-p/|k_0|}} \\ &= \frac{\rho^p}{2^{l_0+l}}.\end{aligned}$$

Therefore,

$$\|z_j\|_{p,n}^p < \left(1 - \frac{1}{2^l}\right) \frac{\rho^p}{2^{l_0}} + \frac{\rho^p}{2^{l_0+l}} = \frac{\rho^p}{2^{l_0}}.$$

Hence, $\|z_j\|_{p,n} < \rho/2^{l_0/p}$ and, consequently, $z_j \in V$. Therefore, $\nu(z_j) \in K$.

For every $k \in M_N$, by Lemma 2.9,

$$\begin{aligned}\widetilde{R}_k(z_j) &= \widetilde{R}_k\left(S_{[1/2^l,1]}(y_\zeta)\right) + \sum_{s=1}^{j} \alpha_s^{|k_0|} \widetilde{R}_k\left(S_{[1/2^{l+s},1/2^{l+s-1}]}(y_\delta)\right) \\ &= \left(1 - \frac{1}{2^l}\right) \widetilde{R}_k(y_\zeta) + \sum_{s=1}^{j} \alpha_s^{|k_0|} \left(\frac{1}{2^{l+s-1}} - \frac{1}{2^{l+s}}\right) \widetilde{R}_k(y_\delta) \\ &= \left(1 - \frac{1}{2^l}\right) \zeta_k + \sum_{s=1}^{j} \alpha_s^{|k_0|} \frac{1}{2^{2l+s}} \delta_k.\end{aligned} \tag{8}$$

Let us show that $\varkappa(z_j) \in B(\zeta', \varepsilon)$. For $k \in M_N \setminus \{k_0\}$, by (8),

$$\widetilde{R}_k(z_j) = \left(1 - \frac{1}{2^l}\right) \zeta_k.$$

Therefore, taking into account (7),

$$\begin{aligned}\left\| \left(\widetilde{R}_k(z_j)\right)_{k\in M_N\setminus\{k_0\}} - \zeta' \right\|_\infty &= \max_{k\in M_N\setminus\{k_0\}} \left| \left(1 - \frac{1}{2^l}\right) \zeta_k - \zeta_k \right| \\ &= \frac{1}{2^l} \max_{k\in M_N\setminus\{k_0\}} |\zeta_k| < \varepsilon.\end{aligned}$$

Thus, $\varkappa(z_j) \in B(\zeta', \varepsilon)$.

Let us show that $\varkappa^{-1}(B(\zeta', \varepsilon))$ is unbounded. By (8),

$$\widetilde{R}_{k_0}(z_j) = \left(1 - \frac{1}{2^l}\right) \zeta_{k_0} + \sum_{s=1}^{j} \alpha_s^{|k_0|} \frac{1}{2^{2l+s}} \frac{1}{2}.$$

Note that

$$\sum_{s=1}^{j} \alpha_s^{|k_0|} \frac{1}{2^{2l+s+1}} = \frac{1}{2^{2l+1}} \sum_{s=1}^{j} \frac{\alpha_s^{|k_0|}}{2^s} = \frac{\beta}{2^{2l+1}} \sum_{s=1}^{j} \frac{1}{s}.$$

Therefore, $\widetilde{R}_{k_0}(z_j) = (1 - 1/2^l)\zeta_{k_0} + \beta/2^{2l+1} \cdot \sum_{s=1}^{j} 1/s \to \infty$ as $j \to +\infty$. Thus, $\varkappa^{-1}(B(\zeta', \varepsilon))$ is unbounded. Since $B(\zeta', \varepsilon) \subset U$, it follows that $\varkappa^{-1}(U)$ is unbounded too.

Note that

$$\sup_{y\in V} |Q(y)| = \sup_{\substack{y\in D \\ \|y\|_{p,n}<\rho/2^{l_0/p}}} |P(y)| \leqslant \sup_{\substack{y\in (L_p)^n \\ \|y\|_{p,n}<\rho/2^{l_0/p}}} |P(y)| = \left(\frac{\rho}{2^{l_0/p}}\right)^N \|P\|.$$

Thus, Q is bounded on V. Consequently, q is bounded on K. By Lemma 2.5, q does not depend on ξ_{k_0}. Therefore, in the case $N > p$, the polynomial q does not depend on ξ_k such that $\lfloor p \rfloor < |k| \leqslant N$.

Thus, in the general case, q depends only on ξ_k such that $k \in M'$, where

$$M' = \left\{ k \in \mathbb{Z}_+^n : 1 \leqslant |k| \leqslant \min\{\lfloor p \rfloor, N\} \right\}.$$

Let $\widehat{q}$ be the restriction of q to $\mathbb{C}^{M'}$. By (6),

$$Q(y) = \widehat{q}\left((\widetilde{R}_k(y))_{k \in M'} \right)$$

for every $y \in D$. Let us show that

$$P(y) = \widehat{q}\,((R_k(y))_{k \in M'})$$

for every $y \in (L_p)^n$. Since D is dense in $(L_p)^n$, for every $y \in (L_p)^n$ there exists a sequence $\{y_j\}_{j=1}^{+\infty} \subset D$ that converges to y. Since P is continuous,

$$P(y) = \lim_{j \to +\infty} Q(y_j) = \lim_{j \to +\infty} \widehat{q}\left((\widetilde{R}_k(y_j))_{k \in M'} \right).$$

Since $\widehat{q}$ is a polynomial on a finite-dimensional space, it follows that it is continuous. Therefore,

$$\lim_{j \to +\infty} \widehat{q}\left((\widetilde{R}_k(y_j))_{k \in M'} \right) = \widehat{q}\Big(\big(\lim_{j \to +\infty} \widetilde{R}_k(y_j) \big)_{k \in M'} \Big).$$

