Oleh Lopushansky
Andriy Zagorodnyuk

Infinite Dimensional Holomorphy

Spectra and Hilbertian Structures

AGH

Oleh Lopushansky
Andriy Zagorodnyuk

Infinite Dimensional Holomorphy

Spectra and Hilbertian Structures

AGH UNIVERSITY OF SCIENCE AND TECHNOLOGY PRESS KRAKOW 2013

Published by AGH University of Science and Technology Press

Reviewers:

prof. dr hab. Anatolij Plichko

prof. dr hab. Michał Zariczny

Author of the monograph is an employee of:

Oleh Lopushansky, University of Rzeszów

Andriy Zagorodnyuk, Precarpathian National University and Institute
for Applied Problems of Mechanics and Mathematics
Ukrainian National Academy of Sciences, Ukraine

Desktop publishing: *Oleh Lopushansky* and *Andriy Zagorodnyuk*

Cover: *Zofia Łucka*

ISBN 978-83-7464-643-7

Wydawnictwa AGH
al. A. Mickiewicza 30, 30-059 Kraków
tel. 12 617 32 28, tel./fax 12 636 40 38
e-mail: redakcja@wydawnictwoagh.pl
www.wydawnictwa.agh.edu.pl

Contents

Introduction 2

Chapter 1. Background on infinite-dimensional holomorphy 3
1.1. Continuous polynomials and symmetric tensor products 3
1.2. Linear subspaces in zeros of polynomials 8
1.3. Analytic functions 9
1.4. The Aron-Berner extension 12
1.5. Concept of regularity 14
1.6. Hilbert-Schmidt polynomials 16
1.7. Reproducing kernels 20

Chapter 2. Descriptions of topological spectra and applications 22
2.1. Spectra of algebras of polynomials 23
2.2. Applications for symmetric polynomials 29
2.3. Polynomials on tensor products 31
2.4. The spectrum of $\mathcal{H}_b(X)$ endowed with the Gelfand topology 34
2.5. The Gelfand transformation and linear structures on M_b 42
2.6. Linearity of topologies on spectra 47
2.7. Discontinuous complex homomorphisms and Michael's problem 50
2.8. Continuous homomorphisms 51
2.9. Continuous derivations 53
2.10. Ball algebras of analytic functions 55
2.11. C^*-algebras of continuous functions 57

Chapter 3. Hardy spaces associated with topological groups 59
3.1. Hardy spaces on compact infinite-dimensional group orbits 60
3.2. Symmetric Fock spaces associated with matrix unitary groups 74
3.3. Hardy spaces associated with infinite-dimensional unitary groups 86
3.4. Hardy spaces on irreducible orbits of locally compact groups 98

Chapter 4. Reproducing kernel spaces of analytic functions 111
4.1. Abstract Hardy spaces 111
4.2. Hilbert-Schmidt analytic functions 123
4.3. Holomorphicity and generalized symmetric Fock spaces 130

Bibliography 138

Introduction

This book is intended to be a survey of recent authors' results in infinite-dimensional holomorphy.

During last years, various spaces and algebras of analytic functions on Banach spaces have been considered by a number of authors. The interplay between function theory on infinite-dimensional domains, geometric properties of Banach spaces, and Banach and Fréchet algebras lead us to investigation of uniform algebras of analytic functions, their spectra (sets of characters), homomorphisms and derivations. The spectra of such algebras was studied first in [**6**], [**7**], [**9**], [**44**], [**77**] by R. Aron, B. Cole, T. Gamelin, P. Galindo, D. García, M. Maestre, J. Mujica and others. In particular, in [**6**] it is shown that the spectrum of the algebra of bounded type entire functions on a Banach space X contains the second dual X'' as point-evaluation functionals of the Aron-Berner extension of entire functions. However, X'' does not exhaust the spectrum in general (see e.g. [**9**]).

Next vital topic related to infinite-dimensional holomorphy is construction Hardy type spaces of analytic functions with infinitely many variables. Integral representations of such Hardy spaces was studied in [**30, 83, 80**] by B. Cole, T. Gamelin, B. Ørted, K. Neeb, D. Pinasco, I. Zalduendo and others. The main difficulty is that, in the general case, we have no good enough invariant measure on a ball in a Banach space.

The book is organized as follows. Chapter 1 contains a background on infinite-dimensional complex analysis. Chapter 2 is devoted to description of spectra of algebras of entire functions on Banach spaces using an approach developed in [**96**], [**97**]. In Chapter 3 we consider Hilbertian Hardy type spaces associated with various infinite-dimensional groups of isometric operators on a separable Hilbert space and applications to symmetric Fock spaces. It presents an approach developed in [**60**], [**61**], [**65**]. In Chapter 4 we give a general construction of reproducing kernel Hilbert space of analytic functions on ℓ_2 and their relations to abstract Fock spaces. It mainly summarizes the works [**56**], [**59**].

Oleh Lopushansky and Andriy Zagorodnyuk,

Rzeszów – Ivano-Frankivsk,

September 2013

CHAPTER 1

Background on infinite-dimensional holomorphy

We begin with a brief summary of important notions and facts used in the theory of polynomials and analytic functions with infinite-dimensional domains in Banach and Hilbert spaces; some with proofs, some without.

1.1. Continuous polynomials and symmetric tensor products

Let X and Y be complex Banach vector spaces. For every positive integer numbers $n, m \in \mathbb{N}$ let X^nY^m will denote the Cartesian product of n copies of X and m copies of Y, and x^ny^m will denote the element $(x, \ldots, x, y, \ldots, y)$ from X^nY^m.

For $n \in \mathbb{N}$ we denote by $\mathscr{L}(^nX, Y)$ the vector space of all continuous n-linear mappings F from X to Y endowed with the norm of uniform convergence on the unit ball of X^n. An n-linear mapping F is called symmetric if

$$F(x_1, \ldots, x_n) = F\left(x_{s(1)}, \ldots, x_{s(n)}\right), \qquad s \in \mathfrak{S}_n,$$

where $\mathfrak{S}_n$ means all permutations

$$s \colon \{1, \ldots, n\} \longmapsto \{s(1), \ldots, s(n)\}.$$

The subspace in $\mathscr{L}(^nX, Y)$ of all continuous symmetric n-linear maps will be denoted by $\mathscr{L}_s(^nX, Y)$. Clearly, $\mathscr{L}(^nX, Y)$ and $\mathscr{L}_s(^nX, Y)$ are Banach spaces. Further in the previous notations we will not write the index $n = 1$. In particular, $\mathscr{L}(X)$ denotes the algebra of all continuous linear operators and $\mathscr{L}(X, \mathbb{C}) := X'$ denotes the dual space of X.

Definition 1.1. Let us denote by Δ_n the natural embeddings called *diagonal mappings* from X to X^n defined as

$$\Delta_n \colon X \longrightarrow X^n$$
$$x \longmapsto (x, \ldots, x).$$

A mapping P from X to Y is called a continuous n-homogeneous polynomial if

$$P(x) = (F \circ \Delta_n)(x) \quad \text{for some} \quad F \in \mathscr{L}(^nX, Y).$$

Let $\mathcal{P}(^nX, Y)$ denote the vector space of all continuous n-homogeneous polynomials endowed with the norm of uniform convergence on the unit ball B of X, i.e.,

$$\|P\| = \sup_{x \in B} \|P(x)\|$$

with $P \in \mathcal{P}(^nX, Y)$.

THEOREM 1.2. *The map*

$$\begin{aligned} \mathscr{L}_s(^nX, Y) &\longrightarrow \mathcal{P}(^nX, Y) \\ F &\longmapsto F \circ \Delta_n \end{aligned}$$

is an isomorphism between the Banach space $\mathscr{L}_s(^nX, Y)$ *and the normed space* $\mathcal{P}(^nX, Y)$, *and*

$$\|F \circ \Delta_n\| \le \|F\| \le \frac{n^n}{n!}\|F \circ \Delta_n\|. \tag{1.1}$$

PROOF. The main tool of the proof is *the polarization formula* (see [**35**, p. 8]):

$$F(x_1, \dots, x_n) = \frac{1}{2^n n!} \sum_{\delta_i = \pm 1} \delta_1 \dots \delta_n F \circ \Delta_n \Big(\sum_{j=1}^{n} \delta_i x_j \Big). \tag{1.2}$$

By the polarization formula

$$\begin{aligned} \|F\| &\le \frac{1}{2^n n!} \sum_{1 \le i \le n} \sum_{\delta_i = \pm 1} \sup_{\|x_i\| \le 1} \Big\| F \circ \Delta_n \Big(\sum_{j=1}^{n} \delta_i x_i \Big) \Big\| \\ &= \frac{n^n}{2^n n!} \sum_{1 \le i \le n} \sum_{\delta_i = \pm 1} \sup_{\|x_i\| \le 1} \Big\| F \circ \Delta_n \Big(\frac{1}{n} \sum_{j=1}^{n} \delta_i x_j \Big) \Big\| \le \frac{n^n}{n!}\|F \circ \Delta_n\|. \end{aligned}$$

The left-hand side of inequality (1.1) is trivial. □

COROLLARY 1.3. $\mathcal{P}(^nX, Y)$ is a Banach space and for any $P \in \mathcal{P}(^nX, Y)$ there is a unique n-linear symmetric map $F_P \in \mathscr{L}_s(^nX, Y)$, so-called the *associated with* P *n-linear map*, such that $P = F_P \circ \Delta_n$.

For a positive integer n and a Banach space X let

$$c(n, X) := \inf\Big\{ M > 0\colon \|F\| \le M\|F \circ \Delta_n\| \text{ for all } F \in \mathscr{L}_s(^nX, Y) \Big\}. \tag{1.3}$$

We call $c(n, X)$ the nth *polarization constant* of X. From (1.1) it follows that

$$1 \le c(n, X) \le \frac{n^n}{n!}. \tag{1.4}$$

It is well known that $c(n, \ell_1) = n^n/n!$ and $c(n, \ell_2) = 1$ (see [**36**, p. 45] for details).

We say that a class $\mathcal{F}(X, Y)$ of nonlinear mappings from X to Y *admits a linearization* if there is a vector space W_X and an injective map

$$J_{\mathcal{F}(X,Y)}\colon X \longrightarrow W_X$$

such that for any $F \in \mathcal{F}(X, Y)$ there is a linear operator $L_F \in \mathscr{L}(W_X, Y)$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad F \quad} & Y \\ {\scriptstyle J_{\mathcal{F}(X,Y)}} \big\downarrow & \nearrow {\scriptstyle L_F} & \\ W_X & & \end{array} \tag{1.5}$$

commutes. The map $J_{\mathcal{F}(X,Y)}$ is called *the canonical map* associated with the linearization.

PROPOSITION 1.4. The space $\mathscr{L}(^nX,Y)$ admits a linearization.

PROOF. Denote by $X^{(n)}$ the vector space of all formal finite sums

$$\sum_{i_1,\dots,i_n\in\mathbb{N}} \lambda_{i_1,\dots,i_n}(x_{i_1},\dots,x_{i_n}), \qquad \lambda_{i_1,\dots,i_n}\in\mathbb{C},\ (x_{i_1},\dots,x_{i_n})\in X^n.$$

Let $X^{(0)}$ denote the subspace of $X^{(n)}$ generated by the elements

$$(x_{i_1},\dots,x_{i_k}+x_{i'_k},\dots,x_{i_n})-(x_{i_1},\dots,x_{i_k},\dots,x_{i_n})-(x_{i_1},\dots,x_{i'_k},\dots,x_{i_n}),$$
$$(x_{i_1},\dots,\lambda x_{i_k},\dots,x_{i_n})-\lambda(x_{i_1},\dots,x_{i_k},\dots,x_{i_n}), \qquad 1\le k\le n,\ \lambda\in\mathbb{C}.$$

Define the n-fold *tensor product* $\otimes^n X$ of X with itself, as the quotient vector space $X^{(n)}/X^{(0)}$. Let us put $x_{i_1}\otimes\dots\otimes x_{i_n}:=(x_{i_1},\dots,x_{i_n})+X^{(0)}$ and denote by J_n the n-linear mapping from X^n into $\otimes^n X$ such that

$$J_n\colon (x_{i_1},\dots,x_{i_n})\longmapsto x_{i_1}\otimes\dots\otimes x_{i_n}.$$

For any n-linear mapping $F\in\mathscr{L}(^nX,Y)$,

$$J_n^*(F)\Big(\sum_{i_1,\dots,i_n\in\mathbb{N}}\lambda_{i_1,\dots,i_n}(x_{i_1}\otimes\dots\otimes x_{i_n})\Big):=\sum_{i_1,\dots,i_n}\lambda_{i_1,\dots,i_n}F(x_{i_1},\dots,x_{i_n}),$$

where the sum is finite. Hence the linear map J_n^* is well defined on $\otimes^n X$ and

$$J_n^*(F)(x_{i_1}\otimes\dots\otimes x_{i_n})=F(x_{i_1},\dots,x_{i_n}).$$

Thus, if $\mathcal{F}(X,Y)=\mathscr{L}(^nX,Y)$, then $L_F=J_n^*(F)$ and $J_{\mathcal{F}(X,Y)}=J_n$. Now in (1.5) we can put $W_X=\otimes^n X$. □

THEOREM 1.5. *The space $\mathscr{L}(^nX,Y)$ is isometrically isomorphic to the space $\mathscr{L}(\otimes^n_\pi X,Y)$ of linear continuous operators from the* projective tensor product $\otimes^n_\pi X$ *to the space Y, where $\otimes^n_\pi X$ means the completion of $\otimes^n X$ by the projective tensor norm*

$$\|w\|=\inf\Big\{\sum_{i_1,\dots,i_n\in\mathbb{N}}\|x_{i_1}\|\dots\|x_{i_n}\|\colon w=\sum_{i_1,\dots,i_n\in\mathbb{N}}x_{i_1}\otimes\dots\otimes x_{i_n}\in\otimes^n X\Big\},$$

where the infimum is taken over all above representations of $w\in\otimes^n X$.

Let us define the symmetric tensor product $\odot^n X$ of X to itself as the subspace of $\otimes^n X$ generated by the vectors

$$x_1\odot\dots\odot x_n:=\frac{1}{n!}\sum_{s\in\mathfrak{S}_n}x_{s(1)}\otimes\dots\otimes x_{s(n)},$$

where $x_i\in X$ and $\mathfrak{S}_n$ is the group of permutations of the set $\{1,\dots,n\}$. For tensor powers of vectors we will use the short notation

$$x^{\otimes n}:=\overbrace{x\otimes\ldots\otimes x}^{n\text{ times}}\in\odot^n X, \qquad x\in X.$$

The symmetric projective tensor product $\odot^n_\pi X$ we define as the closure of $\otimes^n X$ in the space $\otimes^n_\pi X$.

PROPOSITION 1.6. The space $\odot_\pi^n X$ is complemented in $\otimes_\pi^n X$ and the map

$$S_n\Big(\sum_{i_1,\dots,i_n\in\mathbb{N}} x_{i_1}\otimes\cdots\otimes x_{i_n}\Big) = \sum_{i_1,\dots,i_n\in\mathbb{N}} x_{i_1}\odot\cdots\odot x_{i_n}$$

is a continuous projection.

COROLLARY 1.7. The following isomorphism holds

$$\mathscr{L}(\odot_\pi^n X, Y)\simeq \mathscr{L}_s(^n X, Y).$$

From the polarization formula (1.2) and Corollary 1.7 it follows that

$$x_1\odot\cdots\odot x_n = \frac{1}{2^n}\sum_{1\le i\le n}\sum_{\delta_i=\pm1}\delta_1\dots\delta_n\Big(\sum_{j=1}^n \delta_i x_i\Big)^{\otimes n} \tag{1.6}$$

(see e.g. [**41**, 1.5]). Therefore for each vector $w_n\in\odot^n X$ there are representations $w_n=\sum_i u_i^{\otimes n}$ with $u_i\in X$, for which we define an equivalent norm with the projective tensor norm,

$$|\!|\!|w_n|\!|\!| := \inf\Big\{\sum_{i\in\mathbb{N}}\|u_i\|^n : w_n=\sum_{i\in\mathbb{N}} u_i^{\otimes n}\in\odot^n X\Big\}, \tag{1.7}$$

where the infimum is taken over all such representations. Then for any mapping $F\in\mathscr{L}_s(^n X, Y)$,

$$\|F\| = \sup_{|\!|\!|w_n|\!|\!|\le1}\|J_n^*(F)(w_n)\| = \|F\circ\Delta_n\|.$$

Thus we have proved the following theorem.

THEOREM 1.8. *There is an equivalent norm $|\!|\!|\cdot|\!|\!|$ on $\odot_\pi^n X$ such that the space $\mathscr{L}((\odot_\pi^n X, |\!|\!|\cdot|\!|\!|), Y)$ is isometric to $\mathcal{P}(^n X, Y)$ for every Banach space Y. In particular, the following isometry holds*

$$(\odot_\pi^n X, |\!|\!|\cdot|\!|\!|)'\simeq\mathcal{P}^n(X).$$

From the polarization inequality (1.4) and formula (1.7) we have the next polarization inequality for tensor products:

$$\|w\|\le|\!|\!|w|\!|\!|\le c(n,X)\|w\|,\qquad w\in\odot_\pi^n X. \tag{1.8}$$

A map $P\colon X\longrightarrow Y$ is said to be a *continuous polynomial of degree* n if

$$P=P_0+P_1+\dots+P_n$$

with

$$P_0\in Y,\quad P_k\in\mathcal{P}(^k X, Y),\quad P_n\ne0.$$

The space of all continuous polynomials from X into Y will be denoted by $\mathcal{P}(X,Y)$. Usually $\mathcal{P}(X,Y)$ endowed with the uniform norm

$$\|P\| = \sup_{\|x\|\le1}\|P(x)\|.$$

We will use notations $\mathcal{P}(^{\le n}X, Y)$ and $\mathcal{P}(^{\le n}X)$ for spaces of Y-valued and $\mathbb{C}$-valued respectively, all continuous m-degree polynomials on X with $m\le n$. We will shortly denote the spaces $\mathcal{P}(^k X,\mathbb{C})$ and $\mathcal{P}(X,\mathbb{C})$ by $\mathcal{P}(^k X)$ and $\mathcal{P}(X)$ respectively.

Note that $\mathcal{P}(X)$ is a topological algebra with the locally convex topology of uniform convergence on bounded sets.

We will denote by

$$\mathcal{P}_w(X,\cdot), \quad \mathcal{P}_w(^nX,\cdot), \quad \mathcal{P}_w(^{\leq n}X,\cdot)$$

the spaces of all polynomials, n-homogeneous, and m-degree polynomials respectively, which are continuous on bounded subsets of X endowed with the weak topology $w(X,X')$.

An element $P \in \mathcal{P}(X)$ is called a *polynomial of finite type* if it is a finite sum of finite products of linear functionals. More generally, if $P \in \mathcal{P}(X,Y)$, then P is a polynomial of finite type if for every linear functional $h \in Y'$ the composition $h \circ P$ is a polynomial of finite type. The space of n-homogeneous polynomials of finite type is denoted by $\mathcal{P}_f(^nX,Y)$. The closure of $\mathcal{P}_f(^nX,Y)$ in the topology of uniform convergence on bounded sets is called the space of *approximable polynomials* and denoted by $\mathcal{P}_a(^nX,Y)$. Each approximable polynomial is weakly continuous on bounded sets. The following theorem is proved in **[10]** by Aron and Prolla.

THEOREM 1.9. *The dual space X' has the approximation property if and only if for every n the space $\mathcal{P}_f(^nX,Y)$ coincides with the space of all n-homogeneous weakly continuous polynomials $\mathcal{P}_w(^nX,Y)$ for an arbitrary Banach space Y.*

It is unknown does equality

$$\mathcal{P}_f(^nX) = \mathcal{P}_w(^nX)$$

implies the approximation property of the dual X'. However, Aron, Cole and Gamelin in [7] show that if X is a reflexive Banach space without the approximation property, then

$$\mathcal{P}_f(^2X \oplus X') \neq \mathcal{P}_w(^2X \oplus X').$$

The symmetric tensor power $\odot^n X'$ of a dual Banach space X' can be endowed with the injective norm

$$\|x\|_{\odot^n_\varepsilon X'} = \sup_{\|y\|\leq 1} \Big| \sum_{j\in\mathbb{N}} \lambda_j \langle y \mid x_j\rangle^n \Big|$$

for any

$$x = \sum_j \lambda_j \, (\otimes^n x_j) \in \odot^n X',$$

where $y \in X$, $x_j \in X'$, $\lambda_j \in \mathbb{C}$. Let $\odot^n_\varepsilon X'$ denote the corresponding completion. The space $\odot^n_\varepsilon X'$ coincides with a subspace in $\mathcal{P}^n(X)$ of approximable n-homogeneous polynomials on X which are weakly star continuous on the bounded sets and the embedding

$$\mathcal{P}^n_\varepsilon(X) = \odot^n_\varepsilon X' \hookrightarrow \mathcal{P}^n(X)$$

is isometric (see e.g. **[36**, p.112], **[41]**). Further $\mathcal{P}_\varepsilon(X)$ denotes the algebra of all *approximable polynomials* on X which are weakly star continuous on the bounded subsets of X.

1.2. Linear subspaces in zeros of polynomials

If X is an arbitrary complex vector space (not necessarily normed), we define a n-homogeneous complex polynomial by the formula

$$P(x) = (F \circ \Delta_n)(x) \qquad x \in X,$$

where F is a complex n-linear (not necessarily continuous) functional on X.

It is clear that the kernel (i.e. the set of zeros) of an n-homogeneous complex polynomial P on X, where $n > 0$ and $\dim X > 1$, consists of one-dimensional subspaces. Now we will show that it consists of infinite-dimensional subspaces if $\dim X = \infty$.

THEOREM 1.10. *Let X be an infinite-dimensional complex vector space and P is a complex n-homogeneous polynomial on X. Then there exists an infinite-dimensional subspace X_0 such that*

$$X_0 \subset \ker P.$$

LEMMA 1.11. *Let Theorem 1.10 be proved for every homogeneous polynomial of degree $\leq n$. Then for arbitrary homogeneous polynomials $P_1, \cdots, P_m$ of degree $\leq n$ there exists a subspace*

$$X_0 \subset \ker P_1 \cap \ldots \cap \ker P_m$$

such that $\dim X_0 = \infty$.

PROOF. Let $X_1 \subset \ker P_1$ with $\dim X_1 = \infty$. Then there exists a subspace $X_2 \subset X_1 \cap \ker P_2$ such that $\dim X_2 = \infty$. Continuing this process, we get the subspace

$$X_0 = X_m \subset X_{m-1} \subset \cdots \subset X_1$$

with $X_0 \subset \ker P_1 \cap \cdots \cap \ker P_m$ and $\dim X_0 = \infty$. □

Proof of Theorem 1.10. We will construct X_0 using the induction on n. Evidently that the theorem is true for linear functionals. Suppose that it is true for homogeneous polynomials of degree $< n$.

Let $x_1 \in X$ is chosen such that $P(x_1) \neq 0$ (if such x_1 does not exist then the assertion of theorems is true automatically). By the induction hypothesis and by Lemma 1.11 there exists a subspace $X_1 \subset X$ with $\dim X_1 = \infty$, on which each of the homogeneous polynomials

$$P_{x_1}(x) := F\left(x_1, x^{n-1}\right),$$
$$P_{x_1^2}(x) := F\left(x_1^2, x^{n-2}\right),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$P_{x_1^{n-1}}(x) := F\left(x_1^{n-1}, x\right)$$

vanish for all $x \in X_1$, where F is the symmetric n-linear functional associated with the n-homogeneous polynomial P.

On second step we choose an element $x_2 \in X_1$ such that $P(x_2) \neq 0$ (if x_2 does not exist then $X_1 \subset \ker P$ and the theorem is proved at once). By the induction hypothesis and by Lemma 1.11 there exists a subspace $X_2 \subset X_1$ with $\dim X_2 = \infty$ on which each homogeneous polynomials

$$P_{x_1^k, x_2^l}(x) := F\left(x_1^k, x_2^l, x^{n-k-l}\right), \qquad 0 < k + l < n$$

vanish for all $x \in X_2$.

We continue this process in the way written above. If it finishes on the i-th step (i.e. $P(X_i) = 0$), then the theorem is proved. If it does not finish then we will get an infinite sequence (x_i) consisting of linearly independent terms such that $P(x_i) \neq 0$ for every $i \in \mathbb{N}$ and

$$F\left(x_1^{k_1}, x_2^{k_2}, \dots, x_i^{k_i}\right) = 0$$

if $0 < k_i < n$ at least for one k_i.

Consequently, it follows that for any finite sequence of scalars (a_i),

$$P\Big(\sum_i a_i x_i\Big) = \sum_i a_i^n P(x_i).$$

Put $y_i = x_i / P(x_i)$ for all $i \in \mathbb{N}$. Then P vanishes on the linear span of elements

$$y_1 + \sqrt[n]{-1}\, y_2, \;\; y_3 + \sqrt[n]{-1}\, y_4, \;\; y_5 + \sqrt[n]{-1}\, y_6, \;\; \dots .$$

The theorem is proved. □

COROLLARY 1.12. For every polynomial functional P on a complex infinite dimensional vector space, for which $P(0) = 0$, there exists an infinite dimensional linear subspace X_0 such that $X_0 \subset \ker P$.

The corollary is proved in the same way as Lemma 1.11. Applying Corollary 1.12 to the polynomial $P_{x_0}(x) = P(x_0 + x)$, we obtain the following assertion.

COROLLARY 1.13. If P is a polynomial functional on a complex infinite dimensional vector space and $P(x_0) = 0$, then there exists an infinite dimensional affine subspace $X_0 \subset \ker P$ with $x_0 \in X_0$.

1.3. Analytic functions

Let Ω be a subset of a Banach space X. We say that Ω is *finitely open* if for any finite dimensional affine subspace V of X, endowed with the Euclidean topology, $V \cap \Omega$ is open in V.

DEFINITION 1.14. We say that a map

$$f \colon \Omega \longrightarrow Y$$

into a Banach space Y is *G-analytic* (*Gâteaux-analytic*), and write $f \in \mathcal{H}_G(\Omega, Y)$, if for any finite dimensional affine subspace V (or, equivalently, for any complex line $V \subset X$) the restriction of f onto $V \cap \Omega$ is analytic.

A G-analytic map $f \colon \Omega \longrightarrow Y$, defined on an open subset $\Omega \subset X$, is called *analytic*, which is written $f \in \mathcal{H}(\Omega, Y)$, if f is continuous.

If $\Omega = X$ and $f \in \mathcal{H}(X, Y)$ then the function f is called entire.

We will short write $\mathcal{H}(\cdot,\mathbb{C}) = \mathcal{H}(\cdot)$ for spaces of $\mathbb{C}$-valued functions.

Every analytic function $f \in \mathcal{H}(\Omega, Y)$ can be locally represented by its Taylor's series expansion

$$f(a+x) = \sum_{n\in\mathbb{Z}_+} f_n(x), \qquad f_n(x) = \frac{d_a^n f(x)}{n!}$$

uniformly convergent on a neighborhood of $a \in \Omega$, in which $d_a^n f(x)$ denotes the nth Fréchet derivation of f at a by the direction $x \in X$ and belongs to the space of n-homogeneous polynomials $\mathcal{P}(^nX, Y)$.

PROPOSITION 1.15. Let (f_n) be a sequence of n-homogeneous polynomials such that $f_n \in \mathcal{P}(^nX,Y)$ for all $n \in \mathbb{N}$. A necessary and sufficient condition for existence of a function $f \in \mathcal{H}(X,Y)$ such that $f_n = d_0^n f/n!$ is that for any given $\varepsilon > 0$ each $x \in X$ has a neighborhood U such that

$$\sup_U \|f_n\|^{1/n} \le \varepsilon$$

for n large enough.

Let Ω be an open subset of X and $f \in \mathcal{H}(\Omega, Y)$. The *radius of uniform convergence* $\varrho_a(f)$ of the function f at $a \in \Omega$ is defined as a supremum of all number $\lambda \in \mathbb{C}$ such that

$$a + \lambda B \subset \Omega$$

and the Taylor series of f at a converges to f uniformly on $a + \lambda B$, where B is the unit ball of X. Note that if X is an infinite dimensional Banach space, then there exists a $\mathbb{C}$-valued entire function f on X, such that $\varrho_a(f) < \infty$ for every $a \in X$ (see e.g. [**35**, p.169]).

The *radius of boundedness* of f at $a \in \Omega$ is defined as a supremum of all $\lambda \in \mathbb{C}$ such that f is bounded on $a + \lambda B$.

THEOREM 1.16. *The radii of uniform convergence and boundedness of any function $f \in \mathcal{H}(\Omega, Y)$ at $a \in \Omega$ coincide and*

$$\varrho_a(f) = \left(\limsup_{n\to\infty} \|f_n\|^{1/n}\right)^{-1},$$

where

$$f_n = \frac{d_a^n f}{n!}, \qquad \|f_n\| = \sup_{x\in B} \|f_n(x)\|.$$

Denote by $\mathcal{H}_b(X)$ *the space of all entire $\mathbb{C}$-valued functions of bounded type*, i.e., of all entire complex functions on X which are bounded on bounded subsets (having the radius of boundedness equal to infinity).

The space $\mathcal{H}_b(X)$ is a Fréchet topological algebra endowed with the seminorms

$$\|f\|_r = \sup_{x\in rB} |f(x)|, \qquad f \in \mathcal{H}_b(X),$$

where $r > 0$ is a rational number. Consider its dual space $\mathcal{H}_b(X)'$. Each linear functional $\phi \in \mathcal{H}_b(X)'$ is bounded with respect to the norm of uniform convergence on some ball in X.

The radius *function* $R(\phi)$ of the functional $\phi \in \mathcal{H}_b(X)'$ is defined as the infimum of all numbers $r > 0$ such that ϕ is bounded with respect to the norm of uniform convergence on the ball rB.

Denote by ϕ_n the restriction of $\phi \in \mathcal{H}_b(X)'$ to the subspace of n-homogeneous polynomials $\mathcal{P}(^nX)$. Then ϕ_n is a bounded linear functional on $\mathcal{P}(^nX)$ and

$$\|\phi_n\| = \sup\Big\{\, |\phi(P)| : P \in \mathcal{P}(^nX), \|P\| \leq 1\Big\}.$$

THEOREM 1.17. *The radius function R on the dual space $\mathcal{H}_b(X)'$ is given by*

$$R(\phi) = \limsup_{n\to\infty} \|\phi_n\|^{1/n}.$$

PROOF. Suppose that

$$0 < t < \limsup_{n\to\infty} \|\phi_n\|^{1/n}.$$

Then there is a sequence of homogeneous symmetric polynomials (P_j) of degree $n_j \to \infty$ such that $\|P_j\| = 1$ and $|\phi(P_j)| > t^{n_j}$. If $0 < r < t$, then by homogeneity,

$$\|P_j\|_r = \sup_{x\in rB} |P_j(x)| = r^{n_j},$$

so that

$$|\phi(P_j)| > (t/r)^{n_j} \|P_j\|_r,$$

and ϕ is not continuous with respect to the norm of uniform convergence on rB. It follows that $R(\phi) \geq r$, and in view of the arbitrary choice of r we obtain

$$R(\phi) \geq \limsup_{n\to\infty} \|\phi_n\|^{1/n}$$

Let now $s > \limsup_{n\to\infty} \|\phi_n\|^{1/n}$ so that $s^m \geq \|\phi_m\|$ for m large. Then there is $c \geq 1$ such that $\|\phi_m\| \leq cs^m$ for every m. If $r > s$ is arbitrary and $f \in H_b(X)$ has Taylor series expansion $f = \sum_{n\in\mathbb{Z}_+} f_n$, then

$$r^m \|f_m\| = \|f_m\|_r \leq \|f\|_r, \quad m \in \mathbb{Z}_+.$$

Hence

$$|\phi(f_m)| \leq \|\phi_m\| \|f_m\| \leq \frac{cs^m}{r^m} \|f\|_r$$

and so

$$\|\phi(f)\| \leq c\bigg(\sum_{m\in\mathbb{Z}_+} \frac{s^m}{r^m}\bigg) \|f\|_r.$$

Thus ϕ is continuous with respect to the uniform norm on rB, and $R(\phi) \leq r$. Since r and s are arbitrary,

$$R(\phi) \leq \limsup_{n\to\infty} \|\phi_n\|^{1/n}.$$

The theorem is proved. □

Consider now the dual space $\mathcal{P}(^nX)'$ of the space $\mathcal{P}(^nX)$.

THEOREM 1.18. *Suppose that $\phi_n \in \mathcal{P}(^nX)'$ for any $n \in \mathbb{Z}_+$, and suppose that*

$$\|\phi_n\| \leq cs^n$$

for some $c, s > 0$. Then there is a unique functional $\phi \in \mathcal{H}_b(X)'$ whose restriction to $\mathcal{P}(^nX)$ coincides with ϕ_n for all $n \in \mathbb{Z}_+$.

1.4. The Aron-Berner extension

Let X be a complex Banach space and X'' its second dual. Any given continuous n-linear mapping $F\colon X \times \cdots \times X \longrightarrow \mathbb{C}$ can be extended to a continuous n-linear mapping $\widetilde{F}\colon X'' \times \cdots \times X'' \longrightarrow \mathbb{C}$ as follows

$$\widetilde{F}(x_1'', \ldots, x_n'') = \lim_{\alpha_1} \ldots \lim_{\alpha_n} F(x_{\alpha_1}, \ldots, x_{\alpha_n}), \tag{1.9}$$

where for each k, (x_{α_k}) is a net in X weakly-star converging to x_k''.

Let $P \in \mathcal{P}(^nX)$ and $F_P \in \mathscr{L}_s(^nX, \mathbb{C})$ be the n-linear symmetric form associated with P. Then the *Aron-Berner extension* $\widetilde{P}$ of P is defined as

$$\widetilde{P}(x) := \widetilde{F}_P(x, \ldots, x), \qquad x \in X''.$$

THEOREM 1.19. *Let (x_α) be a net in the unit ball of X that weakly-star converges to $z \in X''$ with $\|z\| < 1$. Then there is a net (y_β) in the unit ball B of X such that each y_β is an arithmetic mean of a finite number of x_α's, and*

$$P(y_\beta) \to \widetilde{P}(z)$$

for every polynomial P on X.

THEOREM 1.20. *Let $f \in \mathcal{H}_b(X)$ and*

$$f = \sum_{n \in \mathbb{Z}_+} f_n$$

is its Taylor series expansion. Then there exists $\widetilde{f} \in \mathcal{H}_b(X'')$ with the Taylor series expansion

$$\widetilde{f} = \sum_{n \in \mathbb{Z}_+} \widetilde{f}_n$$

such that $\widetilde{f}_n$ is the Aron-Berner extension of f_n. Moreover,

$$\|\widetilde{f}_n\| = \|f_n\| \quad \textit{for every } n$$

and the operator $f \longmapsto \widetilde{f}$ is a homomorphism between the Fréchet algebras $\mathcal{H}_b(X)$ and $\mathcal{H}_b(X'')$, i.e.,

$$\mathcal{H}_b(X) \simeq \mathcal{H}_b(X'').$$

Let $\mathcal{I}$ be a set of indexes and $(X_i)_{i\in\mathcal{I}}$ with $X_i = X$ be a $\mathcal{I}$-fold family of X. Denote by $\ell_\infty(X,\mathcal{I})$ the ℓ_∞-direct sum of X's, that is, the space of all elements

$$(x_i)_{i\in\mathcal{I}} \in \bigtimes_{i\in\mathcal{I}} X_i$$

such that the sequence $\left(\|x_i\|\right)_{i\in\mathcal{I}}$ is bounded with the ℓ_∞-norm

$$\|(x_i)_{i\in\mathcal{I}}\|_\infty := \sup_{i\in\mathcal{I}} \|x_i\|.$$

Let $\mathfrak{U}$ be an ultrafilter on $\mathcal{I}$ and $(x_i)_{i\in\mathcal{I}} \in \ell_\infty(X,\mathcal{I})$. We define the *ultrapower* of X with respect to $\mathfrak{U}$ as the quotient space $\ell_\infty(X,\mathcal{I})/\mathcal{N}_\mathfrak{U}$ equipped with the quotient ℓ_∞-norm, where

$$\mathcal{N}_\mathfrak{U} := \left\{(x_i) \in \ell_\infty(X,\mathcal{I})\colon \lim_{\mathfrak{U}} \|x_i\| = 0\right\}.$$

Evidently, $\mathcal{N}_\mathfrak{U}$ is a closed vector subspace of $\ell_\infty(X,\mathcal{I})$. We will denote

$$X^\mathfrak{U} := \ell_\infty(X,\mathcal{I})/\mathcal{N}_\mathfrak{U}.$$

Thus, the ultrapower $X^\mathfrak{U}$ consists of elements $(x_i)_\mathfrak{U} \in \ell_\infty(X,\mathcal{I})/\mathcal{N}_\mathfrak{U}$, where $x_i \in X$ for every $i \in \mathcal{I}$ and

$$(x_i)_\mathfrak{U} = (y_i)_\mathfrak{U} \quad \text{if} \quad \lim_{\mathfrak{U}} x_i = \lim_{\mathfrak{U}} y_i$$

in the Banach space X.

There are two approaches to construct of ultrapower extensions of polynomials.

Let $P \in \mathcal{P}(^nX)$ and F_P be the symmetric n-linear forms associated with P. Then we define an n-linear forms on $X^\mathfrak{U}$ by

$$\widetilde{F}_P(x_1,\ldots,x_n) = \lim_{i_1,\mathfrak{U}} \ldots \lim_{i_n,\mathfrak{U}} \widetilde{F}_P\left(x^{(1)}_{i_1},\ldots,x^{(n)}_{i_n}\right)$$

for $x_k = \left(x^{(k)}_i\right)_\mathfrak{U}$. It is easy to see that $\widetilde{F}_P$ is well defined, $\widetilde{F}_P$ is an extension of $\widetilde{F}_P$ and that $\|\widetilde{F}_P\| = \|F_P\|$. Thus, we can define an extension of P to $X^\mathfrak{U}$ by

$$\widetilde{P}\big((x_i)_\mathfrak{U}\big) = \widetilde{F}_P\big((x_i)_\mathfrak{U},\ldots,(x_i)_\mathfrak{U}\big).$$

Note that if F_P is symmetric, it does not necessary follow that $\widetilde{F}_P$ is symmetric.

The ultrafilter $\mathfrak{U}$ associated with the weak convergence on the space X is called a *local ultrafilter* on X.

If $\mathfrak{U}$ is the local ultrafilter on X then the restriction of $\widetilde{P}$ to the canonical image of X'' in $X^\mathfrak{U}$ coincides with the *Aron-Berner extension* of P onto X''.

THEOREM 1.21. *The following assertions are equivalent:*

1. *For every ultrafilter $\mathfrak{U}$ and every continuous symmetric bilinear form F on X, the ultrapower extension $\widetilde{F}_P$ is symmetric.*
2. *For every ultrafilter $\mathfrak{U}$ and every continuous symmetric n-linear form F on X, the ultrapower extension $\widetilde{F}_P$ is symmetric.*
3. *For local ultrafilter on X and every continuous symmetric bilinear form F on X, the ultrapower extension $\widetilde{F}_P$ from X onto X'' is symmetric.*
4. *Every continuous symmetric linear operator from X into X' is weakly compact.*

(5) *Every continuous symmetric bilinear form on X extends to a separately weakly-star continuous bilinear form on X''.*

A Banach space X is said to be *symmetrically regular* if the assertions (1)-(5) of Theorem 1.21 hold.

Since every polynomial $P \in \mathcal{P}(^nX)$ is bounded on bounded nets, we can define

$$\bar{P}((x_i)_{\mathfrak{U}}) := \lim_{\mathfrak{U}} P(x_i)$$

and we have $\|P\| = \|\bar{P}\|$. Note that, in general, $\tilde{P} \neq \bar{P}$.

A closed subspace Y of a Banach space X is *locally complemented* in X if there is a constant M such that whenever E is a finite-dimensional subspace of Y there is a linear map (depending on the given finite-dimensional subspace) $T\colon E \longrightarrow X$ so that $\|T\| \leq M$ and $Tx = x$ for all $x \in E \cap X$.

For instance, Lindenstrauss-Rosenthal Principle of Local Reflexivity says that every Banach space is locally complemented in its second dual. Also, it is well-known that every Banach space is locally complemented in its ultrapowers.

THEOREM 1.22. *Let Y be a subspace of X. Then there exists a linear extension operator*

$$\mathcal{P}(^nY) \longrightarrow \mathcal{P}(^nX)$$

for all (or some) $n \geq 1$ if and only if Y is locally complemented in X.

1.5. Concept of regularity

A bilinear form F on a Banach space X is called *Arens regular* if the following two extensions of F onto $X'' \times X''$ coincide:

$$\widetilde{F}_{[12]}(x'', y'') = \lim_{\alpha} \lim_{\beta} F(x_\alpha, y_\beta),$$

$$\widetilde{F}_{[21]}(x'', y'') = \lim_{\beta} \lim_{\alpha} F(x_\alpha, y_\beta),$$

where (x_α) (resp. (y_β)) is a net in X weakly-star converging to $x'' \in X''$ (resp. $y'' \in X''$). Evidently,

$$\widetilde{F}_{[12]}(x'', y'') = \widetilde{F}_{[21]}(y'', x'').$$

Note that F is Arens regular if and only if the extension $\widetilde{F} = \widetilde{F}_{[12]}$ is separately weakly-star continuous in each variable.

A Banach space X is called *regular* if every continuous bilinear form on $X \times X$ is Arens regular.

From Theorem 1.21 it follows that X is symmetrically regular if and only if every symmetric continuous bilinear form $F \in \mathscr{L}_s(^2X, \mathbb{C})$ is Arens regular. In other words, if

$$\widetilde{F}(x'', y'') := \widetilde{F}_{[12]}(x'', y'') \text{ and } \widetilde{F}(y'', x'') := \widetilde{F}_{[21]}(x'', y'')$$

Thus the symmetric regularity of X means that the extension $\widetilde{F}(x'', y'')$ of any symmetric continuous bilinear form $F(x,y)$ is symmetric. Evidently, if X is symmetrically regular then it is regular.

For a given bilinear form $F \in \mathscr{L}(^2X, \mathbb{C})$ we consider the linear operator $T = T_F$ from X to X' determined by the formula

$$F(x,y) = \langle Tx \mid y \rangle, \qquad x, y \in X,$$

where $Tx(y) = \langle Tx \mid y \rangle$ is the value of $Tx \in X'$ in $y \in X$. If F is a symmetric bilinear form, then T is a symmetric operator, that is

$$\langle Tx \mid y \rangle = \langle Ty \mid x \rangle, \qquad x, y \in X.$$

PROPOSITION 1.23. Let X, Y be a Banach spaces. The Cartesian product $X \times Y$ is regular if and only if every map in any of the following four spaces

$$\mathscr{L}(X, X'), \quad \mathscr{L}(X, Y'), \quad \mathscr{L}(Y, X'), \quad \mathscr{L}(Y, Y')$$

is weakly compact.

PROOF. Suppose that the above four spaces possess the specified property. Let $T = (T_1, T_2)\colon X \times Y \longrightarrow X' \times Y'$ be a continuous linear mapping. Then

$$T(x,y) = (T_1(x,0),0) + T_1((0,y),0) + (0, T_2(x,0)) + (0, T_2(0,y))$$

for every $(x,y) \in X \times Y$. We define

$$\begin{aligned}
R_1 &\colon X \longrightarrow X' &&\text{by} \quad R_1(x) := T_1(x,0), && x \in X,\\
R_2 &\colon Y \longrightarrow X' &&\text{by} \quad R_2(y) := T_1(0,y), && y \in Y,\\
R_3 &\colon X \longrightarrow Y' &&\text{by} \quad R_3(x) := T_2(x,0), && x \in X,\\
R_4 &\colon Y \longrightarrow Y' &&\text{by} \quad R_4(y) := T_2(0,y), && y \in Y.
\end{aligned}$$

Since R_j $(j = 1,2,3,4)$ are weakly compact operators and since

$$T(x,y) = (R_1(x),0) + (R_2(y),0) + (0, R_3(x)) + (0, R_4(y))$$

for all $(x,y) \in X \times Y$, the conclusion holds. The adversary implication is clear. □

COROLLARY 1.24.

(1) If X is regular, then $X \times X$ is regular too.
(2) If X is regular, then $X \times \mathbb{C}$ is regular too.
(3) If X is not reflexive, then $X \times X'$ is not regular.

PROPOSITION 1.25. For a given Banach space X, the Cartesian product $X \times X$ is regular if and only if it is symmetrically regular.

Note that from this propositions it follows that if X is isomorphic to its square, then it is regular if and only if it is symmetrically regular. In [**9**] is noted that the dual of James space is symmetrically regular but it is not regular.

For a given linear operator T from X to X' we denote by T^t the *transposed* (adjoint) operator, acting from X'' to X'. Throughout further we suppose that each Banach space is naturally embedded into its second dual. In particularly, it means that T^{tt} is well defined on X and

$$T^{tt}x = Tx, \qquad x \in X.$$

PROPOSITION 1.16. Let for a bilinear form $F \in \mathscr{L}(^2X, \mathbb{C})$ corresponds the linear operator $T: X \to X'$. Then

$$(1.10) \qquad \widetilde{F}_{[12]}(x'', y'') = \langle T^{tt}x'' \mid y''\rangle,$$

$$(1.11) \qquad \widetilde{F}_{[21]}(x'', y'') = \langle J_1 T^{tt}x'' \mid y''\rangle,$$

where J_1 is the canonical projection of X''' onto the embedded image of X' in X'''.

PROOF. For each $x \in X$ the functional

$$y'' \longmapsto \langle T^{tt}x \mid y''\rangle = \langle Tx \mid y\rangle$$

is weakly-star continuous on X''. Also, for each y'', the functional

$$x'' \longmapsto \langle T^{tt}x'' \mid y''\rangle = \langle x'' \mid T^t y''\rangle$$

is weakly-star continuous on X''. So,

$$\widetilde{F}_{[12]}(x'', y'') = \lim_{\alpha}\lim_{\beta}\langle Tx_\alpha \mid y_\beta\rangle = \lim_{\alpha}\langle T^{tt}x_\alpha \mid y''\rangle$$
$$= \lim_{\alpha}\langle x_\alpha \mid T^t y''\rangle = \langle T^{tt}x'' \mid y''\rangle$$

and the equation (1.10) is proved.

Similarly, to establish (1.11) is sufficient to observe that for each $y \in X$, the functional

$$x'' \longmapsto \langle J_1 T^{tt}x'' \mid y\rangle = \langle T^{tt}x'' \mid y\rangle = \langle x'' \mid T^t y\rangle$$

is weakly-star continuous on X'' and for any fixed $x'' \in X''$, the functional

$$y \longmapsto \langle J_1 T^{tt}x'' \mid y\rangle$$

is weakly-star continuous on X''. □

1.6. Hilbert-Schmidt polynomials

Let E be a separable Hilbert complex space with an orthonormal basis $(\mathfrak{e}_i)_{i\in\mathbb{N}}$, endowed with the scalar product $\langle x \mid y\rangle_{\mathsf{E}}$ and the norm

$$\|x\|_{\mathsf{E}} = \langle x \mid y\rangle_{\mathsf{E}}^{1/2}, \qquad x, y \in \mathsf{E}.$$

Clearly, for all $n \in \mathbb{N}$ the nth tensor power $\otimes^n\mathsf{E}$ is defined to be a complex linear span of elements

$$\{x_1 \otimes \ldots \otimes x_n : x_1, \ldots, x_n \in \mathsf{E}\}.$$

It is well-known (see e.g. [**33**] p. 351 or [**15**]) that it is possible to define a norm $\|\cdot\|_{\otimes^n_{\mathfrak{h}}\mathsf{E}}$ on the vector space $\otimes^n\mathsf{E}$ such that the corresponding completion $\otimes^n_{\mathfrak{h}}\mathsf{E}$ is a Hilbert space. More exactly, the scalar product on $\otimes^n_{\mathfrak{h}}\mathsf{E}$ is defined by the equality

$$\langle x_1 \otimes \ldots \otimes x_n \mid y_1 \otimes \ldots \otimes y_n\rangle_{\otimes^n_{\mathfrak{h}}\mathsf{E}} := \langle x_1 \mid y_1\rangle_{\mathsf{E}} \ldots \langle x_n \mid y_n\rangle_{\mathsf{E}}$$

for all $x_i, y_i \in \mathsf{E}$. Let $\{i\}$ denotes a multi-index $(i_1, \ldots, i_n) \in \mathbb{N}^n$. Since the system

$$\{\mathfrak{e}_{i_1} \otimes \ldots \otimes \mathfrak{e}_{i_n} \in \otimes^n\mathsf{E} : \{i\} \in \mathbb{N}^n\}$$

forms an orthonormal basis in $\otimes_{\mathfrak{h}}^n \mathsf{E}$, each vector $w \in \otimes_{\mathfrak{h}}^n \mathsf{E}$ can be represented as the Fourier series

$$w = \sum_{\{i\} \in \mathbb{N}^n} \lambda_{\{i\}} \mathfrak{e}_{i_1} \otimes \cdots \otimes \mathfrak{e}_{i_n}, \qquad \lambda_{\{i\}} \in \mathbb{C}$$

and it has the norm

$$\|w\|_{\otimes_{\mathfrak{h}}^n \mathsf{E}} = \langle w \mid w \rangle_{\otimes_{\mathfrak{h}}^n \mathsf{E}}^{1/2} = \left(\sum_{\{i\} \in \mathbb{N}^n} |\lambda_{\{i\}}|^2 \right)^{1/2}.$$

It is clear that the above norm, generated by the scalar product, is a cross-norm on $\otimes_{\mathfrak{h}}^n \mathsf{E}$, that is,

$$\|x_1 \otimes \ldots \otimes x_n\|_{\otimes_{\mathfrak{h}}^n \mathsf{E}} = \|x_1\|_{\mathsf{E}} \ldots \|x_n\|_{\mathsf{E}}.$$

PROPOSITION 1.27. There exists a unique continuous orthogonal projection S_n on $\otimes_{\mathfrak{h}}^n \mathsf{E}$ such that

$$S_n (\mathfrak{e}_{i_1} \otimes \ldots \otimes \mathfrak{e}_{i_n}) = \frac{1}{n!} \sum_{s \in \mathfrak{S}_n} \mathfrak{e}_{i_{s(1)}} \otimes \ldots \otimes \mathfrak{e}_{i_{s(n)}}.$$

PROOF. The equality $S_n^2(w) = S_n(w)$ and orthogonality of S_n are evident on a dense set of finite sums $w = \sum_{\{i\}} \lambda_{\{i\}} \mathfrak{e}_{i_1} \otimes \cdots \otimes \mathfrak{e}_{i_n}$. The continuity of S_n follows from the inequality

$$\begin{aligned}
\sup_{\|w\|_{\otimes_{\mathfrak{h}}^n \mathsf{E}} \le 1} \|S_n(w)\|_{\otimes_{\mathfrak{h}}^n \mathsf{E}} &= \sup_{\|w\|_{\otimes_{\mathfrak{h}}^n \mathsf{E}} \le 1} \left\| \frac{1}{n!} \sum_{s \in \mathfrak{S}_n} \sum_{\{i\}} \lambda_{i_{s(1)}, \ldots, i_{s(n)}} \mathfrak{e}_{i_{s(1)}} \otimes \cdots \otimes \mathfrak{e}_{i_{s(n)}} \right\|_{\otimes_{\mathfrak{h}}^n \mathsf{E}} \\
&\le \sup_{\|w\|_{\otimes_{\mathfrak{h}}^n \mathsf{E}} \le 1} \frac{1}{n!} \sum_{s \in \mathfrak{S}_n} \left(\sum_{\{i\}} \left| \lambda_{i_{s(1)}, \ldots, i_{s(n)}} \right|^2 \|\mathfrak{e}_{i_{s(1)}}\|_{\mathsf{E}}^2 \ldots \|\mathfrak{e}_{i_{s(n)}}\|_{\mathsf{E}}^2 \right)^{1/2} \\
&= \frac{1}{n!} \sup_{\|w\|_{\otimes_{\mathfrak{h}}^n \mathsf{E}} \le 1} n! \, \|w\|_{\otimes_{\mathfrak{h}}^n \mathsf{E}} = 1.
\end{aligned}$$

Uniqueness of S_n is a consequence of the basis property of $(\mathfrak{e}_i)$ and $\mathfrak{e}_{i_1} \otimes \ldots \otimes \mathfrak{e}_{i_n}$ in the spaces E and $\odot_{\mathfrak{h}}^n \mathsf{E}$, respectively. □

The symmetric Hilbertian tensor product $\odot_{\mathfrak{h}}^n \mathsf{E}$ is defined to be the codomain of orthogonal projection S_n, setting on the space $\otimes_{\mathfrak{h}}^n \mathsf{E}$, that is,

$$\odot_{\mathfrak{h}}^n \mathsf{E} := S_n(\otimes_{\mathfrak{h}}^n \mathsf{E}).$$

We will use the following notations

$$(k) := (k_1, \ldots, k_n) \in \mathbb{Z}_+^n, \quad |(k)| := \sum_i k_i, \quad (k)! := \prod_i k_i!$$

PROPOSITION 1.28. (i) The system of elements forms an orthogonal basis in $\odot_{\mathfrak{h}}^n\mathsf{E}$, at that

$$\left\|\mathfrak{e}_{\{i\}}^{\otimes(k)}\right\|_{\otimes_{\mathfrak{h}}^n\mathsf{E}}=\sqrt{\frac{(k)!}{n!}},\qquad n=|(k)|. \tag{1.12}$$

(ii) The symmetric Hilbertian tensor product $\odot_{\mathfrak{h}}^n\mathsf{E}$ is the closure in $\otimes_{\mathfrak{h}}^n\mathsf{E}$ of the complex linear span of elements

$$\{x^{\otimes n}\colon x\in\mathsf{E}\}.$$

PROOF. The assertion (i) see e.g. in [**15**, 2.2.2].