Note that $\widetilde{R}_k$ is the restriction of R_k. Therefore, $\widetilde{R}_k(y_j) = R_k(y_j)$. By Theorem 2.1, R_k is continuous. Therefore,

$$\lim_{j \to +\infty} \widetilde{R}_k(y_j) = \lim_{j \to +\infty} R_k(y_j) = R_k(y).$$

Thus, $P(y) = \widehat{q}\,((R_k(y))_{k \in M'})$ for every $y \in (L_p)^n$. □

Corollary 2.11 *The set of polynomials* $\{R_k : k \in \mathbb{Z}_+^n,\ 0 \leqslant |k| \leqslant \lfloor p \rfloor\}$ *is an algebraic basis of the algebra* $\mathcal{P}_{\mathrm{s}}((L_p)^n)$.

Proof Let $P = P_0 + P_1 + \cdots + P_N$ be a symmetric continuous complex-valued polynomial on $(L_p)^n$, where $P_0 \in \mathbb{C}$ and P_j is a j-homogeneous polynomial for $j \in \{1, \ldots, N\}$. Note that $P_0 = P_0 R_{(0,\ldots,0)}$. By Lemma 2.2, for every $j \in \{1, \ldots, N\}$, the polynomial P_j is a symmetric continuous j-homogeneous polynomial. Therefore, by Theorem 2.10, P_j can be uniquely represented as an algebraic combination of polynomials R_k, where $k \in \mathbb{Z}_+^n$ is such that $1 \leqslant |k| \leqslant \min\{\lfloor p \rfloor, j\}$. Consequently, P can be represented as a sum of these algebraic combinations. □

References

1. Alencar, R., Aron, R., Galindo, P., Zagorodnyuk, A.: Algebras of symmetric holomorphic functions on ℓ_p. Bull. London Math. Soc. **35**(2), 55–64 (2003)
2. Aron, R., Galindo, P., Pinasco, D., Zalduendo, I.: Group-symmetric holomorphic functions on a Banach space. Bull. London Math. Soc. **48**(5), 779–796 (2016)
3. Chernega, I.: Symmetric polynomials and holomorphic functions on infinite dimensional spaces. J. Vasyl Stefanyk Precarpathian Natl. Univ. **2**(4), 23–49 (2015)
4. Chernega, I., Galindo, P., Zagorodnyuk, A.: Some algebras of symmetric analytic functions and their spectra. Proc. Edinburgh Math. Soc. **55**(1), 125–142 (2012)
5. Chernega, I., Galindo, P., Zagorodnyuk, A.: The convolution operation on the spectra of algebras of symmetric analytic functions. J. Math. Anal. Appl. **395**(2), 569–577 (2012)
6. Chernega, I., Galindo, P., Zagorodnyuk, A.: A multiplicative convolution on the spectra of algebras of symmetric analytic functions. Rev. Mat. Complut. **27**(2), 575–585 (2014)
7. Galindo, P., Vasylyshyn, T., Zagorodnyuk, A.: The algebra of symmetric analytic functions on L_∞. Proc. Roy. Soc. Edinburgh Sect. A **147**(4), 743–761 (2017)
8. Galindo, P., Vasylyshyn, T., Zagorodnyuk, A.: Symmetric and finitely symmetric polynomials on the spaces ℓ_∞ and $L_\infty[0, +\infty)$. Math. Nachr. (2018). https://doi.org/10.1002/mana.201700314
9. González, M., Gonzalo, R., Jaramillo, J.A.: Symmetric polynomials on rearrangement invariant function spaces. J. London Math. Soc. **59**(2), 681–697 (1999)
10. Kravtsiv, V.: The analogue of Newton's formula for block-symmetric polynomials. Int. J. Math. Anal. **10**(7), 323–327 (2016)
11. Kravtsiv, V.V., Zagorodnyuk, A.V.: Representation of spectra of algebras of block-symmetric analytic functions of bounded type. Carpathian Math. Publ. **8**(2), 263–271 (2016)
12. Kravtsiv, V., Vasylyshyn, T., Zagorodnyuk, A.: On algebraic basis of the algebra of symmetric polynomials on $\ell_p(\mathbb{C}^n)$. J. Funct. Spaces **2017**, # 4947925 (2017)
13. Mujica, J.: Complex Analysis in Banach Spaces. Dover, Mineola (2010)
14. Nemirovskii, A.S., Semenov, S.M.: On polynomial approximation of functions on Hilbert space. Mat. Sb. (N.S.) **92(134)**, 257–281 (1973)
15. Vasylyshyn, T.V.: Symmetric continuous linear functionals on complex space $L_\infty[0, 1]$. Carpathian Math. Publ. **6**(1), 8–10 (2014)
16. Vasylyshyn, T.V.: Continuous block-symmetric polynomials of degree at most two on the space $(L_\infty)^2$. Carpathian Math. Publ. **8**(1), 38–43 (2016)
17. Vasylyshyn, T.V.: Topology on the spectrum of the algebra of entire symmetric functions of bounded type on the complex L_∞. Carpathian Math. Publ. **9**(1), 22–27 (2017)
18. Vasylyshyn, T.V.: Metric on the spectrum of the algebra of entire symmetric functions of bounded type on the complex L_∞. Carpathian Math. Publ. **9**(2), 198–201 (2017)
19. Vasylyshyn, T.: Symmetric polynomials on the space of bounded integrable functions on the semi-axis. Int. J. Pure Appl. Math. **117**(3), 425–430 (2017)