(ii) The system $x_{\{i\}}^{\otimes n}\in\odot_{\mathfrak{h}}^n\mathsf{E}$ with

$$x_{\{i\}}=\delta_1\mathfrak{e}_{i_1}+\ldots+\delta_n\mathfrak{e}_{i_n}\in\mathsf{E},$$

where $\{i\}\in\mathbb{N}^n$ and $\delta_i=\pm1$, is total in $\odot_{\mathfrak{h}}^n\mathsf{E}$. Indeed, using the polarization formula (1.2), we have

$$\mathfrak{e}_{\{i\}}^{\otimes(k)}=\frac{1}{2^n n!}\sum_{i=1}^{n}\sum_{\delta_i=\pm1}\delta_1\cdot\ldots\cdot\delta_n\,x_{\{i\}}^{\otimes n}\qquad\text{with}\quad n=|(k)|.$$

If an element $\psi_n\in\odot_{\mathfrak{h}}^n\mathsf{E}$ satisfies the conditions $\langle x_{\{i\}}^{\otimes n}\mid\psi_n\rangle_{\otimes_{\mathfrak{h}}^n\mathsf{E}}=0$ for all $\{i\}\in\mathbb{N}^n$ then $\langle\mathfrak{e}_{\{i\}}^{\otimes(k)}\mid\psi_n\rangle_{\otimes_{\mathfrak{h}}^n\mathsf{E}}=0$ for all indexes $\{i\}\in\mathbb{N}^n$ and $(k)\in\mathbb{Z}_+^n$ such that $|(k)|=n$. However, elements $\mathfrak{e}_{\{i\}}^{\otimes(k)}$ form a basis in $\odot_{\mathfrak{h}}^n\mathsf{E}$, hence $\psi_n=0$. Thus, the set $x_{\{i\}}^{\otimes n}$ is total in the space $\odot_{\mathfrak{h}}^n\mathsf{E}$, if $\{i\}$ runs all such indexes. □

Using the Riesz theorem, the Hermitian dual of a Hilbert space E we can to define by the relation

$$\mathsf{E}^*=\{y^*:=\langle\cdot\mid y\rangle_{\mathsf{E}}\colon y\in\mathsf{E}\}.$$

Since the natural unitary isometries

$$(\otimes_{\mathfrak{h}}^n\mathsf{E})^*=\otimes_{\mathfrak{h}}^n\mathsf{E}^*,\qquad(\odot_{\mathfrak{h}}^n\mathsf{E})^*=\odot_{\mathfrak{h}}^n\mathsf{E}^*$$

hold, the similar argument allows us to note

$$\otimes_{\mathfrak{h}}^n\mathsf{E}^*=\{w^*:=\langle\cdot\mid w\rangle_{\otimes_{\mathfrak{h}}^n\mathsf{E}}\colon w\in\otimes_{\mathfrak{h}}^n\mathsf{E}\},$$
$$\odot_{\mathfrak{h}}^n\mathsf{E}^*=\{w^*:=\langle\cdot\mid w\rangle_{\otimes_{\mathfrak{h}}^n\mathsf{E}}\colon w\in\odot_{\mathfrak{h}}^n\mathsf{E}\}.$$

Then the restriction $w^*\,|_{\{x^{\otimes n}\colon x\in\mathsf{E}\}}$ of any $w^*\in\odot_{\mathfrak{h}}^n\mathsf{E}^*$, which we denote as

$$P_{w^*}(x)=\langle x^{\otimes n}\mid w\rangle_{\otimes_{\mathfrak{h}}^n\mathsf{E}},\qquad x\in\mathsf{E},$$

is an n-homogeneous polynomial on E. Since

$$|P_{w^*}(x)|\le\|w^*\|_{\otimes_{\mathfrak{h}}^n\mathsf{E}^*}\|x^{\otimes n}\|_{\otimes_{\mathfrak{h}}^n\mathsf{E}}=\|w\|_{\otimes_{\mathfrak{h}}^n E}\|x\|_{\mathsf{E}}^n,\qquad x\in\mathsf{E},$$

the polynomial P_{w^*} is bounded and so $P_{w^*}\in\mathcal{P}(^n\mathsf{E})$.

Let us denote by $\mathcal{P}_{\mathfrak{h}}(^n\mathsf{E})$ the subspace of all n-homogeneous continuous polynomials $\{P_{w^*}:w\in\odot_{\mathfrak{h}}^n\mathsf{E}\}$ with the norm

$$\|P_{w^*}\|=\|w\|_{\otimes_{\mathfrak{h}}^n E}=\|w^*\|_{\otimes_{\mathfrak{h}}^n\mathsf{E}^*},$$

which are called *Hilbert-Schmidt polynomials.* For any Hilbert space Y the Y-valued polynomial $P \in \mathcal{P}({}^n\mathsf{E}, Y)$ is called Hilbert-Schmidt n-homogeneous polynomial if

$$\psi \circ P \in \mathcal{P}_{\mathfrak{h}}({}^n\mathsf{E})$$

for every $\psi \in Y^*$. The previous notes can be precisely formulated as follows.

PROPOSITION 1.29. The Hermitian dual space $\odot_{\mathfrak{h}}^n \mathsf{E}^*$ is isometrically isomorphic to a subspace $\mathcal{P}_{\mathfrak{h}}({}^n\mathsf{E})$ of the space $\mathcal{P}({}^n\mathsf{E})$ of all n-homogeneous continuous polynomials on E.

PROOF. Proposition 1.28(ii) implies that the system $\{x^{\otimes n} : x \in \mathsf{E}\}$ is total in $\odot_{\mathfrak{h}}^n \mathsf{E}$. This yields the one-to-one linear mapping

$$\phi \rightleftarrows P_\phi$$

between $\odot_{\mathfrak{h}}^n \mathsf{E}^*$ and $\mathcal{P}_{\mathfrak{h}}({}^n\mathsf{E})$. Hence, $\odot_{\mathfrak{h}}^n \mathsf{E}^*$ is isometric to $\mathcal{P}_{\mathfrak{h}}({}^n\mathsf{E})$. □

Thus, for any element $w \in \odot_{\mathfrak{h}}^n \mathsf{E}$ uniquely assists the continuous linear functional $w^* := \langle \cdot \mid w \rangle_{\otimes_{\mathfrak{h}}^n \mathsf{E}}$ belonging to $\odot_{\mathfrak{h}}^n \mathsf{E}^*$, which can be identified with the n-homogeneous Hilbert-Schmidt polynomial P_{w^*}. Further we will use the short notation

$$P_{w^*} : \mathsf{E} \ni x \longmapsto w^*(x) := \langle x^{\otimes n} \mid w \rangle_{\otimes_{\mathfrak{h}}^n \mathsf{E}}.$$

It is clear that every polynomial of finite type is Hilbert-Schmidt. Moreover, since every element $w \in \odot_{\mathfrak{h}}^n \mathsf{E}$ has a form $w = \sum_{\{i\}} \lambda_{\{i\}} \mathfrak{e}_{i_1} \otimes \cdots \otimes \mathfrak{e}_{i_n}$, it is a limit of finite sum. So, every Hilbert-Schmidt polynomial is approximated by polynomials of finite type. Thus, if P is a Hilbert-Schmidt polynomial then P is weakly continuous on bounded sets [**8**]. The converse is not true.

PROPOSITION 1.30. There is a polynomial which is weakly continuous on bounded sets and not Hilbert-Schmidt.

PROOF. It is known that if E^* has the approximation property then the space of n-homogeneous weakly continuous polynomials on bounded sets $\mathcal{P}_w({}^n\mathsf{E})$ is isomorphic to the injective symmetric tensor product $\odot_{\varepsilon}^n \mathsf{E}^*$ of E^* [**36**, p. 112]. On the other hand,

$$\mathcal{P}_{\mathfrak{h}}({}^n\mathsf{E}) = (\odot_{\mathfrak{h}}^n \mathsf{E})^* \subset \odot_{\varepsilon}^n \mathsf{E}^* = \mathcal{P}_w({}^n\mathsf{E}).$$

This embedding is proper because $(\odot_{\mathfrak{h}}^n \mathsf{E})^* = \odot_{\mathfrak{h}}^n \mathsf{E}$ is a reflexive space but $\odot_{\varepsilon}^n \mathsf{E}^*$ is not. More exactly, the adjoint space to the injective symmetric tensor product of the Hilbert space is the symmetric projective tensor product of the Hilbert space [**36**, p. 112] which contains a copy of ℓ_1. Note that since $\mathcal{P}_{\mathfrak{h}}({}^n\mathsf{E})$ contains all polynomials of finite type and it is a proper subspace of $\mathcal{P}_w({}^n\mathsf{E})$, the closure of $\mathcal{P}_{\mathfrak{h}}({}^n\mathsf{E})$ in $\mathcal{P}({}^n\mathsf{E})$ coincides with $\mathcal{P}_w({}^n\mathsf{E})$. □

A polynomial P is said to be an *integral* polynomial if there exists a regular Borel measure μ of finite variation on $\bar{B}^*$, endowed with the weak star topology, and such that

$$P(x) = \int_{B^*} (\phi(x))^n \, d\mu(\phi) \tag{1.13}$$

for all $x \in \mathsf{E}$, where $\phi \in B^*$ and B^* is the closed unit ball of E^*. The space of all n-homogeneous integral polynomials with the norm

$$\|P\|_I := \inf\left\{\|\mu\| : \mu \text{ is a regular finite Borel measure satisfying (1.13)}\right\}$$

is denoted by $\mathcal{P}_I({}^n\mathsf{E})$.

PROPOSITION 1.31. The space $\mathcal{P}_I({}^n\mathsf{E})$ is a proper dense subspace of $\mathcal{P}_\mathfrak{h}({}^n\mathsf{E})$.

PROOF. Since $\mathcal{P}_I({}^n\mathsf{E}) = (\odot^n_\varepsilon \mathsf{E})^*$ and $\odot^n_\mathfrak{h}\mathsf{E}$ is a proper subspace of $\odot^n_\varepsilon\mathsf{E}$, thus $\mathcal{P}_I({}^n\mathsf{E})$ is a proper subspace of $(\odot^n_\mathfrak{h}\mathsf{E})^* = \mathcal{P}_\mathfrak{h}({}^n\mathsf{E})$. The density of $\mathcal{P}_I({}^n\mathsf{E})$ follows from the fact that $\mathcal{P}_I({}^n\mathsf{E})$ contains all polynomials

$$\left\{\left\langle \cdot \mid e^{\otimes(k)}_{\{i\}} \right\rangle_{\otimes^n_\mathfrak{h}\mathsf{E}} : e^{\otimes(k)}_{\{i\}} \in \mathscr{E}_n\right\}$$

that form an orthonormal basis in the Hilbert space

$$\odot^n_\mathfrak{h}\mathsf{E}^* \simeq \mathcal{P}_\mathfrak{h}({}^n\mathsf{E})$$

and their linear span as well. □

Therefore Proposition 1.30 and Proposition 1.31 imply that the space of Hilbert-Schmidt polynomials lies strictly between the space of integral and weakly (sequentially) continuous polynomials.

1.7. Reproducing kernels

Let $\mathcal{Q}$ be an abstract set and $\mathcal{H}$ be a Hilbert space of complex valued functions on $\mathcal{Q}$ equipped with the scalar product $\langle \cdot \mid \cdot \rangle_\mathcal{H}$.

DEFINITION 1.32. A function $K(x,z)$ defined on $\mathcal{Q} \times \mathcal{Q}$ is called *reproducing kernel* of a closed subspace $\mathcal{H}_K \subset \mathcal{H}$ if:

(i) for any fixed $z \in \mathcal{Q}$, the kernel $K(x,z)$ belongs to $\mathcal{H}_K$ as a function of $x \in \mathcal{Q}$;
(ii) for any $f \in \mathcal{H}_K$ and for any $z \in \mathcal{Q}$,
$$f(z) = \langle f(\cdot) \mid K(\cdot, z)\rangle_\mathcal{H}.$$

The space $\mathcal{H}_K$ is called a *reproducing kernel Hilbert space.*

THEOREM 1.33. *If a $\mathcal{H}$-valued function $h\colon \mathcal{Q} \longrightarrow \mathcal{H}$ is such that*

$$f(x) = \langle f(\cdot) \mid h(x)\rangle_\mathcal{H}, \qquad x \in \mathcal{Q}$$

for every $f \in \mathcal{H}_K$, then the function

$$K(x,z) = \langle h(z) \mid h(x)\rangle_\mathcal{H}, \qquad x \in \mathcal{Q}$$

is a reproduction kernel of $\mathcal{H}_K$.

THEOREM 1.34. *For a Hilbert space $\mathcal{H}$ of functions*

$$\mathcal{Q} \ni x \longmapsto f(x)$$

there exists a reproducing kernel $K(x,z)$ of $\mathcal{H}$ if and only if for any point $x \in \mathcal{Q}$ the point evaluation

$$\delta_x \colon f \longmapsto f(x), \qquad f \in \mathcal{H}$$

is a continuous linear functional on $\mathcal{H}$.

The next theorems deliver some elementary properties of reproducing kernels.

THEOREM 1.35. *If a reproducing kernel Hilbert space $\mathcal{H}_K$ is a subspace of $\mathcal{H}$, then*

$$f(x) = \langle f(\cdot) \mid K(\cdot, x)\rangle_{\mathcal{H}}, \qquad x \in \mathcal{Q}$$

gives a projection from $\mathcal{H}$ onto $\mathcal{H}_K$.

THEOREM 1.36. *For any subspace $\mathcal{H}_0$ of a reproducing kernel Hilbert space $\mathcal{H}_K$, there exists the reproducing kernel $K_0(x,z)$ for $\mathcal{H}_0$ and it is given by*

$$K_0(x,z) = \langle P_0 K(\cdot, z) \mid K(x, \cdot)\rangle_{\mathcal{H}_K}$$

for the orthogonal projection P_0 from $\mathcal{H}_K$ onto $\mathcal{H}_0$.

Note also that the sum $K_1 + K_2$ of reproducing kernels K_1 and K_2 for the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is a reproducing kernel for the orthogonal Hilbertian sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ and the product $K_1 \cdot K_2$ is a reproducing kernel for the Hilbertian tensor product $\mathcal{H}_1 \otimes_{\mathfrak{h}} \mathcal{H}_2$.

Notes and remarks. For general theory of analytic functions on Banach and locally convex spaces we refer the reader to [**1, 35, 36, 44, 50, 76**].

We essentially use some classical results. Among them the relationship between n-homogeneous polynomials and n-linear mappings with the following application to the Taylor series, established by A.D. Michael and his successors [**71, 72, 73**], based on the polarization formula, which has been independently established by R. Martin [**67**] and by S. Mazur and W. Orlicz [**68, 69**]. Detailed descriptions can be found in the classic book [**52**] and in the historical notes [**74**].

The spectral theory of algebras of analytic functions developed in this book essentially uses a nonlinear version of the Hahn-Banach theorem, which was established in [**4**]. A relationship between the continuity and local boundedness, using in the theory of bounded type analytic functions, was proved in [**18**].

Proofs of basic properties of polynomials and tensor products can be find in [**36**]. Theorem 1.10 was proved in [**85**]. In [**6**] was introduced the radius function of linear functionals on $\mathcal{H}_b(X)$ and proved Theorem 1.17 and Theorem 1.18. Theorem 1.19 and Theorem 1.20 are partial cases of results in [**32**]. Extensions of polynomials to ultrapowers were investigated in [**54**]. Propositions 1.23, 1.25, 1.26 and Corollary 1.24 were proved in [**9**].

Apparently, first applications of the Hilbert–Schmidt type analytic functions appeared in [**39**]. The abstract reproducing kernel theory was developed by N. Aronzajn [**11**], see also Saitoh book [**90**] for proofs of Theorems 1.33, 1.34, 1.35 and 1.36.

CHAPTER 2

Descriptions of topological spectra and applications

Let $\mathcal{A}$ be a complex commutative topological algebra. Let us denote by $M(\mathcal{A})$ the topological spectrum (set of continuous characters, which also are called continuous complex-valued homomorphisms) of $\mathcal{A}$. It is well-known from the theory of commutative algebras that there is a bijective correspondents between maximal ideals of $\mathcal{A}$ and its complex continuous homomorphisms. So, we can identify $M(\mathcal{A})$ with the set of all closed maximal ideals of $\mathcal{A}$.

Recall that $\mathcal{A}$ is a *semisimple* algebra if the complex homomorphisms in $M(\mathcal{A})$ separate points of $\mathcal{A}$. Let $\mathcal{A}$ be a semisimple commutative Fréchet algebra. Then $\mathcal{A}$ is isomorphic to some subalgebra of continuous functions on $M(\mathcal{A})$ endowed with a natural topology. More exactly, for every $a \in \mathcal{A}$ there exists a function

$$\widehat{a}\colon M(\mathcal{A}) \longrightarrow \mathbb{C}, \qquad \widehat{a}(\phi) := \phi(a)$$

with $\phi \in M(\mathcal{A})$. The weakest topology on $M(\mathcal{A})$ such that all functions $\widehat{a}$ with $a \in \mathcal{A}$ are continuous is called the *Gelfand topology.* The Gelfand topology coincides with the weak-star topology of the strong dual space $\mathcal{A}'$, restricted to $M(\mathcal{A})$. If $\mathcal{A}$ is a Banach algebra, $M(\mathcal{A})$ is a weak-star compact subset of the unit ball of $\mathcal{A}'$.

The map

$$\mathcal{A} \ni a \rightsquigarrow \widehat{a} \in C(M(\mathcal{A}))$$

is called the *Gelfand transform* of $\mathcal{A}$, where $C(M(\mathcal{A}))$ is the algebra of all continuous functions on $M(\mathcal{A})$.

If $\mathcal{A}$ is a uniform algebra of continuous complex functions f on a metric space $\mathcal{Q}$ then for every $x \in \mathcal{Q}$ the *point evaluation functional*

$$\delta_x\colon f \longmapsto f(x) \qquad \text{(denoted also as } \delta(x) := \delta_x)$$

belongs to $M(\mathcal{A})$.

Let us consider several important examples of spectra. Let $\mathcal{Q}$ be a metric spaces and $C_b(\mathcal{Q})$ be the uniform Banach algebra of all bounded continuous functions on $\mathcal{Q}$. Then the topological spectrum of $C_b(\mathcal{Q})$ coincides with the the Czech-Stone compactification $\beta\mathcal{Q}$ of $\mathcal{Q}$. That is, every function $f \in C_b(\mathcal{Q})$ can be extended to a continuous function $\widehat{f}$ on $\beta\mathcal{Q}$ and for every point $x \in \beta\mathcal{Q}$ the map

$$f \longmapsto \widehat{f}(x)$$

is a complex homomorphism of $C_b(\mathcal{Q})$.

Let $\mathcal{A}(\Omega)$ be a uniform algebra of all analytic functions on an open domain $\Omega \subset \mathbb{C}^n$ which are continuous on the closure $\bar{\Omega}$. Then $M(\mathcal{A}(\Omega))$ is the *polynomially convex hull* $[\Omega]$ of Ω (see [43] for details), where $[\Omega]$ is defined as a subset of all point $x \in \mathbb{C}^n$

such that for every polynomial $p \in \mathcal{P}(\mathbb{C}^n)$, the inequality $|p(x)| \leq \sup_{z\in\Omega} |p(z)|$ holds. A set is *polynomially convex* if it coincides with its polynomially convex hull.

If Ω is convex, then its polynomially convex hull $[\Omega]$ coincides with the closure $\bar{\Omega}$. In particular, if $\Omega = \mathbb{C}^n$, then $\mathcal{A}(\Omega)$ is the algebra $\mathcal{H}(\mathbb{C}^n)$ of all entire functions on $\mathbb{C}^n$ and its topological spectrum coincides with all point evaluation functionals $\delta(x)$ defined by $x \in \mathbb{C}^n$.

Previous examples show that we can think the topological spectrum of a uniform algebra as a maximal natural domain such that all elements of this algebra can be considered as a continuous function on this domain.

This chapter is devoted to study of spectra in the case of uniform algebras of complex analytic functions with domains in infinite-dimensional Banach spaces.

2.1. Spectra of algebras of polynomials

LEMMA 2.1. (Aron, Cole, Gamelin). *Let Y be a complex vector space. Let $F = (f_1, \ldots, f_n)$ be a map from Y to $\mathbb{C}^n$ such that the restriction of each f_j to any finite dimensional subspace of Y is a polynomial. Then the closure of the range of F is an algebraic variety.*

PROOF. Let Y_0 be a finite dimensional subspace of Y. It is well known to algebraic geometry that the closure $F(Y_0)^-$ of $F(Y_0)$ is an irreducible algebraic variety of dimension $k \leq n$. Without loss of generality, we can assume that Y_0 is chosen so that the dimension k of $F(Y_0)^-$ is a maximum. If Y_1 is any finite dimensional subspace of Y such that

$$Y_1 \supseteq Y_0$$

then $F(Y_1)^-$ is also an irreducible algebraic variety of dimension k, which contains $F(Y_0)^-$. It follows that

$$F(Y_1)^- = F(Y_0)^-,$$

and we conclude that $F(Y_0)^- = F(Y)^-$. □

THEOREM 2.2. (Aron, Cole, Gamelin). *Let Y be a complex vector space. Let $\mathcal{A}$ be an algebra of functions on Y such that the restriction of each $f \in \mathcal{A}$ to any finite dimensional subspace of Y is an analytic polynomial. Let J be a proper ideal in $\mathcal{A}$. Then there is a net (y_α) in Y such that*

$$f(y_\alpha) \to 0 \quad \text{for all} \quad f \in J.$$

PROOF. Suppose that the conclusion fails. Then there are $(f_1, \ldots, f_n) \in J$ such that

$$\max\big(|f_1(y)|, \ldots, |f_n(y)|\big) \geq 1, \qquad y \in Y.$$

Let F be the map from Y to $\mathbb{C}^n$ having components $f_1, \ldots, f_n$. Let V be an algebraic variety which does not contain 0. Hence there is a polynomial p on $\mathbb{C}^n$ such that $p = 0$ on V and $p(0) = 1$. Since the functions p together with the coordinate functions $z_1, \ldots, z_n$ have no common zero, the ideal generated by them in the polynomial ring on $\mathbb{C}^n$ is not proper (via the Hilbert Nullstellensatz). So, there exist polynomials $q_0, q_1, \ldots, q_n$ on $\mathbb{C}^n$ such that

$$pq_0 + z_1 q_1 + \cdots + z_n q_n = 1 \quad \text{on } \mathbb{C}^n,$$

implying

$$z_1 q_1 + \cdots + z_n q_n = 1 \quad \text{on } V.$$

Now let $g_1, \ldots, g_n \in \mathcal{A}$ be the compositions of $q_1, \ldots, q_n$ with F, respectively. Then

$$f_1 g_1 + \cdots + f_n g_n = 1,$$

and the ideal J is not proper. □

COROLLARY 2.3. Let ϕ be any (possibly discontinuous) complex-valued homomorphism on the Fréchet algebra of entire $\mathbb{C}$-valued functions $\mathcal{H}_b(X)$ on the Banach space X. Then there is a net (x_α) in X such that

$$P(x_\alpha) \to \phi(P)$$

for all analytic polynomials P on X.

For a given algebra $\mathcal{A}$ of continuous functions on a Banach space X we define an *$\mathcal{A}$-topology* on X as the weakest topology such that all functions of $\mathcal{A}$ are continuous. That is $\mathcal{A}$-topology is the restriction of the Gelfand topology to X.

We say that a net x_α is $\mathcal{A}$-convergent (notation $x_\alpha \xrightarrow{\mathcal{A}} \phi$) if $f(x_\alpha)$ is convergent for every $f \in \mathcal{A}$.

PROPOSITION 2.4. Let $\mathcal{P}_0(X)$ be a subalgebra of the algebra $\mathcal{P}(X)$ on the Banach space X. Then for every bounded $\mathcal{P}_0$-convergent net $(x_\alpha) \in X$ there is a continuous complex-valued homomorphism ϕ on $\mathcal{P}_0(X)$ such that

$$P(x_\alpha) \to \phi(P)$$

for each $P \in \mathcal{P}_0(X)$.

PROOF. It is easy to see that

$$\phi(P) := \lim_\alpha P(x_\alpha)$$

is a complex-valued homomorphism on $\mathcal{P}_0(X)$. From the boundedness of (x_α) it follows that ϕ is continuous. □

THEOREM 2.5. *Let $\mathcal{P}_0(X)$ be a subalgebra of $\mathcal{P}(X)$ with unity which contains all finite type polynomials. Let J be an ideal in $\mathcal{P}_0(X)$ which is generated by a finite sequence of polynomials $P_1, \ldots, P_n \in \mathcal{P}_0(X)$. If the polynomials $P_1, \ldots, P_n$ have no common zeros, then J is not proper.*

PROOF. According to Lemma 2.1 there exists a finite dimensional subspace $Y_0 = \mathbb{C}^m \subset X$ such that

$$F(Y_0)^- = F(X)^-,$$

where $F(x) = (P_1(x), \ldots P_n(x))$.

Let $e_1, \ldots, e_m$ be a basis in Y_0 and $e_1^*, \ldots, e_m^*$ be the coordinate functionals. Denote by $P_k\,|_{Y_0}$ the restriction of P_k to Y_0. Since $\dim Y_0 = m < \infty$, there exists a continuous projection $T\colon X \longrightarrow Y_0$. So, any polynomial $Q \in \mathcal{P}(Y_0)$ can be extended to a polynomial $\hat{Q} \in \mathcal{P}_0(X)$ by the formula

$$\hat{Q}(x) = Q\big(T(x)\big).$$

Then $\hat{Q}$ belongs to $\mathcal{P}_0(X)$ because it is a finite type polynomial. Let us consider the map

$$G(x) = \Big(P_1(x), \ldots, P_n(x), \widehat{e_1^*}(x), \ldots, \widehat{e_m^*}(x)\Big) : X \longrightarrow \mathbb{C}^{m+n}.$$

By definition of G, we have $G(X)^- = G(Y_0)^-$.

Suppose that J is a proper ideal in $\mathcal{P}_0(X)$, and so, J is contained in a maximal ideal J_M. Let ϕ be a complex homomorphism such that

$$J_M = \ker \phi.$$

By Theorem 2.2 there exists a $\mathcal{P}_0$-convergent net (x_α) such that

$$\phi(P) = \lim_\alpha P(x_\alpha)$$

for every $P \in \mathcal{P}_0(X)$. Since $G(X)^- = G(Y_0)^-$, there is a net $(z_\beta) \subset Y_0$ such that

$$\lim_\alpha G(x_\alpha) = \lim_\beta G(z_\beta).$$

Note that each polynomial $Q \in \mathcal{P}(Y_0)$ is generated by the coordinate functionals. Thus,

$$\lim_\beta Q(z_\beta) = \lim_\alpha \hat{Q}(x_\alpha) = \phi(Q).$$

Also

$$\lim_\beta P_k \,|_{Y_0}\, (z_\beta) = \lim_\alpha P_k(x_\alpha) = \phi(P_k)$$

for $k = 1, \ldots, n$. On the other hand, every $\mathcal{P}_0$-convergent net on a finite dimensional subspace is weakly convergent and so it converges to a point $x_0 \in Y_0 \subset X$. Thus,

$$P_k(x_0) = 0, \qquad 1 \le k \le n$$

that contradicts the assumption that $P_1, \ldots, P_n$ have no common zeros. □

Note that we also proved that each complex homomorphism

$$\phi\colon \mathcal{P}_0(X) \longrightarrow \mathbb{C}$$

is a *local evaluation*, which means that for a given $P_1, \ldots, P_n \in \mathcal{P}_0(X)$ there exists $x_0 \in X$ such that

$$\phi(P_k) = P_k(x_0), \qquad k = 1, \ldots, n.$$

For an ideal $J \subset \mathcal{P}_0(X)$, let $V(J) \subset X$ denotes the *zero* of J, that is, the common set of zeros of all polynomials belonging to J.

Let G be a subset of X and $I(G)$ denotes the *hull* of G, that is, a set of all polynomials in $\mathcal{P}_0(X)$ which vanish on G.

The set $\operatorname{Rad} J \subset \mathcal{P}(X)$ is called the *radical* of $J \subset \mathcal{P}_0(X)$ if $P^k \in J$ for some positive integer k implies $P \in \operatorname{Rad} J$.

A polynomial $P \in \mathcal{P}(X)$ is called a *radical* if it can be represented by a product of mutually different irreducible polynomials. In this case $\operatorname{Rad} P$ is generated by P, i.e.,

$$\operatorname{Rad} P = (P).$$

A subalgebra $\mathcal{A}_0$ of an algebra $\mathcal{A}$ is called *factorial* if for every $f \in \mathcal{A}_0$ the equality $f = f_1 f_2$ implies that $f_1 \in \mathcal{A}_0$ and $f_2 \in \mathcal{A}_0$.

Using a standard idea from Algebraic Geometry, now we can prove the next theorem which is a generalization of the well known Hilbert Nullstellensatz for algebras of polynomials of infinitely many variables.

THEOREM 2.6. *Let $\mathcal{P}_0(X)$ be a factorial subalgebra in $\mathcal{P}(X)$ which contains all polynomials of finite type and J be an ideal in $\mathcal{P}_0(X)$ which is generated by a finite sequence of polynomials $P_1, \dots, P_n \in \mathcal{P}_0(X)$. Then*

$$\operatorname{Rad} J \subset \mathcal{P}_0(X)$$

and the equality

$$I[V(J)] = \operatorname{Rad} J$$

holds in $\mathcal{P}_0(X)$.

PROOF. Since $\mathcal{P}_0(X)$ is factorial,

$$\operatorname{Rad} J \subset \mathcal{P}_0(X)$$

for every ideal $J \subset \mathcal{P}_0(X)$. Evidently, $I[V(J)] \supset \operatorname{Rad} J$. Let $P \in \mathcal{P}_0(X)$ and $P(x) = 0$ for every $x \in V(J)$. Let $y \in \mathbb{C}$ be an additional "independent variable" which is associated with a basis vector e of an extra dimension. Consider a Banach space

$$X \oplus \mathbb{C}e = \{x + ye \colon x \in X, y \in \mathbb{C}\}.$$

We denote by $\mathcal{P}_0(X) \otimes \mathcal{P}(\mathbb{C})$ the algebra of polynomials on $X \oplus \mathbb{C}e$ such that every polynomial in $\mathcal{P}_0(X) \otimes \mathcal{P}(\mathbb{C})$ belongs to $\mathcal{P}_0(X)$ for arbitrary $y \in \mathbb{C}$. The polynomials $P_1, \dots, P_n, Py - 1$ have no common zeros. By Theorem 2.5 there are polynomials $Q_1, \dots, Q_{n+1} \in \mathcal{P}_0(X) \otimes \mathcal{P}(\mathbb{C})$ such that

$$\sum_{i=1}^{n} P_i Q_i + (Py - 1) Q_{n+1} \equiv 1.$$

Since it is an identity, it will be still true for all vectors x such that $P(x) \neq 0$, if we substitute $y = 1/P(x)$. Thus

$$\sum_{i=1}^{n} P_i(x) Q_i\big(x, 1/P(x)\big) = 1.$$

Taking a common denominator, we find that for some positive integer N,

$$\sum_{i=1}^{n} P_i(x) Q_i'(x) P^{-N}(x) = 1 \quad \text{or} \quad \sum_{i=1}^{n} P_i(x) Q_i'(x) = P^N(x), \tag{2.1}$$

where $Q'(x) = Q(x, P^{-1}) P^N(x) \in \mathcal{P}_0(X)$. The equality (2.1) holds on an open subset $X \setminus \ker P$, so it holds for every $x \in X$. But it means that P^N belongs to J. So, $P \in \operatorname{Rad} J$. $\square$

COROLLARY 2.7. Suppose that $\ker P$ with $P \in \mathcal{P}(X)$ contains a linear subspace Z of codimension one. Then there exists a polynomial $Q \in \mathcal{P}(X)$ and a linear functional L such that

$$P = QL.$$

PROOF. Let L be a linear functional on X such that $\ker L = Z$. By Theorem 2.6 L divides P^N for some positive integer N. So, L divides P. □

COROLLARY 2.8. Suppose that $\ker P$ with $P \in \mathcal{P}(X)$ is a union of a finite sequence of linear subspaces. Then P is a product of a finite sequence of linear functionals.

PROOF. From the Hahn-Banach Theorem it follows that $\ker P$ is contained in a finite union of one codimensional linear subspaces. So, P is a factor of a product of linear functionals. Thus, P is a product of a finite sequence of linear functionals. □

THEOREM 2.9. *Let $\mathcal{P}_0(X)$ be a factorial subalgebra in $\mathcal{P}(X)$ which contains all polynomials of finite type and has the following property: if $Q \in \mathcal{P}_0(X)$ and*

$$Q = Q_1 + \cdots + Q_n$$

is the (necessary unique) representation of Q by homogeneous polynomials, then all Q_k belong to $\mathcal{P}_0(X)$.

If $P \in \mathcal{P}(X)$ is continuous in the weakest topology on X, with respect to which all polynomials in $\mathcal{P}_0(X)$ are continuous, then $P \in \mathcal{P}_0(X)$.

PROOF. Without loss of generality, we can assume that $P \in \mathcal{P}(X)$ is m-homogeneous and irreducible. By theorem assumptions, the polynomial P must be bounded on the set

$$\{x \in X : |P_1(x)| < 1, \ldots, |P_n(x)| < 1\}$$

with some $P_1, \ldots, P_n \in \mathcal{P}_0(X)$.

Let J be an ideal generated by $P_1, \ldots, P_n$. If $x_0 \in V(J)$, then $tx_0 \in V(J)$ for every number t. So, P is bonded on the subspace $\{tx_0 : t \in \mathbb{C}\}$. But this is possible only if P is an identical zero on this subset. Hence,

$$V(J) \subset \ker P.$$

Denote by $\mathcal{A}_0$ a minimal factorial algebra which contains $\mathcal{P}_0(X)$ and P. By Theorem 2.6 there are $Q_1, \ldots, Q_n \in \mathcal{A}_0$ such that

$$P_1 Q_1 + \cdots + P_n Q_n = P.$$

We can assume that Q_k with $k = 1, \ldots, n$ are homogeneous and

$$\begin{cases} \deg Q_k + \deg P_k = m & \text{if} \quad \deg P_k \leq m \\ Q_k = 0 & \text{if} \quad \deg P_k > m. \end{cases}$$

Indeed, let $Q_k = \sum_j Q_k^j$ is the decomposition of Q_k by j-homogeneous polynomials. Then

$$\sum_{k=1}^{n} P_k Q_k = \sum_{k=1}^{n} P_k Q_k^{m-\deg P_k} + \sum_{k=1}^{n} P_k \sum_{j/=m-\deg P_k} Q_k^j = P.$$

Since the partial sum

$$\sum_{k=1}^{n} P_k \sum_{j/=m-\deg P_k} Q_k^j$$

contains no m-homogeneous polynomials and $\deg P = m$, we obtain

$$\sum_{k=1}^{n} P_k \sum_{j/=m-\deg P_k} Q_k^j = 0.$$

Putting $Q_k = Q_k^{m-\deg P_k}$, we have the required restrictions for Q_k. Since P is irreducible and $\deg Q_k < \deg P = m$, the polynomial Q_k belongs to $\mathcal{P}_0(X) \subset \mathcal{A}_0$ for every k. Therefore $P \in \mathcal{P}_0(X)$. □

We say that a set $\mathcal{V} \subset X$ is an *algebraic set of finite type* if $\mathcal{V}$ is the set of common zeros of some finite sequence of polynomials $P_1, \dots, P_n \in \mathcal{P}(X)$. $\mathcal{V}$ is called an *algebraic variety of finite type* if the ideal $(P_1, \dots, P_n)$, generated by $P_1, \dots, P_n$, is prime.

Let $\mathcal{V} = V(P_1, \dots, P_n)$ be an algebraic set of finite type. We can define an algebra of polynomials on $\mathcal{V}$ as a quotient algebra

$$\mathcal{P}(\mathcal{V}) := \mathcal{P}(X)/I(\mathcal{V}).$$

From Theorem 2.6 it follows that a polynomials P is the identical zero in $\mathcal{P}(\mathcal{V})$ if and only if $P^N \in (P_1, \dots, P_n)$ for some N and $\mathcal{P}(\mathcal{V})$ is an integral domain if and only if the ideal $(P_1, \dots, P_n)$ is prime.

THEOREM 2.10. *Let ϕ be a complex homomorphism (possible discontinuous) of $\mathcal{P}(\mathcal{V})$. Then there is a net $(x_\alpha) \subset \mathcal{V}$ such that*

$$\phi(P) = \lim_{\alpha} P(x_\alpha)$$

for every $P \in \mathcal{P}(\mathcal{V})$.

PROOF. Note first that each complex homomorphism of $\mathcal{P}(\mathcal{V})$ is a local evaluation at $\mathcal{V}$. Indeed, if ϕ is a complex homomorphism of $\mathcal{P}(\mathcal{V})$, then ϕ may be considered as a complex homomorphism of $\mathcal{P}(X)$ which vanishes on $I(\mathcal{V})$. As we have indicated, ϕ must be a local evaluation at points of x, that is, for every polynomials $P_1, \dots, P_n \in \mathcal{P}(X)$ there exists $x_0 \in X$ such that

$$\phi(P_k) = P_k(x_0).$$

Since ϕ vanishes on $I(\mathcal{V})$, we have $x_0 \in \mathcal{V}$. Thus, for every $Q_1, \dots, Q_n \in \mathcal{P}(\mathcal{V})$ there exists $x_0 \in \mathcal{V}$ such that $\phi(Q_k) = Q_k(x_0)$ with $1 \le k \le n$.

Consider the set of zeros of all finitely generated ideals in $\mathcal{P}(\mathcal{V})$:

$$\left\{ V_\alpha = \bigcap_{k=1}^{m} \ker\left[P_{\alpha,k} - \phi(P_{\alpha,k})\right] : P_{\alpha,k} \in \mathcal{P}(X) \right\}.$$

Each V_α is nonempty and the set $\{V_\alpha\}$ is naturally ordered by inclusion. Let $(x_\alpha) \subset \mathcal{V}$ be a net such that $x_\alpha \in V_\alpha$. It is clear, $\phi(P) = \lim_\alpha P(x_\alpha)$ for every $P \in \mathcal{P}(\mathcal{V})$. □

2.2. Applications for symmetric polynomials

Let $\mathcal{G}$ be a group of linear isometries of a Banach space X. A subset V of X is said to be *$\mathcal{G}$-symmetric* if it is invariant under the action of $\mathcal{G}$ on X. A function with a $\mathcal{G}$-symmetric domain is *$\mathcal{G}$-symmetric* if

$$f(\sigma(x)) = f(x)$$

for every $\sigma \in \mathcal{G}$, where $x \in X$. It is clear that the kernel of a $\mathcal{G}$-symmetric polynomial is $\mathcal{G}$-symmetric. We consider the question: under which conditions a polynomial with a $\mathcal{G}$-symmetric set of zeros is $\mathcal{G}$-symmetric?

First we observe that if $P \in \mathcal{P}(X)$ is an irreducible polynomial then $P \circ \sigma$ is irreducible for every $\sigma \in \mathcal{G}$. Indeed, if $P(\sigma(x)) = P_1(x)P_2(x)$, then

$$P(x) = P_1(\sigma^{-1}(x))P_2(\sigma^{-1}(x)), \qquad x \in X.$$

Recall that a group homomorphism from $\mathcal{G}$ to $S^1 = \{e^{i\vartheta} : 0 \leq \vartheta < 2\pi\}$ is called a *character* of $\mathcal{G}$.

PROPOSITION 2.11. Suppose that $\mathcal{G}$ has no nontrivial characters. If $P \in \mathcal{P}(X)$ is radical and $\ker P$ is a $\mathcal{G}$-symmetric set, then P is a $\mathcal{G}$-symmetric polynomial.

PROOF. Since

$$\ker P = \ker(P \circ \sigma)$$

for every $\sigma \in \mathcal{G}$, then, by Theorem 2.6, $P = c(P \circ \sigma)$ for some constant $c = c(\sigma)$. Because σ is an isometry, $|c| = 1$. If $c \neq 1$, then $c = c(\sigma)$ is a nontrivial character of $\mathcal{G}$. So, $c = 1$. □

Suppose for example that $\mathcal{G} = S^1$, i.e., the group acts on X as follows $x \rightsquigarrow e^{i\vartheta}x$. Then a homogeneous polynomial is $\mathcal{G}$-symmetric only if it is a constant. However, zero set of any homogeneous polynomial is S^1-symmetric.

Note that the subset of all $\mathcal{G}$-symmetric polynomials is a subalgebra in $\mathcal{P}(X)$.

THEOREM 2.12. *Suppose that the algebra of $\mathcal{G}$-symmetric polynomials on X is factorial and $\mathcal{G}$ has no nontrivial characters. Then the kernel of a $\mathcal{G}$-symmetric polynomial P is $\mathcal{G}$-symmetric if and only if P is $\mathcal{G}$-symmetric.*

PROOF. Let $k_1, \dots, k_n \in \mathbb{N}$ and

$$P = P_1^{k_1} \dots P_n^{k_n},$$

where $P_1, \dots, P_n$ are mutually different irreducible polynomials, Then the product $P_1 \dots P_n$ has the same set of zero that P. So, if $\ker P$ is $\mathcal{G}$-symmetric, then $P_1 \dots P_n$ is $\mathcal{G}$-symmetric by Proposition 2.11. On the other hand, by the theorem assumption, all polynomials $P_1, \dots, P_n$ must be $\mathcal{G}$-symmetric. So, P is $\mathcal{G}$-symmetric as well. □

Note that if there exist a $\mathcal{G}$-symmetric polynomial $P = P_1P_2$ such that P_1 is not $\mathcal{G}$-symmetric, then $P_1^2P_2$ is a not $\mathcal{G}$-symmetric polynomial with a $\mathcal{G}$-symmetric kernel.

If X is the infinite-dimensional space ℓ_p, $(1 \leq p < \infty)$ of sequences

$$x = (x_i)_{i \in \mathbb{N}} \subset \mathbb{C}$$

and $\mathcal{G}$ is the group of permutations of basis elements, then it is not difficult to see that the algebra of $\mathcal{G}$-symmetric polynomial is factorial and $\mathcal{G}$ has no nontrivial characters. For any finite-dimensional space there exists a nonsymmetric polynomial which has a symmetric kernel. For example, the polynomial

$$P(x) = x_1^2 x_2 \dots x_n$$

has a symmetric kernel in $\mathbb{C}^n$, but it is not symmetric if $n > 1$.

Note that the algebra $\mathcal{P}_s(\ell_p)$ of symmetric polynomials on the space ℓ_p with respect to the group of permutations of basis elements $(\mathfrak{e}_k) \subset \ell_p$ does not satisfy the conditions of Theorem 2.6. However, this theorem is still true for this algebra. For simplicity we consider the case of ℓ_1-space.

THEOREM 2.13. *The elementary symmetric polynomials* $(\Theta_i)_{i\in\mathbb{N}}$,

$$\Theta_i(x) = \sum_{k_1<\dots<k_i} x_{k_1}\dots x_{k_i}, \qquad x = \sum_{i\in\mathbb{N}} x_i \mathfrak{e}_i \in \ell_1, \qquad k_1,\dots,k_n \in \mathbb{N}.$$

form an algebraic basis in the algebra $\mathcal{P}_s(\ell_1)$. *It means that every symmetric polynomial* $Q \in \mathcal{P}_s(\ell_1)$ *can be represented by the way*

$$Q(x) = q(\Theta_1(x),\dots,\Theta_n(x)), \tag{2.2}$$

where q *is a polynomial in* $\mathcal{P}(\mathbb{C}^n)$ *and* $(\Theta_i)_{i\in\mathbb{N}}$, *are algebraically independent, that is, if* $p(\Theta_1(x),\dots,\Theta_n(x)) \equiv 0$ *for some* $p \in \mathcal{P}(\mathbb{C}^n)$, *then* $p \equiv 0$.

PROOF. It is well known from Algebra (see [94]) that for any symmetric complex polynomial $Q^{(m)} \in \mathcal{P}_s(\mathbb{C}^m)$ with $\deg Q^{(m)} = n$ there is a polynomial $q \in \mathcal{P}(\mathbb{C}^n)$ such that

$$Q^{(m)}(x) = q\left(\Theta_1^{(m)}(x),\dots,\Theta_n^{(m)}(x)\right),$$

where

$$\Theta_i^{(m)}(x) = \sum_{k_1<\dots<k_i}^{m} x_{k_1}\dots x_{k_i}.$$

Let $V_m = \operatorname{span}\{\mathfrak{e}_1,\dots,\mathfrak{e}_m\}$ is the m-dimensional subspace in ℓ_1 spanned by the basis elements $\{\mathfrak{e}_1,\dots,\mathfrak{e}_m\}$. We set

$$T_m\colon \sum_{i\in\mathbb{N}} x_i\mathfrak{e}_i \longmapsto \sum_{i=1}^{m} x_i\mathfrak{e}_i$$

the projection from ℓ_1 to V_m. Let $Q \in \mathcal{P}_s(\ell_1)$, $\deg Q = n$. Then there exists a polynomial $q \in \mathcal{P}(\mathbb{C}^n)$ such that for every $m \geq n$ and for every $x \in \ell_1$

$$Q(T_m(x)) = q\left(\Theta_1^{(m)}(x),\dots,\Theta_n^{(m)}(x)\right).$$

Taking the limit as $m \to \infty$ we obtain (2.2).

To show that Θ_j are algebraically independent, we observe that for every $(\xi_1,\dots,\xi_n) \in \mathbb{C}^n$ there exists a vector $x_\xi = (x_1,\dots,x_n,0,0\dots) \in \ell_1$ such that

$$\Theta_1(x_\xi) = \xi_1,\dots,\Theta_n(x_\xi) = \xi_n. \tag{2.3}$$

Indeed, according to the Vieta formula, the solutions of the equation

$$x^n - \xi_1 x^{n-1} + \dots (-1)^n \xi_n = 0$$

satisfy the conditions $\Theta_i(x_1, \dots, x_n) = \xi_i$ and so $x_\xi = (x_1, \dots, x_n)$ is as required. If $g(\xi_1, \dots, \xi_n) \neq 0$ for some $(\xi_1, \dots, \xi_n) \in \mathbb{C}^n$, then $P(\Theta_1(x_\xi), \dots, \Theta_n(x_\xi)) \neq 0$. □

COROLLARY 2.14. Let $P_1, \dots, P_m \in \mathcal{P}_s(\ell_1)$ be such that

$$\ker P_1 \cap \dots \cap \ker P_m = \emptyset.$$

Then there are $Q_1, \dots, Q_m \in \mathcal{P}_s(\ell_1)$ such that

$$\sum_{i=1}^m P_i Q_i \equiv 1.$$

PROOF. Let $n = \max_i(\deg P_i)$. We may assume that

$$P_i(x) = g_i(\Theta_1(x), \dots, \Theta_n(x))$$

for some $g_i \in \mathcal{P}(\mathbb{C}^n)$. Let us suppose that at some point $\xi \in \mathbb{C}^n$, $\xi = (\xi_1, \dots, \xi_n)$, for which $g_i(\xi) = 0$. Then there is $x_\xi \in \ell_1$ such that $\Theta_i(x_0) = \xi_i$ (see formula 2.3). So, the common set of zeros of all g_i is empty. Thus, by the Hilbert Nullstellensatz there are polynomials $q_1, \dots, q_m$ such that

$$\sum_{i=1}^m g_i q_i \equiv 1.$$

Put $Q_i(x) = q_i(\Theta_1(x), \dots, \Theta_n(x))$. □

2.3. Polynomials on tensor products

We need some technical results about tensor products of tensor product spaces and polynomials on tensor products. Let X be a complex Banach space.

Let $P \in \mathcal{P}(^{km}X)$ for some positive integers m and k. Let F_P be the symmetric multilinear form associated with the polynomial P. Consider $F_P(x_1^m, \dots, x_k^m)$ for some $x_1, \dots, x_k \in X$. For any fixed

$$x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_k, \quad 1 \leq j \leq k,$$

$F_P(x_1^m, \dots, x_j^m, \dots, x_k^m)$ is an m-homogeneous polynomial of $x_j \in X$ and so it can be expressed as a value of a continuous linear functional on $\odot_\pi^m X$ at the point $x_j^{\otimes m}$. Since it is true for every $1 \leq j \leq k$, there is a continuous symmetric multilinear map $F_{P_{(m)}} : (\odot_\pi^m X)^m \longrightarrow \mathbb{C}$ such that

$$F_{P_{(m)}}(x_1^{\otimes m}, \dots, x_k^{\otimes m}) = F_P(x_1^m, \dots, x_k^m).$$

Let us denote $P_{(m)}(x^{\otimes m}) := F_{P_{(m)}}(x^{\otimes m}, \dots, x^{\otimes m})$. Since $\|x^{\otimes n}\| \leq 1$ if $\|x\| \leq 1$, we can see that

$$\|P\| \leq \|P_{(m)}\|. \tag{2.4}$$

Note that we have natural embeddings

$$\odot_\pi^{km} X \subset \odot_\pi^k (\odot_\pi^m X) \subset \otimes_\pi^{km} X.$$

Let w be an element in $\odot_\pi^{km} X$. Consider following norms for w. Let $\|w\|$ be the projective tensor norm on $\odot_\pi^{km} X$. That is,

$$\|w\| = \inf\left\{ \sum_{\{j\}\in\mathbb{N}^n} \prod_{k=1}^{n} \|x_{j_k}\| : w = \sum_{\{j\}\in\mathbb{N}^n} x_{j_1} \odot \cdots \odot x_{j_n} \right\},$$

where $n = km$ and infimum is taken under all above representations of w. According to (1.7) we can define

$$|\!|\!| w |\!|\!| := \inf\left\{ \sum_{j\in\mathbb{N}} \|x_j\|^{km} : w = \sum_{j\in\mathbb{N}} x_j^{\otimes km} \right\},$$

where $n = km$ and the infimum is taken over all representations of w. Also, we set

$$\|w\|_{(m)} := \inf\left\{ \sum_{\{j\}\in\mathbb{N}^m} \left\| \sum_{i\in\mathbb{N}} x_{ij_1} \odot \cdots \odot x_{ij_m} \right\|^k : \right.$$

$$\left. w = \sum_{\{j\}\in\mathbb{N}^m} \left(\sum_{i\in\mathbb{N}} x_{ij_1} \odot \cdots \odot x_{ij_m} \right)^{\otimes k} \right\},$$

$$|\!|\!| w |\!|\!|_{(m)} := \inf\left\{ \sum_{j\in\mathbb{N}} \left\| \sum_{i\in\mathbb{N}} x_{ij}^{\otimes m} \right\|^k : w = \sum_{j\in\mathbb{N}} \left(\sum_{i\in\mathbb{N}} x_{ij}^{\otimes m} \right)^{\otimes k} \right\}$$

and finally,

$$\|w\|_{(k)(m)} := \inf \sum_{\{j\}\in\mathbb{N}^m} \left(\sum_{\{i\}\in\mathbb{N}^k} \prod_{l=1}^{m} \|x_{i_1 j_l}\| \right) \cdots \left(\sum_{\{i\}\in\mathbb{N}^k} \prod_{l=1}^{m} \|x_{i_k j_l}\| \right),$$

where the infimum is taken over all representations

$$w = \sum_{\{j\}\in\mathbb{N}^m} \left(\sum_{i_1\in\mathbb{N}} x_{i_1 j_1} \odot \cdots \odot x_{i_1 j_m} \right) \odot \cdots \odot \left(\sum_{i_k\in\mathbb{N}} x_{i_k j_1} \odot \cdots \odot x_{i_k j_m} \right). \tag{2.5}$$

Let us observe that the representation

$$w = \sum_{j\in\mathbb{N}} \left(\sum_{i\in\mathbb{N}} x_{ij}^{\otimes m} \right)^{\otimes k}$$

is a partial case of the representation

$$w = \sum_{j_1,\ldots,j_m\in\mathbb{N}} \left(\sum_{i\in\mathbb{N}} x_{ij_1} \odot \cdots \odot x_{ij_m} \right)^{\otimes k}. \tag{2.6}$$

So, $\|w\|_{(m)} \le |\!|\!| w |\!|\!|_{(m)}$. Let

$$u_j = \sum_{i\in\mathbb{N}} x_{ij_1} \odot \cdots \odot x_{ij_m}. \tag{2.7}$$

Then by the polarization inequality (1.8),

$$\|u_j\| = \inf \sum_{i\in\mathbb{N}} \|x_{ij_1}\| \cdots \|x_{ij_m}\| \ge \frac{1}{c(m, X)} |\!|\!| u_j |\!|\!|,$$

where the infimum is taken over all representations (2.7). Combining (2.6) and (2.7), we can see that $\|w\|_{(m)} \geq |||w|||_{(m)}/\left[c(m,X)\right]^k$ or

$$\|w\|_{(m)} \leq |||w|||_{(m)} \leq \left[c(m,X)\right]^k \|w\|_{(m)}. \tag{2.8}$$

Now we observe that the representation

$$w = \sum_{\{j\}\in\mathbb{N}^m} \left(\sum_{i\in\mathbb{N}} x_{ij_1} \odot \cdots \odot x_{ij_m} \right)^{\otimes k} \tag{2.9}$$

is a partial case of (2.5). So,

$$\|w\|_{(k)(m)} \leq \|w\|_{(m)} \leq c(k, \odot_\pi^m X)\|w\|_{(k)(m)}. \tag{2.10}$$

On the other hand, the representation given by (2.5) is a partial case of

$$w = \sum_{\{j\}\in\mathbb{N}^n} x_{j_1} \odot \cdots \odot x_{j_n}.$$

Hence, for some constant $s_{k,m}$,

$$\|w\| \leq \|w\|_{(k)(m)} \leq s_{k,m}\|w\|. \tag{2.11}$$

Combining formulas (2.8),(2.10),(2.11) and taking into account

$$\|w\| \leq |||w||| \leq c(km, X)\|w\|,$$

we have the following inequality:

$$|||w||| \leq c(km, X)|||w|||_{(m)}.$$

Note that from formula (2.4) it follows that

$$|||w|||_{(m)} \leq |||w|||.$$

So, we have the following theorem.

THEOREM 2.15. *Let* $w \in \odot_\pi^{km} X$ *and* $P \in \mathcal{P}(^{km}X)$. *Then*

$$|||w|||_{(m)} \leq |||w||| \leq c(km, X)|||w|||_{(m)}$$

and

$$\|P\| \leq \|P_{(m)}\| \leq c(km, X)\|P\|.$$

Let now $n = k_1 + 2k_2 + \cdots + mk_m$ for some $k_1, \ldots, k_m \in \mathbb{N}$ and P be an n-homogeneous polynomial. We define a form $F^P_{k_1,\ldots,k_m}$ on the Cartesian product

$$X \times \odot_\pi^2 X \times \cdots \times \odot_\pi^m X$$

such that $F^P_{k_1,\ldots,k_m}\left(x_1, x_2^{\otimes 2}, \ldots, x_j^{\otimes j}, \ldots, x_m^{\otimes m}\right)$ is a k_j-homogeneous polynomial of the variable $x_j^{\otimes j}$ for any $1 \leq j \leq m$ if $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m$ are fixed and

$$F^P_{k_1,\ldots,k_m}(x, x^{\otimes 2}, \ldots, x^{\otimes m}) = P(x). \tag{2.12}$$

The map $F^P_{k_1,\ldots,k_m}$ is well defined and

$$\|F^P_{k_1,\ldots,k_m}\| \geq \|P\|.$$

Let $w \in \odot_\pi^n X$. Then by Theorem 2.15

$$\begin{aligned}\|w\|_{k_1,\dots,k_m} &:= \inf \Big(\sum_{i_1\in\mathbb{N}} \|x_{i_1}\|\Big)^{k_1} \Big(\sum_{i_2\in\mathbb{N}} \|x_{i_2}\|^2\Big)^{k_2} \cdots \Big(\sum_{i_m\in\mathbb{N}} \|x_{i_m}\|^m\Big)^{k_m} \\ &\le c(k_1,X)c(2k_2,X)\cdots c(mk_m,X)\|w\|,\end{aligned}$$

where the infimum taken over all representations

$$w = \Big(\sum_{i_1\in\mathbb{N}} x_{i_1}\Big)^{\otimes k_1} \Big(\sum_{i_2\in\mathbb{N}} x_{i_2}^{\otimes 2}\Big)^{\otimes k_2} \cdots \Big(\sum_{i_m\in\mathbb{N}} x_{i_m}^{\otimes m}\Big)^{\otimes k_m}.$$

Therefore, the next corollary is proved.

COROLLARY 2.16. Let $P \in \mathcal{P}(^nX)$ and $k_1 + \cdots + k_m = m$. Then

$$\|P\| \le \|F^P_{k_1,\dots,k_m}\| \le c(k_1,X)c(2k_2,X)\cdots c(mk_m,X)\|P\|.$$

2.4. The spectrum of $\mathcal{H}_b(X)$ endowed with the Gelfand topology

Let X be a complex Banach space. Let us denote by $\mathcal{A}_n(X)$ the closure of the algebra, generated by polynomials from $\mathcal{P}(^{\le n}X)$ with respect to the uniform topology on bounded subsets of X. It is clear that

$$\mathcal{A}_1(X) \cap \mathcal{P}(^nX) = \mathcal{P}_a(^nX)$$

and $\mathcal{A}_n(X)$ is a Fréchet algebra of entire analytic functions on X for every $n \in \mathbb{N}$.

Recall that $\mathcal{H}_b(X)$ coincides with the closure of algebra of all polynomials $\mathcal{P}(X)$ with respect to the uniform topology on bounded subsets X.

Denote by $\mathcal{H}^\infty_{uc}(B)$ the closure of algebra of all polynomials $\mathcal{P}(X)$ with respect to the uniform topology on the unit ball $B \subset X$. Then $\mathcal{H}^\infty_{uc}(B)$ is the algebra of all analytic functions on B, which are uniformly continuous on B.

We will use short notations M_b and M_{uc} for the topological spectra $M(\mathcal{H}_b(X))$ and $M(\mathcal{H}^\infty_{uc}(B))$, respectively.

LEMMA 2.17. *Let a functional* $\phi \in \mathcal{H}_b(X)'$ *is such that* $\phi(P) = 0$ *for every* $P \in \mathcal{P}(^mX) \cap \mathcal{A}_{m-1}(X)$, *where* m *is a fixed positive integer and* ϕ_m *is the nonzero restriction of* $\phi \in \mathcal{H}_b(X)'$ *to* $\mathcal{P}(^mX)$.

Then there is a linear multiplicative functional $\psi \in M_b$ *such that its restrictions* ψ_k *to* $\mathcal{P}(^kX)$ *satisfy the conditions:* $\psi_k = 0$ *for all* $k < m$ *and* $\psi_m = \phi_m$. *Moreover, the radius function of* ψ *is calculated by the formula*

$$\|\phi_m\|^{1/m} \le R(\psi) \le e\|\phi_m\|^{1/m}.$$

PROOF. Since $\phi_m \neq 0$, there is an element $w \in (\odot_\pi^m X)''$, $w \neq 0$ such that for any m-homogeneous polynomial P,

$$\phi(P) = \phi_m(P) = \widetilde{P}_{(m)}(w), \qquad \|w\| = \|\phi_m\|,$$

where $\widetilde{P}_{(m)}$ is the Aron-Berner extension of linear functional $P_{(m)}$ from $\odot_\pi^m X$ to $(\odot_\pi^m X)''$. For an arbitrary n-homogeneous polynomial $Q \in \mathcal{P}(^n X)$ we set

$$\psi(Q) = \begin{cases} \widetilde{Q}_{(m)}(w) & \text{if } n = mk \text{ for some } k \geq 0 \\ 0 & \text{otherwise,} \end{cases} \tag{2.13}$$

where $\widetilde{Q}_{(m)}$ is the Aron-Berner extension of the k-homogeneous polynomial $Q_{(m)}$ from $\odot_\pi^m X$ to $(\odot_\pi^m X)''$.

Let (u_α) be a net from $\odot_\pi^m X$ which converges to w in the weak-star topology of $(\odot_\pi^m X)''$, where α belongs to an index set $\mathfrak{A}$. We can assume that every u_α has a representation

$$u_\alpha = \sum_{j \in \mathbb{N}} x_{j,\alpha}^{\otimes m} \quad \text{for some} \quad x_{j,\alpha} \in X.$$

Now we will show that

$$\psi(PQ) = \psi(P)\psi(Q)$$

for any homogeneous polynomials P and Q. Let us suppose first that

$$\deg(PQ) = mr + l$$

for some integers $r \geq 0$ and $m > l > 0$. Then P or Q has degree equal to $mk + s$, $k \geq 0$, $m > s > 0$. Thus, by the definition, $\psi(PQ) = 0$ and $\psi(P)\psi(Q) = 0$. Suppose now that

$$\deg(PQ) = mr$$

for some integer $r \geq 0$. If $\deg P = mk$ and $\deg Q = mn$ for $k, n \geq 0$, then $\deg(PQ) = m(k + n)$ and

$$\psi(PQ) = (\widetilde{PQ})_{(m)}(w) = \widetilde{P}_{(m)}(w)\widetilde{Q}_{(m)}(w) = \psi(P)\psi(Q).$$

Let at last $\deg P = mk + l$ and $\deg Q = mn + r$, $l, r > 0$, $l + r = m$. Write

$$\nu = \frac{1}{(\deg P + \deg Q)!} = \frac{1}{(m(k+n+1))!}.$$

Denote by F_{PQ} the symmetric multilinear map, associated with PQ. Then

$$F_{PQ}\left(x_1, \ldots, x_{m(k+n+1)}\right) = \\ = \nu \sum_{\sigma \in \mathfrak{S}_{m(k+n+1)}} F_P\left(x_{\sigma(1)}, \ldots, x_{\sigma(mk+l)}\right) F_Q\left(x_{\sigma(mk+l+1)}, \ldots, x_{\sigma(m(k+n+1))}\right),$$

where $\mathfrak{S}_{m(k+n+1)}$ is the group of permutations on $\{1,\dots,m(k+n+1)\}$. Thus, for $\alpha_1,\dots,\alpha_{k+n+1}\in\mathfrak{A}$ we have

$$\psi(PQ)=(\widetilde{PQ})_{(m)}(w)=\lim_{\alpha_1,\dots,\alpha_{k+n+1}}\widetilde{F}_{PQ(m)}\left(u_{\alpha_1},\dots,u_{\alpha_{k+n+1}}\right)$$

$$=\lim_{\alpha_1,\dots,\alpha_{k+n+1}}\widetilde{F}_{PQ(m)}\left(\sum_{j\in\mathbb{N}}x^{\otimes m}_{j,\alpha_1},\dots,\sum_{j\in\mathbb{N}}x^{\otimes m}_{j,\alpha_{k+n+1}}\right)$$

$$=\nu\sum_{\sigma\in\mathfrak{S}_{m(k+n+1)}}\lim_{\alpha_{\sigma(1)},\dots,\alpha_{\sigma(k+n+1)}}$$

$$\sum_{j_1,\dots,j_{k+n+1}\in\mathbb{N}}F_P\left(x^m_{j_{\sigma(1)},\alpha_{\sigma(1)}},\dots,x^m_{j_{\sigma(k)},\alpha_{\sigma(k)}},x^l_{j_{\sigma(k+1)},\alpha_{\sigma(k+1)}}\right)\times$$

$$\times F_Q\left(x^r_{j_{\sigma(k+1)},\alpha_{\sigma(k+1)}},x^m_{j_{\sigma(k+2)},\alpha_{\sigma(k+2)}},\dots,x^m_{j_{\sigma(k+n+1)},\alpha_{\sigma(k+n+1)}}\right).$$

Fix some $\sigma\in\mathfrak{S}_{m(k+n+1)}$ and fix all $x_{j_{\sigma(i)},\alpha_{\sigma(i)}}$ for $i\le k$ and for $i>k+1$. Then

$$\sum_{j_1,\dots,j_{k+n+1}\in\mathbb{N}}\lim_{\alpha_{\sigma(k+1)}}F_P\left(x^m_{j_{\sigma(1)},\alpha_{\sigma(1)}},\dots,x^m_{j_{\sigma(k)},\alpha_{\sigma(k)}},x^l_{j_{\sigma(k+1)},\alpha_{\sigma(k+1)}}\right)\times$$

$$\times F_Q\left(x^r_{j_{\sigma(k+1)},\alpha_{\sigma(k+1)}},x^m_{j_{\sigma(k+2)},\alpha_{\sigma(n+2)}},\dots,x^m_{j_{\sigma(k+n+1)},\alpha_{\sigma(k+n+1)}}\right)=0,$$

because for a fixed $x_{k_{\sigma(i)},\alpha_{\sigma(i)}}$, $i\le k$,

$$P_\sigma(y):=\sum_{j_1,\dots,j_k,j_{k+2},\dots,j_{k+n+1}\in\mathbb{N}}F_P\left(x^m_{j_{\sigma(1)},\alpha_{\sigma(1)}},\dots,x^m_{j_{\sigma(k)},\alpha_{\sigma(k)}},y^l\right)$$

is an l-homogeneous polynomial and for fixed $x_{k_{\sigma(i)},\alpha_{\sigma(i)}}$, $i>k+1$,

$$Q_\sigma(y):=\sum_{j_1,\dots,j_k,j_{k+2},\dots,j_{k+n+1}\in\mathbb{N}}F_Q\left(y^r,x^m_{j_{\sigma(k+2)},\alpha_{\sigma(n+2)}},\dots,x^m_{j_{\sigma(k+n+1)},\alpha_{\sigma(k+n+1)}}\right)$$

is an r-homogeneous polynomial. Thus, $P_\sigma Q_\sigma\in\mathcal{A}_{m-1}(X)$. Hence,

$$\lim_\alpha(P_\sigma Q_\sigma)_{(m)}(u_\alpha)=\psi(P_\sigma Q_\sigma)=0$$

for every fixed σ. Therefore, $\psi(PQ)=0$. On the other hand, $\psi(P)\psi(Q)=0$ by the definition of ψ. So,

$$\psi(PQ)=\psi(P)\psi(Q).$$

Thus, we have defined the multiplicative function ψ on homogeneous polynomials. We can extend it by linearity and distributivity to a linear multiplicative functional on the algebra of all continuous polynomials $\mathcal{P}(X)$.

If ψ_n is the restriction of ψ to $\mathcal{P}(^nX)$, then $\|\psi_n\|=\|w\|^{n/m}$ if n/m is a positive integer and $\|\psi_n\|=0$ otherwise. Hence, the series

$$\psi=\sum_{n\in\mathbb{N}}\psi_n$$

is a continuous linear multiplicative functional on $\mathcal{H}_b(X)$ by Theorem 1.18 and the radius function of ψ can be computed by

$$R(\psi)=\limsup_{n\to\infty}\|\psi_n\|^{1/n}\ge\limsup_{n\to\infty}\|w\|^{n/mn}=\|w\|^{1/m}=\|\phi_m\|^{1/m}.$$

On the other hand,

$$\|\psi_n\| = \sup_{\|P\|=1} |\psi_n(P)| = \sup_{\|P\|=1} |P_{(m)}(w)|.$$

Since

$$|P_{(m)}(w)| \leq \|w\|^{n/m}\|P_{(m)}\| \leq c(n,X)\|w\|^{n/m}\|P\|,$$

we have

$$\|\psi_n\| \leq c(n,X)\|w\|^{n/m} \leq \frac{n^n}{n!}\|w\|^{n/m} = \frac{n^n}{n!}\|\phi_m\|^{n/m}.$$

So

$$R(\psi) \leq e\|\phi_m\|^{1/m}.$$

□

For each fixed element $x \in X$ *the translation operator* T_x is defined on $\mathcal{H}_b(X)$ to be

$$(T_x f)(y) = f(y+x), \qquad f \in \mathcal{H}_b(X).$$

It is not complicated to check that $T_x f \in \mathcal{H}_b(X)$ and for fixed $\phi \in \mathcal{H}_b(X)'$ the function

$$X \ni x \longmapsto \phi(T_x f)$$

belongs to $\mathcal{H}_b(X)$ (see [**6**]). For fixed $\phi, \theta \in \mathcal{H}_b(X)'$ *the convolution product* $\phi * \theta$ in $\mathcal{H}_b(X)$ is defined to be

$$(\phi * \theta)(f) = \phi\big(\theta(T_x f)\big), \qquad f \in \mathcal{H}_b(X).$$

Let now $\phi, \theta \in M_b$. By Corollary 2.3, there exist nets $(x_\alpha), (y_\beta) \subset X$ such that

(2.14) $$\phi(P) = \lim_\alpha P(x_\alpha), \qquad \theta(P) = \lim_\beta P(y_\beta)$$

for every polynomial $P \in \mathcal{P}(X)$, or in according to our previous notations, $x_\alpha \xrightarrow{\mathcal{P}} \phi$ and $y_\beta \xrightarrow{\mathcal{P}} \theta$. Thus, for every polynomial $P \in \mathcal{P}(X)$ we have

$$(\phi * \theta)(P) = \lim_\beta \lim_\alpha P(x_\alpha + y_\beta).$$

Note that the spectrum M_b is a semigroup with respect to the convolution product and

$$\phi * \theta \neq \theta * \phi$$

in general (see [**9**, Remark 3.5]). We denote $\phi_1 * \cdots * \phi_n$ briefly by

$$\mathop{\ast}_{k=1}^{n} \phi_k.$$

Let I_k be the minimal closed ideal in the algebra $\mathcal{H}_b(X)$, generated by all m-homogeneous polynomials $\mathcal{P}(^{\leq k}X)$, where $0 < m \leq k$. Evidently, I_k is a proper ideal (contains no unit). So, it is contained in a closed maximal ideal (see [**76**, p. 228]). Let

$$\Phi_k := \{\phi \in M_b \colon \ker\phi \supset I_k\}, \qquad k \in \mathbb{N}.$$

We set $\Phi_0 := M_b$. Note that the functional $\delta(0)$, that is point evaluation at zero, belongs to Φ_k for every $k \in \mathbb{N}$.

LEMMA 2.18. *If for some $m \in \mathbb{N}$,* .tions on $\{1,$

$$\mathcal{A}_m(X) \neq \mathcal{A}$$

then there exists a linear multiplicative func$_{+1}$ $\widetilde{F}_{PQ(m)}$ $(u_{m-1}$ *such that $\psi \notin \Phi_m$.*

PROOF. Let $P \in \mathcal{P}(^m X)$ and $P \notin \mathcal{A}_m$ $\sum_{j\in\mathbb{N}} x^{\otimes m}_{j,\alpha_{k+n}}$ $\mathcal{A}_{m-1}(X)$ is a closed subspace of $\mathcal{H}_b(X)$, by the Hahn-Banach Theo ists a linear functional $\phi \in \mathcal{H}_b(X)'$ such that $\phi(Q) = 0$ for every $Q \in$ d $\phi(P) \neq 0$. So, the restriction $\phi_k \equiv 0$ for $k < m$ and $\phi_m(P) \neq 0$. By 7 there exists $\psi \in M_b$ such that $\psi_k = \phi_k$ for $k = 1, \ldots, m$. Thus $\psi \in \Phi$ $\notin \Phi_m$. □

Note that in particular cases of known $c_{\sigma(1)}, \ldots, x^m_{j_{\sigma(k}}$ch spaces X we have

$$\mathcal{A}_1(c_0) = \mathcal{A}_n(c_0) \quad \text{fo} \quad \in \mathbb{N},$$

but for $k \neq m$ $_{k+2),\alpha_{\sigma(k+2)}},$

$$\mathcal{A}_k(\ell_p) = \mathcal{A}_{\sigma(i)} \quad \text{for } i \leq k$$

if and only if $k < p$ and $m < p$.

Moreover, if the Banach space X admit $\ldots, x^m_{j_{\sigma(k)},\alpha_{\sigma(k}}$ial which is not weakly sequentially continuous, then the chain of algel } does not stabilize and if X contains ℓ_1, then (see [**47, 34**]) $,\alpha_{\sigma(n+2)}, \ldots,$

$$\mathcal{A}_k(X) \neq \mathcal{A}_m(X)$$

LEMMA 2.19. *If $\phi, \psi \in M_b$ and $\psi \in \Phi_{k-}$* $x^m_{j_{\sigma(1)},\alpha_{\sigma(1)}}, \ldots$

$$\phi * \psi(P) = \phi(P$$

for every $P \in \mathcal{P}(^k X)$. $x_{k_{\sigma(i)},\alpha_{\sigma(i)}},$

PROOF. Let (x_α) and (y_β) be nets in X $_{+2),\alpha_{\sigma(n+2)}}, \ldots \xrightarrow{\mathcal{P}} \phi$ and $y_\beta \xrightarrow{\mathcal{P}} \theta$. For any fixed y_β and $0 < n < k$, the continuous sym linear form $F_P(x^{k-n}, y^n_\beta)$ associated with the polynomial $P \in \mathcal{P}(^k X)$, $\in \mathcal{A}_{m-1}(X$ homogeneous polynomial. Thus, $\psi(P_\sigma Q_\sigma) = ($

$$\phi\left(F_P(x^{k-n}, y^n_\beta)\right) = \lim_\alpha F \quad = 0.$$

the other har

Therefore, for every $P \in \mathcal{P}(^k X)$,

$$\begin{aligned}
\phi * \psi(P) &= \lim_{\beta,\alpha} P(x_\alpha + y_\beta) \qquad \psi(Q). \\
& \qquad \text{iction } \psi \text{ on h} \\
&= \sum_{n+m=k} \lim_{\beta,\alpha} F_P(x^n_\alpha, y^m_\beta) = \quad \text{y to a linear } F_P(x^n_\alpha, y^m_\beta)\big) \\
& \qquad (X). \\
&= \lim_\beta \Big(\lim_\alpha F_P(x_\alpha, \ldots, x_\alpha) + \quad \|\psi_n\| = \|w\|\Big) = \phi(P) + \psi(P).
\end{aligned}$$

ies

That it was necessary to prove. □

LEMMA 2.20. *If $P \in \mathcal{P}(^k X)$ and $\phi_j \in \Phi_{j-}$* *very $m > k$,* on $\mathcal{H}_b(X)$ by

$$\mathop{*}_{j=1}^{m} \phi_j(P) = \mathop{*}_{j=1}^{k} \phi$$

PROOF. In fact, since $\phi_j \in \Phi_{j-1}$, we have $^{/mn} = \|w\|^{1/}$ r every $j > k$. □

Given a sequence $(\phi_n)_{n=1}^{\infty} \subset M_b$ with $\phi_n \in \Phi_{n-1}$, the infinite convolution $\mathop{\ast}\limits_{n=1}^{\infty} \phi_n$ denotes a linear multiplicative functional on the algebra of all polynomials $\mathcal{P}(X)$ such that

$$\mathop{\ast}_{n=1}^{\infty} \phi_n(P) = \mathop{\ast}_{n=1}^{k} \phi_n(P) \qquad \text{if} \qquad P \in \mathcal{P}(^kX)$$

for an arbitrary $k \in \mathbb{N}$. This multiplicative functional uniquely determines a functional in M_b (which we denote by the same symbol $\mathop{\ast}\limits_{n=1}^{\infty} \phi_n$) if it is continuous.

The point evaluation operator δ maps X into M_b by

$$x \longmapsto \delta(x), \qquad \delta(x)(f) = f(x)$$

for every $x \in X$.

The operator $\tilde{\delta}$ is the extension of δ onto X'', i.e.

$$\tilde{\delta}(x'')(f) = \tilde{f}(x'')$$

for every $x'' \in X''$.

THEOREM 2.21. *There exists a sequence of dual Banach spaces $(E_n)_{n=1}^{\infty}$ and a sequence of maps*

$$\delta^{(n)} \colon E_n \longrightarrow M_b$$

such that

$$E_1 = X'', \qquad E_n = \mathcal{P}(^nX)' \cap I_{n-1}^{\perp}, \qquad \delta^{(1)} = \tilde{\delta}$$

and such that an arbitrary complex homomorphism $\phi \in M_b$ has a representation

$$\phi = \mathop{\ast}_{n=1}^{\infty} \delta^{(n)}(u_n) \tag{2.15}$$

for some $u_n \in E_n$, $n = 1, 2, \ldots$, where is denoted

$$I_{n-1}^{\perp} := \left\{ u \in \mathcal{H}_b(X)' \colon u(f) = 0 \text{ for all } f \in I_{n-1} \right\}.$$

PROOF. Put $E_1 = X''$. Then

$$\delta^{(1)}(x'') = \tilde{\delta}(x'') \in M_b, \qquad x'' \in X''.$$

Suppose that spaces E_k and maps $\delta^{(k)}$ are constructed for $k < n$. Denote

$$E_n := \{\pi_n(\phi) : \phi \in \Phi_{n-1}\},$$

where $\pi_n(\phi) = \phi_n$ is the restriction of ϕ onto subspace $\mathcal{P}(^nX)$. In other words, the space E_n consists of linear continuous functionals on $\mathcal{P}(^nX)$ that vanish on all polynomials in $\mathcal{P}(^nX) \cap \mathcal{A}_{n-1}$. If $\mathcal{A}_n = \mathcal{A}_{n-1}$, then $E_n \equiv 0$. Otherwise, by Lemma 2.18, there are nonzero points in E_n.

By Lemma 2.19, for $P \in \mathcal{P}(^nX)$ and $\phi, \psi \in \Phi_{n-1} \subset M_b$,

$$\pi_n(\phi * \psi)(P) = \phi * \psi(P) = \phi(P) + \psi(P) = \pi_n\phi(P) + \pi_n\psi(P).$$

Consequently,

$$\pi_n(\phi * \psi) = \pi_n(\phi) + \pi_n(\psi).$$

For an arbitrary complex number a, we have $a\phi \in \mathcal{H}_b(X)'$ and $\pi_k(a\phi) = a\pi_k(\phi)$. So, $a\phi$ vanishes on all homogeneous polynomials of degree k less than n. By Lemma

2.17 there exists $\psi \in M_b$ such that $\psi_k = a\phi_k$ for $1 \leq k \leq n$. Thus, $\psi \in \Phi_{n-1}$ and $a\phi_n = \psi_n \in E_n$.

Hence, E_n is a vector space and polynomials from $\mathcal{P}(^nX)$ are acting on E_n as linear functionals. Put

$$W_n = \mathcal{P}(^nX)/(I_{n-1} \cap \mathcal{P}(^nX)).$$

Then W_n is a Banach space of linear functionals on E_n and the functionals from W_n separate points of E_n.

Let us define a norm $\|\cdot\|_n$ on E_n, as the supremum of values of a vector from E_n on the unit ball of W_n. Therefore,

$$W_n' = \left(\mathcal{P}(^nX)/(I_{n-1} \cap \mathcal{P}(^nX))\right)' = \mathcal{P}(^nX)' \cap I_{n-1}^{\perp} \supset E_n.$$

On the other hand, if $u \in \mathcal{P}(^nX)' \cap I_{n-1}^{\perp}$, then by Lemma 2.17 $u = \pi_n(\phi)$ for some $\phi \in M_b$. So, $u \in E_n$. Therefore,

$$E_n = W_n'.$$

For a given $w \in E_n$ let us define

$$\delta^{(n)}(w)(Q) = \psi(Q)$$

on homogeneous polynomials Q by formula (2.13) and extend it to the unique complex homomorphism on $\mathcal{H}_b(X)$ as in Lemma 2.17. So, $\delta^{(n)}$ maps E_n into M_b. For any $\phi \in M_b$ put

$$u_1 := \phi_1 \in X'' = E_1, \qquad u_2 := \phi_2 - \pi_2(\delta^{(1)}(u_1)).$$

It is clear that $u_2 \in E_2$.

Suppose that we have defined $u_k \in E_k$ with $k < n$. Set

$$u_n := \phi_n - \pi_n\left(\mathop{\ast}_{k=1}^{n-1} \delta^{(k)}(u_k)\right). \tag{2.16}$$

Let us show that $u_n \in E_n$. It is enough to check that for every $P \in \mathcal{P}(^nX)$ such that $P = P_kP_m$ with $\deg P_k = k \neq 0$ and $\deg P_n = n \neq 0$ implies $u_n(P) = 0$. Note that for every n-homogeneous polynomials P_n,

$$\phi_n - \pi_n\left(\mathop{\ast}_{k=1}^{n-1} \delta^{(k)}(u_k)\right)(P_n) = \phi_n - \mathop{\ast}_{k=1}^{n-1} \delta^{(k)}(u_k)(P_n).$$

From the multiplicativity of ϕ and Lemma 2.20 it follows that

$$\begin{aligned}
u_n(P) &= \phi_n(P_kP_m) - \mathop{\ast}_{j=1}^{n-1} \delta^{(j)}(u_j)(P_kP_m)\\
&= \phi_k(P_k)\phi_m(P_m) - \left(\mathop{\ast}_{j=1}^{n-1} \delta^{(j)}(u_j)(P_k)\right)\left(\mathop{\ast}_{j=1}^{n-1} \delta^{(j)}(u_j)(P_m)\right)\\
&= \left(u_k(P_k) + \mathop{\ast}_{j=1}^{k-1} \delta^{(j)}(u_j)(P_k)\right)\left(u_m(P_m) + \mathop{\ast}_{j=1}^{m-1} \delta^{(j)}(u_j)(P_m)\right)\\
&\quad - \left(\mathop{\ast}_{j=1}^{k} \delta^{(j)}(u_j)(P_k)\right)\left(\mathop{\ast}_{j=1}^{m} \delta^{(j)}(u_j)(P_m)\right) = 0.
\end{aligned}$$

The last equality holds, because by the induction assumption, $u_k \in E_k$, $u_m \in E_m$ and hence, by Lemma 2.19,

$$u_k(P_k) + \mathop{\ast}_{j=1}^{k-1} \delta^{(j)}(u_j)(P_k) = \mathop{\ast}_{j=1}^{k} \delta^{(j)}(u_j)(P_k) \tag{2.17}$$

and

$$u_m(P_m) + \mathop{\ast}_{j=1}^{m-1} \delta^{(j)}(u_j)(P_m) = \mathop{\ast}_{j=1}^{m} \delta^{(j)}(u_j)(P_m).$$

Let us consider the functional $\mathop{\ast}_{j=1}^{\infty} \delta^{(j)}(u_j)$. Since $u_k \in E_k$, by Lemma 2.19,

$$\mathop{\ast}_{j=1}^{\infty} \delta^{(j)}(u_j)(f) = f(0) + \sum_{n\in\mathbb{N}} \mathop{\ast}_{j=1}^{n} \delta^{(j)}(u_j)(f_n),$$

where $f = \sum f_n$ is the Taylor series of f. Hence, $\mathop{\ast}_{j=1}^{\infty} \delta^{(j)}(u_j)$ is well defined on $\mathcal{P}(X)$. On the other hand, applying (2.16) and (2.17) we obtain

$$\begin{aligned}\left(\phi - \mathop{\ast}_{j=1}^{\infty} \delta^{(j)}(u_j)\right)(P_n) &= \phi_n(P_n) - \mathop{\ast}_{j=1}^{n} \delta^{(j)}(u_j)(P_n) \\ &= u_n(P) + \mathop{\ast}_{j=1}^{n-1} \delta^{(j)}(u_j)(P_n) - \mathop{\ast}_{j=1}^{n} \delta^{(j)}(u_j)(P_n) = 0\end{aligned}$$

for arbitrary $P_n \in \mathcal{P}(^nX)$. Thus $\phi = \mathop{\ast}_{j=1}^{\infty} \delta^{(j)}(u_j)$ on $\mathcal{P}(X)$. Hence $\phi = \mathop{\ast}_{j=1}^{\infty} \delta^{(j)}(u_j)$ on $\mathcal{H}_b(X)$. □

Let us denote by $\mathbb{E}^\infty$ the space of all finite sequences $\mathfrak{u} = (u_1, \ldots, u_m, 0, \ldots)$ with $u_k \in E_k$. According to Theorem 2.21, every finite sequence $(u_1, \ldots, u_m, 0, \ldots)$ defines a character

$$\phi_{\mathfrak{u}} = \mathop{\ast}_{k=1}^{m} \delta^{(k)}(u_k) \in M_b.$$

Thus, $\mathbb{E}^\infty \subset M_b$ and for every $\mathfrak{u}, \mathfrak{v} \in \mathbb{E}^\infty$, we obtain

$$\phi_{\mathfrak{u}+\mathfrak{v}} \in M_b.$$

Moreover, from the density of polynomials in $\mathcal{H}_b(X)$ it follows that $\mathbb{E}^\infty$ is dense in M_b with respect to the Gelfand topology. So, we have proved the following theorem.

THEOREM 2.22. *The spectrum M_b contains the dense vector subspace $\mathbb{E}^\infty$ of all finite subsequences $(u_1, \ldots, u_m, 0, \ldots)$ with $u_k \in E_k$.*

We will use also notation $\mathbb{E}^m$ for the Banach space which is the direct sum of $E_1, \ldots, E_m$, i.e.,

$$\mathbb{E}^m := E_1 \oplus \ldots \oplus E_m.$$

It is clear $\mathbb{E}^m \subset \mathbb{E}^\infty$ for every m. Note that $\mathbb{E}^m$ coincides with the set of maximal ideals of $\mathcal{A}_m(X)$.

2.5. The Gelfand transformation and linear structures on M_b

Using the Gelfand transformation, every element of a commutative Fréchet algebra can be represented by a continuous function on the topological spectrum. We will consider such representations for functions of the algebra $\mathcal{H}_b(X)$ on a Banach complex space X. Since the space X is naturally embedded into the topological spectrum M_b we can say about extensions of functions from X onto M_b.

A given function $f \in \mathcal{H}_b(X)$ by $\widehat{f}$ we denote its Gelfand transformation. Since $\mathbb{E}^m \subset M_b$, we can use notation $\widehat{f}(u_1,\ldots,u_m)$ instead $\widehat{f}(u_1,\ldots,u_m,0,0\ldots)$.

THEOREM 2.23. *Let $P \in \mathcal{P}(^nX)$. Then for every fixed $m \in \mathbb{N}$ and positive integers $k_1,\ldots,k_m$ such that $k_1+2k_2+\cdots+mk_m=n$ there exists a map*

$$\widetilde{F}^P_{k_1,\ldots,k_m}\colon E_1^{k_1}\times\cdots\times E_m^{k_m}\longrightarrow\mathbb{C}$$

such that for every $1\le j\le n$,

$$\widetilde{F}^P_{k_1,\ldots,k_m}(\underbrace{u_1,\ldots,u_1}_{k_1},\ldots,\underbrace{u_j,\ldots,u_j}_{k_j},\ldots,\underbrace{u_m,\ldots,u_m}_{k_m})$$

is a k_j-homogeneous polynomial of u_j for fixed $u_1,\ldots,u_{j-1},u_{j+1},\ldots,u_m$ and

$$\widehat{P}(u_1,\ldots,u_m)=\sum_{k_1+2k_2+\cdots+mk_m=n}\widetilde{F}^P_{k_1,\ldots,k_m}(u_1,\ldots,u_m).$$

Moreover,

$$\|\widetilde{F}^P_{k_1,\ldots,k_m}\|\le c(k_1,X)c(2k_2,X)\cdots c(mk_m,X)\|P\|.$$

PROOF. For $m=1$ it is trivially true. Suppose that the theorem is true for $m-1$. Then

$$\begin{aligned}\widehat{P}(u_1,\ldots,u_m)&=\mathop{*}_{j=1}^{m}\delta^{(j)}(u_j)(P)\\&=\Big[\Big(\mathop{*}_{j=1}^{m-1}\delta^{(j)}(u_j)\Big)*\delta^{(m)}(u_m)\Big](P)=\delta^{(m)}(u_m)\Big(\mathop{*}_{j=1}^{m-1}\delta^{(j)}(u_j)T_x(P)\Big)\\&=\delta^{(m)}(u_m)\Big(\sum_{i=0}^{n-1}\sum_{k_1+\cdots+(m-1)k_{m-1}=n-i}\widetilde{F}^{P_i}_{k_1,\ldots,k_{m-1}}(u_1,\ldots,u_{m-1})+P(x)\Big),\end{aligned}$$

where

$$T_x(P)(z)=\sum_{i=1}^{n-1}P_i(z),\qquad P_i(z)=\binom{n-i}{i}F_P\left(z^{n-i},x^i\right)$$

is an $(n-i)$-homogeneous polynomial by z for any fixed x, the functional $\mathop{*}_{j=1}^{m-1}\delta^{(j)}(u_j)$ is linear, and $\widetilde{F}^{P_i}_{k_1,\ldots,k_{m-1}}(u_1,\ldots,u_{m-1})$ is an i-homogeneous polynomial of x for all $i=1,\ldots,n-1$ with fixed $u_1,\ldots,u_{m-1}$.

By the definition of $\delta^m(u_m)$,

$$\delta^m(u_m)\Big(\widetilde{F}^{P_i}_{k_1,\ldots,k_{m-1}}(u_1,\ldots,u_{m-1})\Big)$$

is an (i/m)-homogeneous polynomial if (i/m) is integer and zero otherwise. Similarly, $\delta^m(u_m)(P)$ is a (n/m)-homogeneous polynomial if (n/m) is integer and zero otherwise. Therefore, if $i = k_m m$, then we can put

$$\widetilde{F}^P_{k_1,\dots,k_m}(u_1,\dots,u_m) := \delta^m(u_m)\Big(\widetilde{F}^{P_i}_{k_1,\dots,k_{m-1}}(u_1,\dots,u_{m-1})\Big)$$

and if $n = k_m m$, then

$$\widetilde{F}^P_{0,\dots,0,k_m}(u_1,\dots,u_m) := \delta^m(u_m)(P).$$

Since E_j is a subspace of $(\odot^j_\pi X)''$, for every $u_j \in E_j$ there exists a net $(w^j_{\alpha_j}) \subset \odot^j_\pi X$ such that

$$\|w^j_{\alpha_j}\| \le \|u_j\| \qquad \text{and} \qquad w^j_{\alpha_j} \to u_j$$

in the weak-star topology of $(\odot^j_\pi X)''$. Thus,

$$\widetilde{F}^P_{k_1,\dots,k_m}(u_1,\dots,u_m) = \lim_{\alpha_1}\dots\lim_{\alpha_m} F^P_{k_1,\dots,k_m}\left(w^1_{\alpha_1},\dots,w^m_{\alpha_m}\right),$$

where $F^P_{k_1,\dots,k_m}$ is defined by (2.12). So, via Corollary 2.16

$$\|\widetilde{F}^P_{k_1,\dots,k_m}\| = \|F^P_{k_1,\dots,k_m}\| \le c(k_1,X)c(2k_2,X)\cdots c(mk_m,X)\|P\|,$$

as was to be shown. □

COROLLARY 2.24. Let $f \in \mathcal{H}_b(X)$. Then for every E_k the restriction $\widehat{f}|_{E_k}$ of $\widehat{f}$ to E_k belongs to $\mathcal{H}_b(E_k)$.

PROOF. Let

$$f(x) = \sum_{n\in\mathbb{Z}_+} P_n(x)$$

be the Taylor series of f. By Theorem 2.23

$$\|\widehat{P}_n|_{E_k}\| \le \|\widetilde{F}^{P_n}_m\| \le c(mk,X)\|P_n\|,$$

where $mk = n$. So each polynomial $\widehat{P}_n|_{E_k}$ is continuous and the radius of boundedness of

$$\widehat{f}|_{E_k} = \sum_{n\in\mathbb{Z}_+} \widehat{P}_n|_{E_k}$$

equals infinity. □

A given positive integer n let $p(n)$ be the number of positive solutions of the Diophantine equation

$$k_1 + 2k_2 + \cdots + nk_n = n.$$

It is known from Combinatorics that $p(n)$ is equal to the number of all partitions on n and asymptotically

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}. \tag{2.18}$$

THEOREM 2.25. *Let $(u_k)_{k=1}^{\infty}$ be a sequence such that $u_k \in E_k$ for every k. Then*

$$\phi = \mathop{\ast}_{k=1}^{\infty} \delta^{(k)}(u_k)$$

is a continuous complex homomorphism in M_b if and only if

$$\sup_{k\in\mathbb{N}} \|u_k\|^{1/k} < \infty.$$

In this case

$$\sup_{k\in\mathbb{N}} \|u_k\|^{1/k} \leq R(\phi) \leq e \sup_{k\in\mathbb{N}} \|u_k\|^{1/k}. \tag{2.19}$$

PROOF. Let $\sup_k \|u_k\|^{1/k} = r < \infty$ for some positive r. Then $\|u_k\| \leq r^k$. For any $P \in \mathcal{P}(^nX)$ such that $\|P\| = 1$ we have

$$\begin{aligned}\|\phi_n(P)\| = \|\phi(P)\| &= \left\|\widehat{P}(u_1, \dots, u_n)\right\| \\ &\leq \left\| \sum_{k_1+2k_2+\cdots+nk_n} \widetilde{F}^P_{k_1,\dots,k_n}(u_1, \dots, u_n)\right\| \\ &\leq \sum_{k_1+2k_2+\cdots+nk_n=n} \left\|\widetilde{F}^P_{k_1,\dots,k_n}(u_1, \dots, u_n)\right\| \\ &\leq m_n \sum_{k_1+2k_2+\cdots+nk_n=n} \|u_1\|^{k_1} \cdots \|u_n\|^{k_n},\end{aligned}$$

where

$$m_n = \max_{k_1+2k_2+\cdots+nk_n=n} \left[c(k_1, X)c(2k_2, X) \cdots c(mk_m, X)\right].$$

We observe that

$$m_n \leq \max_{s_1+\cdots+s_n=n} \frac{s_1^{s_1} \cdots s_n^{s_n}}{s_1! \cdots s_n!}.$$

So, via the Sterling formula,

$$\limsup_{n\to\infty} m_n^{1/n} \leq e.$$

Therefore,

$$\begin{aligned}R(\phi) = \limsup_{n\to\infty} \|\phi_n\|^{1/n} &\leq e \limsup_{n\to\infty} \left(p(n) r^{k_1+2k_2+\cdots+nk_n}\right)^{1/n} \\ &= e\, r \limsup_{n\to\infty} (p(n))^{1/n},\end{aligned}$$

where $p(n)$ is the number of partitions of n. Using the asymptotical formula (2.18), we have that

$$R(\phi) \leq er = e \sup_{k\in\mathbb{N}} \|u_k\|^{1/k} < \infty.$$

So, $\phi \in M_b$.

On the other hand, $\|u_k\| \leq \|\phi_{km}\|$ for every positive integer m. Hence

$$\sup_{k\in\mathbb{N}} \|u_k\|^{1/k} \leq \limsup_{k\to\infty} \|\phi_k\|^{1/k} = R(\phi).$$

Hence, the inequalities (2.19) are proved. □

Let $\mathfrak{u} = (u_1, u_2, \ldots)$, $\mathfrak{v} = (v_1, v_2, \ldots)$ be sequences with $u_k, v_k \in E_k$ for every $k \in \mathbb{N}$ and

$$\sup_{k\in\mathbb{N}} \|u_k\|^{1/k} < \infty, \qquad \sup_{k\in\mathbb{N}} \|v_k\|^{1/k} < \infty.$$

Thus, there are complex homomorphisms of $\mathcal{H}_b(X)$ of the forms

$$\phi_{\mathfrak{u}} = \mathop{\ast}_{k=1}^{\infty} \delta^{(k)}(u_k), \qquad \phi_{\mathfrak{v}} = \mathop{\ast}_{k=1}^{\infty} \delta^{(k)}(v_k).$$

COROLLARY 2.26. For any $\phi_{\mathfrak{u}}$, $\phi_{\mathfrak{v}} \in M_b$ and a constant $c > 0$, we have

$$\phi_{\mathfrak{u}+\mathfrak{v}} = \mathop{\ast}_{k=1}^{\infty} \delta^{(k)}(u_k + v_k) \in M_b, \qquad \phi_{c\mathfrak{u}} = \mathop{\ast}_{k=1}^{\infty} \delta^{(k)}(cu_k) \in M_b.$$

So, the spectrum M_b may be identified with a sequence space

$$\bar{\mathbb{E}}^{\infty} := \Big\{(u_1, u_2, \ldots) \colon u_k \in E_k,\ \sup_{k\in\mathbb{N}} \|u_k\|^{1/k} < \infty\Big\}.$$

PROPOSITION 2.27. The function

$$\rho(\mathfrak{u}, \mathfrak{v}) := \sup_{k\in\mathbb{N}} \|u_k - v_k\|^{1/k}$$

is a metric on $\bar{\mathbb{E}}^{\infty}$, which is invariant with respect to translations.

PROOF. From the definition of ρ it follows

$$\rho(\mathfrak{u} - \mathfrak{v}, 0) = \rho(\mathfrak{u}, \mathfrak{v}).$$

It is enough to check the triangle inequality. Since

$$\|u_k + v_k\|^{1/k} \le \big(\|u_k\| + \|v_k\|\big)^{1/k} \le \|u_k\|^{1/k} + \|v_k\|^{1/k},$$

we have

$$\sup_{k\in\mathbb{N}} \|u_k + v_k\|^{1/k} \le \sup_{k\in\mathbb{N}} \big(\|u_k\|^{1/k} + \|v_k\|^{1/k}\big) \le \sup_{k\in\mathbb{N}} \|u_k\|^{1/k} + \sup_{j\in\mathbb{N}} \|v_j\|^{1/j}$$

and

$$\rho(\mathfrak{u} + \mathfrak{v}, 0) \le \rho(\mathfrak{u}, 0) + \rho(\mathfrak{v}, 0).$$

It is necessary to do replacement $\mathfrak{u}$ by $\mathfrak{u} - \mathfrak{w}$ and $\mathfrak{v}$ by $\mathfrak{w} - \mathfrak{v}$. □

The following proposition easily follows from the completeness of each E_k.

PROPOSITION 2.28. The metric space

$$\bar{\mathbb{E}}^{\infty} = (\bar{\mathbb{E}}^{\infty}, \rho) = \Big\{(u_1, u_2, \ldots) \colon u_k \in E_k,\ \sup_{k\in\mathbb{N}} \|u_k\|^{1/k} < \infty\Big\}$$

is complete.

Note that in the general case both the Gelfand topology and the topology generated by ρ are not linear. We consider this situation in the next section.

Given a sequence of Banach spaces $\big(E_n, \|\cdot\|_n\big)_{n=1}^{\infty}$ and $0 < \rho \le \infty$ the Köthe sequence space $\lambda^1\big(K_\rho; (E_n)\big)$ with $K_\rho = \big\{(r^n)_{n=1}^{\infty} \colon 0 < r < \rho\big\}$ is defined to be the Fréchet vector space

$$\Big\{(x_n)_{n=1}^{\infty} \in \mathop{\times}_{n\in\mathbb{N}} E_n \colon p_r\Big((x_n)_{n=1}^{\infty}\Big) := \sum_{n\in\mathbb{N}} \|x_n\| r^n < \infty \text{ for all } r \in (0, \rho)\Big\},$$

endowed with the topology given by seminorms $\{p_r\}_{0<r<\rho}$. It is clear that

$$\lambda^1(K_{\rho_1};(E_n)) \supset \lambda^1(K_{\rho_2};(E_n)) \quad \text{if} \quad \rho_1 < \rho_2$$

and the topology induced by $\lambda^1(K_{\rho_1};(E_n))$ on $\lambda^1(K_{\rho_2};(E_n))$ is weaker than the original topology of $\lambda^1(K_{\rho_2};(E_n))$. We define on the union

$$\bigcup_{\rho>0} \lambda^1(K_\rho;(E_n))$$

the topology of inductive limits of the Köthe spaces respectively above continuous embeddings, that is denoted

$$\widehat{\mathbb{E}}^\infty := \lim_{\rho\to 0} \operatorname{ind} \lambda^1(K_\rho;(E_n)).$$

THEOREM 2.29. *The map*

$$\mathop{\ast}_{k=1}^{\infty} \delta^{(k)}(u_k) \longmapsto (u_1, u_2, \ldots), \qquad u_k \in E_k,$$

produces a bijection from M_b onto $\widehat{\mathbb{E}}^\infty$. The inductive topology of $\widehat{\mathbb{E}}^\infty$ is stronger than the Gelfand topology transferred by this bijection from M_b, i.e., the bijective mapping

$$\widehat{\mathbb{E}}^\infty \longrightarrow M_b$$

is continuous. The space $\widehat{\mathbb{E}}^\infty$ is a Hausdorff locally convex vector space.

PROOF. Let $\phi = \mathop{\ast}_{k=1}^{\infty} \delta^{(k)}(u_k) \in M_b$. Then by Theorem 2.25

$$\sup_{k\in\mathbb{N}} \|u_k\|^{1/k} < \infty.$$

So, there is some $0 < \varrho < \infty$ such that

$$\limsup_{k\to\infty} \|u_k\|^{1/k} < \varrho.$$

By Cauchy-Hadamard's formula

$$\sum_{n\in\mathbb{N}} \|u_n\| r^n < \infty \quad \text{for} \quad 0 < r < \frac{1}{\varrho}.$$

So, $(u_k)_{k=1}^\infty \subset \lambda^1(K_\rho;(E_n))$, where $\rho = 1/\varrho$.

Now in the opposite side, let $(u_k)_{k=1}^\infty \subset \lambda^1(K_\rho;(E_n))$. Then by the Cauchy-Hadamard formula

$$\limsup_{k\to\infty} \|u_k\|^{1/k} < \frac{1}{\rho}, \qquad \text{consequently,} \qquad \sup_{k\in\mathbb{N}} \|u_k\|^{1/k} < \infty.$$

Let $f \in \mathcal{H}_b(X)$. By Corollary 2.24 the restriction of $\widehat{f}$ to each space E_k is continuous. So, $\widehat{f}$ is continuous on $\widehat{\mathbb{E}}^\infty$. Since the Gelfand topology is the weakest topology such that all functions $\widehat{f}$ with $f \in \mathcal{H}_b(X)$ are continuous, it also is weaker than the topology of $\widehat{\mathbb{E}}^\infty$. As a consequence, the inductive limits of Köthe spaces $\widehat{\mathbb{E}}^\infty$ is a Hausdorff topological space, thus it is a locally convex vector space. □

COROLLARY 2.30. For every $f \in \mathcal{H}_b(X)$ the Gelfand transform $\widehat{f}$ belongs to $\mathcal{H}_b(\widehat{\mathbb{E}}^\infty)$ and the mapping

$$\mathcal{H}_b(X) \ni f \longmapsto \widehat{f} \in \mathcal{H}_b(\widehat{\mathbb{E}}^\infty)$$

is continuous, where $\mathcal{H}_b(\widehat{\mathbb{E}}^\infty)$ means the space of all entire $\mathbb{C}$-valued functions of bounded type on the locally convex space $\widehat{\mathbb{E}}^\infty$ endowed with the topology of uniform convergence on bounded sets.

2.6. Linearity of topologies on spectra

We have proved that $(\bar{\mathbb{E}}^\infty, \rho)$ is a complete metric space. If $\mathcal{H}_b(X) = \mathcal{A}_1(X)$, then $\bar{\mathbb{E}}^\infty$ is isometrically isomorphic to the second dual X''. However, in the general case, $\bar{\mathbb{E}}^\infty$ is not a topological vector space.

THEOREM 2.31. *If the sequence of algebras $\mathcal{A}_n(X)$ does not stabilize, then there exists $\mathrm{u} = (u_1, u_2, \ldots) \in M_b$ such that the operation of multiplication by a constant*

$$\lambda \rightsquigarrow \lambda\mathrm{u} = (\lambda u_1, \lambda u_2, \ldots), \qquad \lambda \in \mathbb{C}$$

is discontinuous as a function from $\mathbb{C}$ to $(\bar{\mathbb{E}}^\infty, \rho)$.

PROOF. Let $\mathrm{u} = (u_1, u_2, \ldots) \in M_b$ such that $\|u_k\| = 1$ for every $k \in \mathbb{N}$. Then for each $0 < \lambda < 1$,

$$\rho(0, \lambda\mathrm{u}) = \sup_{k\in\mathbb{N}} \lambda^{1/k}\|u_k\|^{1/k} = 1.$$

In particular, if $\lambda_n \to 0$, $0 < \lambda_n < 1$, then $\rho(0, \lambda_n\mathrm{u}) = 1$ but $\rho(0, 0\mathrm{u}) = 0$. □

COROLLARY 2.32. Let $\mathrm{u} = (u_1, u_2, \ldots) \in M_b$ such that $\|u_k\| = 1$ for every k. Then the linear map

$$\mathbb{C} \ni \lambda \longmapsto \lambda\mathrm{u} \in (\bar{\mathbb{E}}^\infty, \rho)$$

is discontinuous at every $\lambda \in \mathbb{C}$.

PROOF. Let $\lambda_n \to \lambda_0 \in \mathbb{C}$ as $n \to \infty$. Then $\lambda_n - \lambda$ tends to 0 as $n \to \infty$ but $(\lambda_n - \lambda)\mathrm{u}$ does not. □

PROPOSITION 2.33. The multiplication by a constant

$$(\lambda, \mathrm{u}) \rightsquigarrow \lambda\mathrm{u} = (\lambda u_1, \lambda u_2, \ldots), \qquad \lambda \in \mathbb{C}, \quad \mathrm{u} \in M_b$$

is jointly continuous in the Gelfand topology.

PROOF. Let $\lambda_n \to \lambda_0$ in $\mathbb{C}$ as $n \to \infty$, $\mathrm{u}_\alpha \to \mathrm{u}_0$ in the Gelfand topology and $f \in \mathcal{H}_b(X)$. If $f = \sum_k f_k$ is the Taylor series of f, then $\widehat{f}(\lambda_n\mathrm{u}_\alpha) = \sum_k \lambda_n^k \widehat{f}_k(\mathrm{u}_\alpha)$ and

$$\lim_{n\to\infty} \lim_{\alpha} \sum_{k\in\mathbb{Z}_+} \lambda_n^k \widehat{f}_k(\mathrm{u}_\alpha) = \sum_{k\in\mathbb{Z}_+} \lambda_0^k \widehat{f}_k(\mathrm{u}_0) = \lim_{\alpha} \lim_{n\to\infty} \sum_{k\in\mathbb{Z}_+} \lambda_n^k \widehat{f}_k(\mathrm{u}_\alpha).$$

Since it is true for every $f \in \mathcal{H}_b(X)$, the multiplication by a constant is jointly continuous in the Gelfand topology. □

Let X_Γ be the Banach space X endowed with the finest topology Γ, which coincides on bounded sets with the Gelfand topology of M_b restricted to X. Since every function in $\mathcal{H}_b(X)$ can be approximated by polynomials uniformly on bounded sets, Γ is the finest topology which coincides on bounded sets with the weakest topology on X such that all polynomials in $\mathcal{P}(X)$ are continuous. We consider the question: *Under which conditions the operation of sum $(x,y) \rightsquigarrow x+y$ is jointly continuous in X_Γ?*

It is not difficult to check that the operation of sum is separately continuous in the space X_Γ.

THEOREM 2.34. *If the operation of sum is jointly continuous in X_Γ, then $\odot_\pi^n X$ is symmetrically regular for every positive integer n.*

PROOF. Suppose that $\odot_\pi^n X$ is not symmetrically regular for some n. Then there are $u'', v'' \in (\odot_\pi^n X)''$ and nets $u_\alpha, v_\beta \in \odot_\pi^n X$ such that $u_\alpha \to u''$, $v_\beta \to v''$ in the weak-star topology of $(\odot_\pi^n X)''$ and

$$\lim_\alpha \lim_\beta F(u_\alpha, v_\beta) \neq \lim_\beta \lim_\alpha F(u_\alpha, v_\beta)$$

for a symmetric bilinear form F on $\odot_\pi^n X$. Let

$$u_\alpha = \sum_{i_1,\dots,i_n \in \mathbb{N}} x_\alpha^{i_1} \otimes \cdots \otimes x_\alpha^{i_n} \tag{2.20}$$

and

$$v_\beta = \sum_{j_1,\dots,j_n \in \mathbb{N}} y_\beta^{j_1} \otimes \cdots \otimes y_\beta^{j_n} \tag{2.21}$$

be some representations of u_α and v_β respectively, where $x_\alpha^i, y_\beta^j \in X$. According to the definition of the projective tensor product series (2.20) and (2.21) are absolutely convergent. So, if $\Phi \in (\odot_\pi^n X)'$, then

$$\Phi(v_\beta) = \Phi\Big(\sum_{j_1,\dots,j_n \in \mathbb{N}} y_\beta^{j_1} \otimes \cdots \otimes y_\beta^{j_n}\Big) = \sum_{j_1,\dots,j_n \in \mathbb{N}} \Phi(y_\beta^{j_1} \otimes \cdots \otimes y_\beta^{j_n}).$$

For a fixed index α, $F(u_\alpha, \cdot)$ is a linear functional on $\odot_\pi^n X$. Thus,

$$\begin{aligned}\lim_{\alpha,\beta} F(u_\alpha, v_\beta) &= \lim_{\alpha,\beta} F\Big(\sum_{\{i\}\in \mathbb{N}^n} x_\alpha^{i_1} \otimes \cdots \otimes x_\alpha^{i_n}, \sum_{\{j\}\in \mathbb{N}^n} y_\beta^{j_1} \otimes \cdots \otimes y_\beta^{j_n}\Big) \\ &= \lim_\alpha \lim_\beta \sum_{\{i\}\in \mathbb{N}^n} \sum_{\{j\}\in \mathbb{N}^n} F\left(x_\alpha^{i_1} \otimes \cdots \otimes x_\alpha^{i_n}, y_\beta^{j_1} \otimes \cdots \otimes y_\beta^{j_n}\right).\end{aligned}$$

Since the series

$$\sum_{\{i\}\in \mathbb{N}^n} \sum_{\{j\}\in \mathbb{N}^n} F\left(x_\alpha^{i_1} \otimes \cdots \otimes x_\alpha^{i_n}, y_\beta^{j_1} \otimes \cdots \otimes y_\beta^{j_n}\right)$$

absolutely converges, we have

$$\lim_{\alpha,\beta} F(u_\alpha, v_\beta) = \sum_{\{i\}\in \mathbb{N}^n} \sum_{\{j\}\in \mathbb{N}^n} \lim_\alpha \lim_\beta F\left(x_\alpha^{i_1} \otimes \cdots \otimes x_\alpha^{i_n}, y_\beta^{j_1} \otimes \cdots \otimes y_\beta^{j_n}\right).$$

Hence, for some $\{i\} \in \mathbb{N}^n$ $\{j\} \in \mathbb{N}^n$,

$$\lim_\alpha \lim_\beta F\left(x_\alpha^{i_1} \otimes \cdots \otimes x_\alpha^{i_n}, y_\beta^{j_1} \otimes \cdots \otimes y_\beta^{j_n}\right) \neq$$

$$\neq \lim_\beta \lim_\alpha F\left(x_\alpha^{i_1} \otimes \cdots \otimes x_\alpha^{i_n}, y_\beta^{j_1} \otimes \cdots \otimes y_\beta^{j_n}\right).$$

Let $P_B(x) := B(x \otimes \cdots \otimes x, x \otimes \cdots \otimes x)$. From the Polarization formulas (1.2) and (1.6) it follows that there are constants a_k, b_k, c_k, d_k, h_k such that

$$F_{P_B}\left(x_\alpha^{i_1}, \ldots, x_\alpha^{i_n}, y_\beta^{j_1}, \ldots, y_\beta^{j_n}\right) = \sum_k c_k P_B\left(a_k \sum_{m=1}^n d_m x_\alpha^{i_m} + b_k \sum_{m=1}^n h_m y_\beta^{j_m}\right),$$

where F_{P_B} is the symmetric n-linear form associated with P_B and the right hand sum contains a finite number of terms. Since,

$$F_{P_B}\left(x_\alpha^{i_1}, \ldots, x_\alpha^{i_n}, y_\beta^{j_1}, \ldots, y_\beta^{j_n}\right) = B\left(x_\alpha^{i_1} \otimes \cdots \otimes x_\alpha^{i_n}, y_\beta^{j_1} \otimes \cdots \otimes y_\beta^{j_n}\right)$$

for some k,

$$\lim_\alpha \lim_\beta P_B\left(a_k \sum_{m=1}^n d_m x_\alpha^{i_m} + b_k \sum_{m=1}^n h_m y_\beta^{j_m}\right) \neq$$

$$\neq \lim_\beta \lim_\alpha P_B\left(a_k \sum_{m=1}^n d_m x_\alpha^{i_m} + b_k \sum_{m=1}^n h_m y_\beta^{j_m}\right).$$

Hence the sum is discontinuous in X_Γ. □

PROPOSITION 2.35. If $\odot_\pi^n X$ is symmetrically regular for every n, then the operation of convolution of complex homomorphisms in M_b is commutative.

PROOF. Let P be an n-homogeneous polynomial and F_P be the correspondent n-linear form, associated with P. Let g be an arbitrary functional in X', $g \neq 0$. Consider a map

$$B_g(x^{\otimes n}, y^{\otimes n}) := \sum_{k=1}^{n-1} F_P(x^k, y^{n-k}) g^{n-k}(x) g^k(y),$$

where $x^{\otimes n}, y^{\otimes n} \in \odot_\pi^n X$. It is clear that $B_g(x^{\otimes n}, y^{\otimes n})$ defines a symmetric bilinear form $\odot_\pi^n X$. Let ϕ and ψ be the restriction of some arbitrary characters to $\mathcal{P}(^n X)$ and x_α and y_α be corespondent nets which converge in X_Γ to ϕ and ψ respectively. Then $x_\alpha^{\otimes n}$, $y_\beta^{\otimes n} \in \odot_\pi^n X$ are weakly-star convergent to some elements in $(\odot_\pi^n X)''$. Since $\odot_\pi^n X$ is symmetrically regular,

$$\lim_{\alpha,\beta} B_g\left(x_\alpha^{\otimes n}, y_\beta^{\otimes n}\right) = \lim_{\beta,\alpha} B_g\left(x_\alpha^{\otimes n}, y_\beta^{\otimes n}\right).$$

If we put y_β instead $t y_\beta$ with $t \in \mathbb{C}$ we get

$$\sum_{k=1}^{n-1} t^k \left(\lim_{\alpha,\beta} F_P\left(x_\alpha^k, y_\beta^{n-k}\right) - \lim_{\beta,\alpha} F_P\left(x_\alpha^k, y_\beta^{n-k}\right)\right) g^{n-k}(x_\alpha) g^k(y_\beta) = 0.$$

Since it is true for every $t \in \mathbb{C}$ and $g \in X'$, we have

$$\lim_{\alpha,\beta} F_P\left(x_\alpha^k, y_\beta^{n-k}\right) = \lim_{\beta,\alpha} F_P\left(x_\alpha^k, y_\beta^{n-k}\right)$$

for an arbitrary k. Taking into account that

$$P(x_\alpha + y_\beta) = \sum_{k=0}^{n} \binom{n}{n-k} F_P(x_\alpha, \dots, x_\alpha, y_\beta, \dots, y_\beta)$$

and that

$$\phi * \psi(P) = \lim_\alpha \lim_\beta P(x_\alpha + y_\beta),$$

we have that the convolution is commutative. □

2.7. Discontinuous complex homomorphisms and Michael's problem

E.Michael [**75**] posed the following problem in 1952 which is still open:

Is every complex homomorphism of a commutative Fréchet algebra continuous?

In [**76**, p. 240] Mujica proved that the Michael problem can be reduced to the case of the algebra $\mathcal{H}_b(X)$ for an arbitrary Banach space X with a Schauder basis. However, a dense subalgebra of $\mathcal{H}_b(X)$ may admit a discontinuous complex homomorphism. Dixon [**37**] has given an example of an algebra of polynomials of infinitely many variables which admits discontinuous scalar-valued homomorphisms. In [**42**] Galindo et al. gave a construction of a discontinuous scalar-valued homomorphism of algebra of polynomials on arbitrary infinite-dimensional Banach space. Their idea is to take a discontinuous functional on X' and extend it to a functional on $\mathcal{P}(X)$. The next proposition shows that the restriction of a discontinuous complex homomorphism on $\mathcal{A}_n(X) \cap \mathcal{P}(X)$ can be continuous for every $n \in \mathbb{N}$.

PROPOSITION 2.36. If the sequence of algebras $\mathcal{A}_n(X)$ does not stabilize, then there is a discontinuous complex homomorphism ζ on $\mathcal{P}(X)$ such that the restriction of ζ on $\mathcal{A}_n(X) \cap \mathcal{P}(X)$ is a continuous complex homomorphism for every $n \in \mathbb{N}$.

PROOF. By Corollary 2.18 and Theorem 2.21 there exists an infinity sequence $(u_k)_{k=1}^{\infty}$ with $u_k \in E_k$, $u_k \neq 0$. Since each E_k is a vector space, we can choose u_k such that $\sup_k \|u_k\|_k^{1/k} = \infty$. If

$$\zeta = \mathop{*}_{k=1}^{\infty} \delta^{(k)}(u_k) \quad \text{then} \quad \zeta(f) = \mathop{*}_{k=1}^{n} \delta^{(k)}(u_k)(f)$$

for every $f \in \mathcal{A}_n(X)$. So, ζ is well defined and continuous on $\mathcal{A}_n(X) \cap \mathcal{P}(X)$. If ζ is continuous on $\mathcal{P}(X)$, then it can be extended to a continuous complex homomorphism on $\mathcal{H}_b(X)$. But it contradicts Theorem 2.25. □

A discontinuous complex homomorphism of $\mathcal{H}_b(X)$ (if it exists) eventually, need not to be discontinuous on $\mathcal{P}(X)$.

PROPOSITION 2.37. If there exists a discontinuous complex homomorphism ϕ of $\mathcal{H}_b(X)$, then there exists a discontinuous complex homomorphism ψ of $\mathcal{H}_b(X)$ such that the restriction of ϕ on X' is discontinuous.

PROOF. Let (f_n) be a sequence in $\mathcal{H}_b(X)$ such that $\|f_n\|_r \to 0$ as $n \to \infty$ for every $r > 0$ and

$$\phi(f_n) > 4^n.$$

Let $(\mathfrak{e}_n)$ be a normalized basis sequence in X with a normalized biorthogonal sequence $(\mathfrak{e}_n^*) \subset X$. Put

$$F(x) := \sum_{n\in\mathbb{N}} \frac{1}{2^n} f_n(x)\mathfrak{e}_n.$$

It is easy to check that $F \in \mathcal{H}_b(X, X)$. So, the composition operator

$$T_F \colon f \longmapsto f \circ F$$

is a continuous homomorphism from $\mathcal{H}_b(X)$ to itself. We set $\psi := \phi \circ F$. Then ψ is a complex homomorphism of $\mathcal{H}_b(X)$ and

$$|\psi(\mathfrak{e}_n^*)| = \left|\frac{\phi(f_n)}{2^n}\right| > 2^n.$$

Hence, ψ is a discontinuous complex homomorphism of $\mathcal{H}_b(X)$. □

2.8. Continuous homomorphisms

Recall that $\mathbb{E}^n \subset \mathbb{E}^\infty \subset M_b$, where

$$\mathbb{E}^n := E_1 \times \cdots \times E_n = \{(u_1, \ldots, u_n) \colon u_k \in E_k, 1 \le k \le n\}.$$

PROPOSITION 2.38. Let Θ be a continuous homomorphism from $\mathcal{H}_b(X)$ to itself. Then for every positive integer n there exists a map

$$F_n \colon \mathbb{E}^n \longrightarrow \mathbb{E}^n$$

such that for every $f \in \mathcal{A}_n(X)$,

$$\Theta(f) = \widehat{f} \circ F_n.$$

PROOF. If $\mathfrak{u} = (u_1, \ldots, u_n) \in \mathbb{E}^n$. Then

$$\phi_\mathfrak{u} \circ \Theta = \mathop{*}_{k=1}^{m} \delta^{(k)}(u_k) \circ \Theta \in M_b.$$

By Theorem 2.21 there exists a point $\mathfrak{v} = (v_1, v_2, \ldots) \in M_b$ such that

$$\phi_\mathfrak{u} \circ \Theta(f) = \widehat{f}(\mathfrak{v}).$$

If $f \in \mathcal{A}_n(X)$, then $\widehat{f}(\mathfrak{v}) = \widehat{f}(v_1, \ldots, v_n)$. So, we can assume that $\mathfrak{v} \in \mathbb{E}^n$. Put $F_n(\mathfrak{u}) := \mathfrak{v}$. Thus, we have constructed the required mapping $\mathfrak{u} \longmapsto F_n(\mathfrak{u})$ with the property $\Theta(f) = \widehat{f} \circ F_n$. □

We notice that the above map F_n need not to be analytic in $\mathbb{E}^n$. For example, let $0 \neq u_2 \in E_2$ and g be a linear functional on X. We define $F\colon X \longrightarrow E_2$ by $F(x) := \sqrt{g(x)}u$. Then

$$\Theta_F(f)(x) := f \circ F(x) = \sum_{n \in \mathbb{Z}_+} \big(g(x)\big)^n f_{2n}(u_2),$$

for an arbitrary Taylor series $f = \sum f_n \in \mathcal{H}_b(X)$. It is easy to see that Θ_F is a continuous homomorphism of $\mathcal{H}_b(X)$ to itself but F is not holomorphic.

A homomorphism Θ from $\mathcal{H}_b(X)$ to itself is called *AB-composition homomorphism* [**23**] if there exists $F \in \mathcal{H}_b(X'', X'')$ such that

$$\widetilde{\Theta(f)}(x'') = \tilde{f}\big(F(x'')\big), \qquad x'' \in X'',$$

where $\tilde{f}$ is the Aron-Berner extension of f.

THEOREM 2.39. *Every polynomial on X is approximable if and only if every homomorphism on $\mathcal{H}_b(X)$ is an AB-composition homomorphism.*

PROOF. Suppose that every polynomial on X is approximable. Then $\mathcal{H}_b(X) = \mathcal{A}_1(X)$. By Proposition 2.38 for every homomorphism

$$\Theta\colon \mathcal{H}_b(X) \longrightarrow \mathcal{H}_b(X)$$

there exists a mapping $F\colon X'' \longrightarrow X''$ such that

$$\Theta(f) = \hat{f} \circ F = \tilde{f} \circ F, \qquad f \in \mathcal{H}_b(X).$$

In particular, for every $f \in X'$, we have $\tilde{f} \circ F \in \mathcal{H}_b(X)$. So, we can say that F is weak-star analytic map on X''. By a classical result of Dunford [**38**] and Grothendieck [**49**] on weak-star analytic mappings, we obtain that F is analytic on X''. Since $\tilde{f} \circ F$ is bounded on bounded sets of X'' for every $f \in X'$ and weak-star boundedness implies boundedness, we have $F \in H_b(X'', X'')$.

Suppose now that $\mathcal{A}_n(X) \neq \mathcal{A}_1(X)$ for some n. Let us choose $u_n \in E_n$ with $u_n \neq 0$ and $l \in X'$ with $l \neq 0$. Put

$$F(x) := l(x)u_n, \qquad \Theta(f)(x) := \hat{f}\big(F(x)\big), \qquad x \in X.$$

Since $F \in \mathcal{H}_b(X, \mathbb{E}^n)$, we obtain that $\Theta(f)(x) \in \mathcal{H}_b(X)$. But Θ is not an AB-composition homomorphism, because $\Theta \neq 0$ and $\Theta(f) = 0$ for every $f \in \mathcal{A}_1$. ☐

Since the approximation property of X' implies that every weakly continuous on bounded sets polynomial is approximable [**10**], we have the following corollary.

COROLLARY 2.40. (c.f. [**23**]). Let X' have the approximation property. Then every polynomial on X is weakly continuous on bounded sets if and only if every homomorphism on $\mathcal{H}_b(X)$ is an AB-composition homomorphism.

The result of Theorem 2.39 can be improved for a reflexive Banach space.

THEOREM 2.41. (Mujica [**77**]). *If $\mathcal{P}(X) = \mathcal{P}_a(X)$ for a reflexive Banach space X, then for every continuous homomorphism*

$$\Theta\colon \mathcal{H}_b(X) \longrightarrow \mathcal{H}_b(X)$$

there is a unique map $F \in \mathcal{H}_b(X, X)$ *such that*

$$\Theta(f) = f \circ F.$$

COROLLARY 2.42. Let X be a reflexive Banach space with $\mathcal{P}(X) = \mathcal{P}_a(X)$ and $F \in \mathcal{H}_b(X, X)$. Suppose that $\Theta(f) = f \circ F$ is an isomorphism of $\mathcal{H}_b(X)$. Then F is invertible and $F^{-1} \in \mathcal{H}_b(X, X)$.

PROOF. By Theorem 2.41 there exists a map $G \in \mathcal{H}_b(X, X)$ such that $\Theta^{-1}(f) = f \circ G$. It is easy to see that $G = F^{-1}$. □

2.9. Continuous derivations

Let $u_k \in E_k$. According to Theorem 2.21 we can define a complex homomorphism

$$\phi = \delta^{(k)}(u_k) \in M_b, \qquad \phi(f) = \widehat{f}(u_k)$$

for every $f \in \mathcal{H}_b(X)$, where $\widehat{f}$ denotes the Gelfand transform of f.

However, the element u_k belongs to $(\odot_\pi^k X)''$ and so there is an another natural way to define a linear functional on $\mathcal{H}_b(X)$, associated with u_k. Let

$$\theta = \theta(u_k) = \sum_{m \in \mathbb{Z}_+} \theta_m \in \mathcal{H}_b(X)'$$

such that $\theta_k(P) = \widehat{P}(u_k)$ if $P \in \mathcal{P}(^k X)$ and $\theta_m = 0$ if $m \neq k$. Recall that here θ_m is the restriction of θ to $\mathcal{P}(^m X)$. It is easy to see that θ is not a homomorphism if $u_k \neq 0$. We define a linear operator $\partial_{(k)}(u_k)$ on $\mathcal{H}_b(X)$ by

$$\partial_{(k)}(u_k)(f)(x) := \theta(u_k) \circ T_x(f), \qquad f \in \mathcal{H}_b(X).$$

For the multilinear form F_P associated with an n-homogeneous polynomial P we denote by $\widehat{F}_P\left(x^{n-k}, u_k\right)$ the value of the Gelfand transform of the k-homogeneous polynomial $E_k \ni u_k \longmapsto F_P\left(x^{n-k}, \cdot\right)$ with a fixed $x \in X$.

THEOREM 2.43. *Let* $u_k \in E_k$. *The operator* $\partial_{(k)}(u_k)$ *is a continuous derivation on* $\mathcal{H}_b(X)$,

$$\partial_{(k)}(u_k)(P)(x) = \binom{n}{k} \widehat{F}_P\left(x^{n-k}, u_k\right), \qquad x \in X \tag{2.22}$$

for every $P \in \mathcal{P}(^n X)$ *and*

$$\delta^{(k)}(u_k)(f)(x) = \sum_{m \in \mathbb{Z}_+} \frac{(k!)^m}{(mk)!} \partial_{(k)}^m(u_k)(f)(x), \qquad x \in X \tag{2.23}$$

for every $f \in \mathcal{H}_b(X)$.

PROOF. To prove the formula (2.22) we observe that

$$P(z+x)=\sum_{m=0}^{n}\binom{n}{m}F_P\left(x^{n-m},z^m\right).$$

So, for a fixed $x\in X$, we obtain

$$\partial_{(k)}(u_k)(P)(x)=\theta(u_k)\big(P(z+x)\big)=\binom{n}{k}\widehat{F}_P\left(x^{n-k},u_k\right).$$

Note that if $\deg P\le k$, then $\partial_{(k)}(u_k)(P)(x)=0$ for every $x\in X$ by the definitic of $\partial_{(k)}(u_k)$.

Let $P\in\mathcal{P}(^nX)$ and $Q\in\mathcal{P}(^mX)$. The multilinear form $F_{PQ}\left(x^{nm-k},z^k\right)$ ass ciated with PQ can be represented by

$$F_{PQ}\left(x^{nm-k},z^k\right)=F^1_{PQ}\left(x^{nm-k},z^k\right)+F^2_{PQ}\left(x^{nm-k},z^k\right)+F^3_{PQ}\left(x^{nm-k},z^k\right),$$

where

$$F^1_{PQ}\left(x^{n-k},z^k\right):=F_P\left(x^{n-k},z^k\right)F_Q\left(x^m\right),$$
$$F^2_{PQ}\left(x^{n-k},z^k\right):=F_P\left(x^n\right)F_Q\left(z^k,x^{m-k}\right)$$

and

$$F^3_{PQ}\left(x^{n-k},z^k\right):=\frac{1}{k-1}\sum_{s=1}^{k-1}F_P\left(x^{n-s}z^s\right)F_Q\left(z^{k-s},x^{m-k+s}\right).$$

If $n\le k$ (resp. $m\le k$), then F^1_{PQ} (resp. F^2_{PQ}) is equal to zero. By definitions $\theta(u_k)$ and u_k, we have

$$\theta(u_k)F^3_{PQ}\left(x^{n-k},z^k\right)=0$$

for any fixed $x\in X$. So,

$$\partial_{(k)}(u_k)(PQ)(x)=\partial_{(k)}(u_k)(P)(x)Q(x)+P(x)\partial_{(k)}(u_k)(Q)(x).$$

Since $\partial_{(k)}(u_k)$ is linear, it is a differentiation on the algebra $\mathcal{H}_b(X)$. The continuit of $\partial_{(k)}(u_k)$ follows from the continuity of $\theta(u_k)$ and the translation T_x.

Let $P\in\mathcal{P}(^nX)$ and $n=km$. From (2.22) we have that

$$\partial^m_{(k)}(u_k)(P)=\binom{km}{k}\binom{k(m-1)}{k}\cdots\binom{k}{k}\widehat{P}(u_k)=\frac{(mk)!}{(k!)^m}\,\delta^{(k)}(u_k)(P).$$

Thus,

$$\delta^{(k)}(u_k)=\sum_{m\in\mathbb{Z}_+}\frac{(k!)^m}{(mk)!}\,\partial^m_{(k)}(u_k),$$

so, the formula (2.22) is proved. □

This approach can be generalized by the following way. Let $v_p\ne 0$ be an arbitrar element in E_p for some positive integer p. Denote by T_{v_p} the following operator o $\mathcal{H}_b(X)$,

$$T_{v_p}(f):=\widehat{f}(\cdot+v_p).$$

We can write

$$\partial_{(k,p)}(u_k)(\widehat{f})(v_p):=\theta(u_k)\circ T_{v_p}(f).$$

Repeating arguments of Theorem 2.43, we can see that for every $P \in \mathcal{P}(^{km}X)$,

$$\partial_{(k,k)}(u_k)(\widehat{P})(v_k) = m\widehat{F}_P\left(v_k^{m-1}, u_k\right).$$

Moreover, if $f = \sum f_n \in \mathcal{H}_b(X)$, then

$$\widehat{f}(v_k + u_k) = \sum_{m \in \mathbb{Z}_+} \frac{\partial^m_{(k,k)}(u_k)(\widehat{f_{km}})(v_k)}{m!}.$$

Aron, Cole and Gamelin in [6] considered the operation $\partial_{(k)}(u_k)$ for the case when $k = 1$ and so $u_k = u_1 = z$ for some $z \in X''$. They used notation

$$(z)T_x f = (*z)f(x)$$

instead our $\partial_{(1)}(z)f(x)$. For this special case using this notation formula 2.23 can be rewritten as

$$\delta^{(1)}(z)f = \widetilde{\delta}(z)f = \sum_{m \in \mathbb{Z}_+} \frac{1}{m!} z^{*m} = \exp(*z).$$

2.10. Ball algebras of analytic functions

In this section we consider maximal ideals of uniform algebras of analytic complex functions on the ball rB for some $r > 0$, where B is the unit open ball of a Banach complex space X. We will analyze the following uniform algebras:

$$\mathcal{H}^\infty(rB), \qquad \mathcal{H}^\infty_{uc}(rB), \qquad \text{and} \qquad \mathcal{H}^\infty_c(rB),$$

where $\mathcal{H}^\infty(rB)$ is the algebra of all bounded analytic complex functions on rB, $\mathcal{H}^\infty_{uc}(rB)$ is the algebra of all uniformly continuous analytic complex functions on rB, and $\mathcal{H}^\infty_c(rB)$ is the algebra of all bounded analytic complex functions on rB, which are continuous on the closure $r\bar{B}$. It is clear that

$$\mathcal{H}_b(X) \subset \mathcal{H}^\infty_{uc}(rB) \subset \mathcal{H}^\infty_c(rB) \subset \mathcal{H}^\infty(rB).$$

It is easy to check at once that $\mathcal{H}^\infty_{uc}(rB)$ precisely consists with the uniform limit on rB of functions from $\mathcal{H}_b(X)$.

Since the set of $\phi \in M_b$, satisfying the inequality $R(\phi) \le r$, is the $\mathcal{H}_b(X)$-convex hull of rB in M_b, we obtain the following theorem.

THEOREM 2.44. *For each fixed $r > 0$, the compact set*

$$\{\phi \in M_b \colon R(\phi) \le r\}$$

coincides with the spectrum of $\mathcal{H}^\infty_{uc}(rB)$, in particular,

$$M(\mathcal{H}^\infty_{uc}(B)) = \{\phi \in M_b \colon R(\phi) \le 1\}.$$

COROLLARY 2.45. The spectrum of $\mathcal{H}^\infty_{uc}(B)$ contains unit balls of E_k for every positive integer k.

Let now $\mathcal{H}$ be a uniform algebra such that

$$\mathcal{H}^\infty_{uc}(rB) \subset \mathcal{H} \subset \mathcal{H}^\infty(rB)$$

and $M_\mathcal{H}$ be its spectrum. There is a natural projection

$$\iota\colon M_\mathcal{H} \longrightarrow M_b$$

such that $\iota(\psi)$ is the restriction of $\psi \in M_\mathcal{H}$ to $\mathcal{H}_b(X)$.

Now we expand the definition of *the radius function* R to any functional $\psi \in M_\mathcal{H}$ by declaring $R(\psi)$ to be the smallest value of r, $0 \le r \le 1$, such that ψ is continuous with respect to the norm of uniform convergence on rB.

THEOREM 2.46. *Let $\mathcal{H}$ be a uniform algebra between $\mathcal{H}^\infty_{uc}(B)$ and $\mathcal{H}^\infty(B)$. The image $\iota(M_\mathcal{H})$ of the projection ι precisely coincides with the set*

$$\{\phi \in M_b\colon R(\phi) \le 1\}.$$

PROOF. If $\psi \in M_\mathcal{H}$ and

$$|\psi(f)| \le \sup_{rB} |f| \quad \text{for all} \quad f \in \mathcal{H},$$

then this inequality holds, in particular, for all $f \in \mathcal{H}_b(X)$. Therefore,

$$R\big(\iota(\psi)\big) \le R(\psi) \quad \text{for all} \quad \psi \in M_\mathcal{H}.$$

Suppose that $\phi \in M_b$ satisfies $R(\phi) < 1$. Then ϕ is continuous on $\mathcal{H}_b(X)$ with respect to the norm of uniform convergence on $R(\phi)B$. Each $f \in \mathcal{H}^\infty(B)$ is a uniform limit on any ball rB, $0 < r < 1$ of the partial sums of its Taylor series. Hence, ϕ extends uniquely to f and determine a unique $\psi \in M_\mathcal{H}$ such that $\iota(\psi) = \phi$ and $R(\psi) < 1$. As a consequence, we obtain that

$$R(\phi) = R(\psi).$$

Suppose now that $\phi \in M_b$ satisfies the equality $R(\phi) = 1$. Let

$$\phi = \mathop{\ast}_{k=1}^{\infty} \delta^{(k)}(u_k)$$

for some (u_k). For $|\xi| < 1$, consider the homomorphism

$$\phi^\xi := \mathop{\ast}_{k=1}^{\infty} \delta^{(k)}(\xi u_k).$$

Since $R(\phi^\xi) = |\xi| < 1$, the homomorphism ϕ^ξ extends to a homomorphism in $M_\mathcal{H}$. If ψ is any cluster point in $M_\mathcal{H}$ of the extension of ϕ^ξ as $\xi \to 1$ with $|\xi| < 1$, then

$$\iota(\psi) = \phi \text{ and } R(\phi) = R(\psi).$$

Thus, the image of ι is precisely coincided with $\{\phi \in M_b\colon R(\phi) \le 1\}$. □

Comparing Theorem 2.46 and Theorem 2.44 we can see that if $\mathcal{H} = \mathcal{H}^\infty_{uc}(B)$, then the projection ι is one-to-one.

THEOREM 2.47. *Let $\mathcal{H}$ be a uniform algebra between $\mathcal{H}^\infty_{uc}(B)$ and $\mathcal{H}^\infty(B)$. Then the natural projection of the spectrum $M_\mathcal{H}$ of $\mathcal{H}$ onto*

$$\{\phi \in M_b\colon R(\phi) \le 1\}$$

is one-to-one if and only if $\mathcal{H} = \mathcal{H}^\infty_{uc}(B)$.

PROOF. Suppose that $f \in \mathcal{H}$ is not uniformly continuous. Then there are $\varepsilon > 0$, and sequences (x_n) and (y_n) in B such that $\|x_n - y_n\| \to 0$, while $|f(x_n) - f(y_n)| \geq \varepsilon$ for all $n \in \mathbb{N}$. A subnet (x_{n_α}) converges in M_b to some ϕ satisfying $R(\phi) \leq 1$. The net (y_{n_α}) then also converges in M_b to ϕ. Since $|f(x_n) - f(y_n)| \geq \varepsilon$, we see that (x_{n_α}) and (y_{n_α}) have cluster points θ and θ' in $M_{\mathcal{H}}$ such that $f(\theta) \neq f(\theta')$. However, θ and θ' both coincide with ϕ on $\mathcal{H}^\infty_{uc}(B)$, that is θ and θ' both project onto ϕ. □

We notice that in [**6**] is proved that if X is an infinite-dimensional Banach space, then

$$\mathcal{H}^\infty_{uc}(B) \neq \mathcal{H}^\infty_c(B).$$

2.11. C^*-algebras of continuous functions

For a given complex Banach space X we denote by $X^{\Re}$ a Banach space which coincides with X as a point set but endowed with the real structure. In the other words, $X^{\Re}$ is X where we allow real scalar multiplication only. Evidently $X = X^{\Re}$ as topological spaces and each continuous function f on X is well defined and continuous on $X^{\Re}$. We will denote by $f^{\Re}$ the act of f on $X^{\Re}$.

DEFINITION 2.48. A mapping $Q\colon X \longrightarrow \mathbb{C}$ is called an n-degree $*$-polynomial if $Q^{\Re}\colon X^{\Re} \longrightarrow \mathbb{C}$ is a complex-valued polynomial of n degree on the real Banach space $X^{\Re}$.

We denote by $\mathcal{P}^*(X)$ the algebra of all $*$-polynomials on X and by $C_{\mathcal{P}}(B)$ the completion of $\mathcal{P}^*(X)$ in the uniform topology on the open unit ball B of X. $C_{\mathcal{P}}(B)$ contains all continuous polynomials on X and all continuous *anti-polynomials* on X, where anti-polynomials are just complex conjugates to polynomials. Let us denote by $C_a(B)$ a minimal closed subalgebra of $C_{\mathcal{P}}(B)$ which contains all continuous polynomials on X and all continuous anti-polynomials. Notice that $C_{\mathcal{P}}(B) \neq C_a(B)$ in the general case. For example it is easy to check that a $*$-polynomial Q on ℓ_2,

$$Q\Big(\sum_{n\in\mathbb{N}} x_n e_n\Big) = \sum_{n\in\mathbb{N}} x_n \bar{x}_n$$

belongs to $C_{\mathcal{P}}(B)$ but does not belong to $C_a(B)$.

THEOREM 2.49. *The spectrum $M(C_a(B))$ of $C_a(B)$ consists of all characters ϕ of $\mathcal{H}^\infty_{uc}(B)$ for which there are nets $(x_\alpha) \subset B$ such that*

$$\phi(P) = \lim_\alpha P(x_\alpha), \qquad \forall P \in \mathcal{P}(X). \tag{2.24}$$

PROOF. Let $\phi \in \mathcal{H}^\infty_{uc}(B)$ such that (2.24) holds for some $(x_\alpha) \subset B$. Then $\phi(\bar{P}) := \bar{\phi}(P)$ is well defined for every $P \in \mathcal{P}(X)$. If Q is in an algebraic span of polynomials and antipolynomials, $|\phi(Q)| \leq \sup_\alpha |Q(x_\alpha)| \leq \|Q\|$. So, ϕ can be extended by continuity to a character on $C_a(B)$.

Let now ϕ be a character on $C_a(B)$. Since $C_a(B)$ is a C^*-algebra, $M(C_a(B))$ is the Czech-Stone compactification of B in the Gelfand topology of $C_a(B)$ on B.

Hence, B is dense in $\beta B = M(\mathcal{C}_a(B))$, that is, there exists a net $(x_\alpha) \subset B$ such that $\phi(f) = \lim_\alpha f(x_\alpha)$ for every $f \in \mathcal{C}_a(B)$. So (2.24) holds. □

By the theorem we can write $M(\mathcal{C}_a(B)) \subset M(\mathcal{H}^\infty_{uc}(B))$. Since, $M(\mathcal{H}^\infty_{uc}(B)) = \{\phi \in M_b\colon R(\phi) \le 1\}$, we can apply Theorem 2.21 and Theorem 2.25.

Corollary 2.50. Let $\phi \in M(\mathcal{C}_a(B))$. Then there exists a sequence $(u_k)_{k=1}^\infty$, $u_k \in E_k$ such that $\sup_k \|u_k\|^{1/k} \le e$ and

$$\phi(f) = \mathop{\ast}_{k=1}^{\infty} \delta^{(k)}(u_k)(f) \quad \text{and} \quad \phi(\overline{f}) = \mathop{\ast}_{k=1}^{\infty} \overline{\delta^{(k)}(u_k)(f)}$$

for every $f \in \mathcal{H}^\infty_{uc}(B)$.

A given positive integer m we denote by Q_m a $*$-polynomial on ℓ_{2m} as

$$Q_m(x) = Q_m\Big(\sum_{n\in\mathbb{N}} x_n e_n\Big) = \sum_{n\in\mathbb{N}} x_n^m \bar{x}_n^m.$$

Let x_α be a weakly polynomially zero net in ℓ_{2m} with $\|x_\alpha\| = 1$, where α belongs to an index set $\mathfrak{A}$. Let $\mathcal{U}$ be a free ultrafilter on $\mathfrak{A}$. We set

$$\psi(f) = \lim_{\mathcal{U}} f(x_\alpha).$$

It is clear that $\psi(f) = f(0)$ if $f \in \mathcal{C}_a(B)$ but $\psi(Q_m) = 1$. So we can see that $\mathcal{C}_a(B) \neq \mathcal{C}_\mathcal{P}(B)$ in ℓ_{2m} and there exists a character ψ in $M(\mathcal{C}_\mathcal{P}(B))$ which vanishes on homogeneous polynomials of $\mathcal{C}_a(B)$.

Notes and Remarks. The problem of description of the spectrum of $H_b(X)$ was first studied by Aron, Cole and Gamelin [**6, 7**]. Using the Aron-Berner extension operation [**4, 32**], they showed, in particular that X'' belongs to the spectrum of $H_b(X)$. In [**9**] it is proved that this inclusion is proper if there exists a polynomial on X which is not weakly continuous on bounded sets. This approach was generalized for algebra-valued analytic functions by García et al. in [**46**]. Some analytic structure on the set of maximal ideals was considered in [**9**] (a generalization for functions with values in an algebra is given in [**45**]). In [**77**] Mujica investigated ideals of analytic functions of bounded type on Tsirelson's space T and showed that each character on $H_b(T)$ is a point evaluation functional. Homomorphisms of H_b was studied by Carando, García and Maestre in [**23**]. Further developing in this direction is in [**5, 24, 25, 70**].

Main results of this chapter were basically obtained in [**96, 97**]. Some of them were generalized for Wiener type algebras generated by (p,q)-polynomials by Vasylyshyn in [**93**].

Note that in [**2, 27, 28, 29**] were investigated spectra of algebras of *symmetric* analytic functions on ℓ_p using some another approach. Locally convex algebras of analytic functions on nuclear spaces were investigated in [**58, 62**].

CHAPTER 3

Hardy spaces associated with topological groups

The classic theory of unitary Hardy spaces $\mathcal{H}^2_\chi(\mathsf{B})$ of complex analytic functions

$$\left\{ f\colon \sup_{r\in[0,1)} \int_{\mathbb{T}} |f(re^{\mathrm{i}\vartheta})|^2\, d\chi(e^{\mathrm{i}\vartheta}) < \infty \right\},$$

defined on the 1-dimensional open complex ball B, essentially uses of invariant properties of probability Haar's measure χ on the complex cyclic group

$$\mathbb{T} = \{ e^{\mathrm{i}\vartheta}\colon \vartheta \in [0, 2\pi) \}$$

irreducibly and unitarily acting in $\mathbb{C}$. The $\mathbb{T}$-invariancy provides that $\mathcal{H}^2_\chi(\mathsf{B})$ is unitary equivalent to the Hermitian dual ℓ^{2*} of the ℓ^2-space Taylor coefficients $(d_0^n f/n!)$ at the origin i.e.,

$$\mathcal{H}^2_\chi(\mathsf{B}) \simeq \ell^{2*}.$$

Moreover, every function $f \in \mathcal{H}^2_\chi(\mathsf{B})$ can be uniquely defines by its radial boundary values on $\mathbb{T}$ with the help of the integral Cauchy formula

$$f(\xi) = \int_{\mathbb{T}} \left(e^{\mathrm{i}\vartheta} - \xi\right)^{-1} f(e^{\mathrm{i}\vartheta})\, d\chi(e^{\mathrm{i}\vartheta}), \qquad |\xi| < 1.$$

There is a natural question: *is it possible to replace in this theory the cyclic group* $\mathbb{T}$ *by a infinite-dimensional topological group?* The purpose of this chapter is to show that this is possible in many cases.

Namely, we analyze various cases of infinite-dimensional topological groups $\mathcal{G}$ that have an unitary representation over an infinite-dimensional Hilbert complex space E. For these groups we give definitions of appropriate unitary Hardy spaces $\mathcal{H}^2_\chi$ defined by $\mathcal{G}$-invariant probability measures χ, which coincide in the 1-dimensional case with the classic unitary Hardy space. These spaces $\mathcal{H}^2_\chi$ consist of complex functions that have unique analytic extensions on a domain in E.

Also, we show that for infinite-dimensional cases instead of the space ℓ^{2*} of Taylor coefficients $(d_0^n f/n!)$ it is necessary to use the Hermitian dual F^* of the symmetric Fock space F generated by a given Hilbert space E. Similarly as in the 1-dimensional case, the integral Cauchy type formulas use reproducing kernels which are generated by an orthogonal basis of the symmetric Fock space F associated with E.

3.1. Hardy spaces on compact infinite-dimensional group orbits

3.1.1. Hilbert spaces of compact group representations. Denote by $\mathscr{U}_r = \mathscr{U}(n_r)$ the group of all linear unitary operators in an n_r-dimensional complex Hilbert space $\mathbb{C}^{n_r}$ with the scalar product $\langle\cdot\mid\cdot\rangle_{\mathbb{C}^{n_r}}$ and an orthonormal basis

$$\mathscr{E}_r := \left\{\mathfrak{e}_{j_r(1)},\dots,\mathfrak{e}_{j_r(n_r)}\right\}.$$

A given subsequence $\{n_r : r\in\mathbb{N}\}$ of natural numbers we correspond the Cartesian product

$$\mathcal{G} := \bigtimes_{r\in\mathbb{N}} \mathscr{U}_r = \left\{U = (U_r)\colon U_r \in \mathscr{U}_r\right\},$$

endowed with the product's topology, which is an infinite-dimensional compact topological group. As is well known (see e.g. [**51**]), the compact group $\mathcal{G}$ can be unitary represented on the countable orthogonal Hilbertian sum

$$\mathsf{E} := \ell^2_{\mathcal{G}},$$

$$\ell^2_{\mathcal{G}} = \bigoplus_{r\in\mathbb{N}} \mathsf{E}_r = \left\{x = (x_r)\colon x_r \in \mathbb{C}^{n_r},\ \|x\| := \Big(\sum_{r\in\mathbb{N}} \|x_r\|^2_{\mathbb{C}^{n_r}}\Big)^{1/2} < \infty\right\}$$

endowed with the scalar product $\langle x\mid y\rangle := \sum_r \langle x_r \mid y_r\rangle_{\mathbb{C}^{n_r}}$, where $\{\mathsf{E}_r : r\in\mathbb{N}\}$ is a sequence of $\mathcal{G}$-irreducible subspaces such that

$$\mathsf{E}_r \bigcap \mathsf{E}_s = \{0\} \qquad \text{for all} \qquad r \neq s$$

and each E_r is unitary equivalent to $\mathbb{C}^{n_r}$ for the corresponding $r\in\mathbb{N}$. For simplicity we identify any element $x_r \in \mathbb{C}^{n_r}$ with its image $(0,\dots,0,x_r,0,\dots)\in\mathsf{E}_r$ under the canonical embedding

$$\mathbb{C}^{n_r} \hookrightarrow \mathsf{E}.$$

So, we can consider in E the orthonormal basis

$$\mathscr{E} := \bigcup_{r\in\mathbb{N}} \mathscr{E}_r = \left\{\mathfrak{e}_j\right\}_{j\in\mathbb{N}}$$

indexed such that $j < i$ for all $\mathfrak{e}_j \in \mathsf{E}_r$ and $\mathfrak{e}_i \in \mathsf{E}_{r+1}$. Let $\mathsf{B} = \{x \in \mathsf{E}\colon \|x\| < 1\}$ denote the open Hilbertian ball.

The symmetric tensor elements

$$\mathscr{E}_n := \left\{\mathfrak{e}_{\{j\}}^{\otimes(k)} := \mathfrak{e}_{j_1}^{\otimes k_1}\cdot\ldots\cdot\mathfrak{e}_{j_n}^{\otimes k_n} :\quad \{j\}\in\mathbb{N}^n,\quad (k)\in\mathbb{Z}^n_+,\quad |(k)| = n\right\}$$

form an orthogonal basis in $\odot^n_{\mathfrak{h}}\mathsf{E}$. If $n = |(k)| = 0$, we set $\mathfrak{e}_{\{j\}}^{\otimes(k)} \equiv 1$. So,

$$\mathscr{E}_0 = \{1\} \qquad \text{and} \qquad \mathscr{E}_1 = \mathscr{E}.$$

In the symmetric Fock space $\mathsf{F} = \bigoplus_n \left(\odot^n_{\mathfrak{h}}\mathsf{E}\right)$, generated by the Hilbert space E, the system

$$\mathscr{E} := \left\{\mathscr{E}_n \colon n\in\mathbb{Z}_+\right\}$$

forms an orthogonal basis (see Proposition 1.28).

We also consider the corresponding basis of Hilbert-Schmidt homogeneous polynomials

$$\mathscr{E}^* := \{\mathscr{E}_n^* \colon n \in \mathbb{Z}_+\},$$
$$\mathscr{E}_n^* := \Big\{\mathfrak{e}_{\{j\}}^{*(k)} = \mathfrak{e}_{j_1}^{*k_1} \ldots \mathfrak{e}_{j_n}^{*k_n} \in \mathcal{P}_h({}^n\mathsf{E}) \colon \mathfrak{e}_{\{j\}}^{\otimes(k)} \in \mathscr{E}_n,\ |(k)| = n\Big\},$$

generated by generated by $\mathscr{E}$, where

$$\mathfrak{e}_{\{j\}}^{*(k)}(x) := \langle x^{\otimes n} \mid \mathfrak{e}_{\{j\}}^{\otimes(k)}\rangle_{\otimes_\mathfrak{h}^n \mathsf{E}} = \langle x \mid \mathfrak{e}_{j_1}\rangle_{\mathsf{E}}^{k_1} \ldots \langle x \mid \mathfrak{e}_{j_n}\rangle_{\mathsf{E}}^{k_n}, \qquad x \in \mathsf{E}.$$

Clearly such polynomials form an orthogonal basis in the Hermitian dual symmetric Fock space F^*.

3.1.2. Representing invariant measures. Consider the Banach space

$$\ell_\mathcal{G}^\infty = \Big\{x = (x_r) \in \bigtimes_{r\in\mathbb{N}} \mathbb{C}^{n_r} \colon \|x\|_{\ell_\mathcal{G}^\infty} = \sup_{r\in\mathbb{N}} \|x_r\|_{\mathbb{C}^{n_r}} < \infty\Big\}$$

and the compact metric spaces

$$\mathsf{S}_\mathcal{G}^\infty := \bigtimes_{r\in\mathbb{N}} \mathsf{S}_r, \qquad \mathsf{S}_r := \{x_r \in \mathbb{C}^{n_r} \colon \|x_r\|_{\mathbb{C}^{n_r}} = 1\},$$
$$\mathsf{K}_\mathcal{G}^\infty := \bigtimes_{r\in\mathbb{N}} \mathsf{K}_r, \qquad \mathsf{K}_r := \{x_r \in \mathbb{C}^{n_r} \colon \|x_r\|_{\mathbb{C}^{n_r}} \leq 1\},$$

endowed with the product topologies. It is easy to see that $\mathsf{K}_\mathcal{G}^\infty$ coincides with a norm closed unit ball of $\ell_\mathcal{G}^\infty$ endowed with the weak-star topology. The contractive embedding

$$\mathsf{E} \hookrightarrow \ell_\mathcal{G}^\infty, \qquad \|x\|_{\ell_\mathcal{G}^\infty} \leq \|x\|, \qquad x \in \mathsf{E}$$

holds. A norm open unit ball in $\ell_\mathcal{G}^\infty$ we denote by

$$\mathsf{B}_\mathcal{G}^\infty := \{x \in \ell_\mathcal{G}^\infty \colon \|x\|_{\ell_\mathcal{G}^\infty} < 1\}.$$

As well we consider the uniform algebra of all continuous complex functions ψ on $\mathsf{K}_\mathcal{G}^\infty$,

$$C(\mathsf{K}_\mathcal{G}^\infty) \qquad \text{with the norm} \qquad \|\psi\|_{C(\mathsf{K}_\mathcal{G}^\infty)} = \sup_{x\in\mathsf{K}_\mathcal{G}^\infty} |\psi(x)|.$$

Clearly $C(\mathsf{K}_\mathcal{G}^\infty)$ contains the unity function $1_{\mathsf{K}_\mathcal{G}^\infty}$.

Remark 3.1. Note that each linear functional $\mathfrak{e}_j^* \in \mathscr{E}^*$ can be uniquely extended to a weak-star continuous linear functional on $\ell_\mathcal{G}^\infty$ which we denote by the same symbol. Up to this extension we can write

$$\mathscr{E}^* \subset C(\mathsf{K}_\mathcal{G}^\infty).$$

Let $\mathcal{A}(\mathsf{K}_\mathcal{G}^\infty)$ be a closure in $C(\mathsf{K}_\mathcal{G}^\infty)$ of the complex linear span of extended Hilbert-Schmidt polynomials $\mathscr{E}^*$. Clearly, $\mathcal{A}(\mathsf{K}_\mathcal{G}^\infty)$ is a uniform subalgebra in the algebra $C(\mathsf{K}_\mathcal{G}^\infty)$.

Recall that an element $x \in \mathsf{K}_\mathcal{G}^\infty$ is a peak point if there is a function $f \in \mathcal{A}(\mathsf{K}_\mathcal{G}^\infty)$ such that $f(x) = 1$ and $|f(y)| < 1$ for all $y \in \mathsf{K}_\mathcal{G}^\infty$ if $y \neq x$.

PROPOSITION 3.2. Let $\mathfrak{P}$ be the set of peak points and $\partial\mathcal{A}$ denotes the Choquet boundary of $\mathcal{A}(\mathsf{K}_{\mathcal{G}}^{\infty})$. Then

$$\mathsf{S}_{\mathcal{G}}^{\infty} = \mathfrak{P} = \partial\mathcal{A}.$$

For the uniform algebra

$$\mathcal{A}(\mathsf{S}_{\mathcal{G}}^{\infty}) := \mathcal{A}(\mathsf{K}_{\mathcal{G}}^{\infty})\,|_{\mathsf{S}_{\mathcal{G}}^{\infty}}$$

endowed with the uniform norm $\sup_{x\in\mathsf{S}_{\mathcal{G}}^{\infty}} |f(x)|$ the isometry

$$\mathcal{A}(\mathsf{K}_{\mathcal{G}}^{\infty}) = \mathcal{A}(\mathsf{S}_{\mathcal{G}}^{\infty}) \tag{3.1}$$

holds.

PROOF. As is well-known (see [**6**], [**44**])), the algebra $\mathcal{A}(\mathsf{K}_{\mathcal{G}}^{\infty})$, as an uniform closure of the linear span of finite type continuous polynomials, consists of complex analytic functions in the norm open unit ball $\mathsf{B}_{\mathcal{G}}^{\infty}$ having the form

$$\mathsf{B}_{\mathcal{G}}^{\infty} = \bigtimes_{r\in\mathbb{N}} \mathsf{B}_r, \qquad \mathsf{B}_r := \Big\{x_r \in \mathbb{C}^{n_r} \colon \|x_r\|_{\mathbb{C}^{n_r}} < 1\Big\}.$$

Hence, if $a \in \mathfrak{P}$, then $a \in \mathsf{S}_{\mathcal{G}}^{\infty}$ via Maximum Principle for analytic functions. Therefore, $\mathfrak{P} \subset \mathsf{S}_{\mathcal{G}}^{\infty}$.

In the other hand, for every fixed $a_r \in \mathsf{S}_r$ there exists an analytic in B_r and continuous on B_r function f_r such that

$$f_r(a_r) = 1, \qquad \text{and} \qquad |f_r(x_r)| < 1 \qquad \text{for all} \qquad x_r \in \mathsf{K}_r \setminus \{a_r\}$$

(see e.g. [**43**]). Then for each $a \in \mathsf{S}_{\mathcal{G}}^{\infty}$ such that the orthogonal projection of a onto the subspace $\mathsf{E}_r \simeq \mathbb{C}^{n_r}$ is equal to a fixed a_r, the analytic function

$$f = f_r \cdot \prod_{j\in\mathbb{N}\setminus\{r\}} \mathbf{1}_j \in \mathcal{A}(\mathsf{K}_{\mathcal{G}}^{\infty})$$

satisfies the conditions

$$f(a) = 1, \qquad \text{and} \qquad |f(x)| < 1 \qquad \text{for all} \qquad x \in \mathsf{K}_{\mathcal{G}}^{\infty} \setminus \{a\},$$

where $\mathbf{1}_j$ denotes the identically unit function on K_j, Hence, $a \in \mathfrak{P}$ and the embedding $\mathsf{S}_{\mathcal{G}}^{\infty} \subset \mathfrak{P}$ is proved. Therefore, $\mathfrak{P} = \mathsf{S}_{\mathcal{G}}^{\infty}$ and so we have the isometrical isomorphism (3.1).

Finally, since $\mathsf{K}_{\ell_{\mathcal{G}}^{\infty}}$ is a compact metric separable space, the set of peak points $\mathfrak{P}$ of $\mathcal{A}(\mathsf{K}_{\mathcal{G}}^{\infty})$ is a G_δ-subset in $\mathsf{K}_{\mathcal{G}}^{\infty}$. Hence the equality

$$\partial\mathcal{A} = \mathfrak{P}$$

is also true (for detail see [**43**, Section II.11.2]). □

As is well-known there exists a probability Haar measure χ on the group $\mathcal{G}$ exists a probability Haar measure χ (respectively, there exists a probability Haar measure

χ_r on the subgroup $\mathscr{U}_r$) for which

$$\chi(\phi) := \int_{\mathcal{G}} \phi(U)\ d\chi(U) = \int_{\mathcal{G}} \phi(VU)\ d\chi(U) = \int_{\mathcal{G}} \phi(UV)\ d\chi(U)$$

with all $U, V \in \mathcal{G}$ and $\phi \in C(\mathcal{G}$ such that $\chi(\mathcal{G}) = \|\chi\|$, where $C(\mathcal{G})$ stands for the uniform algebra of continuous complex functions on $\mathcal{G}$ (similarly for the measure χ_r and the uniform algebra $C(\mathscr{U}_r)$).

The unitary group $\mathcal{G}$ on the compact set $\mathsf{S}_{\mathcal{G}}^{\infty}$ acts continuously. This group generates a group of linear operators on the algebra $C(\mathsf{S}_{\mathcal{G}}^{\infty})$:

$$C(\mathsf{S}_{\mathcal{G}}^{\infty}) \ni \varphi \longmapsto \varphi \circ U.$$

For a fixed $a \in \mathsf{S}_{\mathcal{G}}^{\infty}$ the mapping $\mathcal{G} \ni U \longmapsto Ua \in \mathsf{S}_{\mathcal{G}}^{\infty}$ is continuous and surjective. Hence, the function $U \longmapsto (\varphi \circ U)(a)$ belongs to $C(\mathcal{G})$ for all $\varphi \in C(\mathsf{S}_{\mathcal{G}}^{\infty})$. Therefore,

$$\sup_{U \in \mathcal{G}} \left|(\varphi \circ U)(a)\right| = \sup_{x \in \mathsf{S}_{\mathcal{G}}^{\infty}} |\varphi(x)|.$$

The Riesz representation theorem implies that the Haar measure χ uniquely defines a probability $\mathcal{G}$-invariant measure ς on the $\mathcal{G}$-orbit $\mathsf{S}_{\mathcal{G}}^{\infty} = \{Ua \colon U \in \mathcal{G}\}$ by the formula

$$\varsigma(\varphi) := \int_{\mathsf{S}_{\mathcal{G}}^{\infty}} \varphi\ d\varsigma = \int_{\mathcal{G}} \varphi(Ua)\ d\chi(U), \qquad \varphi \in C(\mathsf{S}_{\mathcal{G}}^{\infty}), \tag{3.2}$$

where ς does not depend on a via transitivity of $\mathcal{G}$ on the $\mathcal{G}$-orbit. Recall that a probability measure ς on $\mathsf{S}_{\mathcal{G}}^{\infty}$ is $\mathfrak{G}$-invariant, if ς satisfies the relation $\varsigma = \varsigma \circ U$ for all $U \in \mathcal{G}$.

For a given E_r let

$$\mathsf{E}_r^{\perp} := \{x = (x_m) \in \ell_{\mathcal{G}}^{\infty} : x_r = 0\}.$$

Then $\ell_{\mathcal{G}}^{\infty} = \mathsf{E}_r^{\perp} \oplus \mathsf{E}_r$ and for every $a \in \ell_{\mathcal{G}}^{\infty}$ we have

$$a = a_r^{\perp} + a_r \in \ell_{\mathcal{G}}^{\infty},$$

where $a_r^{\perp} \in \mathsf{E}_r^{\perp}$, $a_r \in \mathsf{E}_r$ and $a \longmapsto a_r$ is a projection.

For a given n_r-dimensional subgroup $\mathscr{U}_r$ the mappings

$$a \longmapsto U_r(a_r) + a_r^{\perp} \quad \text{with} \quad U_r \in \mathscr{U}_r$$

generate the linear operators

$$T_r\varphi(a) := \varphi\left(U_r(a_r) + a_r^{\perp}\right), \qquad \varphi \in C(\mathsf{S}_{\mathcal{G}}^{\infty})$$

acting in the algebra $C(\mathsf{S}_{\mathcal{G}}^{\infty})$. We will use the following useful formulas.

PROPOSITION 3.3. For any $r_1, \ldots, r_m \in N$ the equality

$$\int_{\mathsf{S}_{\mathcal{G}}^{\infty}} \varphi\ d\varsigma = \int_{S_{\mathcal{G}}^{\infty}} d\varsigma(a) \prod_{i=1}^{m} \int_{\mathscr{U}_{r_i}} T_{r_i}\varphi(a)\ d\chi_{r_i}(U_{r_i}), \qquad \varphi \in C(\mathsf{S}_{\mathcal{G}}^{\infty}) \tag{3.3}$$

holds. Thereto, for any compact subgroup $\mathcal{G}_0 \subset \mathcal{G}$ with the probability Haar measure ς_0 the equality

$$\int_{\mathsf{S}_{\mathcal{G}}^\infty} \varphi\ d\varsigma = \int_{\mathsf{S}_{\mathcal{G}}^\infty} d\varsigma(a) \int_{\mathcal{G}_0} \varphi(Ua)\ d\varsigma_0(U), \qquad \varphi \in C(\mathsf{S}_{\mathcal{G}}^\infty) \tag{3.4}$$

holds.

PROOF. For each $\varphi \in C(\mathsf{S}_{\mathcal{G}}^\infty)$ the function

$$(a, U_{r_1}, \dots, U_{r_m}) \longmapsto T_{r_1} \cdot \ldots \cdot T_{r_m}\varphi(a)$$

is continuous on the Cartesian product

$$\mathsf{S}_{\mathcal{G}}^\infty \times \mathscr{U}_{r_1} \times \ldots \times \mathscr{U}_{r_m}.$$

By the Fubini theorem, we have

$$\int_{\mathsf{S}_{\mathcal{G}}^\infty} d\varsigma(a) \prod_{i=1}^m \int_{\mathscr{U}_{r_i}} T_{r_i}\varphi(a)\ d\chi_{r_i} = \prod_{i=1}^m \int_{\mathscr{U}_{r_i}} d\chi_{r_i} \int_{\mathsf{S}_{\mathcal{G}}^\infty} T_{r_i}\varphi(a)\ d\varsigma(a).$$

However, the internal integrals on the right side does not depend of $T_{r_1}, \dots, T_{r_m}$. Therefore, taking into account that $\int_{\mathscr{U}_{r_i}} d\chi_{r_i} = 1$, we obtain (3.3). The formula (3.4) can be proved similarly. □

PROPOSITION 3.4. The $\mathcal{G}$-invariant measure ς represents the character $\delta_0(f) = f(0)$ of the algebra $\mathcal{A}(\mathsf{K}_{\mathcal{G}}^\infty)$ i.e. it satisfies the following relation

$$\delta_0(f) = \int_{\mathsf{S}_{\mathcal{G}}^\infty} f\ d\varsigma, \qquad f \in_{\mathcal{A}} (\mathsf{K}_{\mathcal{G}}^\infty). \tag{3.5}$$

PROOF. By the formula (3.4) for any $\mathfrak{e}_{\{j\}}^{*(k)} \in \mathscr{E}_n^*$ we obtain

$$\begin{aligned} \int_{\mathsf{S}_{\mathcal{G}}^\infty} \mathfrak{e}_{\{j\}}^{*(k)}\ d\varsigma &= \frac{1}{2\pi} \int_{\mathsf{S}_{\mathcal{G}}^\infty} d\varsigma(a) \int_0^{2\pi} \mathfrak{e}_{\{j\}}^{*(k)}(\exp(\mathrm{i}\vartheta)a)\ d\vartheta \\ &= \frac{1}{2\pi} \int_{\mathsf{S}_{\mathcal{G}}^\infty} \mathfrak{e}_{\{j\}}^{*(k)}(a)\ d\varsigma(a) \int_0^{2\pi} \exp(\mathrm{i}n\vartheta)a\ d\vartheta \\ &= \begin{cases} 0 & : \ n \neq 0 \\ 1 & : \ n = 0. \end{cases} \end{aligned}$$

Uniformly approaching any function $f \in \mathcal{A}(\mathsf{K}_{\mathcal{G}}^\infty)$ by polynomials $\mathscr{E}^*$ and using the linearity and continuity on $\mathcal{A}(\mathsf{K}_{\mathcal{G}}^\infty)$ of the integral with the measure ς, we come to the formula (3.5). □

3.1.3. Hardy spaces on orbits of compact groups. Let the probability $\mathcal{G}$-invariant measure ς, defined by the formula (3.2), be given. Then the functional

$$\|f\|_{L^2_\varsigma} = \left(\int_{\mathsf{S}_{\mathcal{G}}^\infty} |f|^2\ d\varsigma \right)^{1/2}, \qquad f \in C(\mathsf{S}_{\mathcal{G}}^\infty)$$

is a Hilbertian norm on the space $C(\mathsf{S}_{\mathcal{G}}^{\infty})$. Indeed, let f be a nonzero function in $C(\mathsf{S}_{\mathcal{G}}^{\infty})$ and

$$\varphi := |f|^2.$$

If we suppose that $\varsigma(\varphi) = 0$ then (3.2) implies that for a fixed $a \in \mathsf{S}_{\mathcal{G}}^{\infty}$,

$$(\varphi \circ U)(a) \overset{\text{a.e.}}{=} 0, \qquad U \in \mathcal{G}$$

with respect to the Haar measure χ defined on $\mathcal{G}$. The function

$$U \longmapsto (\varphi \circ U)(a)$$

belongs to $C(\mathcal{G})$, hence $(\varphi \circ U)(a) \equiv 0$ as a function of $U \in \mathcal{G}$. Since the mapping

$$\mathcal{G} \ni U \longmapsto U(a) \in \mathsf{S}_{\mathcal{G}}^{\infty}$$

is surjective, we obtain $\varphi \equiv 0$ on $\mathsf{S}_{\mathcal{G}}^{\infty}$. Consequently $f \equiv 0$ on $\mathsf{S}_{\mathcal{G}}^{\infty}$ which contradicts the assumption.

Consider the Hilbert space $L^2_\varsigma = L^2(d\varsigma)$ of all quadratically ς-integrable complex functions with the scalar product

$$\langle f \mid g \rangle_{L^2_\varsigma} = \int_{\mathsf{S}_{\mathcal{G}}^{\infty}} f\,\bar{g}\; d\varsigma, \qquad f, g \in L^2(d\varsigma).$$

DEFINITION 3.5. The Hardy type space $\mathcal{H}^2(d\varsigma)$ we define as a closure of the algebra $\mathcal{A}(\mathsf{K}_{\mathcal{G}}^{\infty})$ (or $\mathcal{A}(\mathsf{S}_{\mathcal{G}}^{\infty})$, that is the same) in the space $L^2(d\varsigma)$ endowed with the L^2_ς-norm.

Note that the embedding $\mathcal{A}(\mathsf{K}_{\mathcal{G}}^{\infty}) \hookrightarrow \mathcal{H}^2(d\varsigma)$ is continuous, since

$$\|f\|_{L^2_\varsigma} \le \|f\|_{C(\mathsf{S}_{\mathcal{G}}^{\infty})}, \qquad f \in \mathcal{A}(\mathsf{K}_{\mathcal{G}}^{\infty}). \tag{3.6}$$

For a fixed $n \in \mathbb{Z}_+$ let $\mathcal{H}^2_n$ be a closure in the space $L^2(d\varsigma)$ of the complex linear span of homogenous Hilbert-Schmidt polynomials $\mathscr{E}^*_n$ (extended on $\mathsf{K}_{\mathcal{G}}^{\infty}$) and $\mathcal{H}^2_0 := \mathbb{C}$.

THEOREM 3.6. *The sequence of homogeneous Hilbert-Schmidt polynomials $\mathscr{E}^*$ forms an orthogonal basis in $\mathcal{H}^2(d\varsigma)$. In particular, the subsequence $\mathscr{E}^*_n$ forms the same basis in $\mathcal{H}^2_n$ for any $n \in \mathbb{Z}_+$ and*

$$\mathcal{H}^2_n \perp \mathcal{H}^2_m$$

in $\mathcal{H}^2(d\varsigma)$ whenever $n \neq m$.

PROOF. Every element $a = \sum_{i\in\mathbb{N}} \mathfrak{e}^*_i(a)\mathfrak{e}_i \in \ell^{\infty}_{\mathcal{G}}$ can be written as

$$a = a^{\perp}_s + \mathfrak{e}^*_s(a)\mathfrak{e}_s,$$

where $a^{\perp}_s$ denotes a projection of a onto the complementing subspace

$$\mathfrak{e}^{\perp}_s := \{a \in \ell^{\infty}_{\mathcal{G}} : \mathfrak{e}^*_s(a) = 0\}, \qquad s \in \mathbb{N}.$$

Consider the 1-dimensional subgroups in $\mathcal{G}$ of linear transformations

$$U_s(\vartheta)a := \exp(\mathrm{i}\vartheta)\mathfrak{e}^*_s(a)\mathfrak{e}_s + a^{\perp}_s,$$

$$U_0(\vartheta)a := \exp(\mathrm{i}\vartheta)a$$

with $a \in \mathsf{S}_{\mathcal{G}}^{\infty}$ and $\vartheta \in (-\pi, \pi]$. We assign to these transformations the 1-parameter groups of linear operators on the algebra $C(\mathsf{S}_{\mathcal{G}}^{\infty})$

$$\vartheta \longmapsto T_s(\vartheta)f := f(U_s(\vartheta)a),$$
$$\vartheta \longmapsto T_0(\vartheta)f := f(U_0(\vartheta)a)$$

with $f \in C(\mathsf{S}_{\mathcal{G}}^{\infty})$ and $a \in \mathsf{S}_{\mathcal{G}}^{\infty}$. The formulas (3.3) and (3.4) imply that

$$\int_{\mathsf{S}_{\mathcal{G}}^{\infty}} f\, d\varsigma = \frac{1}{2\pi}\int_{\mathsf{S}_{\mathcal{G}}^{\infty}} d\varsigma(a) \int_{-\pi}^{\pi} T(\vartheta)f(a)\, d\vartheta \tag{3.7}$$

for any $T \in \{T_0, T_s : s \in \mathbb{N}\}$. If $|(k)| \neq |(l)|$ then from (3.7) it follows that

$$\int_{\mathsf{S}_{\mathcal{G}}^{\infty}} \mathfrak{e}_{\{j\}}^{*(k)} \cdot \overline{\mathfrak{e}}_{\{i\}}^{*(l)}\, d\varsigma = \int_{\mathsf{S}_{\mathcal{G}}^{\infty}} \mathfrak{e}_{\{j\}}^{*(k)}(\exp(\mathrm{i}\vartheta)a)\, \overline{\mathfrak{e}}_{\{i\}}^{*(l)}(\exp(\mathrm{i}\vartheta)a)\, d\varsigma(a)$$
$$= \frac{1}{2\pi}\int_{\mathsf{S}_{\mathcal{G}}^{\infty}} \mathfrak{e}_{\{j\}}^{*(k)} \cdot \overline{\mathfrak{e}}_{\{i\}}^{*(l)}\, d\varsigma \int_{-\pi}^{\pi} \exp\left(\mathrm{i}(|(k)| - |(l)|)\vartheta\right) d\vartheta = 0.$$

So, $\mathfrak{e}_{\{j\}}^{*(k)} \perp \mathfrak{e}_{\{i\}}^{*(l)}$ in $L^2(d\varsigma)$ if $|(k)| \neq |(l)|$ for all $\{j\}, \{i\} \in \mathbb{N}^n$.

If $|(k)| = |(l)|$ and the corresponding elements $\mathfrak{e}_{\{j\}}^{*(k)}$ with $\{j\} = (j_1, \ldots, j_n)$ and $\mathfrak{e}_{\{i\}}^{*(l)}$ with $\{i\} = (i_1, \ldots, i_m)$ are different, then there exists an index $j_s \in \{j_1, \ldots, j_n\}$ such that $j_s \notin \{i_1, \ldots, i_m\}$. Now (3.7) implies that

$$\int_{\mathsf{S}_{\mathcal{G}}^{\infty}} \mathfrak{e}_{\{j\}}^{*(k)} \cdot \overline{\mathfrak{e}}_{\{i\}}^{*(l)}\, d\varsigma = \int_{\mathsf{S}_{\mathcal{G}}^{\infty}} T_{j_s}(\vartheta)\mathfrak{e}_{\{j\}}^{*(k)} \cdot \overline{T_{j_s}(\vartheta)\mathfrak{e}_{\{i\}}^{*(l)}}\, d\varsigma$$
$$= \frac{1}{2\pi}\int_{\mathsf{S}_{\mathcal{G}}^{\infty}} \mathfrak{e}_{\{j\}}^{*(k)} \cdot \overline{\mathfrak{e}}_{\{i\}}^{*(l)}\, d\varsigma \int_{-\pi}^{\pi} \exp\left(\mathrm{i}k_s\vartheta\right) d\vartheta = 0,$$

hence, $\mathfrak{e}_{\{i\}}^{*(l)} \perp \mathfrak{e}_{\{j\}}^{*(k)}$ in $L^2(d\varsigma)$ as well. □

Further in this section we use the following notations.

Let $\{j\}_r := (j_{r(1)}, \ldots, j_{r(n_r)}) \in \mathbb{N}^{n_r}$ denote a sub-index of the multi-index $\{j\} = (j_1, \ldots, j_n) \in \mathbb{N}^n$ such that $j_{r(1)} < \ldots < j_{r(n_r)}$.

Let $(k)_r := (k_{r(1)}, \ldots, k_{r(n_r)}) \in \mathbb{Z}_+^{n_r}$ stand for a sub-index of the index $(k) = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$ with $n_r \leq n$.

As is usual, $|(k)_r| := k_{r(1)} + \cdots + k_{r(n_r)}$ and $(k)_r! := k_{r(1)}! \ldots k_{r(n_r)}!$.

THEOREM 3.7. *If a Hilbert-Schmidt polynomial*

$$\mathfrak{e}_{\{j\}}^{*(k)} = \mathfrak{e}_{j_1}^{*k_1} \ldots \mathfrak{e}_{j_n}^{*k_n} \in \mathscr{E}_n^*, \qquad \{j\} \in \mathbb{N}^n, \quad (k) \in \mathbb{Z}_+^n, \quad |(k)| = n$$

is of the form

$$\mathfrak{e}_{\{j\}}^{*(k)} = \mathfrak{e}_{\{j\}_{r(1)}}^{*(k)_{r(1)}} \cdots \mathfrak{e}_{\{j\}_{r(t)}}^{*(k)_{r(t)}}$$

with block-indexes $\{j\} = (\{j\}_{r(1)}, \ldots, \{j\}_{r(t)})$ *and* $(k) = ((k)_{r(1)}, \ldots, (k)_{r(t)})$ *such that*

$$n_{r(1)} + \ldots + n_{r(t)} = n,$$

where

$$\{\mathfrak{e}_{j_{r(1)}},\dots,\mathfrak{e}_{j_{r(n_r)}}\}=\{\mathfrak{e}_{j_1},\dots,\mathfrak{e}_{j_n}\}\bigcap \mathsf{E}_r$$

and $t\in\mathbb{N}$ *stands for the number of all such sub-indexes in* $\{j\}$, *then*

$$\|\mathfrak{e}^{*(k)}_{\{j\}}\|^2_{L^2_\xi}=\prod_{r\in\{r_1,\dots,r_t\}}\frac{(n_r-1)!(k)_r!}{(n_r-1+|(k)_r|)!}. \tag{3.8}$$

PROOF. Use that

$$T_r|\mathfrak{e}^{*(k)_r}_{\{j\}_r}|^2(a)=|\mathfrak{e}^{*(k)_r}_{\{j\}_r}|^2(U_r(a_r))$$

for any $a=(a_r)\in \mathsf{S}^\infty_{\mathfrak{G}}$ with $a_r\in\mathsf{S}_r$. As is well-known [88, 1.4.9],

$$\int_{\mathscr{U}_r}T_r|\mathfrak{e}^{*(k)_r}_{\{j\}_r}|^2(a)\;d\chi_r=\int_{\mathscr{U}_r}|\mathfrak{e}^{*(k)_r}_{\{j\}_r}|^2(U_r(a_r))\;d\chi_r(U_r)$$
$$=\frac{(n_r-1)!(k)_r!}{(n_r-1+|(k)_r|)!}$$

with the Haar measure χ_r on $\mathscr{U}_r$. Thus formula (3.3) immediately implies (3.8). □

3.1.4. Cauchy kernels for compact group orbits. Let us define the following auxiliary Banach space, associated with the group $\mathcal{G}$,

$$\ell^1_{n_r}:=\left\{x=(x_r)\in\bigtimes_{r\in\mathbb{N}}\mathbb{C}^{n_r}:\;\|x\|_{\ell^1_{n_r}}:=\sum_{r\in\mathbb{N}}n_r\|x_r\|_{\mathbb{C}^{n_r}}<\infty\right\}.$$

Note that $\mathfrak{e}_j\in\ell^\infty_{\mathcal{G}}\bigcap\ell^1_{n_r}$ for all $j\in\mathbb{N}$ and the group

$$\mathcal{G}\ni U\longrightarrow Ux=(U_rx_r)_{r\in\mathbb{N}}$$

acts isometrically in both $\ell^\infty_{\mathcal{G}}$ and $\ell^1_{n_r}$. Since the embedding

$$\ell^1_{n_r}\hookrightarrow\ell^\infty_{\mathcal{G}}$$

is continuous, the set $\mathsf{B}^\infty_{\mathcal{G}}\bigcap\ell^1_{n_r}$ is open and the set $\mathsf{K}^\infty_{\mathcal{G}}\bigcap\ell^1_{n_r}$ is closed in $\ell^1_{n_r}$.

Let us examine the Cauchy type kernel

$$\mathfrak{C}(x,a):=\prod_{r\in\mathbb{N}}\frac{1}{(1-\langle x_r\mid a_r\rangle_{\mathbb{C}^{n_r}})^{n_r}},\qquad a\in\mathsf{S}^\infty_{\mathcal{G}} \tag{3.9}$$

which is a priori Gâteaux analytic mapping of x running over the finitely open ball

$$\bigcup_{r\in\mathbb{N}}\mathsf{B}_1\times\ldots\times\mathsf{B}_r$$

with values in the uniform algebra $\mathcal{A}(\mathsf{S}^\infty_{\mathcal{G}})$.

PROPOSITION 3.8. The Cauchy type kernel $\mathfrak{C}$ is a well defined analytic $\mathcal{A}(\mathsf{S}^\infty_{\mathcal{G}})$-valued mapping

$$\mathsf{B}^\infty_{\mathcal{G}}\bigcap\ell^1_{n_r}\ni x\longmapsto\mathfrak{C}(x,a),\qquad a\in\mathsf{S}^\infty_{\mathcal{G}}.$$

PROOF. For every $\varrho \in (0,1)$ the series

$$\ln(1-\zeta)^{-r} = -r\sum_{n\in\mathbb{N}}\frac{\zeta^n}{n} = -r\zeta\sum_{n\in\mathbb{N}}\frac{\zeta^{n-1}}{n}, \qquad r\in\mathbb{N}$$

is convergent absolutely for all $|\zeta|\le\varrho$. Therefore the estimation

$$(3.10)\qquad |\ln(1-\zeta)^{-r}| \le c_\varrho r|\zeta| \quad \text{for all} \quad |\zeta|\le\varrho$$

with

$$c_\varrho := \sum_{n\in\mathbb{N}}\frac{\varrho^{n-1}}{n} < \infty$$

holds. Denote by $\mathsf{B}^1_{n_r}$ and $\mathsf{K}^1_{n_r}$ the open and the closed unit balls in the space $\ell^1_{n_r}$, respectively. Consider the 1-parametr families of balls

$$\mathsf{K}_\varepsilon := \varepsilon\mathsf{K}^\infty_{\mathcal{G}} \bigcap \frac{1}{1-\varepsilon}\mathsf{K}^1_{n_r},$$

$$\mathsf{B}_\varepsilon := \varepsilon\mathsf{B}^\infty_{\mathcal{G}} \bigcap \frac{1}{1-\varepsilon}\mathsf{B}^1_{n_r}$$

with $\varepsilon\in(0,1)$. Clearly, K_ε and B_ε are closed and open sets in the space $\ell^1_{n_r}$ respectively, because the embedding

$$\ell^1_{n_r} \hookrightarrow \ell^\infty_{\mathcal{G}}$$

is continuous.

Let $x=(x_r)\in\mathsf{K}_\varepsilon$ and $a=(a_r)\in\mathsf{S}^\infty_{\mathcal{G}}$. Then we obtain for instance

$$\sup_{\|a_r\|=1}\left|\langle x_r \mid a_r\rangle_{\mathbb{C}^{n_r}}\right| \le \frac{\varepsilon}{n_r} < 1.$$

Hence, the inequality (3.10) implies

$$\sum_{r\in\mathbb{N}}\left|\ln\frac{1}{(1-\langle x_r\mid a_r\rangle_{\mathbb{C}^{n_r}})^{n_r}}\right| \le \sum_{r\in\mathbb{N}} c_{\varrho(r)}n_r\left|\langle x_r\mid a_r\rangle_{\mathbb{C}^{n_r}}\right|$$

with $\varrho(r) := \dfrac{\varepsilon}{n_r}$. Since $c_{\varrho(r)}\le c_\varepsilon$ for any $r\in\mathbb{N}$, it follows that

$$\begin{aligned}\sup_{\|a\|_{\ell^\infty_{\mathcal{G}}}=1}\sum_{r\in\mathbb{N}}\left|\ln\frac{1}{(1-\langle x_r\mid a_r\rangle_{\mathbb{C}^{n_r}})^{n_r}}\right| &\le c_\varepsilon \sup_{\|a\|_{\ell^\infty_{\mathcal{G}}}=1}\sum_{r\in\mathbb{N}} n_r\left|\langle x_r\mid a_r\rangle_{\mathbb{C}^{n_r}}\right| \\ &\le c_\varepsilon\sum_{r\in\mathbb{N}} n_r \sup_{\|a_r\|_{\mathbb{C}^{n_r}}=1}\left|\langle x_r\mid a_r\rangle_{\mathbb{C}^{n_r}}\right| \\ &= c_\varepsilon\sum_{r\in\mathbb{N}} n_r\|x_r\|_{\mathbb{C}^{n_r}} = c_\varepsilon\|x\|_{\ell^1_{n_r}}.\end{aligned}$$

Consequently, the series of $\mathcal{A}(\mathsf{S}^\infty_{\mathcal{G}})$-valued functions

$$(3.11)\qquad \mathsf{K}_\varepsilon \ni x \longmapsto \sum_{r\in\mathbb{N}}\ln\frac{1}{(1-\langle x_r\mid a_r\rangle_{\mathbb{C}^{n_r}})^{n_r}} \in \mathcal{A}(\mathsf{S}^\infty_{\mathcal{G}}), \qquad a\in\mathsf{S}^\infty_{\mathcal{G}}$$

converges absolutely and uniformly on K_ε. Hence, its sum represents a bounded continuous $\mathcal{A}(\mathsf{S}^\infty_\mathcal{G})$-valued function on K_ε for any $\varepsilon \in (0,1)$. Moreover, one is Gâteaux-analytic on the open domain B_ε, since its restriction to any 1-dimensional affine subset is obviously analytic. Thus, the function (3.11) is analytic on B_ε. By the analyticity of the exponential function, the following map

$$\mathsf{K}_\varepsilon \ni x \longmapsto \mathfrak{C}_\varepsilon(x,a) := \exp \sum_{r\in\mathbb{N}} \ln \frac{1}{(1-\langle x_r \mid a_r\rangle_{\mathbb{C}^{n_r}})^{n_r}} \in \mathcal{A}(\mathsf{S}^\infty_\mathcal{G}) \tag{3.12}$$

is a bounded continuous function, which is analytic on B_ε. As is easy to see, for any $a \in \mathsf{S}^\infty_\mathcal{G}$ and $x \in \mathsf{B}_\varepsilon$ we have

$$\mathfrak{C}_\varepsilon(x,a) = \prod_{r\in\mathbb{N}} \frac{1}{(1-\langle x_r \mid a_r\rangle_{\mathbb{C}^{n_r}})^{n_r}}.$$

If $\varepsilon_1 < \varepsilon_2$ then $\mathsf{B}_{\varepsilon_1} \subset \mathsf{B}_{\varepsilon_2}$ and the function $\mathfrak{C}_{\varepsilon_1}$ defined on $\mathsf{B}_{\varepsilon_1}$ has a unique analytic extension $\mathfrak{C}_{\varepsilon_2}$ on $\mathsf{B}_{\varepsilon_2}$ such that

$$\mathfrak{C}_{\varepsilon_2}|_{\mathsf{B}_{\varepsilon_1}} = \mathfrak{C}_{\varepsilon_1},$$

by virtue of Uniqueness Principle for analytic functions. Therefore, the Cauchy kernel $\mathfrak{C}$, defined by (3.9), has a unique $\mathcal{A}(\mathsf{S}^\infty_\mathcal{G})$-valued analytic extension on the open domain

$$\mathsf{B}^\infty_\mathcal{G} \bigcap \ell^1_{n_r} = \bigcup_{\varepsilon\in(0,1)} \mathsf{B}_\varepsilon,$$

which we also denote by $\mathfrak{C}$. □

3.1.5. Hilbertian extensions of Cauchy kernels. Now we define a Hilbert space, associated with $\mathcal{G}$,

$$\ell^2_{\sqrt{n_r}} := \left\{ x = (x_r) \in \bigtimes_{r\in\mathbb{N}} \mathbb{C}^{n_r} : \|x\|_{\ell^2_{\sqrt{n_r}}} := \Big(\sum_{r\in\mathbb{N}} n_r \|x_r\|^2_{\mathbb{C}^{n_r}}\Big)^{1/2} < \infty \right\},$$

where the group $\mathcal{G}$ acts isometrically. The inequality

$$\sum_{r\in\mathbb{N}} n_r \|x_r\|^2_{\mathbb{C}^{n_r}} \le \Big(\sum_{r\in\mathbb{N}} \sqrt{n_r}\|x_r\|_{\mathbb{C}^{n_r}}\Big)^2 \le \Big(\sum_{r\in\mathbb{N}} n_r \|x_r\|_{\mathbb{C}^{n_r}}\Big)^2$$

implies that the following continuous embedding is true

$$\ell^1_{n_r} \hookrightarrow \ell^2_{\sqrt{n_r}}.$$

Consider the Banach space

$$\ell^{2,\infty}_{\sqrt{n_r}} := \ell^\infty_\mathcal{G} \bigcap \ell^2_{\sqrt{n_r}}$$

with the norm

$$\|x\|_{\ell^{2,\infty}_{\sqrt{n_r}}} = \max\left\{ \|x\|_{\ell^\infty_\mathcal{G}}, \|x\|_{\ell^2_{\sqrt{n_r}}} \right\}.$$

In each of the above spaces $\mathcal{G}$ acts isometrically. Since, the embedding

$$\ell^{2,\infty}_{\sqrt{n_r}} \hookrightarrow \ell^\infty_\mathcal{G}$$

is continuous, the set $\mathsf{B}^\infty_\mathcal{G} \bigcap \ell^2_{\sqrt{n_r}}$ is open and the set $\mathsf{K}^\infty_\mathcal{G} \bigcap \ell^2_{\sqrt{n_r}}$ is closed in $\ell^{2,\infty}_{\sqrt{n_r}}$.

PROPOSITION 3.9. The Cauchy kernel $\mathfrak{C}$ possesses a unique analytic $\mathcal{A}(\mathsf{S}_{\ell^\infty_\mathcal{G}})$-valued extension

$$\mathsf{B}^\infty_\mathcal{G} \bigcap \ell^2_{\sqrt{n_r}} \ni x \longmapsto \mathfrak{C}(x,a), \qquad a \in \mathsf{S}^\infty_\mathcal{G}.$$

PROOF. Let $x = (x_r) \in \ell^\infty_\mathcal{G}$ with $x_r \in \mathsf{E}_r$ and we denote

$$\hat{x} := (\hat{x}_r) \qquad \text{with} \qquad \hat{x}_r := v_r x_r, \qquad v_r = \frac{1}{\sqrt{n_r 2^r}}.$$

Note that if $x = (x_r) \in \mathsf{S}^\infty_\mathcal{G}$ with $x_r \in \mathsf{S}_r$ then $\hat{x} \in \ell^2_{\sqrt{n_r}}$ and

$$\|\hat{x}\|^2_{\ell^2_{\sqrt{n_r}}} = \sum_{r\in\mathbb{N}} \frac{1}{2^r}\|x_r\|^2_{\mathbb{C}^{n_r}} = 1.$$

Consider the linear mapping

$$\hat{v}\colon \ell^\infty_\mathcal{G} \ni x \longmapsto \hat{x} \in \ell^2_{\sqrt{n_r}}.$$

The mapping $\hat{v}$ is continuous, since

$$\|\hat{x}\|^2_{\ell^2_{\sqrt{n_r}}} = \sum_{r\in\mathbb{N}} \frac{1}{2^r}\|x_r\|^2_{\mathbb{C}^{n_r}} \le \|x\|^2_{\ell^\infty_\mathcal{G}}.$$

Moreover, from

$$\|x\|^2_{\ell^\infty_\mathcal{G}} = \sup_{r\in\mathbb{N}} \|x_r\|^2_{\mathbb{C}^{n_r}} \le \sum_{r\in\mathbb{N}} n_r\|x_r\|^2_{\mathbb{C}^{n_r}} = \|x\|^2_{\ell^2_{\sqrt{n_r}}}$$

we come to the continuous embedding

$$\ell^2_{\sqrt{n_r}} \hookrightarrow \ell^\infty_\mathcal{G}.$$

Note that the restriction $\hat{v}\,|_{\ell^2_{\sqrt{n_r}}}$ maps continuously from $\ell^2_{\sqrt{n_r}}$ into $\ell^1_{n_r}$. In fact, from the Cauchy-Schwartz Inequality it follows that

$$\|\hat{x}\|_{\ell^1_{n_r}} = \sum_{r\in\mathbb{N}} n_r\|x_r\|_{\mathbb{C}^{n_r}} v_r \le \|x\|_{\ell^2_{\sqrt{n_r}}},$$

since $\sum_{r\in\mathbb{N}} n_r v_r^2 = 1$ and $\|\hat{x}\|_{\ell^\infty_\mathcal{G}} \le \|x\|_{\ell^\infty_\mathcal{G}}$. Hence, the mapping

$$\hat{v}\colon \ell^\infty_\mathcal{G} \bigcap \ell^2_{\sqrt{n_r}} \ni x \longmapsto \hat{x} \in \ell^\infty_\mathcal{G} \bigcap \ell^1_{n_r},$$

is continuous as well. By Proposition 3.8 the mapping

$$\mathsf{B}^\infty_\mathcal{G} \bigcap \ell^1_{n_r} \ni z \longmapsto \mathfrak{C}(z,a) \quad \text{with} \quad a \in \mathsf{S}^\infty_\mathcal{G}$$

is an analytic $\mathcal{H}^2(d\varsigma)$-valued function. Hence, putting $z = \hat{x}$ with an element $x \in \mathsf{B}^\infty_\mathcal{G} \bigcap \ell^2_{\sqrt{n_r}}$, we obtain that the mapping

$$\mathsf{B}^\infty_\mathcal{G} \bigcap \ell^2_{\sqrt{n_r}} \ni x \longmapsto \mathfrak{C}(\hat{x},a)$$

is also analytic. Note that

$$\frac{1}{(1 - \langle \hat{x}_r \mid a_r\rangle_{\mathbb{C}^{n_r}})^{n_r}} = \frac{1}{(1 - \langle x_r \mid \hat{a}_r\rangle_{\mathbb{C}^{n_r}})^{n_r}}.$$

Thus we have the following equalities

$$
\begin{aligned}
\mathfrak{C}(\hat{x}, a) &= \prod_{r \in \mathbb{N}} \frac{1}{(1 - \langle \hat{x}_r \mid a_r \rangle_{\mathbb{C}^{n_r}})^{n_r}} \\
&= \prod_{r \in \mathbb{N}} \frac{1}{(1 - \langle x_r \mid \hat{a}_r \rangle_{\mathbb{C}^{n_r}})^{n_r}} = \mathfrak{C}(x, \hat{a}),
\end{aligned} \tag{3.13}
$$

which is true for all $x \in \mathsf{B}_{\mathcal{G}}^{\infty} \bigcap \ell^2_{\sqrt{n_r}}$ and for a suitable vector such that the right side product in (3.13) converges. Let us check that it converges for every $a \in \mathsf{S}_{\mathcal{G}}^{\infty}$. Using notations from the proof of Proposition 3.8 and the Cauchy-Schwartz Inequality we obtain

$$
\begin{aligned}
\sup_{\|a\|_{\ell_{\mathcal{G}}^{\infty}} = 1} \sum_{r \in \mathbb{N}} \left| \ln \frac{1}{(1 - \langle x_r \mid \hat{a}_r \rangle_{\mathbb{C}^{n_r}})^{n_r}} \right| &\le c_\varepsilon \sup_{\|a\|_{\ell_{\mathcal{G}}^{\infty}} = 1} \sum_{r \in \mathbb{N}} n_r \left| \langle x_r \mid \hat{a}_r \rangle_{\mathbb{C}^{n_r}} \right| \\
&\le c_\varepsilon \Big(\sum_{r \in \mathbb{N}} n_r \|x_r\|^2_{\mathbb{C}^{n_r}} \Big)^{1/2} \\
&= c_\varepsilon \|x\|_{\ell^2_{\sqrt{n_r}}}
\end{aligned}
$$

for all $a = (a_r) \in \mathsf{S}_{\mathcal{G}}^{\infty}$ and $x = (x_r) \in \ell^2_{\sqrt{n_r}} \bigcap \varepsilon \mathsf{K}_{\mathcal{G}}^{\infty}$ with $\varepsilon \in (0, 1)$.

From the density of $\hat{v}(\mathsf{S}_{\mathcal{G}}^{\infty})$ in $\mathsf{S}_{\mathcal{G}}^{\infty}$ it follows that the previous inequality has a unique continuous extension to $\mathsf{S}_{\mathcal{G}}^{\infty}$ i.e.,

$$
\sup_{\|a\|_{\ell_{\mathcal{G}}^{\infty}} = 1} \sum_{r \in \mathbb{N}} \left| \ln \frac{1}{(1 - \langle x_r \mid a_r \rangle_{\mathbb{C}^{n_r}})^{n_r}} \right| \le c_\varepsilon \|x\|_{\ell^2_{\sqrt{n_r}}}
$$

for all $a = (a_r) \in \mathsf{S}_{\mathcal{G}}^{\infty}$ and $x = (x_r) \in \ell^2_{\sqrt{n_r}} \bigcap \varepsilon \mathsf{K}_{\mathcal{G}}^{\infty}$ with $\varepsilon \in (0, 1)$. Consequently, the following product of $\mathcal{A}(\mathsf{S}_{\mathcal{G}}^{\infty})$-valued functions

$$
\begin{aligned}
\ell^2_{\sqrt{n_r}} \bigcap \varepsilon \mathsf{K}_{\mathcal{G}}^{\infty} \ni x \longmapsto \exp \sum_{r \in \mathbb{N}} \ln \frac{1}{(1 - \langle x_r \mid a_r \rangle_{\mathbb{C}^{n_r}})^{n_r}} &= \\
= \prod_{r \in \mathbb{N}} \frac{1}{(1 - \langle x_r \mid a_r \rangle_{\mathbb{C}^{n_r}})^{n_r}} &\in \mathcal{A}(\mathsf{S}_{\mathcal{G}}^{\infty})
\end{aligned}
$$

with $a \in \mathsf{S}_{\mathcal{G}}^{\infty}$ converges absolutely and uniformly. Finally, this product represents a bounded continuous $\mathcal{A}(\mathsf{S}_{\mathcal{G}}^{\infty})$-valued function on $\ell^2_{\sqrt{n_r}} \bigcap \varepsilon \mathsf{K}_{\mathcal{G}}^{\infty}$ for all $\varepsilon \in (0, 1)$ and therefore it has a unique analytic $\mathcal{A}(\mathsf{S}_{\mathcal{G}}^{\infty})$-valued extension on the open domain

$$
\mathsf{B}_{\mathcal{G}}^{\infty} \bigcap \ell^2_{\sqrt{n_r}} = \bigcup_{\varepsilon \in (0,1)} \ell^2_{\sqrt{n_r}} \bigcap \varepsilon \mathsf{K}_{\mathcal{G}}^{\infty},
$$

which we also denote by $\mathfrak{C}$. The proposition is proved. □

3.1.6. The Cauchy integral formula on compact group orbits. Now we can already formulate and prove a first main result of this section.

THEOREM 3.10. *Every function*

$$f = \sum_{n\in\mathbb{Z}_+} f_n \in \mathcal{H}^2(d\varsigma) \qquad \text{with} \qquad f_n \in \mathcal{H}^2_n$$

has an analytic extension onto the open domain $\mathsf{B}^\infty_\mathcal{G} \bigcap \ell^2_{\sqrt{n_r}}$, *which can be represented by the Cauchy type integral formula*

$$\mathfrak{C}[f](x) := \int_{\mathsf{S}^\infty_\mathcal{G}} f(a)\,\mathfrak{C}(x,a)\ d\varsigma(a), \qquad x \in \mathsf{B}^\infty_\mathcal{G} \bigcap \ell^2_{\sqrt{n_r}}. \tag{3.14}$$

The corresponding Cauchy type kernel $\mathfrak{C}$ *can be represented, in turn, by the series*

$$\mathfrak{C}(x,a) = \sum_{n\in\mathbb{Z}_+} \mathfrak{C}_n(x,a), \qquad \mathfrak{C}_n(x,a) := \sum_{\substack{(k)\in\mathbb{Z}^n_+\\ |(k)|=n}} \sum_{\{j\}\in\mathbb{N}^n} \frac{\mathfrak{e}^{*(k)}_{\{j\}}(x)\bar{\mathfrak{e}}^{*(k)}_{\{j\}}(a)}{\|\mathfrak{e}^{*(k)}_{\{j\}}\|^2_{L^2_\varsigma}} \tag{3.15}$$

with $a \in \mathsf{S}^\infty_\mathcal{G}$, *weakly convergent in* $\mathcal{H}^2(d\varsigma)$ *for all* $x \in \mathsf{B}^\infty_\mathcal{G} \bigcap \ell^2_{\sqrt{n_r}}$. *The Taylor coefficients at origin are uniquely defined by the formula*

$$\frac{d^n_0\mathfrak{C}[f](x)}{n!} = \int_{\mathsf{S}^\infty_\mathcal{G}} f_n(a)\,\mathfrak{C}_n(x,a)\ d\varsigma(a), \qquad x \in \ell^\infty_\mathcal{G} \bigcap \ell^2_{\sqrt{n_r}}. \tag{3.16}$$

PROOF. For any $f \in \mathcal{H}^2(d\varsigma)$ the linear functional

$$\varsigma_f : \mathcal{H}^2(d\varsigma) \ni g \longmapsto \int fg\ d\varsigma$$

is continuous. Since $\mathfrak{C}[f](x) = \varsigma_f \circ \mathfrak{C}(x,\cdot)$, the function $\mathfrak{C}[f]$ defined by the formula (3.14) is analytic on $\mathsf{B}^\infty_\mathcal{G} \bigcap \ell^2_{\sqrt{n_r}}$ via Proposition 3.9.

Let $a = (a_r) \in \mathsf{S}^\infty_\mathcal{G}$ with $a_r \in \mathsf{S}_r$ and $x = (x_r) \in \mathsf{B}^\infty_\mathcal{G} \bigcap \ell^2_{\sqrt{n_r}}$ with $x_r \in \mathbb{C}^{n_r}$. Consider the polynomial $\mathfrak{e}^{*(k)}_{\{j\}} \in \mathscr{E}^*_n$ of the form

$$\mathfrak{e}^{*(k)}_{\{j\}} = \mathfrak{e}^{*(k)_{r_1}}_{\{j\}_{r_1}} \cdots \mathfrak{e}^{*(k)_{r_t}}_{\{j\}_{r_t}},$$

cited in Theorem 3.7. Since

$$x_r = \mathfrak{e}^*_{j_{r(1)}}(x_r)\mathfrak{e}_{j_{r(1)}} + \ldots + \mathfrak{e}^*_{j_{r(n_r)}}(x_r)\mathfrak{e}_{j_{r(n_r)}},$$

and $\|x_r\|_{\mathbb{C}^{n_r}} < 1$, $\|a_r\|_{\mathbb{C}^{n_r}} = 1$, we obtain

$$\begin{aligned}
\frac{1}{(1-\langle x_r \mid a_r\rangle_{\mathbb{C}^{n_r}})^{n_r}} &= \sum_{n\in\mathbb{Z}_+} \frac{(n_r-1+n)!}{(n_r-1)!n!} \langle x_r \mid a_r\rangle^n_{\mathbb{C}^{n_r}} \\
&= \sum_{n\in\mathbb{Z}_+} \frac{(n_r-1+n)!}{(n_r-1)!n!} \Big(\sum_{i=1}^{n_r} \mathfrak{e}^*_{j_{r(i)}}(x_r)\bar{\mathfrak{e}}^*_{j_{r(i)}}(a_r)\Big)^n \\
&= \sum_{n\in\mathbb{Z}_+} \sum_{|(k)_r|=n} \frac{(n_r-1+n)!}{(n_r-1)!(k)_r!} \mathfrak{e}^{*(k)_r}_{\{j\}_r}(x_r)\bar{\mathfrak{e}}^{*(k)_r}_{\{j\}_r}(a_r).
\end{aligned}$$

Taking into account Theorem 3.7, it follows that

$$\mathfrak{C}(x,a) = \prod_{r\in\mathbb{N}} \frac{1}{(1-\langle x_r \mid a_r\rangle_{\mathbb{C}^{n_r}})^{n_r}} = \sum_{n\in\mathbb{Z}_+} \mathfrak{C}_n(x,a)$$

with

$$\mathfrak{C}_n(x,a) = \sum_{\{j\}\in\mathbb{N}^n} \sum_{\substack{(k)\in\mathbb{Z}_+^n \\ |(k)|=n}} \frac{\mathfrak{e}^{*(k)}_{\{j\}}(x)\bar{\mathfrak{e}}^{*(k)}_{\{j\}}(a)}{\|\mathfrak{e}^{*(k)}_{\{j\}}\|^2_{L^2_\varsigma}} \tag{3.17}$$

$$= \sum_{\substack{(\{j\}_{r_1},\dots,\{j\}_{r_t})\in\mathbb{N}^n \\ |(k)_{r_1}|+\dots+|(k)_{r_t}|=n}} \prod_{i=1}^{t} \frac{(n_{r_i}-1+n)!}{(n_{r_i}-1)!(k)_{r_i}!}\, \mathfrak{e}^{*(k)_{r_i}}_{\{j\}_{r_i}}(x_{r_i})\bar{\mathfrak{e}}^{*(k)_{r_i}}_{\{j\}_{r_i}}(a_{r_i}).$$

On the other hand, the equality (3.17) implies that for any $x \in \ell^\infty_{\mathcal{G}} \bigcap \ell^2_{\sqrt{n_r}}$

$$\int_{\mathsf{S}^\infty_{\mathcal{G}}} \mathfrak{e}^{*(k)}_{\{j\}}(a)\,\mathfrak{C}_n(x,a)\; d\varsigma(a) = \mathfrak{e}^{*(k)}_{\{j\}}(x), \qquad \mathfrak{e}^{*(k)}_{\{j\}} \in \mathscr{E}^*_n.$$

Since $\mathscr{E}^*_n$ forms an orthogonal basis in $\mathcal{H}^2_n$, the kernel $\mathfrak{C}_n$ produces the identity mapping in $\mathcal{H}^2_n$. It follows that for any $f_n \in \mathcal{H}^2_n$

$$f_n(x) = \int_{\mathsf{S}^\infty_{\mathcal{G}}} f_n(a)\,\mathfrak{C}_n(x,a)\; d\varsigma(a), \qquad x \in \ell^\infty_{\mathcal{G}} \bigcap \ell^2_{\sqrt{n_r}}.$$

Using that $f_n \perp \mathfrak{C}_l(x,\cdot)$ at $n \neq l$, we obtain

$$\mathfrak{C}[f](x) = \int_{\mathsf{S}^\infty_{\mathcal{G}}} f(a)\,\mathfrak{C}(x,a)\; d\varsigma(a) = \sum_{n\in\mathbb{Z}_+} \varrho^n f_n(y)$$

for all $x = \varepsilon y \in \ell^\infty_{\mathcal{G}} \bigcap \ell^2_{\sqrt{n_r}}$ with $\|y\|_{\ell^{2,\infty}_{\sqrt{n_r}}} = 1$ and $\varepsilon \in [0,1)$. Now the equality

$$f_n(y) = \frac{1}{n!}\frac{d^n_0\mathfrak{C}[f](\varepsilon y)}{d\varepsilon^n}\Big|_{\varepsilon=0} \tag{3.18}$$

implies that f_n is a Taylor coefficient of $\mathfrak{C}[f]$.

Finally, the relation

$$\mathfrak{C}[f](x) = \varsigma_f \circ \mathfrak{C}(x,\cdot)$$

with $x \in \mathsf{B}^\infty_{\mathcal{G}} \bigcap \ell^2_{\sqrt{n_r}}$ implies that for any $f \in \mathcal{H}^2(d\varsigma)$ the series (3.15) is pointwise by $x \in \mathsf{B}^\infty_{\mathcal{G}} \bigcap \ell^2_{\sqrt{n_r}}$ weakly convergent in $\mathcal{H}^2(d\varsigma)$, as a function of the variable $a \in \mathsf{S}^\infty_{\mathcal{G}}$. Clearly, $d^n_0\mathfrak{C}[f]$ in (3.18) can be extended on $\ell^\infty_{\mathcal{G}} \bigcap \ell^2_{\sqrt{n_r}}$, as a continuous polynomial. Thus, the formula (3.16) is true.

It remains to note that Taylor coefficients $\dfrac{d^n_0\mathfrak{C}[f]}{n!} = f_n$ uniquely define the analytic function $\mathfrak{C}[f]$ on the open domain $\mathsf{B}^\infty_{\mathcal{G}} \bigcap \ell^2_{\sqrt{n_r}}$. □

COROLLARY 3.11. For every $x \in \mathsf{B}^\infty_{\mathcal{G}} \bigcap \ell^2_{\sqrt{n_r}}$ the point-evaluation functional

$$\delta_x(f)\colon f \longmapsto f(x)$$

is continuous on $\mathcal{H}^2(d\varsigma)$.

PROOF. From Theorem 3.10 we have

$$\delta_x(f) = f(x) = \langle \mathfrak{C}(x,\cdot) \mid f(\cdot)\rangle_{\mathcal{H}^2(d\varsigma)}$$

and Proposition 3.9 implies that $\mathfrak{C}(x,\cdot) \in \mathcal{A}(\mathsf{S}_{\ell^\infty_{\mathcal{G}}}) \subset \mathcal{H}^2(d\varsigma)$. □

REMARK 3.12. In the partial case if $n_r = 1$ for all $r \in \mathbb{N}$, the ball $\mathsf{K}_{\mathcal{G}}^{\infty}$ has a polydisk form. For this case the Cauchy type formula has been established in [**80**].

3.2. Symmetric Fock spaces associated with matrix unitary groups

3.2.1. Unitarily-invariant integral. We consider the compact group

$$U = U(d), \qquad d \in \mathbb{N}$$

of unitary $(d \times d)$-matrices with the unit $\mathbb{1} = \mathbb{1}_d$, equipped with the probability Haar measure χ. On U we define the right action

$$u.g = w^{-1}uv, \qquad u \in U, \quad g = (v, w) \in U \times U.$$

Recall some known invariant properties under the right action of integrals over unitary groups.

Let $C(U)$ denote the space of continuous functions $f\colon U \longrightarrow \mathbb{C}$ endowed with the uniform norm $\|f\|_{C(U)} = \sup_{u \in U} |f(u)|$. Consider in $C(U)$ the group of shifts

$$\Theta_g f(u) = f(u.g)$$

generated by the right action. The unitary invariance of χ instantly yields the equality

$$\int_U f\, d\chi = \int_U \Theta_g f(u)\, d\chi(u), \qquad f \in C(U).$$

The application of Fubini's theorem for double integrals leads to the following factorization formulas under various scalar-type matrix subgroups in U. Namely, for the scalar matrix subgroup

$$\Theta_0 = \{g(\vartheta) = \exp(\mathrm{i}\vartheta)\mathbb{1} : \ \vartheta \in (-\pi, \pi]\}$$

we have

$$\int_U f\, d\chi = \frac{1}{2\pi} \int_U d\chi(u) \int_{-\pi}^{\pi} \Theta_{g(\vartheta)} f(u)\, d\vartheta.$$

Let $U_\imath \subset U$ $(\imath = 1, \dots, m)$ mean the subgroup of $(\imath \times \imath)$-matrices with the unit $\mathbb{1}_\imath$ equipped with the probability Haar measure $\chi_\imath$. Consider the corresponding matrix subgroups

$$\Theta_\imath = \{g_\imath(\vartheta) = \mathbb{1}_{\imath-1} \otimes \exp(\mathrm{i}\vartheta) \otimes \mathbb{1}_{d-\imath} : \ \vartheta \in (-\pi, \pi]\}$$

for all indexes $\imath = 1, \dots, d$.

Similarly, the unitary invariance of $\chi_\imath$ yields the equalities

$$\int_U f\, d\chi = \int_U d\chi(u) \int_{U_\imath \times U_\imath} \Theta_g f(u)\, d(\chi_\imath \otimes \chi_\imath)(g),$$

for all elements $g \in U_\imath \times U_\imath$ and indexes $\imath = 1, \dots, d$.

Let L^2_χ stand for the space of square χ-integrable complex functions f on the compact space U with the norm $\|f\|_{L^2_\chi} = \left(\int_U |f|^2 d\chi\right)^{1/2}$. Since χ is a probability measure, the contractive embedding $C(U) \hookrightarrow L^2_\chi$ holds, i.e.,

$$\|f\|_{L^2_\chi} \le \|f\|_{C(U)}, \qquad f \in C(U).$$

3.2.2. Homogenous polynomial functions. We endow the d-dimensional complex space

$$\mathbb{C}^d = \{x = (\mathfrak{x}_1, \dots, \mathfrak{x}_d) : \mathfrak{x}_\imath \in \mathbb{C}, \imath = 1, \dots, d\}$$

with the Euclidean norm $\|x\|_{\mathbb{C}^d} = \langle x \mid x\rangle_{\mathbb{C}^d}^{1/2}$ and the corresponding scalar product $\langle \cdot \mid \cdot \rangle_{\mathbb{C}^d}$. Consider in the complex space $\mathbb{C}^d$ the canonical orthonormal basis

$$\mathscr{E}(\mathbb{C}^d) = \{\mathfrak{e}_1, \dots, \mathfrak{e}_d\}, \qquad \mathfrak{e}_\imath = (\overbrace{0, \dots, 0, 1}^{\imath}, 0, \dots, 0) \in \mathbb{C}^d.$$

The unit sphere and open ball in $\mathbb{C}^d$ we denote by

$$\mathbb{S}^d = \{x \in \mathbb{C}^d : \|x\|_{\mathbb{C}^d} = 1\}, \qquad \mathbb{B}^d = \{x \in \mathbb{C}^d : \|x\|_{\mathbb{C}^d} < 1\}.$$

Let $\otimes_{\mathfrak{h}}^n \mathbb{C}^d$ be the nth tensor power of the complex space $\mathbb{C}^d$ endowed with the norm $\|\psi_n\|_{\otimes_{\mathfrak{h}}^n \mathbb{C}^d} = \langle \psi_n \mid \psi_n \rangle_{\otimes_{\mathfrak{h}}^n \mathbb{C}^d}^{1/2}$ for all elements $\psi_n = \sum_\imath y_{1\imath} \otimes \dots \otimes y_{n\imath}$, where $y_{1\imath} \otimes \dots \otimes y_{n\imath} \in \otimes_{\mathfrak{h}}^n \mathbb{C}^d$ with $y_{t\imath} \in \mathbb{C}^d$ at $t = 1, \dots, n$. Put $\otimes_{\mathfrak{h}}^0 \mathbb{C} = 1$. If $d = 1$ then $\otimes_{\mathfrak{h}}^n \mathbb{C} = \mathbb{C}$.

Let us define the nth symmetric tensor power $\odot_{\mathfrak{h}}^n \mathbb{C}^d$ as the image of the mapping

$$S_n : \otimes_{\mathfrak{h}}^n \mathbb{C}^d \ni x_1 \otimes \dots \otimes x_n \longmapsto x_1 \odot \dots \odot x_n,$$

linearly extended to the whole space $\otimes_{\mathfrak{h}}^n \mathbb{C}^d$ (see Proposition 1.29).

Let us use the short notations $(k) = (k_1, \dots, k_m) \in \mathbb{Z}_+^d$, $|(k)| = k_1 + \dots + k_m$, $(k)! = k_1! \cdot \ldots \cdot k_m!$, $x^{(k)} := \mathfrak{x}_1^{k_1} \cdot \ldots \cdot \mathfrak{x}_d^{k_d}$, $\{d\} = \{1, \dots, d\}$.

Proposition 1.28 implies that the system of symmetric tensor elements with a fixed n, indexed by (k),

$$\mathscr{E}\left(\odot_{\mathfrak{h}}^n \mathbb{C}^d\right) = \left\{\mathfrak{e}_{\{d\}}^{\otimes(k)} := \mathfrak{e}_1^{\otimes k_1} \odot \dots \odot \mathfrak{e}_d^{\otimes k_d} : (k) \in \mathbb{Z}_+^d;\ |(k)| = n\right\},$$

forms an orthogonal basis in $\odot_{\mathfrak{h}}^n \mathbb{C}^d$ such that $\left\|\mathfrak{e}_{\{d\}}^{\otimes(k)}\right\|_{\otimes_{\mathfrak{h}}^n \mathbb{C}^d}^2 = (k)!/n!$.

Fix an element $\mathfrak{a} \in \mathbb{S}^d$. Then for any element $\psi_n = \sum_\imath y_{1\imath} \otimes \dots \otimes y_{n\imath} \in \otimes_{\mathfrak{h}}^n \mathbb{C}^d$ we can define the n-homogeneous Hilbert-Schmidt polynomial

$$\psi_n^* : \mathbb{C}^d \ni x \longmapsto \langle x^{\otimes n} \mid \psi_n \rangle_{\otimes_{\mathfrak{h}}^n \mathbb{C}^d} = \sum_\imath \prod_{t=1}^n \langle x \mid y_{t\imath} \rangle_{\mathbb{C}^d}.$$

With this polynomial is uniquely associated the corresponding n-homogenous complex function

$$\psi_n^\circ : U \ni u \longmapsto \langle \check{u}^{\otimes n} \mid \psi_n \rangle_{\otimes_{\mathfrak{h}}^n \mathbb{C}^d} = \sum_\imath \prod_{t=1}^n \langle u(\mathfrak{a}) \mid y_{t\imath} \rangle_{\mathbb{C}^d},$$

where an arbitrary element of U-orbit of $\mathfrak{a} \in \mathbb{S}^d$ is denoted by

$$\check{u} := u(\mathfrak{a}) \in \mathbb{S}^d.$$

We denote by

$$\mathcal{P}_\chi^n = \{\psi_n^\circ : \psi_n \in \otimes_{\mathfrak{h}}^n \mathbb{C}^d\}, \qquad \mathcal{P}_\chi^0 = \{1\}$$

the space of all functions ψ_n° on U with a fixed integer $n \in \mathbb{Z}_+$ endowed with the L_χ^2-norm.

The following theorem is main in this section.

THEOREM 3.13. *Every function* $\psi_n^\diamond \in \mathcal{P}_\chi^n$ *is independent of an element* $\mathfrak{a} \in \mathbb{S}^d$ *and the one-to-one anti-linear correspondence*

$$\mathcal{P}_\chi^n \ni \psi_n^\diamond \rightleftarrows \psi_n \in \odot_\mathfrak{h}^n \mathbb{C}^d$$

satisfying the equality

$$\binom{n+d-1}{n} \int_U \phi_n^\diamond \bar{\psi}_n^\diamond \, d\chi = \langle \psi_n \mid \phi_n \rangle_{\otimes_\mathfrak{h}^n \mathbb{C}^d}, \qquad \psi_n, \phi_n \in \odot_\mathfrak{h}^n \mathbb{C}^d \tag{3.19}$$

holds. As a consequence, the subspace $\mathcal{P}_\chi^n$ *is closed in the space* L_χ^2 *and the following system of* n*-homogenous complex functions on the group* U,

$$\begin{aligned} \mathscr{E}_n^\diamond &= \left\{ \mathfrak{e}_{\{d\}}^{\diamond(k)} = \mathfrak{e}_1^{\diamond k_1} \cdot \ldots \cdot \mathfrak{e}_d^{\diamond k_d} : \mathfrak{e}_{\{d\}}^{\otimes(k)} \in \otimes_\mathfrak{h}^n \mathbb{C}^d;\ |(k)| = n \right\}, \\ \mathscr{E}_0^\diamond &= \{1\}, \end{aligned} \tag{3.20}$$

forms an orthogonal basis in $\mathcal{P}_\chi^n$.

PROOF. First verify that the sesquilinear form defined by the above integral is continuous. The probability property of χ yields the inequalities

$$\begin{aligned} \left| \int_U \phi_n^\diamond \bar{\psi}_n^\diamond \, d\chi \right| &\le \sup_{u \in U} \left| \langle \breve{u}^{\otimes n} \mid \phi_n \rangle_{\otimes_\mathfrak{h}^n \mathbb{C}^d} \right| \cdot \left| \langle \breve{u}^{\otimes n} \mid \psi_n \rangle_{\otimes_\mathfrak{h}^n \mathbb{C}^d} \right| \\ &\le \sup_{u \in U} \|\breve{u}\|_{\mathbb{C}^d}^{2n} \, \|\phi_n\|_{\otimes_\mathfrak{h}^n \mathbb{C}^d} \, \|\psi_n\|_{\otimes_\mathfrak{h}^n \mathbb{C}^d} \\ &= \|\phi_n\|_{\otimes_\mathfrak{h}^n \mathbb{C}^d} \, \|\psi_n\|_{\otimes_\mathfrak{h}^n \mathbb{C}^d} \end{aligned}$$

for all $\phi_n, \psi_n \in \odot_\mathfrak{h}^n \mathbb{C}^d$. Hence, there exists a linear operator A_n over $\odot_\mathfrak{h}^n \mathbb{C}^d$ such that

$$\langle A_n(\psi_n) \mid \phi_n \rangle_{\otimes_\mathfrak{h}^n \mathbb{C}^d} = \int_U \phi_n^\diamond \, \bar{\psi}_n^\diamond \, d\chi.$$

Let us show that this operator A_n commutes with all operators

$$\left\{ v^{\otimes n} \in \mathscr{L}\left(\odot_\mathfrak{h}^n \mathbb{C}^d\right) : v \in U \right\}$$

defined by the formula $v^{\otimes n}(x^{\otimes n}) = [v(x)]^{\otimes n}$. The unitarily-invariant property of Haar's measure χ under the right action yields

$$\begin{aligned} \langle (A_n \circ v^{\otimes n})(\psi_n) \mid \phi_n \rangle_{\otimes_\mathfrak{h}^n \mathbb{C}^d} &= \int_U \langle \breve{u}^{\otimes n} \mid \phi_n \rangle_{\otimes_\mathfrak{h}^n \mathbb{C}^d} \overline{\langle \breve{u}^{\otimes n} \mid v^{\otimes n}(\psi_n) \rangle}_{\otimes_\mathfrak{h}^n \mathbb{C}^d} d\chi(u) \\ &= \int_U \langle [v^{-1}(\breve{u})]^{\otimes n} \mid (v^{-1})^{\otimes n}(\phi_n) \rangle_{\otimes_\mathfrak{h}^n \mathbb{C}^d} \overline{\langle [v^{-1}(\breve{u})]^{\otimes n} \mid \psi_n \rangle}_{\otimes_\mathfrak{h}^n \mathbb{C}^d} d\chi(u) \\ &= \int_U \langle \breve{u}^{\otimes n} \mid (v^{-1})^{\otimes n}(\phi_n) \rangle_{\otimes_\mathfrak{h}^n \mathbb{C}^d} \overline{\langle \breve{u}^{\otimes n} \mid \psi_n \rangle}_{\otimes_\mathfrak{h}^n \mathbb{C}^d} d\chi(u) \\ &= \langle A_n(\psi_n) \mid (v^{-1})^{\otimes n}(\phi_n) \rangle_{\otimes_\mathfrak{h}^n \mathbb{C}^d} \\ &= \langle (v^{\otimes n} \circ A_n)\psi_n \mid \phi_n \rangle_{\otimes_\mathfrak{h}^n \mathbb{C}^d} \end{aligned}$$

for all elements $\phi_n, \psi_n \in \otimes_\mathfrak{h}^n \mathbb{C}^d$. Hence, the equality

$$A_n \circ v^{\otimes n} = v^{\otimes n} \circ A_n \tag{3.21}$$

holds for any $v \in U$. Let us check that the operator A_n, satisfying (3.21), is proportional to the identity operator $\mathbb{1}_{\odot_{\mathfrak{h}}^n \mathbb{C}^d}$ on $\odot_{\mathfrak{h}}^n \mathbb{C}^d$. For this goal we form the n-fold tensor power of the unitary group $U = U(d)$,

$$U^{\otimes n} = \left\{ v^{\otimes n} \in \mathscr{L}\left(\odot_{\mathfrak{h}}^n \mathbb{C}^d\right) : v \in U \right\}, \qquad U^{\otimes 0} = 1,$$

invariantly acting on the space $\odot_{\mathfrak{h}}^n \mathbb{C}^d$ by the formula

$$v^{\otimes n}(x_1 \odot \ldots \odot x_n) = v(x_1) \odot \ldots \odot v(x_n)$$

for all $x_1 \otimes \ldots \otimes x_n \in \odot_{\mathfrak{h}}^n \mathbb{C}^d$. Clearly, $U^{\otimes n}$ is a unitary group over $\odot_{\mathfrak{h}}^n \mathbb{C}^d$ as a restriction of the unitary group $U^{\otimes n}$, acting over $\otimes_{\mathfrak{h}}^n \mathbb{C}^d$. Check that the corresponding unitary representation

$$U \ni v \longmapsto v^{\otimes n} \in \mathscr{L}\left(\odot_{\mathfrak{h}}^n \mathbb{C}^d\right) \tag{3.22}$$

is irreducible. This means that there is no subspace $\odot_{\mathfrak{h}}^n \mathbb{C}^d$ other than $\{0\}$ and the whole space which is invariant under the action of $U^{\otimes n}$.

Suppose, on the contrary, that there is an element $\psi_n \in \odot_{\mathfrak{h}}^n \mathbb{C}^d$ such that the corresponding n-homogenous complex function ψ_n° satisfies the equality

$$\psi_n^\circ(v) = \langle \bar{v}^{\otimes n} \mid \psi_n \rangle_{\otimes_{\mathfrak{h}}^n \mathbb{C}^d} = 0$$

for all elements $\bar{v} = v\,(\mathfrak{a}) \in \mathbb{S}^d$ dependent on the variable $v \in U$. The unitary group U acts surjectivity over the unit sphere $\mathbb{S}^d$. Hence, by n-homogeneity of ψ_n°, we obtain

$$\langle x^{\otimes n} \mid \psi_n \rangle_{\otimes_{\mathfrak{h}}^n \mathbb{C}^d} = 0$$

for all elements $x \in \mathbb{C}^d$. Let us apply the polarization formula for symmetric tensor products (see (1.6))

$$z_1 \odot \cdots \odot z_n = \frac{1}{2^n n!} \sum_{1 \le \imath \le n} \sum_{\theta_\imath = \pm 1} \theta_1 \ldots \theta_n \, x^{\otimes n}, \qquad x = \sum_{\imath=1}^{n} \theta_\imath z_\imath^* \in \mathbb{C}^d.$$

As a result, we obtain

$$\langle z_1 \odot \cdots \odot z_n \mid \phi_n \rangle_{\otimes_{\mathfrak{h}}^n \mathbb{C}^d} = 0$$

for all $z_1, \ldots, z_n \in \mathbb{C}^d$. It follows that $\psi_n = 0$, because the set of elements $z_1 \odot \cdots \odot z_n$ is total in $\odot_{\mathfrak{h}}^n \mathbb{C}^d$. As a consequence, the unitary representation (3.22) is irreducible.

Now we can apply to the unitary representation (3.22) the well known Schur Lemma [**51**, Theorem 21.30]: a non-zero matrix which commutes with all of the matrices of an irreducible representation is a constant multiple of the unit matrix. So, we obtain that the operator A_n, satisfying the equality (3.21), is proportional to the identity operator on $\odot_{\mathfrak{h}}^n \mathbb{C}^d$ i.e.

$$A_n = \alpha_n \mathbb{1}_{\odot_{\mathfrak{h}}^n \mathbb{C}^d}$$

with a constant $\alpha_n > 0$. In other words, we have

$$\int_U \phi_n^\circ \, \bar{\psi}_n^\circ \, d\chi = \alpha_n \langle \psi_n \mid \phi_n \rangle_{\otimes_{\mathfrak{h}}^n \mathbb{C}^d}, \qquad \phi_n, \psi_n \in \odot_{\mathfrak{h}}^n \mathbb{C}^d. \tag{3.23}$$

It yields in particular that the system of functions $\mathscr{E}_n^{\diamond}$ forms an orthogonal basis in the space $\mathcal{P}_\chi^n$, because the system of tensors $\mathscr{E}\left(\odot_\mathfrak{h}^n\mathbb{C}^d\right)$ forms an orthogonal basis in the spaces $\odot_\mathfrak{h}^n\mathbb{C}^d$. The subspace $\mathcal{P}_\chi^n$ in L_χ^2 is closed, because it is isometric to the complete space $\odot_\mathfrak{h}^n\mathbb{C}^d$.

Taking into account (1.12), we can choose

$$\phi_n = \psi_n = \mathfrak{e}_{\{d\}}^{\otimes(k)}\sqrt{\frac{n!}{(k)!}}$$

with $|(k)| = n$ in (3.23). As a result, we obtain

$$\alpha_n = \frac{n!}{(k)!}\left\|\mathfrak{e}_{\{d\}}^{\diamond(k)}\right\|_{L_\chi^2}^2, \qquad |(k)| = n \in \mathbb{Z}_+. \tag{3.24}$$

Using the well known formula [88, 1.4.9] for the unitary d-dimensional group U,

$$\left\|\mathfrak{e}_{\{d\}}^{\diamond(k)}\right\|_{L_\chi^2}^2 = \frac{(k)!(d-1)!}{(n+d-1)!}, \qquad |(k)| = n,$$

we see that the constant α_n does not depend on indexes (k) such that $|(k)| = n$ and

$$\alpha_n = \frac{n!}{(k)!}\int_U\left|\mathfrak{e}_{\{d\}}^{\diamond(k)}\right|^2 d\chi = \frac{n!}{(k)!}\,\frac{(k)!(d-1)!}{(n+d-1)!} = \frac{n!(d-1)!}{(n+d-1)!}. \tag{3.25}$$

Thus, the isometry (3.19) is proven. □

Using the fact that any function $\psi_n^{\diamond} \in \mathcal{P}_\chi^n$ is independent of an element $\mathfrak{a} \in \mathbb{S}^d$, in what follows in (3.20) we take

$$\mathfrak{a} = \mathfrak{e}_d$$

and denote

$$\mathfrak{e}_{\{d\}}^{\diamond(k)}(u) := \left\langle [u(\mathfrak{e}_d)]^{\otimes n} \mid \mathfrak{e}_{\{d\}}^{\otimes(k)}\right\rangle_{\otimes_\mathfrak{h}^n\mathbb{C}^d} = \prod_{r=1}^{d}\langle u(\mathfrak{e}_d) \mid \mathfrak{e}_r\rangle_{\mathbb{C}^d}^{k_r}$$

for all $(k) \in \mathbb{Z}_+^d$ so that $|(k)| = n$.

COROLLARY 3.14. Symmetric tensors $\mathscr{E}\left(\odot_\mathfrak{h}^n\mathbb{C}^d\right)$ uniquely define the system $\mathscr{E}_n^{\diamond}$ of n-homogeneous polynomial functions of the variable $u \in U$ with norms

$$\left\|\mathfrak{e}_{\{d\}}^{\otimes(k)}\right\|_{\otimes_\mathfrak{h}^n\mathbb{C}^d} = \binom{n+d-1}{n}^{-1/2}\left\|\mathfrak{e}_{\{d\}}^{\diamond(k)}\right\|_{L_\chi^2}, \tag{3.26}$$

In particular,

$$\|\mathfrak{e}_\imath^{\diamond}\|_{L_\chi^2} = \frac{1}{\sqrt{d}}$$

for all $\imath = 1, \ldots, d$.

The dimension of the space $\mathcal{P}_\chi^n$ is equal to

$$\dim\mathcal{P}_\chi^n = \frac{(n+d-1)!}{n!(d-1)!} = \binom{n+d-1}{n}.$$

3.2.3. Polynomial reproducing kernels. Let the orbits in the space $\odot_{\mathfrak{h}}^n \mathbb{C}^d$ of the last basis element $\mathfrak{e}_d \in \mathscr{E}(\mathbb{C}^d)$ under the unitary representation (3.22) is denoted by

$$\left\{ \breve{u}^{\otimes n} : \breve{u} = u(\mathfrak{e}_d),\ u \in U \right\}.$$

Note that it is total in $\odot_{\mathfrak{h}}^n \mathbb{C}^d$.

LEMMA 3.15. *The reproducing kernel of the space $\mathcal{P}_\chi^n$ endowed with the L_χ^2-norm has the form*

$$\begin{aligned} \binom{n+d-1}{n} \langle \breve{v} \mid \breve{u} \rangle_{\mathbb{C}^d}^n &= \sum_{|(k)|=n} \left\| \mathfrak{e}_{\{d\}}^{\diamond(k)} \right\|_{L_\chi^2}^{-2} \mathfrak{e}_{\{d\}}^{\diamond(k)}(v)\, \overline{\mathfrak{e}}_{\{d\}}^{\diamond(k)}(u) \\ &= \sum_{|(k)|=n} \frac{(n+m-1)!}{(k)!(m-1)!} \mathfrak{e}_{\{d\}}^{\diamond(k)}(v)\, \overline{\mathfrak{e}}_{\{d\}}^{\diamond(k)}(u) \end{aligned} \tag{3.27}$$

for all $u, v \in U$.

PROOF. Expand any vector $\breve{u} \in \mathbb{S}^d$ into the sum of the basis elements

$$\breve{u} = \sum_{j\in\mathbb{N}} \mathfrak{e}_j^{\diamond}(u) \mathfrak{e}_j.$$

Then the Tensor Multinomial Theorem yields the Fourier expansion

$$\breve{u}^{\otimes n} = \Big(\sum_{j\in\mathbb{N}} \mathfrak{e}_j^{\diamond}(u)\mathfrak{e}_j \Big)^{\otimes n} = \sum_{|(k)|=n} \frac{n!}{(k)!} \mathfrak{e}_{\{d\}}^{\diamond(k)}(u)\, \mathfrak{e}_{\{d\}}^{\otimes(k)}$$

with respect to the orthogonal basis $\mathscr{E}\left(\odot_{\mathfrak{h}}^n \mathbb{C}^d\right)$ in the space $\odot_{\mathfrak{h}}^n \mathbb{C}^d$. Using the equality (1.12), we obtain

$$\begin{aligned} \langle \breve{v} \mid \breve{u} \rangle_{\mathbb{C}^d}^n &= \left\langle \breve{v}^{\otimes n} \mid \breve{u}^{\otimes n} \right\rangle_{\otimes_{\mathfrak{h}}^n \mathbb{C}^d} \\ &= \sum_{|(k)|=n} \left(\frac{n!}{(k)!} \right)^2 \left\langle \mathfrak{e}_{\{d\}}^{\otimes(k)} \mid \mathfrak{e}_{\{d\}}^{\otimes(k)} \right\rangle_{\otimes_{\mathfrak{h}}^n \mathbb{C}^d} \mathfrak{e}_{\{d\}}^{\diamond(k)}(v)\, \overline{\mathfrak{e}}_{\{d\}}^{\diamond(k)}(u) \\ &= \sum_{|(k)|=n} \frac{n!}{(k)!} \mathfrak{e}_{\{d\}}^{\diamond(k)}(v)\, \overline{\mathfrak{e}}_{\{d\}}^{\diamond(k)}(u) \end{aligned}$$

for all $n \in \mathbb{Z}_+$. Multiplying both sides by

$$\binom{n+d-1}{n} = \frac{(n+d-1)!}{n!(d-1)!},$$

we get the equality (3.27).

Via Theorem 3.32 the system $\mathscr{E}_n^{\diamond}(U)$ forms an orthogonal basis in the space $\mathcal{P}_\chi^n$. Hence, applying the equality (3.27), we have

$$\binom{n+d-1}{n} \int_U \langle \breve{v} \mid \breve{u} \rangle_{\mathbb{C}^d}^n \mathfrak{e}_{\{d\}}^{\diamond(k)}(u)\, d\chi(u) = \mathfrak{e}_{\{d\}}^{\diamond(k)}(v)$$

for all $v \in U$, i.e., the integral operator with the kernel (3.27) acts as the identity mapping over $\mathcal{P}^n_\chi$. So, the equality

$$\binom{n+d-1}{n} \int_U \langle \bar{v} \mid \bar{u} \rangle^n_{\mathbb{C}^d} \psi^\diamond_n(u)\, d\chi(u) = \psi^\diamond_n(v), \qquad v \in U \tag{3.28}$$

holds for all $\psi^\diamond_n \in \mathcal{P}^n_\chi$. Consequently, the kernel (3.27) is reproducing in $\mathcal{P}^n_\chi$. □

3.2.4. Segal-Bargmann space. Consider the standard Gaussian measure μ on the complex space $\mathbb{C}^d$, centered at 0 and normalized so that $\mu(\mathbb{C}^d) = 1$. In terms of the Lebesgue measure $dx = dp\,dq$ ($x = p + \mathrm{i}q$ with $p, q \in \mathbb{R}^n$) on the corresponding real space $\mathbb{R}^{2n} \simeq \mathbb{C}^d$ we can write

$$d\mu(x) = \pi^{-d} \exp\left(-\|x\|^2_{\mathbb{C}^d}\right) dx, \qquad x = (\mathfrak{x}_1, \ldots, \mathfrak{x}_d) \in \mathbb{C}^d.$$

Let L^2_μ be the space of square μ-integrable complex functions f on $\mathbb{C}^d$ with the norm $\|f\|_{L^2_\mu} = \left(\int_{\mathbb{C}^d} |f|^2 d\mu\right)^{1/2}$. As is well-known [40, Theorem 1.63]

$$\int_{\mathbb{C}^d} x^{(k)} \bar{x}^{(l)} d\mu(x) = \begin{cases} (k)! & : k = l, \\ 0 & : k \neq l \end{cases}$$

for any $(k), (l) \in \mathbb{Z}^d_+$.

Recall that the the Segal-Bargmann space $\mathcal{H}^2_\mu(\mathbb{C}^d)$ (see e.g. [**12, 14**]) is defined to be the subspace in L^2_μ of all entire analytic functions. It is well known that $\mathcal{H}^2_\mu(\mathbb{C}^d)$ is the L^2_μ-closure of all polynomials in the variables $\mathfrak{x}_1, \ldots, \mathfrak{x}_d \in \mathbb{C}$. The reproducing and normalized reproducing kernels of the space $\mathcal{H}^2_\mu$ have the forms, respectively

$$K_y(x) = \exp\langle x \mid y \rangle_{\mathbb{C}^d},$$

$$k_y(x) = \frac{K_y(x)}{\sqrt{K_y(y)}} = \exp\left(\langle x \mid y \rangle_{\mathbb{C}^d} - \frac{\|y\|^2_{\mathbb{C}^d}}{2}\right)$$

for all $x, y \in \mathbb{C}^d$.

The space of all n-homogenous polynomials on $\mathbb{C}^d$, generated by the system

$$\left\{x^{(k)} \colon |(k)| = n,\ x \in \mathbb{C}^d\right\}$$

with a fixed $n \in \mathbb{Z}_+$ and endowed with the L^2_μ-norm, we denote by $\mathcal{P}^n_\mu$. Clearly, this system forms an orthogonal basis in $\mathcal{P}^n_\mu$. Whereas, the system $\left\{x^{(k)}\right\}$ forms an orthogonal basis in the whole Segal-Bargmann space $\mathcal{H}^2_\mu(\mathbb{C}^d)$. So, the following orthogonal decomposition holds

$$\mathcal{H}^2_\mu(\mathbb{C}^d) = \mathbb{C} \oplus \mathcal{P}^1_\mu \oplus \mathcal{P}^2_\mu \oplus \ldots. \tag{3.29}$$

Consider the Hilbertian orthogonal sum

$$\Gamma(\mathbb{C}^d) = \left\{\psi = \bigoplus_{n \in \mathbb{Z}_+} \psi_n \colon \psi_n \in \odot^n_\mathfrak{h} \mathbb{C}^d,\ \|\psi\|_\Gamma < \infty\right\}$$

endowed with the scalar product and norm, respectively

$$\langle \psi \mid \phi \rangle_\Gamma = \sum_{n \in \mathbb{Z}_+} n!\, \langle \psi_n \mid \phi_n \rangle_{\mathbb{C}^d} \quad \text{and} \quad \|\psi\|_\Gamma = \langle \psi \mid \psi \rangle^{1/2}_\Gamma,$$

called the symmetric Fock space generated by the complex space $\mathbb{C}^d$. The polarization formula (1.6) implies that the set of so-called exponential vectors

$$\left\{\varepsilon(x) = \bigoplus_{n\in\mathbb{Z}_+} \frac{x^{\otimes n}}{n!} : x \in \mathbb{C}^d\right\}, \qquad x^{\otimes 0} = 1$$

is total in the symmetric Fock space $\Gamma(\mathbb{C}^d)$ and

$$\|\varepsilon(x)\|_\Gamma^2 = \sum_{n\in\mathbb{Z}_+} \frac{\|x\|_{\mathbb{C}^d}^{2n}}{n!} = \exp\left(\|x\|_{\mathbb{C}^d}^2\right) < \infty$$

for all elements $x \in \mathbb{C}^d$.

LEMMA 3.16. *For any $(k),(l) \in \mathbb{Z}_+^d$ such that $|(k)| = |(l)| = n$, we have*

$$\begin{aligned}\int_{\mathbb{C}^d} \frac{x^{(k)}\bar{x}^{(l)}}{\sqrt{(k)!(l)!}}\, d\mu(x) &= \frac{n!\,\langle \mathfrak{e}_{\{d\}}^{\otimes(k)} \mid \mathfrak{e}_{\{d\}}^{\otimes(l)}\rangle_{\otimes_\mathfrak{h}^n\mathbb{C}^d}}{\sqrt{(k)!(l)!}} \\ &= \frac{(n+m-1)!}{(m-1)!}\int_U \frac{\mathfrak{e}_{\{d\}}^{\circ(l)}\mathfrak{e}_{\{d\}}^{\circ(k)}}{\sqrt{(l)!(k)!}}\,d\chi \\ &= \begin{cases} 1: & k = l, \\ 0: & k \neq l.\end{cases}\end{aligned}$$

PROOF. In fact, for all $x = (\mathfrak{x}_1, \ldots, \mathfrak{x}_m) \in \mathbb{C}^d$ such that

$$x = \{\alpha\breve{u}\colon \alpha \in \mathbb{C},\, u \in U\}$$

we have

$$x^{(k)} = \mathfrak{x}_1^{k_1}\cdot\ldots\cdot\mathfrak{x}_m^{k_m} = \left\langle x^{\otimes|(k)|} \mid \mathfrak{e}_{\{d\}}^{\otimes(k)}\right\rangle_{\otimes_\mathfrak{h}^n\mathbb{C}^d} = \alpha^{|(k)|}\mathfrak{e}_{\{d\}}^{\circ(k)}(u). \tag{3.30}$$

To complete the proof of lemma it remains to apply Theorem 3.32. □

THEOREM 3.17. *The mapping*

$$\Xi\colon \mathcal{H}_\mu^2(\mathbb{C}^d) \ni f \longmapsto \sum_{n\in\mathbb{Z}_+} \psi_n \in \Gamma(\mathbb{C}^d), \qquad \psi_n \in \odot_\mathfrak{h}^n\mathbb{C}^d$$

defined by the equalities

$$\Xi\left(x^{(k)}\right) = \mathfrak{e}_{\{d\}}^{\otimes(k)}$$

for all indexes $(k) \in \mathbb{Z}_+^d$ produces the anti-linear isometry

$$\Gamma(\mathbb{C}^d) \simeq \mathcal{H}_\mu^2(\mathbb{C}^d) \tag{3.31}$$

such that the following Taylor analytic expansion at the origin

$$f(x) = \langle\varepsilon(x) \mid \Xi(f)\rangle_\Gamma = \sum_{n\in\mathbb{Z}_+} \psi_n^*(x),$$

$$\psi_n^*(x) = \frac{1}{n!}\int_{\mathbb{C}^d} \langle x \mid y\rangle_{\mathbb{C}^d}^n\, f(y)\, d\mu(y)$$

holds for all $x \in \mathbb{C}^d$.

PROOF. By (3.29) for every $f \in \mathcal{H}^2_\mu$ there exists a unique sequence of elements $f_n \in \mathcal{P}^n_\mu$ such that the series $\sum_n f_n$ is convergent to f in L^2_μ. By Lemma 3.16 for every $f_n \in \mathcal{P}^n_\mu$ there exists a unique element $\psi_n \in \odot^n_\mathfrak{h}\mathbb{C}^d$ such that $f_n = \psi^*_n$ and conversely. Therefore, the isometry (3.31) follows from the orthogonal property

$$\psi^*_l \perp (x^{\otimes n})^*$$

for all $l \neq n$. Applying the n-homogeneity of ψ^*_n, we obtain

$$d^n_0 f(x) = \frac{d^n}{d\alpha^n} \sum \alpha^n \psi^*_n(x) \mid_{\alpha=0} = n!\psi^*_n(x)$$

for all $\alpha x \in \mathbb{C}^d$ with $\alpha \in \mathbb{C}$. So, the Taylor analytic expansion holds. □

3.2.5. Orthogonal decomposition. In this section we denote by $\mathcal{H}^2_\chi$ and call the Hardy space on U the L^2_χ-closure of the complex linear span of all complex continuous functions

$$\mathscr{E}^\diamond = \{\mathscr{E}^\diamond_n : n \in \mathbb{Z}_+\}, \qquad \mathscr{E}^\diamond_n = \left\{\mathfrak{e}^{\diamond(k)}_{\{d\}} : |(k)| = n\right\}.$$

THEOREM 3.18. *The system $\mathscr{E}^\diamond$ forms an orthogonal basis in $\mathcal{H}^2_\chi$ and the following orthogonal decomposition holds,*

$$\mathcal{H}^2_\chi = \mathbb{C} \oplus \mathcal{P}^1_\chi \oplus \mathcal{P}^2_\chi \oplus \ldots. \tag{3.32}$$

The reproducing kernel of the Hardy space $\mathcal{H}^2_\chi$ has the form

$$\begin{aligned} \frac{1}{(1 - \langle \bar{v} \mid \bar{u}\rangle_{\mathbb{C}^d})^d} &= \sum_{n\in\mathbb{Z}_+} \binom{n+d-1}{n} \langle \bar{v} \mid \bar{u}\rangle^n_{\mathbb{C}^d} \\ &= \sum_{n\in\mathbb{Z}_+} \sum_{|(k)|=n} \frac{(n+m-1)!}{(k)!(m-1)!} \mathfrak{e}^{\diamond(k)}_{\{d\}}(v)\, \bar{\mathfrak{e}}^{\diamond(k)}_{\{d\}}(u) \end{aligned} \tag{3.33}$$

for all $u, v \in U$.

PROOF. Applying the equality (3.19) in the case of any indexes $(k), (l) \in \mathbb{Z}^d_+$ such that

$$n = |(k)| \neq |(l)| = m,$$

we obtain

$$\int_U \mathfrak{e}^{\diamond(k)}_{\{d\}}\, \bar{\mathfrak{e}}^{\diamond(l)}_{\{d\}}\, d\chi = \frac{1}{2\pi} \int_U \mathfrak{e}^{\diamond(k)}_{\{d\}}\, \bar{\mathfrak{e}}^{\diamond(l)}_{\{d\}}\, d\chi \int_{-\pi}^{\pi} \exp\left[\mathrm{i}(|(k)| - |(l)|)\vartheta\right] d\vartheta = 0. \tag{3.34}$$

This yields the orthogonal property

$$\mathfrak{e}^{\diamond(k)}_{\{d\}} \perp \mathfrak{e}^{\diamond(l)}_{\{d\}},$$

i.e., $\mathcal{P}^m_\chi \perp \mathcal{P}^n_\chi$ in L^2_χ. As a result, the system $\mathscr{E}^\diamond$ forms an orthogonal basis in the Hardy space $\mathcal{H}^2_\chi$, because every system $\mathscr{E}^\diamond_n$ forms the same basis in the corresponding subspace $\mathcal{P}^n_\chi$.

The first equality in (3.33) is well known (see e.g. [88, 1.4.10]), whereas the second equality follows from (3.27). By Theorem 3.18 every element $f \in \mathcal{H}^2_\chi$ is equal to the L^2_χ-convergent orthogonal series $\sum_n f_n$, where $f_n \in \mathcal{P}^n_\chi$ is the orthogonal projection of f in the decomposition (3.32). Applying (3.28) from Theorem 3.15 and the L^2_χ-orthogonal property

$$\langle \check{v} \mid \cdot \rangle^m_{\mathbb{C}^d} \perp f_n$$

with $n \neq m$, we obtain that for any element $v \in U$ the equalities

$$\begin{aligned}\int_U \frac{f(u)\, d\chi(u)}{(1 - \langle \check{v} \mid \check{u} \rangle_{\mathbb{C}^d})^d} &= \int_U \sum_{m \in \mathbb{Z}_+} \binom{m+d-1}{m} \langle \check{v} \mid \check{u} \rangle^m_{\mathbb{C}^d} \sum_{n \in \mathbb{Z}_+} f_n(u)\, d\chi(u) \\ &= \sum_{n \in \mathbb{Z}_+} f_n(v) = f(v)\end{aligned}$$

hold for all $f \in \mathcal{H}^2_\chi$. Hence, the kernel (3.33) is reproducing in $\mathcal{H}^2_\chi$. □

3.2.6. Weighted symmetric Fock space. Consider the symmetric Fock subspace

$$\Phi(\mathbb{C}^d) = \left\{ \psi = \bigoplus_{n \in \mathbb{Z}_+} \psi_n \in \Gamma(\mathbb{C}^d) \colon \psi_n \in \odot^n_{\mathfrak{h}} \mathbb{C}^d,\ \|\psi\|_\Phi < \infty \right\}$$

endowed with the scalar product and norm, respectively

$$\langle \psi \mid \phi \rangle_\Phi = \sum_{n \in \mathbb{Z}_+} \frac{n!(d-1)!}{(n+d-1)!} \langle \psi_n \mid \phi_n \rangle_{\mathbb{C}^d}, \qquad \|\psi\|_\Phi = \langle \psi \mid \psi \rangle^{1/2}_\Phi,$$

called the weighted symmetric Fock space generated by the complex space $\mathbb{C}^d$.

By Proposition 1.28 the system of symmetric tensors $\mathscr{E}\left(\odot^n_{\mathfrak{h}} \mathbb{C}^d\right)$ forms an orthogonal baseis in $\odot^n_{\mathfrak{h}} \mathbb{C}^d$. As a consequence, the system

$$\mathscr{E} = \left\{ \mathscr{E}\left(\odot^n_{\mathfrak{h}} \mathbb{C}^d\right) : n \in \mathbb{Z}_+ \right\}$$

forms an orthogonal basis in the space $\Phi(\mathbb{C}^d)$. Moreover, for any tensor element $\mathfrak{e}^{\otimes(k)}_{\{d\}} \in \mathscr{E}$ with $n = |(k)|$,

$$\begin{aligned}\left\| \mathfrak{e}^{\otimes(k)}_{\{d\}} \right\|_\Phi &= \left(\frac{n!(d-1)!}{(n+d-1)!} \right)^{1/2} \left\| \mathfrak{e}^{\otimes(k)}_{\{d\}} \right\|_{\otimes^n_{\mathfrak{h}} \mathbb{C}^d} \\ &= \left(\frac{n!(d-1)!}{(n+d-1)!} \right)^{1/2} \left(\frac{(k)!}{n!} \right)^{1/2} = \left\| \mathfrak{e}^{\diamond(k)}_{\{d\}} \right\|_{L^2_\chi}. \end{aligned} \tag{3.35}$$

Thus, applying Lemma 3.16, we obtain the following statement.

THEOREM 3.19. *The mapping*

$$\Upsilon \colon \mathcal{H}^2_\chi \ni f \longmapsto \sum_{n \in \mathbb{Z}_+} \psi_n \in \Phi(\mathbb{C}^d), \qquad \psi_n \in \odot^n_{\mathfrak{h}} \mathbb{C}^d$$

determined by the equality

$$\Upsilon\left(\mathfrak{e}^{\diamond(k)}_{\{d\}} \right) = \mathfrak{e}^{\otimes(k)}_{\{d\}}, \qquad |(k)| = n$$

produces the following isometric equalities

$$\mathcal{H}^2_\chi \simeq \Phi(\mathbb{C}^d), \qquad \Upsilon\left(\mathcal{P}^n_\chi\right) \simeq \left(\odot^n_{\mathfrak{h}} \mathbb{C}^d, \|\cdot\|_\Phi\right).$$

It is also easy to see that the diagonal matrix

$$\operatorname{diag}\left\{\sqrt{\frac{(d-1)!}{(n+d-1)!}} : n \in \mathbb{Z}_+\right\}$$

isometrically maps the Fock space $\Gamma(\mathbb{C}^d)$ onto the weighted Fock space $\Phi(\mathbb{C}^d)$.

3.2.7. Holomorphic extension. The polarization formula (1.6) implies that the set of so-called geometric vectors

$$\left\{\omega(x) = \bigoplus_{n\in\mathbb{Z}_+} x^{\otimes n} : x \in \mathbb{B}^m\right\}, \qquad x^{\otimes 0} = 1$$

is total in the weighted symmetric Fock space $\Phi(\mathbb{C}^m)$. Note that

$$\|\omega(x)\|_\Phi^2 = \sum_{n\in\mathbb{Z}_+} \frac{n!(m-1)!}{(n+m-1)!}\|x\|_{\mathbb{C}^m}^{2n} \le \frac{1}{1-\|x\|_{\mathbb{C}^m}^2} < \infty, \qquad x \in \mathbb{B}^m.$$

Hence, the orthogonal series $\omega(x)$ is absolutely convergent in $\Phi(\mathbb{C}^m)$ for all $x \in \mathbb{B}^m$ i.e., the $\Phi(\mathbb{C}^m)$-valued function

$$\omega\colon \mathbb{B}^m \ni x \longmapsto \omega(x) \in \Phi(\mathbb{C}^m)$$

is analytic.

THEOREM 3.20. *For every $f \in \mathcal{H}_\chi^2$ there exists a unique element*

$$\Upsilon(f) = \bigoplus_{n\in\mathbb{Z}_+} \varphi_n \in \Phi(\mathbb{C}^m)$$

with $\varphi_n \in \odot_\mathfrak{h}^n\mathbb{C}^m$ such that the Cauchy integral formula

$$(3.36)\qquad \begin{aligned} \tilde{f}(x) &= \int_U \frac{f(u)\,d\chi(u)}{(1-\langle x \mid \check{u}\rangle_{\mathbb{C}^m})^m}, \\ \tilde{f}(x) &:= \langle \omega(x) \mid \Upsilon(f)\rangle_\Phi = \sum_{n\in\mathbb{Z}_+} \frac{n!(m-1)!}{(n+m-1)!}\varphi_n^\star(x), \end{aligned}$$

for all $x \in \mathbb{B}^m$ uniquely define an analytic function on $\mathbb{B}^m$ with the Taylor coefficients at the origin

$$(3.37)\qquad \frac{n!(m-1)!}{(n+m-1)!}\varphi_n^\star(x) = \int_U \langle x \mid \check{u}\rangle_{\mathbb{C}^m}^n f(u)\,d\chi(u), \qquad x \in \mathbb{C}^m.$$

PROOF. By Theorem 3.18 for every $f \in \mathcal{H}_\chi^2$ there exists a unique sequence of elements $f_n \in \mathcal{P}_\chi^n$ such that the series $\sum_n f_n$ is convergent to f in L_χ^2. By Theorem 3.12 for every $f_n \in \mathcal{P}_\chi^n$ there exists a unique element $\varphi_n \in \odot_\mathfrak{h}^n\mathbb{C}^m$ such that $f_n = \varphi_n^\star$ and conversely. Consequently, the equality (3.37) it follows from the orthogonal property

$$\varphi_l^\star \perp \check{v}^{\star n}$$

for any $l \neq n$ and the equality (3.28) with using the n-homogeneity of φ_n^*. Summing (3.37) with respect to n, we get (3.36). It follows in particular that

$$\tilde{f}(\alpha\check{v}) = \langle \omega(\alpha\check{v}) \mid \Lambda(f)\rangle_\Phi = \sum_{n\in\mathbb{Z}_+} \alpha^n \frac{n!(m-1)!}{(n+m-1)!}\varphi_n^*(\check{v})$$

for any $x = \alpha\check{v}$, $\alpha \in \mathbb{B}^1$, $v \in U$. The $\Phi(\mathbb{C}^m)$-valued function ω is analytic on $\mathbb{B}^m$ and $\tilde{f}$ is equal to the composition of $\omega(\cdot)$ with the linear continuous functional $\langle\cdot \mid \Upsilon(f)\rangle_\Phi$ on $\Phi(\mathbb{C}^m)$. Consequently, $\tilde{f}$ is analytic on $\mathbb{B}^m$ as well. Differentiating $\tilde{f}$ at $x = 0$ and using the n-homogeneity of derivatives, we obtain

$$\frac{1}{n!}d_0^m\tilde{f} = \frac{1}{n!}\frac{d^n}{d\alpha^n}\sum_{n\in\mathbb{Z}_+} \alpha^n \frac{n!(m-1)!}{(n+m-1)!}\varphi_n^*\Big|_{\alpha=0} = \frac{n!(m-1)!}{(n+m-1)!}\varphi_n^*$$

for all $\alpha \in \mathbb{B}^1$. Hence, the functions φ_n^* coincide with the Taylor coefficients of the analytic function $\tilde{f}$ at the origin, which are uniquely defined on $\mathbb{B}^m$. □

DEFINITION 3.21. Following [88, V.6] we define the space of analytic extension on $\mathbb{B}^m$

$$\mathcal{H}_\chi^2(\mathbb{B}^m) := \left\{\tilde{f} = \mathfrak{C}(f) \colon f \in \mathcal{H}_\chi^2\right\}$$

endowed with the norm

$$\|\tilde{f}\|_{\mathcal{H}_\chi^2(\mathbb{B}^m)} = \sup_{|\alpha|<1}\left(\int_U \left|\tilde{f}(\alpha\check{u})\right|^2 d\chi(u)\right)^{1/2}.$$

Then the well known Cauchy integral formula can be written as the transform

$$\mathfrak{C} \colon \mathcal{H}_\chi^2 \ni f \longmapsto \tilde{f} \in \mathcal{H}_\chi^2(\mathbb{B}^m).$$

THEOREM 3.22. *The linear isometry and the anti-linear isometry, respectively*

$$\mathcal{H}_\chi^2 \simeq \mathcal{H}_\chi^2(\mathbb{B}^m), \qquad \mathcal{H}_\chi^2(\mathbb{B}^m) \simeq \Phi(\mathbb{C}^m), \tag{3.38}$$

produced by the mappings

$$\mathfrak{C} \colon f \longmapsto \tilde{f} := \mathfrak{C}(f), \qquad \Upsilon \circ \mathfrak{C}^{-1} \colon \tilde{f} \longmapsto \Upsilon(f),$$

hold.

PROOF. For any $x = \alpha\check{v}$ with $\alpha \in \mathbb{B}^1$ and $v \in U$ from (3.36) it follows that

$$\tilde{f}(\alpha\check{v}) = \int_U \frac{f(u)\,d\chi(u)}{(1 - \langle\alpha\check{v} \mid \check{u}\rangle_{\mathbb{C}^m})^m},$$

as a function of the variable $v \in U$, belongs to $\mathcal{H}_\chi^2$. In fact, using the homogenous property of coefficients φ_n^*, we have

$$\tilde{f}(\alpha\check{v}) = \sum_{n\in\mathbb{Z}_+} \alpha^n\varphi_n^*(\check{v}).$$

The equalities (3.28) and (3.33) and the orthogonal property of $\{\varphi_n^*\}$ yield

$$\int_U \left|\widetilde{f}(\alpha\check{v})\right|^2 d\chi(v) = \int_U \left| \sum_{n\in\mathbf{Z}_+} \alpha^n \int_U \frac{(n+m-1)!}{n!(m-1)!} \langle \check{v} \mid \check{u}\rangle_{\mathbb{C}^m}^n \varphi_n^*(\check{u})\, d\chi(u)\right|^2 d\chi(v)$$

$$= \int_U \left| \sum_{n\in\mathbf{Z}_+} \alpha^n \varphi_n^*(\check{v})\right|^2 d\chi(v) = \sum_{n\in\mathbf{Z}_+} |\alpha|^{2n} \|\varphi_n^*\|_{L^2_\chi}^2$$

for all $\alpha \in \mathbb{B}^1$. This implies that

$$\|\widetilde{f}\|_{\mathcal{H}^2_\chi(\mathbb{B}^m)} = \sup_{|\alpha|<1} \sum_{n\in\mathbf{Z}_+} |\alpha|^{2n} \|\varphi_n^*\|_{L^2_\chi}^2$$

$$= \sum_{n\in\mathbf{Z}_+} \|\varphi_n^*\|_{L^2_\chi}^2 = \|f\|_{L^2_\chi}^2$$

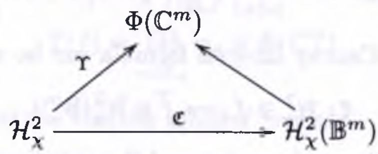
for all $f \in \mathcal{H}^2_\chi$. Hence, the first equality in (3.38) is valid. It is easy to see that

$$\|\widetilde{f}\|_{\mathcal{H}^2_\chi(\mathbb{B}^m)} = \sum_{n\in\mathbf{Z}_+} \|\varphi_n^*\|_{L^2_\chi}^2$$

$$= \sum_{n\in\mathbf{Z}_+} \frac{n!(m-1)!}{(n+m-1)!} \|\varphi_n\|_{\otimes_\mathfrak{h}^n \mathbb{C}^m}^2 = \|\Lambda(f)\|_\Phi$$

via (3.19). The theorem is proven. □

COROLLARY 3.23. As a consequence, we get instantly that the following diagram



is commutative.

3.3. Hardy spaces associated with infinite-dimensional unitary groups

3.3.1. Invariant integrals with respect to infinite dimensional groups. We consider the infinite-dimensional unitary matrix groups:

$$U(\infty) = \bigcup \{U(m) : m \in \mathbb{N}\}, \qquad U^2(\infty) := U(\infty) \times U(\infty),$$

where $U(m)$ is the group of unitary $(m \times m)$-matrices which is identified with the subgroup in $U(m+1)$ fixing the $(m+1)$th basis vector. In other words, $U(\infty)$ is the group of infinite unitary matrices

$$u = [u_{ij}]_{i,j\in\mathbb{N}}$$

with finitely many matrix entries u_{ij} distinct from δ_{ij}. We equip every group $U(m)$ with the probability Haar measure χ_m. We endow the group $U(\infty)$ with the inductive topology under embeddings

$$U(m) \hookrightarrow U(\infty).$$

Every matrix $u_m \in U(m)$ with $m > 1$ we write in the block matrix form

$$u_m = \begin{bmatrix} z_{m-1} & a \\ b & t \end{bmatrix}, \qquad z_{m-1} \in U(m-1), \quad t \in \mathbb{C}$$

corresponding to the partition

$$(m-1)+1 \times (m-1)+1.$$

Over the group $U(\infty)$ (respectively, $U(m)$) the right action is well defined:

$$u.g = w^{-1}uv, \tag{3.39}$$

where u belongs to $U(\infty)$ (respectively, to $U(m)$) and

$$g = (v, w)$$

belongs to $U^2(\infty)$ (respectively, to $U^2(m) := U(m) \times U(m)$).

Let us give some basic known facts about invariant integrals with respect to the group $U(\infty)$. In [**78**, Proposition 0.1], [**79**, Lemma 3.1] it is proven the following claim.

LEMMA 3.24. *The Livšic-type mapping*

$$\pi^m_{m-1} \colon U(m) \longrightarrow U(m-1)$$

of the form

$$\pi^m_{m-1} \colon u_m = \begin{bmatrix} z_{m-1} & a \\ b & t \end{bmatrix} \longmapsto u_{m-1} = \begin{cases} z_{m-1} - a(1+t)^{-1}b: & t \neq -1 \\ z_{m-1}: & t = -1 \end{cases}$$

(which is not a group homomorphism) is Borel and surjective onto $U(m-1)$, and commutes with the right action of $U^2(m-1)$.

LEMMA 3.25 ([**78**, Theorem 1.6]). *The pullback of the probability Haar measure χ_{m-1} on $U(m-1)$ under the mapping π^m_{m-1} is the probability Haar measure χ_m on $U(m)$, i.e.,*

$$\chi_{m-1} \circ \pi^m_{m-1} = \chi_m. \tag{3.40}$$

Let $U'(m) \subset U(m)$ be the subset of unitary matrices which do not have $\{-1\}$ as an eigenvalue. Then $U'(m)$ is open in $U(m)$ and the complement $U(m) \setminus U'(m)$ is a χ_m-negligible set.

LEMMA 3.26 ([**79**, Lemma 3.11]). *The restricted mapping*

$$\pi^m_{m-1} \colon U'(m) \longrightarrow U'(m-1)$$

is continuous and surjective.

Consider the projective limits, taken with respect to the surjective Borel projections π^m_{m-1} and their continuous restrictions $\pi^m_{m-1}|_{U'(m)}$, respectively,

$$\mathfrak{U} = \varprojlim U(m), \qquad \mathfrak{U}' = \varprojlim U'(m),$$

called the spaces of virtual unitary matrices. Notice that $\mathfrak{U}$ is a Borel subset in the Cartesian product

$$\bigtimes_{m\in\mathbb{N}} U(m) = \{u = (u_m) : \ u_m \in U(m)\}$$

endowed with the product topology, because all mapping π^m_{m-1} are Borel. Moreover, the canonical projections

$$\pi_m : \mathfrak{U} \longrightarrow U(m), \qquad \pi_m : \mathfrak{U}' \longrightarrow U'(m),$$

such that $\pi_{m-1} = \pi^m_{m-1} \circ \pi_m$, are surjective by surjectivity of π^m_{m-1} and $\pi^m_{m-1}|_{U'(m)}$.

Applying the well known Kolmogorov consistent theorem, we uniquely define a probability Radon measure χ on $\mathfrak{U}'$ as the projective limit under the mapping (3.40),

$$\chi = \varprojlim \chi_m, \tag{3.41}$$

which satisfies the equality

$$\chi = \chi_m \circ \pi_m$$

for all $m \in \mathbb{N}$. On $\mathfrak{U} \setminus \mathfrak{U}'$ the measure χ is zero, because χ_m is zero on $U(m) \setminus U'(m)$ for all $m \in \mathbb{N}$.

Using (3.39), right actions of the group $U^2(\infty)$ on the space of virtual unitary matrices $\mathfrak{U}$ can be defined as follows:

$$\pi_m(u.g) = w^{-1}\pi_m(u)v, \qquad u \in \mathfrak{U}, \tag{3.42}$$

where m is so large that $g = (v, w) \in U^2(m)$.

The canonical dense embedding

$$\imath : U(\infty) \hookrightarrow \mathfrak{U}$$

to any element $u_m \in U(m)$ assigns the unique sequence $u = (u_l)_{l\in\mathbb{N}}$, such that

$$\imath : U(m) \ni u_m \longmapsto (u_l) \in \mathfrak{U},$$

$$u_l = \begin{cases} \pi^{l+1}_l \circ \ldots \circ \pi^m_{m-1}(u_m) & : \ l < m, \\ u_m & : \ l = m, \\ \begin{bmatrix} u_m & 0 \\ 0 & \mathbb{1}_{l-m} \end{bmatrix} & : \ l > m, \end{cases}$$

where $\mathbb{1}_{l-m}$ is the unit in $U(l-m)$. So, the image $\imath \circ U(\infty)$ consists of stabilizing sequences in $\mathfrak{U}$.

In what follows, we endow the space of virtual unitary matrices $\mathfrak{U}$ with the measure (3.41).

LEMMA 3.27 ([**78**, Proposition 3.2]). *The measure* χ *is* $U^2(\infty)$*-invariant under the right actions* (3.42) *over the space* $\mathfrak{U}$.

DEFINITION 3.28 ([79, Definition 4.5]). A complex function on $\mathfrak{U}$ is called cylindrical if it has the form

$$f(u) = (f_m \circ \pi_m)(u), \qquad u \in \mathfrak{U} \tag{3.43}$$

for a certain $m \in \mathbb{N}$ and a certain complex function f_m on $U(m)$.

Any continuous bounded function f on $\mathfrak{U}'$ has a unique χ-essentially bounded extension on $\mathfrak{U}$, because the set $\mathfrak{U} \setminus \mathfrak{U}'$ is χ-negligible. Therefore, if the function

$$U'(m) \ni \pi_m(u) \longmapsto f_m[\pi_m(u)]$$

in the definition (3.43) is continuous and bounded, then the corresponding cylindrical function f is χ-essentially bounded.

By $\mathcal{L}^\infty_\chi$ we denote closure of the algebraic hull of all cylindrical χ-essentially bounded functions (3.43) with respect to the norm

$$\|f\|_{\mathcal{L}^\infty_\chi} = \operatorname*{ess\,sup}_{u\in\mathfrak{U}} |f(u)|.$$

LEMMA 3.29. *The measure* $\chi = \varprojlim \chi_m$ *on* $\mathfrak{U}$ *is a Radon probability measure such that*

$$\int_{\mathfrak{U}} f(u.g)\, d\chi(u) = \int_{\mathfrak{U}} f(u)\, d\chi(u)$$

for all $g \in U^2(\infty)$ *and* $f \in \mathcal{L}^\infty_\chi$.

The measure χ *has the property: for all compact set* K *in* $U(\infty)$ *such that* $K \subset U(m)$ *with an index* $m \in \mathbb{N}$,

$$(\chi \circ \imath)(K) = \chi_m(K). \tag{3.44}$$

PROOF. First recall the known Prohorov criterion (see [21, IX.4.2, Theorem 1] or [98, Theorem 6]). Adapting to our notation, it has the form: there exists a Radon probability measure χ' on $\mathfrak{U}'$ such that

$$\chi' = \chi_m \circ \pi_m|_{\mathfrak{U}'} \quad \text{for all} \quad m \in \mathbb{N} \tag{3.45}$$

if and only if for every $\varepsilon > 0$ there exists a compact set $\mathcal{K}$ in $\mathfrak{U}'$ such that the inequality

$$(\chi_m \circ \pi_m)(\mathcal{K}) \geq 1 - \varepsilon \quad \text{for all} \quad m \in \mathbb{N}$$

holds; in this case χ' is uniquely determined by means of the formula

$$\chi'(\mathcal{K}) = \inf_{m\in\mathbb{N}} (\chi_m \circ \pi_m)(\mathcal{K}),$$

where $\mathcal{K}$ is a compact set in $\mathfrak{U}'$.

Let $K_n \subset U'(n)$ be a compact set with a fixed n. Putting

$$K_{n-1} = \pi^n_{n-1}(K_n),$$

we have

$$\chi_{n-1}(K_{n-1}) = \left(\chi_{n-1} \circ \pi^n_{n-1}\right)(K_n) = \chi_n(K_n).$$

On the other hand, if we put

$$K_{n+1} = \begin{bmatrix} K_n & 0 \\ 0 & 1 \end{bmatrix}$$

then via (3.40),

$$\chi_{n+1}(K_{n+1}) = \left(\chi_n \circ \pi_n^{n+1}\right)(K_{n+1})$$
$$= \left(\chi_n \circ \pi_n^{n+1}\right)\begin{bmatrix} K_n & 0 \\ 0 & 1 \end{bmatrix} = \chi_n(K_n).$$

As a consequence, the compact set $\mathcal{K} = (K_m)$ in $\mathfrak{U}'$, generated by a compact set $K_n \subset U'(n)$ with the help of mappings π_{n-1}^n, satisfies the condition:

$$\chi_n(K_n) = \chi_m(K_m) \quad \text{for all} \quad m \in \mathbb{N}. \tag{3.46}$$

The probability Haar measure χ_n is regular on $U(n)$, and the complement $U(n) \setminus U'(n)$ is a negligible set. Hence, if K_n runs over all compact sets in $U'(n)$, then

$$\sup_{K_n \subset U'(n)} \chi_n(K_n) = 1.$$

Therefore, for every $\varepsilon > 0$ there exists a compact set $K_n \subset U'(n)$ such that

$$\chi_n(K_n) \geq 1 - \varepsilon.$$

From (3.46) it follows that for every $\varepsilon > 0$ the compact set $\mathcal{K} = (K_m)$ satisfies the hypothesis of Prohorov's criterion:

$$(\chi_m \circ \pi_m)(\mathcal{K}) = \chi_m(K_m) \geq 1 - \varepsilon \quad \text{for all} \quad m \in \mathbb{N}.$$

So, in view of this criterion, there exists a unique Radon probability measure χ' on $\mathfrak{U}'$ which satisfies the condition (3.45). However, on the projective limits

$$\mathfrak{U}' = \varprojlim U'(m)$$

there exists a unique $U^2(\infty)$-invariant Radon measure χ, determined by the equality (3.41). Using the uniqueness property of projective limits, we obtain

$$\chi' = \chi.$$

The measure χ on $\mathfrak{U} \setminus \mathfrak{U}'$ is defined to be zero, because χ_m is zero on $U(m) \setminus U'(m)$.

As a consequence of (3.46), we obtain (3.44), because

$$\chi(\mathcal{K}) = \inf_{m \in \mathbb{N}} \chi_m(K_m) = \chi_n(K_n).$$

By Lemma 3.27 the measure χ is $U^2(\infty)$-invariant under the right actions (3.42) on the space $\mathfrak{U}$. Hence, for every $f \in \mathcal{L}_\chi^\infty$, the equality (3.29) holds. □

Consider in the space $\mathcal{L}_\chi^\infty$ the group of shifts

$$Q_g f(u) = f(u.g), \qquad g \in U^2(\infty) \quad u \in \mathfrak{U}$$

generated by the right action of $U^2(\infty)$ over $\mathfrak{U}$. Choosing instead of $U(\infty)$ a compact subgroup $U(m)$ or the compact subgroups

$$U_0 = \left\{g_0(\vartheta) = \exp(i\vartheta) : \vartheta \in (-\pi, \pi]\right\},$$

and

$$U_j(m) = \left\{g_{mj}(\vartheta) = \mathbb{1}_{j-1} \otimes \exp(i\vartheta) \otimes \mathbb{1}_{m-j} : \vartheta \in (-\pi, \pi]\right\},$$
$$j = 1, \ldots, m,$$

we obtain the corresponding subgroups of shifts Q_g with elements $g \in U^2(m)$ or with elements $g_0(\vartheta) \in U_0^2$ and $g_{mj}(\vartheta) \in U_j^2(m)$, respectively. The following lemma is an infinite-dimensional generalization of integral formulas from Subsection 3.2.1.

LEMMA 3.30. *For any $f \in \mathcal{L}_\chi^\infty$ the following equalities:*

$$\int_{\mathfrak{U}} f\,d\chi = \int_{\mathfrak{U}} d\chi(u) \int_{U^2(m)} Q_g f(u)\,d(\chi_m \otimes \chi_m)(g), \tag{3.47}$$

$$\int_{\mathfrak{U}} f\,d\chi = \frac{1}{2\pi}\int_{\mathfrak{U}} d\chi(u) \int_{-\pi}^{\pi} Q_{g(\vartheta)} f(u)\,d\vartheta \tag{3.48}$$

with $g(\vartheta) \in U_0^2$ or $g(\vartheta) \in U_j^2(m)$ hold.

PROOF. For any $f \in \mathcal{L}_\chi^\infty$, the function

$$(u,g) \longmapsto Q_g f(u) = f(u.g)$$

is integrable on the Cartesian product $\mathfrak{U} \times U^2(m)$. Applying a standard argument based on the Fubini theorem, we obtain

$$\int_{\mathfrak{U}} d\chi(u) \int_{U^2(m)} Q_g f(u)\,d(\chi_m \otimes \chi_m)(g) =$$

$$= \int_{U^2(m)} d(\chi_m \otimes \chi_m)(g) \int_{\mathfrak{U}} Q_g f(u)\,d\chi(u).$$

This equality yields the required formula (3.47), because the internal integral on the right-hand side is independent of g and

$$\int_{U^2(m)} d(\chi_m \otimes \chi_m) = 1.$$

In turn, putting instead of $U(m)$ the subgroups U_0 and $U_j(m)$, respectively, we obtain equalities (3.48). □

3.3.2. Homogeneous Hilbert-Schmidt polynomials. Consider the countable orthogonal Hilbertian sum

$$\mathsf{E} := \bigoplus_{m\in\mathbb{N}} \mathbb{C}^m = \left\{ x = (x_m) : x_m \in \mathbb{C}^m,\ \|x\|_{\mathsf{E}} = \Big(\sum_{m\in\mathbb{N}} \|x_m\|_{\mathbb{C}^m}^2\Big)^{1/2} < \infty \right\}$$

with the scalar product

$$\langle x \mid y\rangle_{\mathsf{E}} = \sum_{m\in\mathbb{N}} \langle x_m \mid y_m\rangle_{\mathbb{C}^m}\,,$$

where every coordinate $x_m \in \mathbb{C}^m$ is identified with its image

$$(0,\ldots,0,x_m,0,\ldots) \in \mathsf{E}$$

under the embedding $\mathbb{C}^m \hookrightarrow \mathsf{E}$.

As usually, let $\otimes_{\mathfrak{h}}^n \mathsf{E}$ stand for the complete nth tensor power of the Hilbert subspace E, endowed with the Hilbertian norm,

$$\|\psi_n\|_{\otimes_{\mathfrak{h}}^n \mathsf{E}} = \langle \psi_n \mid \psi_n\rangle_{\otimes_{\mathfrak{h}}^n \mathsf{E}}^{1/2},$$

where $\psi_n = \sum_j y_{1j} \otimes \ldots \otimes y_{nj}$ denotes a finite sum with $y_{1j} \otimes \ldots \otimes y_{nj} \in \otimes_{\mathfrak{h}}^n \mathsf{E}$ and with $x_{tj}, y_{tj} \in \mathsf{E}$ for all $t = 1, \ldots, n$. Put $\otimes_{\mathfrak{h}}^0 \mathsf{E} = \mathbb{C}$.

Replacing the space E by the subspace $\mathbb{C}^m$, we similarly define the tensor product $\otimes_{\mathfrak{h}}^n \mathbb{C}^m$. There is the unitary embedding

$$\otimes_{\mathfrak{h}}^n \mathbb{C}^m \hookrightarrow \otimes_{\mathfrak{h}}^n \mathsf{E}.$$

If $m = 1$ then $\otimes_{\mathfrak{h}}^n \mathbb{C} = \mathbb{C}$.

For any finite sum $\psi_n = \sum_j y_{1j} \otimes \ldots \otimes y_{nj}$ from the space $\otimes_{\mathfrak{h}}^n \mathbb{C}^m$ (or $\otimes_{\mathfrak{h}}^n \mathsf{E}$) we can to define the finite type n-homogeneous Hilbert-Schmidt polynomials

$$\mathbb{C}^m \ni x \longmapsto \langle x^{\otimes n} \mid \psi_n \rangle_{\otimes_{\mathfrak{h}}^n \mathbb{C}^m} = \sum_j \prod_{t=1}^n \langle x \mid y_{tj} \rangle_{\mathbb{C}^m} . \tag{3.49}$$

Consider the canonical orthonormal bases in $\mathbb{C}^m$ and E, respectively,

$$\mathscr{E}(\mathbb{C}^m) = \{\mathfrak{e}_{m1}, \ldots, \mathfrak{e}_{mm}\},$$
$$\mathscr{E}(\mathsf{E}) = \bigcup \{\mathscr{E}(\mathbb{C}^m) : m \in \mathbb{N}\},$$

where $\mathfrak{e}_{ml} = \underbrace{(\overbrace{0, \ldots, 0, 1}^{l}, 0, \ldots, 0)}_{m}$.

Also consider the nth symmetric tensor powers $\odot_{\mathfrak{h}}^n \mathbb{C}^m$ and $\odot_{\mathfrak{h}}^n \mathsf{E}$. Clearly, $\odot_{\mathfrak{h}}^n \mathbb{C}^m$ is a closed subspace in $\odot_{\mathfrak{h}}^n \mathsf{E}$.

Given a pair of numbers $(m, n) \in \mathbb{N} \times \mathbb{Z}_+$, we consider the n-fold tensor power of the canonical mapping $\pi_m : \mathfrak{U} \ni u \longmapsto \pi_m(u) \in U(m)$,

$$\mathfrak{U} \ni u \longmapsto \pi_m^{\otimes n}(u) \in \mathscr{L}\left(\odot_{\mathfrak{h}}^n \mathbb{C}^m\right) \tag{3.50}$$

where

$$\pi_m^{\otimes n}(u) := \underbrace{\pi_m(u) \otimes \ldots \otimes \pi_m(u)}_{n}.$$

If $n = 0$, we put $\pi_m^{\otimes 0}(u) = 1$ for all $u \in \mathfrak{U}$ and $m \in \mathbb{N}$. The mapping (3.50) is Borel and has a continuous restriction to $\mathfrak{U}'$, because π_m has the same property.

Let $\mathfrak{a}_m \in \mathbb{C}^m$ be an arbitrary fixed element such that $\|\mathfrak{a}_m\|_{\mathbb{C}^m} = 1$. Then, $\mathfrak{a}_m^{\otimes n} \in \odot_{\mathfrak{h}}^n \mathbb{C}^m$. Using the mapping (3.50), we can write

$$\left[\pi_m^{\otimes n}(u)\right]\left(\mathfrak{a}_m^{\otimes n}\right) = \underbrace{[\pi_m(u)](\mathfrak{a}_m) \otimes \ldots \otimes [\pi_m(u)](\mathfrak{a}_m)}_{n}.$$

To any n-homogeneous Hilbert-Schmidt polynomial (3.49) with $y_{1j} \otimes \ldots \otimes y_{nj} \in \otimes_{\mathfrak{h}}^n \mathbb{C}^m$, there corresponds the function

$$\psi_n^*(u) := \left\langle \left[\pi_m^{\otimes n}(u)\right]\left(\mathfrak{a}_m^{\otimes n}\right) \mid \psi_n \right\rangle_{\otimes_{\mathfrak{h}}^n \mathbb{C}^m} = \sum_j \prod_{t=1}^n \left\langle [\pi_m(u)](\mathfrak{a}_m) \mid y_{tj} \right\rangle_{\mathbb{C}^m}$$

of the variable $u \in \mathfrak{U}$. Any cylindrical function of the form

$$\mathfrak{U} \ni u \longmapsto \left\langle [\pi_m(u)](\mathfrak{a}_m) \mid y_{tj} \right\rangle_{\mathbb{C}^m}$$

has a continuous bounded restriction to $\mathfrak{U}'$. Therefore, it is χ-essentially bounded on $\mathfrak{U}$, because $\mathfrak{U}\setminus\mathfrak{U}'$ is a χ-negligible set. Consequently, $\psi_n^* \in L_\chi^\infty$ and $\psi_n^*|_{\mathfrak{U}'}$ is continuous and bounded.

DEFINITION 3.31. We define $\mathscr{P}_\mathfrak{h}^n(\mathbb{C}^m)$ to be the space of all functions ψ_n^* of the matrix variable $u \in \mathfrak{U}$, determined by finite type n-homogeneous Hilbert-Schmidt polynomials (3.49).

LEMMA 3.32. *For any element $\mathfrak{a}_m \in \mathbb{C}^m$ such that $\|\mathfrak{a}_m\|_{\mathbb{C}^m} = 1$ the set*

$$\mathsf{S}^m = \{x = [\pi_m(u)]\,(\mathfrak{a}_m) : \ u \in \mathfrak{U}\}$$

coincides with the unit sphere in $\mathbb{C}^m$. As a consequence, the one-to-one anti-linear correspondence

$$\odot_\mathfrak{h}^n\mathbb{C}^m \ni \psi_n \rightleftarrows \psi_n^* \in \mathscr{P}_\mathfrak{h}^n(\mathbb{C}^m) \tag{3.51}$$

holds, and any function ψ_n^* is independent of the choice of an element $\mathfrak{a}_m \in \mathsf{S}^m$.

PROOF. Suppose, on the contrary, that there is an element $\psi_n \in \odot_\mathfrak{h}^n\mathbb{C}^m$ such that $\langle x^{\otimes n} \mid \psi_n\rangle_{\otimes_\mathfrak{h}^n\mathbb{C}^m} = 0$ for all $x = [\pi_m(u)]\,(\mathfrak{a}_m) \in \mathsf{S}^m$ with $u \in \mathfrak{U}$. The mapping

$$\pi_m \colon \mathfrak{U} \ni u \longmapsto \pi_m(u) \in U(m)$$

is surjective by Lemma 3.24. Hence, the set S^m coincides with the unit sphere in $\mathbb{C}^m$ and is independent on the choice of an element $\mathfrak{a}_m$. By n-homogeneity, we have $\langle x^{\otimes n} \mid \psi_n\rangle_{\otimes_\mathfrak{h}^n\mathbb{C}^m} = 0$ for all $x \in \mathbb{C}^m$.

Similarly as in Theorem 3.13, applying the polarization formula (1.6) for symmetric tensor products, we obtain

$$\langle z_1 \odot \cdots \odot z_n \mid \psi_n\rangle_{\otimes_\mathfrak{h}^n\mathbb{C}^m} = 0$$

for all elements $z_1, \ldots, z_n \in \mathbb{C}^m$. Hence $\psi_n = 0$, because the subset of all elements $z_1 \odot \cdots \odot z_n$ is total in $\odot_\mathfrak{h}^n\mathbb{C}^m$. As a consequence, the subset

$$\{x^{\otimes n} = [\pi_m^{\otimes n}(u)]\,(\mathfrak{a}_m^{\otimes n}) : u \in \mathfrak{U}\}$$

is also total in $\odot_\mathfrak{h}^n\mathbb{C}^m$. It immediately yields the correspondence (3.51). ☐

Consider the symmetric Fock space F and its closed subspace F_m, where

$$\mathsf{F} := \mathbb{C} \oplus \mathsf{E} \oplus (\odot_\mathfrak{h}^2\mathsf{E}) \oplus (\odot_\mathfrak{h}^3\mathsf{E}) \oplus \ldots,$$
$$\mathsf{F}_m := \mathbb{C} \oplus \mathbb{C}^m \oplus (\odot_\mathfrak{h}^2\mathbb{C}^m) \oplus (\odot_\mathfrak{h}^3\mathbb{C}^m) \oplus \ldots.$$

We will use here the following notations:

$$(m) := (m1, \ldots, mm), \qquad k_{(m)} := (k_{m1}, \ldots, k_{mm}) \in \mathbb{Z}_+^m,$$
$$|k_{(m)}| := k_{m1} + \ldots + k_{mm}, \quad k_{(m)}! := k_{m1}! \cdot \ldots \cdot k_{mm}!.$$

As is well-known, the system of symmetric tensor elements, indexed by the set $k_{(m)}$,

$$\mathscr{E}\,(\odot_\mathfrak{h}^n\mathbb{C}^m) = \left\{\mathfrak{e}_{(m)}^{\otimes k_{(m)}} = \mathfrak{e}_{m1}^{\otimes k_{m1}} \odot \ldots \odot \mathfrak{e}_{mm}^{\otimes k_{mm}} : k_{(m)} \in \mathbb{Z}_+^m;\ |k_{(m)}| = n\right\}$$

forms an orthogonal basis in the subspace $\odot_{\mathfrak{h}}^{n}\mathbb{C}^{m} \subset \mathsf{F}_{m}$. We will also use the notations

$$
\begin{aligned}
&[m] := \{(11),(21,22)\ldots,(m1,\ldots,mm)\},\\
&\{k\} := \{k_{(1)},\ldots,k_{(m)}\} \in \bigtimes_{r=1}^{m}\mathbb{Z}_{+}^{r},\\
&|\{k\}| := |k_{(1)}| + \ldots + |k_{(m)}|,\\
&\{k\}! := k_{(1)}!\cdot\ldots\cdot k_{(m)}!.
\end{aligned}
$$

Then, the system of symmetric tensor elements with a fixed n, indexed by the sets $[m]$ and $\{k\}$,

$$
\begin{aligned}
\mathscr{E}_{n} &= \bigcup_{m\in\mathbb{N}}\Big\{\mathfrak{e}_{[m]}^{\otimes\{k\}} = \mathfrak{e}_{(1)}^{\otimes k_{(1)}}\otimes\ldots\otimes\mathfrak{e}_{(m)}^{\otimes k_{(m)}}:\\
&\mathfrak{e}_{(1)}^{\otimes k_{(1)}} \in \mathscr{E}\left(\odot_{\mathfrak{h}}^{|k_{(1)}|}\mathbb{C}\right),\ldots,\mathfrak{e}_{(m)}^{\otimes k_{(m)}} \in \mathscr{E}\left(\odot_{\mathfrak{h}}^{|k_{(m)}|}\mathbb{C}^{m}\right);\ |\{k\}| = n\Big\}
\end{aligned}
$$

forms an orthogonal basis in the subspace $\odot_{\mathfrak{h}}^{n}\mathsf{E} \subset \mathsf{F}$. Thus, the system

$$\mathscr{E} = \{\mathscr{E}_{n}:\ n\in\mathbb{Z}_{+}\}$$

forms an orthogonal basis in the symmetric Fock space F.

By virtue of the one-to-one mapping (3.51), the system of symmetric tensor elements $\mathscr{E}\left(\odot_{\mathfrak{h}}^{n}\mathbb{C}^{m}\right)$ uniquely defines the following corresponding system:

$$\mathscr{E}_{m,n}^{*} \subset \mathscr{P}_{\mathfrak{h}}^{n}(\mathbb{C}^{m})$$

of χ_{m}-integrable cylindrical functions

$$
\begin{aligned}
\mathfrak{e}_{(m)}^{*k_{(m)}}(u) &:= \left\langle\left[\pi_{m}^{\otimes n}(u)\right]\left(\mathfrak{e}_{m1}^{\otimes n}\right)\mid\mathfrak{e}_{(m)}^{\otimes k_{(m)}}\right\rangle_{\otimes_{\mathfrak{h}}^{n}\mathbb{C}^{m}}\\
&= \prod_{r=1}^{m}\left\langle(\pi_{m}\circ u)(\mathfrak{e}_{m1})\mid\mathfrak{e}_{mr}\right\rangle_{\mathbb{C}^{m}}^{k_{mr}}
\end{aligned}
\tag{3.52}
$$

of the variable $u\in\mathfrak{U}$, where we take

$$\mathfrak{a}_{m} = \mathfrak{e}_{m1}.$$

Consider the system of functions of the variable $u\in\mathfrak{U}$,

$$
\begin{aligned}
\mathscr{E}_{n}^{*} &= \bigcup_{m\in\mathbb{N}}\Big\{\mathfrak{e}_{[m]}^{*\{k\}} = \mathfrak{e}_{(1)}^{*k_{(1)}}\cdot\ldots\cdot\mathfrak{e}_{(m)}^{*k_{(m)}}:\\
&\mathfrak{e}_{(1)}^{*k_{(1)}} \in \mathscr{E}_{1,|k_{(1)}|}^{*},\ldots,\mathfrak{e}_{(m)}^{*k_{(m)}} \in \mathscr{E}_{m,|k_{(m)}|}^{*}\ \text{with fixed}\ |\{k\}| = n\Big\},
\end{aligned}
$$

generated by the system of symmetric tensor elements $\mathscr{E}_{n}$. All these functions belong to the space $\mathcal{L}_{\chi}^{\infty}$ by their definition. Denote

$$\mathscr{E}^{*} = \{\mathscr{E}_{n}^{*}:\ n\in\mathbb{Z}_{+}\},\qquad \mathscr{E}_{m}^{*} = \{\mathscr{E}_{m,n}^{*}:\ n\in\mathbb{Z}_{+}\}.$$

3.3.3. Hardy-type space. Let L^2_χ be the space of square χ-integrable complex functions f defined on the space of virtual matrices $\mathfrak{U}$. Since χ is a probability measure, the embedding $\mathcal{L}^\infty_\chi \subset L^2_\chi$ holds and

$$\|f\|_{L^2_\chi} \leq \operatorname*{ess\,sup}_{u\in\mathfrak{U}} |f(u)|, \qquad f \in \mathcal{L}^\infty_\chi.$$

Denote by $\mathcal{H}^2_{\chi_m}$ the L^2_χ-closure of complex linear spans of the subsystem $\mathscr{E}^*_m$.

LEMMA 3.33 ([88, Theorem 5.6.8]). *The space $\mathcal{H}^2_{\chi_m}$ is isomorphic to the classic Hardy space $\mathcal{H}^2_{\chi_m}(\mathrm{B}^m)$ of analytic complex functions on the open unit ball*

$$\mathrm{B}^m = \{x_m \in \mathbb{C}^m : \|x_m\|_{\mathbb{C}^m} < 1\}.$$

DEFINITION 3.34. The Hardy-type space $\mathcal{H}^2_\chi$ on the space of virtual unitary matrices $\mathfrak{U}$ is defined to be the L^2_χ-closure of the complex linear span of the system of cylindrical functions $\mathscr{E}^*$.

THEOREM 3.35. *The system $\mathscr{E}^*$ of all cylindrical functions*

$$\mathfrak{e}^{*\{k\}}_{[m]} = \mathfrak{e}^{*k(1)}_{(1)} \cdot \ldots \cdot \mathfrak{e}^{*k(m)}_{(m)}$$

with $m \in \mathbb{N}$, such that $\mathfrak{e}^{*k(r)}_{(r)} \in \mathscr{E}^*_{r,|k_{(r)}|}$ as $r = 1, \ldots, m$, forms an orthogonal basis in the Hardy-type spaces $\mathcal{H}^2_\chi$ with norms

$$\left\|\mathfrak{e}^{*\{k\}}_{[m]}\right\|_{L^2_\chi} = \left(\prod_{r=1}^m \frac{(r-1)!(k)_r!}{(r-1+|(k)_r|)!}\right)^{1/2}.$$

PROOF. If $|\{k\}| \neq |\{q\}|$, then from (3.55) it follows

$$\int_{\mathfrak{U}} \mathfrak{e}^{*\{k\}}_{[m]} \cdot \bar{\mathfrak{e}}^{*\{q\}}_{[n]}\, d\chi = \int_{\mathfrak{U}} \mathfrak{e}^{*\{k\}}_{[m]}\left(\exp(i\vartheta)u\right) \cdot \bar{\mathfrak{e}}^{*\{q\}}_{[n]}\left(\exp(i\vartheta)u\right) d\chi(u)$$
$$= \frac{1}{2\pi}\int_{\mathfrak{U}} \mathfrak{e}^{*\{k\}}_{[m]} \cdot \bar{\mathfrak{e}}^{*\{q\}}_{[n]}\, d\chi \int_{-\pi}^{\pi} \exp\left(i(|\{k\}| - |\{q\}|)\vartheta\right) d\vartheta = 0.$$

So, $\mathfrak{e}^{*\{k\}}_{[m]} \perp \mathfrak{e}^{*\{q\}}_{[n]}$ in the space L^2_χ if $|\{k\}| \neq |\{q\}|$ for all indices $[m], [n]$.

Let $|\{k\}| = |\{q\}|$ and $m > n$ for definiteness. If the elements $\mathfrak{e}^{*\{k\}}_{[m]}$ and $\mathfrak{e}^{*\{q\}}_{[n]}$ are different, then there exists a sub-index

$$ms \in \{11, 21, 22, \ldots, m1, \ldots, mm\}$$

in the block-index $[m] = [(11), (21, 22), \ldots, (m1, \ldots, mm)]$ such that

$$ms \notin \{11, 21, 22, \ldots, n1, \ldots, nn\},$$

where $[n] = [(11), (21, 22) \ldots, (n1, \ldots, nn)]$. The formula (3.48) implies that for the group of shifts $Q_{g_{ms}(\vartheta)}$ generated by elements $g_{ms}(\vartheta) \in U^2_s(m)$ with the sub-index ms,

$$\int_{\mathfrak{U}} \mathfrak{e}^{*\{k\}}_{[m]} \cdot \bar{\mathfrak{e}}^{*\{q\}}_{[n]}\, d\chi = \int_{\mathfrak{U}} Q_{g_{ms}(\vartheta)}\mathfrak{e}^{*\{k\}}_{[m]} \cdot Q_{g_{ms}(\vartheta)}\bar{\mathfrak{e}}^{*\{q\}}_{[n]}\, d\chi$$
$$= \frac{1}{2\pi}\int_{\mathfrak{U}} \mathfrak{e}^{*\{k\}}_{[m]} \cdot \bar{\mathfrak{e}}^{*\{q\}}_{[n]}\, d\chi \int_{-\pi}^{\pi} \exp\left(ik_{ms}\vartheta\right) d\vartheta = 0.$$

Hence, $\mathfrak{e}^{*\{k\}}_{[m]} \perp \mathfrak{e}^{*\{q\}}_{[n]}$ in L^2_χ.

Let now $|\{k\}| = |\{q\}|$ and $m = n$. If $\mathfrak{e}^{*\{k\}}_{[m]} \neq \mathfrak{e}^{*\{q\}}_{[n]}$, then $\{k\} \neq \{q\}$. Hence, there exists a sub-index rs in the block-index $[m] = [n]$ such that $k_{rs} \neq q_{rs}$. Similarly as previous mentioned, applying the formula (3.48) to the group of shifts $Q_{g_{rs}(\vartheta)}$ generated by elements $g_{rs}(\vartheta) \in U^2_s(r)$ with the sub-index rs, we get

$$\int_{\mathfrak{U}} \mathfrak{e}^{*\{k\}}_{[m]} \cdot \bar{\mathfrak{e}}^{*\{q\}}_{[n]}\, d\chi = \frac{1}{2\pi} \int_{\mathfrak{U}} \mathfrak{e}^{*\{k\}}_{[m]} \cdot \bar{\mathfrak{e}}^{*\{q\}}_{[n]}\, d\chi \int_{-\pi}^{\pi} \exp\left(\mathrm{i}\,(k_{rs} - q_{rs})\,\vartheta\right)\, d\vartheta = 0.$$

Hence, in this case also $\mathfrak{e}^{*\{k\}}_{[m]} \perp \mathfrak{e}^{*\{q\}}_{[n]}$ under the measure χ.

Let $g_r = (\mathbb{1}_r, w_r) \in U^2(r)$ and $u \in \mathfrak{U}$. Using (3.42) and (3.52), we have

$$\int_{U^2(r)} Q_{g_r} \left|\mathfrak{e}^{*(k)_r}_{(r)}\right|^2 (u)\, d(\chi_r \otimes \chi_r)(g_r) =$$

$$= \int_{U(r)} \prod_{l=1}^{r} \left| \left\langle [w_r^{-1} \pi_r(u)]\,(\mathfrak{e}_{r1}) \;\middle|\; \mathfrak{e}_{rl} \right\rangle_{\mathbb{C}^r}^{k_{rl}} \right|^2 d\chi_r(w_r).$$

However, the previous integral with the Haar measure χ_r is independent of elements $\pi_r(u) \in U(r)$. It follows that

$$\int_{U^2(r)} Q_{g_r} \left|\mathfrak{e}^{*(k)_r}_{(r)}\right|^2 (u)\, d(\chi_r \otimes \chi_r)(g_r) = \int_{U(r)} \prod_{l=1}^{r} \left| \left\langle w_r^{-1}\,(\mathfrak{e}_{r1}) \;\middle|\; \mathfrak{e}_{rl} \right\rangle_{\mathbb{C}^r}^{k_{rl}} \right|^2 d\chi_r(w_r)$$

$$= \frac{(r-1)!(k)_r!}{(r-1+|(k)_r|)!} = \left\|\mathfrak{e}^{*(k)_r}_{(r)}\right\|^2_{L^2_{\chi_r}}$$

by the well-known formula [88, 1.4.9]. Using the formula (3.47) m-times for $r = 1, \dots, m$, we get

$$\int_{\mathfrak{U}} \left|\mathfrak{e}^{*\{k\}}_{[m]}\right|^2 d\chi = \int_{\mathfrak{U}} d\chi(u) \prod_{r=1}^{m} \int_{U^2(r)} Q_{g_r} \left|\mathfrak{e}^{*(k)_r}_{(r)}\right|^2 (u)\, d(\chi_r \otimes \chi_r)(g_r)$$

$$= \prod_{r=1}^{m} \left\|\mathfrak{e}^{*k_{(r)}}_{(r)}\right\|^2_{L^2_{\chi_r}},$$

because $\int_{\mathfrak{U}} d\chi = 1$. It follows that

$$\left\|\mathfrak{e}^{*\{k\}}_{[m]}\right\|^2_{L^2_\chi} = \prod_{r=1}^{m} \left\|\mathfrak{e}^{*k_{(r)}}_{(r)}\right\|^2_{L^2_{\chi_r}} = \prod_{r=1}^{m} \frac{(r-1)!(k)_r!}{(r-1+|(k)_r|)!}$$

for all basic elements $\mathfrak{e}^{*\{k\}}_{[m]} = \mathfrak{e}^{*k_{(m)}}_{(1)} \cdot \ldots \cdot \mathfrak{e}^{*k_{(m)}}_{(m)}$. □

As is known, the system $\mathscr{E}_m$ of symmetric tensors $\mathfrak{e}^{\otimes(k)_m}_{(m)}$ with a fixed m forms an orthogonal basis in the space F_m with norms

$$\left\|\mathfrak{e}^{\otimes(k)_m}_{(m)}\right\|_{\mathsf{F}_m} = \left(\frac{(k)_m!}{|(k)_m|!}\right)^{1/2}.$$

Similarly, the system $\mathscr{E}$ of all symmetric tensors

$$\mathfrak{e}_{[m]}^{\otimes\{k\}} = \mathfrak{e}_{(1)}^{\otimes(k)_1} \odot \ldots \odot \mathfrak{e}_{(m)}^{\otimes k_{(m)}}$$

with all $m \in \mathbb{N}$, such that $\mathfrak{e}_{(r)}^{\otimes(k)_r} \in \mathscr{E}_{r,|(k)_r|}$ as $r = 1, \ldots, m$, forms an orthogonal basis in the symmetric Fock space F with norms

$$\left\|\mathfrak{e}_{[m]}^{\otimes\{k\}}\right\|_{\mathsf{F}} = \left(\frac{\{k\}!}{|\{k\}|!}\right)^{1/2}.$$

Combining Lemmas 3.32 – 3.33 and Theorems 3.35 , we obtain the following.

THEOREM 3.36. *Anti-linear extensions of the one-to-one mappings between orthonormal bases*

$$\frac{\mathfrak{e}_{(m)}^{\otimes(k)_m}}{\|\mathfrak{e}_{(m)}^{\otimes(k)_m}\|_{\mathsf{F}_m}} \rightleftarrows \frac{\mathfrak{e}_{(m)}^{*(k)_m}}{\|\mathfrak{e}_{(m)}^{*(k)_m}\|_{L^2_{\chi_m}}},$$
$$\frac{\mathfrak{e}_{[m]}^{\otimes\{k\}}}{\|\mathfrak{e}_{[m]}^{\otimes\{k\}}\|_{\mathsf{F}}} \rightleftarrows \frac{\mathfrak{e}_{[m]}^{*\{k\}}}{\|\mathfrak{e}_{[m]}^{*\{k\}}\|_{L^2_\chi}}$$

uniquely define the corresponding anti-linear isometric isomorphisms

$$\mathsf{F}_m \simeq \mathcal{H}^2_{\chi_m}(\mathsf{B}^m), \qquad \mathsf{F} \simeq \mathcal{H}^2_\chi.$$

Theorem directly implies several corollaries. From (3.41) and (3.52) it follows that

$$\left\|\mathfrak{e}_{(m)}^{*(k)_m}\right\|^2_{L^2_{\chi_m}} = \int_{U(m)} \left|\mathfrak{e}_{(m)}^{*(k)} \circ \pi_m^{-1}\right|^2 d\chi_m = \int_{\mathfrak{U}} \left|\mathfrak{e}_{(m)}^{*(k)}\right|^2 d\chi = \left\|\mathfrak{e}_{(m)}^{*(k)_m}\right\|^2_{L^2_\chi}.$$

Hence, the following isometrical embeddings

$$\mathcal{H}^2_{\chi_m}(\mathsf{B}^m) \hookrightarrow \mathcal{H}^2_\chi, \qquad \bigoplus_{m\in\mathbb{N}} \mathcal{H}^2_{\chi_m}(\mathsf{B}^m) \hookrightarrow \mathcal{H}^2_\chi$$

hold.

Reasoning by analogy with the proof of Theorem 3.10, it is easy to show that the Hardy space $\mathcal{H}^2_\chi$ possesses the reproducing kernel of a Cauchy type

$$\begin{aligned}\mathfrak{C}(v,u) &= \sum_{n\in\mathbb{Z}_+} \sum_{|\{k\}|=n} \frac{\mathfrak{e}_{[m]}^{*\{k\}}(v)\bar{\mathfrak{e}}_{[m]}^{*\{k\}}(u)}{\|\mathfrak{e}_{[m]}^{*\{k\}}\|^2_{L^2_\chi}} \\ &= \prod_{m=1}^{\infty} \Big(1 - \big\langle(\pi_m \circ v)(\mathfrak{e}_{m1}) \mid (\pi_m \circ u)(\mathfrak{e}_{m1})\big\rangle_{\mathsf{E}}\Big)^{-m}\end{aligned}$$

with $u, v \in \mathfrak{U}$, where the sum $\sum_{|\{k\}|=n}$ is over all indices

$$\{k\} \in \left\{ \bigtimes_{r=1}^{m} \mathbb{Z}_+^r : m \in \mathbb{N} \right\}$$

such that $|\{k\}| = n$. As a consequence, the integral representation of any function $f \in \mathcal{H}^2_\chi$,

$$f(\lambda v) = \int_{\mathfrak{U}} f(u)\mathfrak{C}(\lambda v, u)\, d\chi(u)$$

gives a unique analytic extension in the complex variable $\lambda \in \mathsf{B}^1$ for all elements $v \in \mathfrak{U}$ such that

$$\sum_{m\in\mathbb{N}} m\,\|(\pi_m \circ v)(\mathfrak{e}_{m1})\|^2_{\mathbb{C}^m} < \infty.$$

Moreover, there exists the isomorphism

$$\mathcal{H}^2_\chi \ni \{f(u) \colon u \in \mathfrak{U}\} \in\!\!\longmapsto \{f_\lambda(v) \colon v \in \mathfrak{U}\} \in \mathcal{H}^2_\chi(\mathsf{B}^1)$$

for all $\lambda \in \mathsf{B}^1$ and elements $v \in \mathfrak{U}$ such that

$$\sum_{m\in\mathbb{N}} m\,\|(\pi_m \circ v)(\mathfrak{e}_{m1})\|^2_{\mathbb{C}^m} < \infty.$$

3.4. Hardy spaces on irreducible orbits of locally compact groups

3.4.1. Polynomial orthogonal systems on irreducible orbits. Let E stand for a complex separable Hilbert space and let $\mathcal{G}$ stand for a locally compact second countable group. Suppose that there exists a unitary representation

$$U \colon \mathcal{G} \ni x \longmapsto U_x \in \mathscr{L}(\mathsf{E}),$$

which is weakly continuous.

Fix an element $\hbar \in \mathsf{E}$ with $\|\hbar\|_{\mathsf{E}} = 1$ and consider its orbit

$$\mathsf{G} := \{U_x\hbar = \zeta \in \mathsf{E} \colon x \in \mathcal{G}\},$$

which as a topological space we identify with the factor-space $\mathcal{G}/\mathcal{G}_\hbar$, where $\mathcal{G}_\hbar := \{v \in \mathcal{G} \colon U_v\hbar = \hbar\}$. The closed unit ball in E endowed with the weak topology, we will denote by K. The weak continuity of U implies that the embedding $\mathsf{G} \hookrightarrow \mathsf{K}$ is continuous. Further we denote by $C(\mathsf{K})$ the uniform algebra of continuous complex functions on K.

Recall that a measure χ on the orbit G is $\mathcal{G}$-invariant if for any $x \in \mathcal{G}$ its shift $\chi \circ U_{x^{-1}}$ is equal to χ, i.e., if

$$\int_{\mathsf{G}} f(\zeta)\, d\chi(U_{x^{-1}}\zeta) = \int_{\mathsf{G}} (f \circ U_x)(\zeta)\, d\chi(\zeta) = \int_{\mathsf{G}} f(\zeta)\, d\chi(\zeta) \tag{3.53}$$

for all $x \in \mathcal{G}$ and χ-integrable complex function f on G. As is well known (see [66]), for any locally compact second countable group $\mathcal{G}$ an invariant measure χ on an orbit G exists and the equality

$$\int_{\mathsf{G}} d\chi(U_x\hbar) \int_{\mathcal{G}_\hbar} \varphi(xv)\, dv = \int_{\mathcal{G}} \varphi(x)\, dx \tag{3.54}$$

uniquely connects it with a Haar measure dx on $\mathcal{G}$. Here φ is any integrable complex function on $\mathcal{G}$ and dv denotes a Haar measure on $\mathcal{G}_\hbar$. Clearly, the choice of a $\mathcal{G}$-invariant measure χ on G depends on the element $\hbar \in \mathsf{S}$.

In what follows we suppose that an element $\hbar \in \mathsf{S}$ and the corresponding $\mathcal{G}$-invariant measure χ on its orbit G are fixed, and that the representation codomain $U_{\mathcal{G}}$ of a group $\mathcal{G}$ contains the complex cyclic subgroup $\mathbb{T}$. Let L^2_χ stand for the Hilbert space of all quadratically χ-integrable complex functions on the orbit G.

First we recall an auxiliary integral formulas which follow from Fubini's theorem and our assumptions about the group. Namely, if χ is a $\mathcal{G}$-invariant measure on G then the equalities

$$\int_{\mathsf{G}} f\,d\chi = \frac{1}{2\pi}\int_{\mathsf{G}} d\chi(\zeta)\int_{-\pi}^{\pi} f(e^{i\vartheta}\zeta)\,d\vartheta, \qquad \int_{\mathsf{G}} \zeta\ d\chi(\zeta) = 0 \tag{3.55}$$

with χ-integrable complex function f on G hold.

As previously, $\otimes^n_{\mathfrak{h}}\mathsf{E}$ and $\odot^n_{\mathfrak{h}}\mathsf{E}$, $(n \in \mathbb{N})$ denote the complete nth tensor and symmetric tensor Hilbertian power of E, respectively. The corresponding symmetric Fock space is defined to be $\mathsf{F} = \mathbb{C} \oplus \mathsf{E} \oplus (\odot^2_{\mathfrak{h}}\mathsf{E}) \oplus (\odot^3_{\mathfrak{h}}\mathsf{E}) \oplus \ldots$. We use

$$\mathsf{E}^* = \{\zeta^* := \langle \cdot \mid \zeta\rangle_{\mathsf{E}} : \zeta \in \mathsf{E}\}$$

to denote the Hermitian dual space for E. Recall the isometries $(\otimes^n_{\mathfrak{h}}\mathsf{E})^* = \otimes^n_{\mathfrak{h}}\mathsf{E}^*$ and $(\odot^n_{\mathfrak{h}}\mathsf{E})^* = \odot^n_{\mathfrak{h}}\mathsf{E}^*$. So, for every element $\psi_n \in \odot^n_{\mathfrak{h}}\mathsf{E}$ uniquely assists the functional $\psi_n^* := \langle \cdot \mid \psi_n\rangle_{\mathsf{F}}$ belonging to $\odot^n_{\mathfrak{h}}\mathsf{E}^*$, which further we identify with the n-homogeneous Hilbert-Schmidt polynomial

$$\psi_n^* : \mathsf{E} \ni \xi \longmapsto \psi_n^*(\xi) := \langle \xi^{\otimes n} \mid \psi_n\rangle_{\mathsf{F}}.$$

For each n-homogeneous polynomial ψ_n^* with $\psi_n \in \odot^n_{\mathfrak{h}}\mathsf{E}$ we assign the polynomial function

$$\hbar_n[\psi_n](\zeta) = \delta_\zeta(\psi_n^*)$$

of the variable $\zeta = U_x\hbar$ with $x \in \mathcal{G}$, generated on the orbit G by all $\mathcal{G}$-shifts of the point evaluation character

$$\delta_\hbar(\psi_n^*) = \psi_n^*(\hbar).$$

In this section let $\mathcal{H}^2_\chi$ denote the closure in the space L^2_χ of all Hilbert-Schmidt polynomials over E and call the Hardy type space on the orbit G.

THEOREM 3.37. *Let an element $\hbar \in \mathsf{S}$ be fixed in such way that the antilinear operators*

$$\hbar_n : \odot^n_{\mathfrak{h}}\mathsf{E} \ni \psi_n \longmapsto \hbar_n[\psi_n] \in L^2_\chi, \qquad (n \in \mathbb{N})$$

are well defined and have the bounded norm $\|\hbar_n\| = \|\hbar_n\|_{\mathscr{L}(\odot^n_{\mathfrak{h}}\mathsf{E},\,L^2_\chi)}$, and let

$$\mathsf{E}^n_{\mathfrak{h}} := \odot^n_{\mathfrak{h}}\mathsf{E} \ominus \ker \hbar_n, \qquad \mathsf{F}_{\mathfrak{h}} := \mathbb{C} \oplus \mathsf{E}^1_{\mathfrak{h}} \oplus \mathsf{E}^2_{\mathfrak{h}} \oplus \mathsf{E}^3_{\mathfrak{h}} \oplus \ldots.$$

Then:

(i) *the corresponding restricted mapping*

$$\widehat{\hbar}_n : \mathsf{E}^n_{\mathfrak{h}} \ni \psi_n \longmapsto \widehat{\psi}_n := \widehat{\hbar}_n[\psi_n] \in L^2_\chi, \qquad \widehat{\hbar}_n := \frac{\hbar_n}{\|\hbar_n\|}.$$

is an isometry between the subspace $\mathsf{E}^n_{\mathfrak{h}}$ and its image $\mathcal{H}^2_n := \widehat{\hbar}_n(\mathsf{E}^n_{\mathfrak{h}})$, so

$$\int_{\mathsf{G}} \widehat{\psi}_n\,\overline{\widehat{\omega}}_n\,d\chi = \langle \omega_n \mid \psi_n\rangle_{\mathsf{F}}, \qquad \psi_n, \omega_n \in \mathsf{E}^n_{\mathfrak{h}}; \tag{3.56}$$

(ii) *the antilinear mapping*

$$\widehat{\hbar}\colon \mathsf{F}_\mathfrak{h} \ni \psi = \sum_{n\in\mathbb{Z}_+} \psi_n \longmapsto \widehat{\psi} := \sum_{n\in\mathbb{Z}_+} \widehat{\hbar}_n[\psi_n] \in \mathcal{H}^2_\chi, \qquad \widehat{\hbar} := (\widehat{\hbar}_n), \tag{3.57}$$

where $\widehat{\psi}_0 = \bar{\psi}_0$ *with* $\psi_0 \in \mathbb{C}$, *is an isometry between the subspace* $\mathsf{F}_\mathfrak{h}$ *and its image* $\mathcal{H}^2_\chi := \widehat{\hbar}(\mathsf{F}_\mathfrak{h})$, *so*

$$\int_\mathsf{G} \widehat{\psi}\,\overline{\widehat{\omega}}\, d\chi = \langle \omega \mid \psi\rangle_\mathsf{F}, \qquad \psi,\omega \in \mathsf{F}_\mathfrak{h}; \tag{3.58}$$

(iii) *the following orthogonal decomposition holds:*

$$\mathcal{H}^2_\chi = \mathbb{C} \oplus \mathcal{H}^2_1 \oplus \mathcal{H}^2_2 \oplus \mathcal{H}^2_3 \oplus \dots.$$

PROOF. Due to boundedness of $\hbar_n$ the following integral

$$\int_\mathsf{G} \hbar_n[\psi_n]\overline{\hbar_n[\omega_n]}\, d\chi = \int_\mathsf{G} (\psi_n^* \circ U_x)(\hbar)\overline{(\omega_n^* \circ U_x)(\hbar)}\, d\chi(U_x\hbar)$$

is an Hermitian continuous form on the Hilbert space $\odot^n_\mathfrak{h}\mathsf{E}$, which is antilinear by $\psi_n \in \odot^n_\mathfrak{h}\mathsf{E}$ and linear by $\omega_n \in \odot^n_\mathfrak{h}\mathsf{E}$. Therefore, there exists a bounded positive linear operator $A_n \in \mathscr{L}(\odot^n_\mathfrak{h}\mathsf{E})$ for which

$$\langle \omega_n \mid A_n\psi_n\rangle_\mathsf{F} = \int_\mathsf{G} \hbar_n[\psi_n]\overline{\hbar_n[\omega_n]}\, d\chi. \tag{3.59}$$

Similarly as in the proof of Theorem 3.13 from the $\mathcal{G}$-invariance properties (3.53) it follows that A_n commutates with all diagonal unitary representations of the form $\{U_y^{\otimes n} \in \mathscr{L}(\odot^n_\mathfrak{h}\mathsf{E}) : y \in \mathcal{G}\}$, i.e., the equality

$$A_n \circ U_y^{\otimes n} = U_y^{\otimes n} \circ A_n, \qquad y \in \mathcal{G}$$

holds, where $U_y^{\otimes n} = U_y \otimes \dots \otimes U_y$ denotes the nth tensor power. In fact, we have

$$\langle \omega_n \mid (A_n \circ U_y^{\otimes n})\psi_n\rangle_{\otimes^n_\mathfrak{h}\mathsf{E}} =$$

$$= \int_\mathsf{G} \langle (U_x\hbar)^{\otimes n} \mid U_y^{\otimes n}\psi_n\rangle_{\otimes^n_\mathfrak{h}\mathsf{E}} \overline{\langle (U_x\hbar)^{\otimes n} \mid \omega_n\rangle_{\otimes^n_\mathfrak{h}\mathsf{E}}}\, d\chi(U_x\hbar).$$

The $\mathcal{G}$-invariancy of the measure χ on G implies that

$$\int_\mathsf{G} \langle (U_x\hbar)^{\otimes n} \mid U_y^{\otimes n}\psi_n\rangle_{\otimes^n_\mathfrak{h}\mathsf{E}} \overline{\langle (U_x\hbar)^{\otimes n} \mid \omega_n\rangle_{\otimes^n_\mathfrak{h}\mathsf{E}}}\, d\chi(U_x\hbar) =$$

$$= \int_\mathsf{G} \langle (U_{y^{-1}x}\hbar)^{\otimes n} \mid \psi_n\rangle_{\otimes^n_\mathfrak{h}\mathsf{E}} \overline{\langle (U_{y^{-1}x}\hbar)^{\otimes n} \mid U_{y^{-1}}^{\otimes n}\omega_n\rangle_{\otimes^n_\mathfrak{h}\mathsf{E}}}\, d\chi(U_x\hbar)$$

$$= \int_\mathsf{G} \langle (U_x\hbar)^{\otimes n} \mid \psi_n\rangle_{\otimes^n_\mathfrak{h}\mathsf{E}} \overline{\langle (U_x\hbar)^{\otimes n} \mid U_{y^{-1}}^{\otimes n}\omega_n\rangle_{\otimes^n_\mathfrak{h}\mathsf{E}}}\, d\chi(U_x\hbar).$$

As a result, we obtain

$$\langle \omega_n \mid (A_n \circ U_y^{\otimes n})\psi_n\rangle_{\otimes^n_\mathfrak{h}\mathsf{E}} = \langle U_{y^{-1}}^{\otimes n}\omega_n \mid A_n\psi_n\rangle_{\otimes^n_\mathfrak{h}\mathsf{E}} = \langle \omega_n \mid (U_y^{\otimes n} \circ A_n)\psi_n\rangle_{\otimes^n_\mathfrak{h}\mathsf{E}}.$$

On the other hand, for any $n \in \mathbb{N}$ the set $\{(U_x\hbar)^{\otimes n} : x \in \mathcal{G}\}$ is total in the subspace $\mathsf{E}^n_\mathfrak{h}$ under its definition. Hence, the corresponding representations $U_y^{\otimes n}$ are irreducible over $\mathsf{E}^n_{\mathfrak{h}}$. By the well-known Schur property [**51**, Theorem 21.30] the restriction $A_n\,|_{\mathsf{E}^n_{\mathfrak{h}}}$ is proportional to the identity operator $\mathbf{1}_{\mathsf{E}^n_{\mathfrak{h}}}$ on $\mathsf{E}^n_{\mathfrak{h}}$, i.e.,

$$A_n\,|_{\mathsf{E}^n_{\mathfrak{h}}} = \aleph_n^{-2}\mathbf{1}_{\mathsf{E}^n_{\mathfrak{h}}}$$

for a constant $\aleph_n^2 \in \mathbb{C}$. Hence, we can rewrite (3.59) as follows

$$(3.60)\qquad \langle \omega_n \mid \psi_n\rangle_{\mathsf{F}} = \aleph_n^2 \int_{\mathsf{G}} \hbar_n[\psi_n]\overline{\hbar_n[\omega_n]}\, d\chi, \qquad \psi_n, \omega_n \in \mathsf{E}^n_{\mathfrak{h}}.$$

This yields in particular the equalities

$$\|\hbar_n\| = \sup_{\|\psi_n\|_{\odot^n_{\mathfrak{h}}\mathsf{E}}=1} \big\|\hbar_n[\psi_n]\big\|_{L^2_\chi} = \frac{1}{\aleph_n}.$$

By formula (3.55), we obtain

$$\int_{\mathsf{G}} \widehat{\psi}_n\, \overline{\widehat{\omega}}_m\, d\chi = \frac{1}{2\pi}\int_{\mathsf{G}} \widehat{\psi}_n\, \overline{\widehat{\omega}}_m\, d\chi \int_{-\pi}^{\pi} e^{\mathrm{i}(n-m)\vartheta}\, d\vartheta = \begin{cases} 0 & : n \neq m \\ \langle \omega_n \mid \psi_n\rangle_{\mathsf{F}} & : n = m \end{cases}$$

for any $\psi_n \in \mathsf{E}^n_{\mathfrak{h}}$ and $\omega_m \in \mathsf{E}^m_{\mathfrak{h}}$. Hence, $\widehat{\psi}_n \perp \widehat{\omega}_m$ in L^2_χ if $n \neq m$. Thus, the orthogonal decomposition (iii) holds. □

Now we will analyze two important cases where operators $\hbar_n$ are bounded.

First consider the case when $\mathsf{E} = L^2_\mu(\mathcal{Q})$ is the Lebesgue space of quadratically integrable complex functions on a measure space $(\mathcal{Q}, \mu)$ with a positive measure μ.

PROPOSITION 3.38. Let $\mathsf{E} = L^2_\mu(\mathcal{Q})$ be a Hilbert space of quadratically integrable complex functions on a measure space $(\mathcal{Q}, \mu)$ and the element $\hbar \in \mathsf{S}$ satisfies the condition

$$(3.61)\qquad c_n = \operatorname*{ess\,sup}_{y_1,\dots,y_n\in\mathcal{Q}} \int_{\mathfrak{G}} \prod_{i=1}^n \left|U_{\mathrm{g}}\hbar(y_i)\right|^2 d\chi(U_{\mathrm{g}}\hbar) < \infty, \qquad n \in \mathbb{N}.$$

Then the inequality

$$(3.62)\qquad \left|\int_{\mathfrak{G}} (\psi_n^* \circ U_{\mathrm{g}})(\hbar)\, \overline{(\omega_n^* \circ U_{\mathrm{g}})(\hbar)}\, d\chi(U_{\mathrm{g}}\hbar)\right| \leq c_n \|\psi_n\|_{\otimes^n_{\mathfrak{h}}\mathsf{E}}\, \|\omega_n\|_{\otimes^n_{\mathfrak{h}}\mathsf{E}}$$

holds for all $\psi_n, \omega_n \in \odot^n_{\mathfrak{h}}\mathsf{E}$.

PROOF. Let $\mathbf{y} = (y_1, \dots, y_n), \mathbf{z} = (z_1, \dots, z_n) \in \mathcal{Q}^n$, where $\mathcal{Q}^n := \mathcal{Q} \times \dots \times \mathcal{Q}$. Due Fubini's theorem and Schwarz's inequality, we have

$$\left|\int_{\mathfrak{G}} \langle (U_{\mathrm{g}}\hbar)^{\otimes n} \mid \psi_n\rangle_{\odot^n_{\mathfrak{h}}\mathsf{E}} \overline{\langle (U_{\mathrm{g}}\hbar)^{\otimes n} \mid \omega_n\rangle_{\odot^n_{\mathfrak{h}}\mathsf{E}}}\, d\chi(U_{\mathrm{g}}\hbar)\right| \leq$$

$$\leq \int_{\mathfrak{G}} d\chi(U_{\mathrm{g}}\hbar) \int_{\mathcal{Q}^n} \prod_{i=1}^n \left|U_{\mathrm{g}}\hbar(y_i)\overline{U_{\mathrm{g}}\hbar}(z_i)\psi_n(\mathbf{y})\bar{\omega}(\mathbf{z})\right| \prod_{i=1}^n d\mu(y_i)\, d\mu(z_i)$$

$$= \int_{\mathcal{Q}^n} |\psi_n(\mathbf{y})\bar{\omega}(\mathbf{z})| \prod_{i=1}^n d\mu(y_i)\, d\mu(z_i) \int_{\mathfrak{G}} \prod_{i=1}^n \left|U_{\mathrm{g}}\hbar(y_i)\overline{U_{\mathrm{g}}\hbar}(z_i)\right| d\chi(U_{\mathrm{g}}\hbar)$$

$$\leq \|\psi_n\|_{\odot^n_{\mathfrak{h}}\mathsf{E}} \|\omega_n\|_{\odot^n_{\mathfrak{h}}\mathsf{E}} \operatorname*{ess\,sup}_{y_1,\dots,y_n\in\mathcal{Q}} \int_{\mathfrak{G}} \prod_{i=1}^n \left|U_{\mathrm{g}}\hbar(y_i)\right|^2 d\chi(U_{\mathrm{g}}\hbar)$$

for all $\psi_n, \omega_n \in \odot_{\mathfrak{h}}^n \mathsf{E}$. □

PROPOSITION 3.39. In the case when $\mathcal{Q} = \mathcal{G}$ let $\mu = \chi$ be a Haar measure on the locally compact group $\mathcal{G}$ and $\mathsf{E} \subset L^2_\chi(\mathcal{G})$. Let

$$\hbar \in \mathsf{S} \cap L^\infty_\chi(\mathcal{G}), \qquad U_{\mathsf{g}} f(\mathsf{q}) = f(\mathsf{g}^{-1}\mathsf{q}), \qquad f \in \mathsf{E}$$

with $\mathsf{g}, \mathsf{q} \in \mathcal{G}$, i.e., the group representation U is regular. Then the inequality (3.62) holds with the constant

$$c_n = c^{n-1}, \qquad c = \operatorname*{ess\,sup}_{\mathsf{g}\in\mathcal{G}} |\hbar(\mathsf{g})|.$$

PROOF. Modifying reasons of the previous proof, we obtain

$$\begin{aligned}
&\left| \int_{\mathcal{G}} \langle (U_{\mathsf{g}}\hbar)^{\otimes n} \mid \psi_n \rangle_{\odot_{\mathfrak{h}}^n \mathsf{E}} \overline{\langle (U_{\mathsf{g}}\hbar)^{\otimes n} \mid \omega_n \rangle_{\odot_{\mathfrak{h}}^n \mathsf{E}}} \, d\chi(\mathsf{g}) \right| \le \\
&\quad \le \int_{\mathcal{G}} d\chi(\mathsf{g}) \int_{\mathcal{G}^{2n}} \prod_{l=1}^{n} |\hbar(\mathsf{g}^{-1}\mathsf{y}_l)\hbar(\mathsf{g}^{-1}\mathsf{z}_l)\, \psi_n(\mathbf{y})\bar{\omega}_n(\mathbf{z})| \prod_{l=1}^{n} d\chi(\mathsf{y}_l)\, d\chi(\mathsf{z}_l) \\
&\quad = \int_{\mathcal{G}^{2n}} |\psi_n(\mathbf{y})\bar{\omega}_n(\mathbf{z})| \prod_{l=1}^{n} d\chi(\mathsf{y}_l)\, d\chi(\mathsf{z}_l) \int_{\mathcal{G}} \prod_{l=1}^{n} |\hbar(\mathsf{g}^{-1}\mathsf{y}_l)\hbar(\mathsf{g}^{-1}\mathsf{z}_l)|\, d\chi(\mathsf{g}) \\
&\quad \le \|\psi_n\|_{\mathsf{F}} \|\omega_n\|_{\mathsf{F}} \operatorname*{ess\,sup}_{\mathbf{y},\mathbf{z}\in\mathcal{G}^n} \int_{\mathcal{G}} \prod_{l=1}^{n} |\hbar(\mathsf{g}^{-1}\mathsf{y}_l)\hbar(\mathsf{g}^{-1}\mathsf{z}_l)| \, d\chi(\mathsf{g}) \\
&\quad \le c^{n-1} \|\psi_n\|_{\mathsf{F}} \|\omega_n\|_{\mathsf{F}} \operatorname*{ess\,sup}_{\mathsf{y}_1,\mathsf{z}_1\in\mathcal{G}} \int_{\mathcal{G}} |\hbar(\mathsf{g}^{-1}\mathsf{y}_1)\hbar(\mathsf{g}^{-1}\mathsf{z}_1)|\, d\chi(\mathsf{g}) \\
&\quad \le c^{n-1} \|\psi_n\|_{\mathsf{F}} \|\omega_n\|_{\mathsf{F}} \operatorname*{ess\,sup}_{\mathsf{y}_1,\mathsf{z}_1\in\mathcal{G}} \left(\int_{\mathcal{G}} |\hbar(\mathsf{g}^{-1}\mathsf{y}_1)|^2 d\chi(\mathsf{g}) \right)^{1/2} \left(\int_{\mathcal{G}} |\hbar(\mathsf{g}^{-1}\mathsf{z}_1)|^2 d\chi(\mathsf{g}) \right)^{1/2} \\
&\quad \le c^{n-1} \|\psi_n\|_{\mathsf{F}} \|\omega_n\|_{\mathsf{F}}
\end{aligned}$$

for all $\psi_n, \omega_n \in \odot_{\mathfrak{h}}^n \mathsf{E}$, since for the convolution

$$(\mathbf{1}_{\mathcal{G}} * |\hbar|^2)(\mathsf{y}_1) = \int |\hbar(\mathsf{g}^{-1}\mathsf{y}_1)|^2 \, d\chi(\mathsf{g})$$

with the identically unit $\mathbf{1}_{\mathcal{G}} \in L^\infty_\chi(\mathcal{G})$, we have

$$\left\| \mathbf{1}_{\mathcal{G}} * |\hbar|^2 \right\|_{L^\infty_\chi} \le \|\mathbf{1}_{\mathcal{G}}\|_{L^\infty_\chi} \left\| |\hbar|^2 \right\|_{L^1_\chi} = 1$$

with the norms in the Lebesgue spaces $L^\infty_\chi(\mathcal{G})$ and $L^1_\chi(\mathcal{G})$. □

Using Proposition 3.38 we will prove that the inequality (3.62) with a finite constant is satisfied for the Schrödinger representation of reduced Heisenberg groups. Clearly, it is satisfied for any finite-dimensional unitary groups, as well.

3.4.2. Cauchy type formula and radial boundary values. Now we establish the Cauchy type formula

$$\mathfrak{C}[f](\xi) = \int_{\mathsf{G}} \mathfrak{C}(\xi,\zeta) f(\zeta)\, d\chi(\zeta), \qquad \xi \in \mathsf{B} \tag{3.63}$$

which for every function $f \in \mathcal{H}^2_\chi$ produces its unique analytic extensions $\mathfrak{C}[f]$ on the open unit ball B in E. Also, we describe the Taylor coefficients and the space of boundary values of these analytic extensions. Namely, we establish that the radial boundary values of $\mathfrak{C}[f]$ are equal to f for every function $f \in \mathcal{H}^2_\chi$. As an example, we consider a reduced Heisenberg group.

Recall, that in Chapter 3.2 it was proved that if $\mathcal{G}$ is the full unitary group of linear operators over the m-dimensional complex space $\mathbb{C}^m$ $(m \in \mathbb{N})$, endowed with the probability Haar measure, then for any $\hbar \in \mathbb{C}^m$ such that $\|\hbar\|_{\mathbb{C}^m} = 1$ the Cauchy kernel with the variables $\xi \in \mathbb{C}^m$, $\|\xi\|_{\mathbb{C}^m} < 1$ and $\zeta \in \mathbb{C}^m$, $\|\zeta\|_{\mathbb{C}^m} = 1$ has the form (see [88, 1.4.9])

$$\mathfrak{C}(\xi,\zeta) = \sum_{n\in\mathbb{Z}_+} \frac{(m-1+n)!}{(m-1)!n!} \langle \xi \mid \zeta\rangle^n_{\mathbb{C}^m} \quad \text{with} \quad \aleph^2_n = \frac{(m-1+n)!}{(m-1)!n!},$$

where the condition

$$\lim_{n\to\infty} \sqrt[n]{\aleph^2_n} = 1$$

is satisfied. This fact justifies that the following kernel

$$\mathfrak{C}\,(\xi,\zeta) = \sum_{n\in\mathbb{Z}_+} \aleph^2_n \,\langle \alpha\xi \mid \zeta\rangle^n_{\mathsf{E}}, \qquad \aleph^2_n = \frac{1}{\|\hbar_n\|^2} \tag{3.64}$$

with $\|\xi\|_{\mathsf{E}} < 1$ and $\|\zeta\|_{\mathsf{E}} \le 1$, for which there exists the limit

$$\lim_{n\to\infty} \sqrt[n]{\aleph^2_n} = 1/\alpha \quad \text{for some constant} \quad \alpha \ge 0, \tag{3.65}$$

we can mean the Cauchy type kernel in more general cases. Now we are going to consider this more carefully.

Put for simplicity $\mathsf{E} = \mathsf{E}^1_{\mathfrak{h}}$ and denote

$$\mathsf{B} := \{\xi \in \mathsf{E}\colon \|\xi\|_{\mathsf{E}} < 1\}, \qquad \mathsf{S} := \{\xi \in \mathsf{E}\colon \|\xi\|_{\mathsf{E}} = 1\}.$$

PROPOSITION 3.40. If the condition (3.65) is satisfied then the kernel $\mathfrak{C}\,(\xi,\zeta)$ with $\zeta \in \mathsf{K}$ is an analytic $C(\mathsf{K})$-valued function by the variable $\xi \in \mathsf{B}$.

PROOF. Calculating a uniform norm by $\zeta \in \mathsf{K}$ of the power series (3.64), we obtain

$$\|\mathfrak{C}\,(\xi,\cdot)\|_{C(\mathsf{K})} \le \sum_{n\in\mathbb{Z}_+} \aleph^2_n \,\|\alpha\xi\|^n_{\mathsf{E}} < \infty$$

for all $\xi \in \mathsf{B}$. Hence, $\mathfrak{C}\,(\xi,\cdot)$ is an analytic $C(\mathsf{K})$-valued function by $\xi \in \mathsf{B}$. □

PROPOSITION 3.41. Let the assumptions of Theorem 3.37 be satisfied. Then for any fixed $r \in [0, 1/\alpha)$ the integral operator

$$\mathfrak{C}[f]\,(\xi) = \int_{\mathsf{G}} \mathfrak{C}\,(\xi,\zeta)\, f(\zeta)\, d\chi(\zeta), \qquad f \in \mathcal{H}^2_\chi \tag{3.66}$$

with $\xi = r\lambda$, $(\lambda \in \mathsf{G})$ belongs to the algebra $\mathscr{L}(\mathcal{H}^2_\chi)$. The function

$$\mathfrak{C}[f]_r \colon \mathsf{G} \ni \lambda \longmapsto \mathfrak{C}[f]\,(r\lambda)$$

with $r \in [0, 1/\alpha)$ belongs to $\mathcal{H}^2_\chi$ and

$$\|f\|_{L^2_\chi} = \sup_{r\in[0,1/\alpha)} \left(\int_{\mathsf{G}} |\mathfrak{C}[f]\,(r\lambda)|^2\, d\chi(\lambda)\right)^{1/2}.$$

PROOF. Let $(\varphi_{j,n})_{j\in\mathbb{N}}$ be an orthonormal basis in the space $\mathsf{E}^n_{\mathfrak{h}}$ with $n \in \mathbb{N}$. Then the system $(\widehat{\varphi}_{j,n})_{j\in\mathbb{N}}$ is an orthonormal basis in $\mathcal{H}^2_n$. Indeed, substituting in (3.60) $\omega_n = \varphi_{j,n}$ and $\psi_n = \varphi_{i,n}$ with $j \neq i$ we have

$$\int_{\mathsf{G}} \widehat{\varphi}_{j,n}\,\overline{\widehat{\varphi}}_{i,n}\, d\chi = \langle \varphi_{i,n} \mid \varphi_{j,n}\rangle_{\mathsf{F}} = 0,$$

i.e., $\widehat{\varphi}_{j,n} \perp \widehat{\varphi}_{i,n}$ in L^2_χ. So, the system $(\widehat{\varphi}_{j,n})$ is orthonormal in the space L^2_χ. If $\xi^{\otimes n} = \sum_j \langle \xi^{\otimes n} \mid \varphi_{j,n}\rangle_{\mathsf{F}}\varphi_{j,n}$ denotes the Fourier expansions under $(\varphi_{j,n})$ of an element $\xi \in \mathsf{E}$ then we have

$$\mathfrak{C}_n(\xi,\zeta) := \aleph^2_n \langle \alpha^n \xi^{\otimes n} \mid \zeta^{\otimes n}\rangle_{\mathsf{F}} = (r\alpha)^n \sum_{j\in\mathbb{N}} \widehat{\varphi}_{j,n}(\lambda)\overline{\widehat{\varphi}}_{j,n}(\zeta),$$

i.e., $\mathfrak{C}_n(\xi,\zeta) = (r\alpha)^n \mathfrak{C}_n(\lambda,\zeta)$ with $\zeta = U_y\hbar$, $\lambda = U_x\hbar \in \mathsf{G}$ for all $x, y \in \mathcal{G}$ and $\xi = r\alpha\lambda$, $(r\alpha = \|\xi\|_{\mathsf{E}})$. So,

$$\mathfrak{C}(\xi,\zeta) = \sum_{n\in\mathbb{Z}_+} (r\alpha)^n \sum_{j\in\mathbb{N}} \widehat{\varphi}_{j,n}(\lambda)\overline{\widehat{\varphi}}_{j,n}(\zeta) = \sum_{n\in\mathbb{Z}_+} (r\alpha)^n \mathfrak{C}_n(\zeta,\lambda).$$

Theorem 3.37 implies that

$$\int_{\mathsf{G}} \widehat{\varphi}_{j,n}(\zeta)\mathfrak{C}_n(\xi,\zeta)\, d\chi(\zeta) = \widehat{\varphi}_{j,n}(\xi) \int_{\mathsf{G}} \widehat{\varphi}_{j,n}(\zeta)\overline{\widehat{\varphi}}_{j,n}(\zeta)\, d\chi(\zeta) = \widehat{\varphi}_{j,n}(\xi)$$

for all $\varphi_{j,n}$ and $\xi \in \mathsf{G}$. Since $(\widehat{\varphi}_{j,n})$ is an orthonormal basis in $\mathcal{H}^2_n$, the integral operator with kernel $\mathfrak{C}_n$ produces the identity mapping on $\mathcal{H}^2_n$.

Let $f = \sum_{n\in\mathbb{Z}_+} f_n \in \mathcal{H}^2_\chi$ with $f_n \in \mathcal{H}^2_n$. Using that $f_n \perp \mathfrak{C}_m$ at $n \neq m$ in L^2_χ, we obtain

$$f(\xi) = \sum_{n\in\mathbb{Z}_+} \int_{\mathsf{G}} \mathfrak{C}_n(\xi,\zeta) f_n(\zeta)\, d\chi(\zeta) = \int_{\mathsf{G}} \mathfrak{C}(\xi,\zeta) f(\zeta)\, d\chi(\zeta)$$

for all $\xi \in \mathsf{G}$. It follows that the series $\mathfrak{C}[f](\xi) = \sum_{n\in\mathbb{Z}_+} \mathfrak{C}[f_n](\xi)$ with

$$\begin{aligned}\mathfrak{C}[f_n](\xi) &= \int_{\mathsf{G}} \mathfrak{C}_n(\xi,\zeta) f_n(\zeta)\, d\chi(\zeta) = \aleph^2_n \int_{\mathsf{G}} \langle \alpha\xi \mid \zeta\rangle^n_{\mathsf{E}} f_n(\zeta)\, d\chi(\zeta)\\ &= (r\alpha)^n \int_{\mathsf{G}} \mathfrak{C}_n(\lambda,\zeta) f_n(\zeta)\, d\chi(\zeta) = (r\alpha)^n f_n(\lambda) = f_n(\xi)\end{aligned}$$

is convergent in $\mathcal{H}^2_\chi$ by the variable $\lambda \in \mathsf{G}$, uniformly by $r \in [0, \varepsilon]$ with $0 < \varepsilon < 1/\alpha$. Applying that $\mathfrak{C}_n \perp f_m$ and $f_n \perp f_m$ at $n \neq m$ in L^2_χ, we have

$$\begin{aligned}\|\mathfrak{C}_r[f]\|^2_{L^2_\chi} &= \int_{\mathsf{G}} \Big| \sum_{n\in\mathbb{Z}_+} (r\alpha)^n \int_{\mathsf{G}} \mathfrak{C}_n(\lambda,\zeta) f_n(\zeta)\, d\chi(\zeta)\Big|^2 d\chi(\lambda) \\ &= \int_{\mathsf{G}} \Big| \sum_{n\in\mathbb{Z}_+} (r\alpha)^n f_n(\lambda)\Big|^2 d\chi(\lambda) \\ &= \Big\| \sum (r\alpha)^n f_n \Big\|^2_{L^2_\chi} = \sum_{n\in\mathbb{Z}_+} (r\alpha)^{2n} \|f_n\|^2_{L^2_\chi}\end{aligned}$$

for any $r < 1/\alpha$. It follows that

$$\sup_{r\in[0,1/\alpha)} \sum (r\alpha)^{2n} \|f_n\|^2_{L^2_\chi} = \sum \|f_n\|^2_{L^2_\chi} = \|f\|^2_{L^2_\chi}.$$

By the Cauchy-Schwarz inequality, we have

$$\|\mathfrak{C}_r[f]\|_{L^2_\chi} \le \frac{1}{(1-r^2\alpha^2)^{1/2}} \Big(\sum_{n\in\mathbb{Z}_+} \|f_n\|^2_{L^2_\chi}\Big)^{1/2} = \frac{\|f\|_{L^2_\chi}}{(1-r^2\alpha^2)^{1/2}}$$

for all $f \in \mathcal{H}^2_\chi$. Hence, the operator (3.66) belongs to $\mathscr{L}(\mathcal{H}^2_\chi)$. □

THEOREM 3.42. *Let the assumptions of Theorem 3.37 and the condition* (3.65) *are simultaneously satisfied. Then for any* $f = \sum_{n\in\mathbb{Z}_+} f_n \in \mathcal{H}^2_\chi$ *with* $f_n \in \mathcal{H}^2_n$ *the integral transform* (3.63) *with the Cauchy type kernel* (3.64) *is a unique analytic extension of the function* f *on the open ball* B *with the Taylor coefficients at the origin*

$$\frac{d^n_0 \mathfrak{C}[f](\xi)}{n!} = \aleph^2_n \int_{\mathsf{G}} \langle \alpha\xi \mid \zeta\rangle^n_{\mathsf{E}} f_n(\zeta)\, d\chi(\zeta), \qquad \xi \in \mathsf{E}. \tag{3.67}$$

For each analytic function $\mathfrak{C}[f]$ *its radial boundary values on the orbit* G *are equal to* f *in the following sense*

$$\lim_{r\to 1/\alpha} \int_{\mathsf{G}} |\mathfrak{C}_r[f] - f|^2\, d\chi = 0, \qquad r \in [0, 1/\alpha). \tag{3.68}$$

PROOF. By Proposition 3.40 $\mathfrak{C}(\xi,\cdot)$ is an analytic $C(\mathsf{K})$-valued function by $\xi \in \mathsf{B}$. Hence, the function $\mathfrak{C}[f]$ determined by (3.63) is also analytic of $\xi \in \mathsf{B}$ in view of [50, 3.1.2]. Differentiating at the origin, we obtain

$$\frac{d^n_0 \mathfrak{C}[f](\xi)}{n!} = \aleph^2_n \int_{\mathsf{G}} \langle \alpha\xi \mid \zeta\rangle^n_{\mathsf{E}} f_n(\zeta)\, d\chi(\zeta) = \mathfrak{C}[f_n](\xi), \qquad \xi \in \mathsf{B}.$$

By the Cauchy-Schwarz inequality,

$$|\mathfrak{C}[f_n](\xi)| \le \aleph^2_n \int_{\mathsf{G}} |\langle \alpha\xi \mid \zeta\rangle^n_{\mathsf{E}} f_n(\zeta)|\, d\chi(\zeta) \le \aleph^2_n \|\alpha\xi\|^n_{\mathsf{E}} \|f_n\|_{L^2_\chi}$$

for all $\xi \in \mathsf{E}$. Hence, any $\mathfrak{C}[f_n]$ is a n-homogenous polynomial on E, which takes the form (3.67). As is well known [50, 2.4.2], continuous Taylor coefficients uniquely define the analytic function $\mathfrak{C}[f]$ on B. So, the uniqueness of the analytic extension $\mathfrak{C}[f]$ is proved. Finally, using the orthogonal property we have

$$\int_{\mathsf{G}} |\mathfrak{C}_r[f] - f|^2 \, d\chi = \sum_{n \in \mathbb{Z}_+} \left(r^{2n}\alpha^{2n} - 1\right) \|f_n\|^2_{L^2_\chi} \longrightarrow 0$$

if $r \to 1/\alpha$. The theorem is proved. □

The space of analytic extensions of functions from $\mathcal{H}^2_\chi$, defined by the formula (3.63), we denote by

$$\mathcal{H}^2_\chi(\mathsf{B}) := \left\{\mathfrak{C}[f] \colon f \in \mathcal{H}^2_\chi\right\}$$

and endow with the norm

$$\left\|\mathfrak{C}[f]\right\|_{\mathcal{H}^2_\chi} = \sup_{r \in [0,1/\alpha)} \left(\int_{\mathsf{G}} \left|\mathfrak{C}[f](r\lambda)\right|^2 d\chi(\lambda)\right)^{1/2}.$$

COROLLARY 3.43. The following antilinear isometry

$$\mathcal{H}^2_\chi(\mathsf{B}) \simeq \mathsf{F}_{\mathfrak{h}}$$

holds.

PROOF. Since $\left\|\mathfrak{C}[f]\right\|_{\mathcal{H}^2_\chi} = \|f\|_{L^2_\chi}$ for all $f \in \mathcal{H}^2_\chi$, the isometry $\mathcal{H}^2_\chi(\mathsf{B}) \simeq \mathcal{H}^2_\chi$ holds. So, the required isometry instantly follows from Theorem 3.37. □

3.4.3. Example of a reduced Heisenberg group. In what follows, we put $\mathcal{G} = \mathbb{H}$, where the Cartesian product

$$\mathbb{H} = \mathbb{R}^2 \times \mathbb{T}$$

stands for the reduced Heisenberg group with the multiplication

$$(x, y, e^{i\vartheta}) \cdot (u, v, e^{i\eta}) = \left(x + u, y + v, e^{i(\vartheta+\eta)} e^{i(xv - yu)/2}\right),$$

endowed with the Haar measure $dx\, dy\, d\tau$, where $\tau = e^{i\vartheta} \in \mathbb{T}$ and $d\tau = \dfrac{d\vartheta}{2\pi}$. We refer to [**91**] for detail about Heisenberg groups.

Let $\mathsf{E} = L^2_{\mathbb{R}}$ be the Hilbert space of quadratically integrable complex functions f on $\mathbb{R}$ endowed with the norm $\|f\|_{L^2_{\mathbb{R}}} = \left(\int_{\mathbb{R}} |f(x)|^2 \, dx\right)^{1/2}$. Consider in $L^2_{\mathbb{R}}$ the orthonormal basis

$$\varphi_j \colon \mathbb{R} \ni t \longmapsto \frac{e^{-t^2/2}}{\sqrt[4]{\pi}} \frac{\phi_{j-1}(t)}{\sqrt{2^{j-1}(j-1)!}}, \qquad \phi_{j-1}(t) = (-1)^{j-1} e^{t^2} \frac{d^{j-1}}{dt^{j-1}} e^{-t^2},$$

where $j \in \mathbb{N}$ and ϕ_{j-1} is the Hermite $(j-1)$-degree polynomial. Note that the space $L^2_{\mathbb{R}^n} = \otimes^n_{\mathfrak{h}} L^2_{\mathbb{R}}$ coincides with the closure of complex linear span of functions

$$\left\{\xi_1(t_1) \ldots \xi_n(t_n) \colon \xi_1, \ldots, \xi_n \in L^2_{\mathbb{R}},\ (t_1, \ldots, t_n) \in \mathbb{R}^n\right\}.$$

Therefore, $\odot^n_{\mathfrak{h}} L^2_{\mathbb{R}}$ is the closed subspace in $L^2_{\mathbb{R}^n}$ of symmetric functions with respect to the permutations of n scalar variables. By (1.12) the following system

$$\varphi^{\otimes(k)}_{(j)} := \varphi^{\otimes k_1}_{j_1} \odot \ldots \odot \varphi^{\otimes k_n}_{j_n}$$

with all $(j) = (j_1, \dots, j_n) \in \mathbb{N}^n$, $j_1 < \dots < j_n$ and $(k) = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$ such that $k_1 + \dots + k_n = n$ forms an orthogonal basis in $\odot_\mathfrak{h}^n L_\mathbb{R}^2$, which is non-orthonormal and $\|\varphi_{(j)}^{\otimes(k)}\|_{\odot_\mathfrak{h}^n L_\mathbb{R}^2} = \sqrt{(k)!/n!}$, where $(k)! := k_1! \dots k_n!$.

The Schrödinger representation U of the group $\mathbb{H}$ into $\mathscr{L}\left(L_\mathbb{R}^2\right)$ is given by

$$U_{x,y,\tau}\xi(t) = \tau e^{ixy/2} e^{iyt} \xi(t+x), \qquad x, y, t \in \mathbb{R}, \quad \tau \in \mathbb{T}, \quad \xi \in L_\mathbb{R}^2$$

which is unitary and irreducible. It is easy to see that the codomain of U contains the complex cyclic group, since

$$\mathbb{T} = \left\{U_{0,0,\tau} \colon (0,0,\tau) \in \mathbb{H}\right\}.$$

REMARK 3.44. By the Stone-von Neumann Theorem every irreducible unitary representation V of $\mathbb{H}$ over any Hilbert space E, satisfying the condition

$$V(0,0,\tau)\xi = \tau\xi$$

for all $\tau \in \mathbb{T}$ and $\xi \in \mathsf{E}$, is unitarily equivalent to the Schrödinger representation U.

The Gauss density function

$$\hbar \colon \mathbb{R} \ni t \longmapsto \pi^{-1/4} e^{-t^2/2} \qquad \text{(i.e.} \quad \hbar = \varphi_1)$$

belongs to the unit sphere $\mathsf{S} \subset L_\mathbb{R}^2$. The $\mathbb{H}$-orbit of $\hbar$

$$\begin{aligned} \mathsf{G} &= \left\{U_{x,y,\tau}\hbar \in L_\mathbb{R}^2 \colon (x,y,\tau) \in \mathbb{H}\right\} \\ &= \left\{g_{x,y,\tau}(t) = \pi^{-1/4} \tau e^{ixy/2} e^{iyt} e^{-(t+x)^2/2} \colon (x,y,\tau) \in \mathbb{H},\ t \in \mathbb{R}\right\} \end{aligned}$$

contains in S, as a function of the variable t for any fixed $(x,y,\tau) \in \mathbb{H}$. In fact, for any fixed $(x,y,\tau) \in \mathbb{H}$ we have

$$\|U_{x,y,\tau}\hbar\|_{L_\mathbb{R}^2} = \left(\int_\mathbb{R} \left|\pi^{-1/4} e^{-t^2/2}\right|^2 dt\right)^{1/2} = 1.$$

The stationary subgroup $\left\{(x,y,\tau) \in \mathbb{H} \colon U_{x,y,\tau}\hbar = \hbar\right\}$ coincides with the group unit $(0,0,1) \in \mathbb{H}$, hence the equality (3.54) has the form

$$\int_\mathsf{G} f\, d\chi = \int_\mathbb{H} (f \circ U_{x,y,\tau})(\hbar)\, dx\, dy\, d\tau,$$

where the $\mathbb{H}$-invariant measure χ on G is defined by the Haar measure $dx\, dy\, d\tau$ on $\mathbb{H}$ and $f \circ U$ is an integrable complex function on $\mathbb{H}$.

Consider the diagonal nth tensor power of Schrödinger's representation

$$\mathbb{H} \ni (x,y,\tau) \longmapsto U_{x,y,\tau}^{\otimes n} \in \mathscr{L}\left(\odot_\mathfrak{h}^n L_\mathbb{R}^2\right), \qquad n \in \mathbb{N}$$

and put $U_{x,y,\tau}^{\otimes 0} = 1$. Let L_χ^2 be the corresponding Hilbert space of quadratically χ-integrable complex functions on the orbit G. Each function

$$\hbar_n\left[\varphi_{(j)}^{\otimes(k)}\right] \colon \mathbb{H} \ni (x,y,\tau) \longmapsto \left\langle (U_{x,y,\tau}\hbar)^{\otimes n} \mid \varphi_{(j)}^{\otimes(k)} \right\rangle_{L_{\mathbb{R}^n}^2}$$

with $k_1 + \dots + k_n = n$, belongs to L_χ^2 and the following operator

$$\hbar_n \colon \odot_\mathfrak{h}^n L_\mathbb{R}^2 \ni \varphi_{(j)}^{\otimes(k)} \longrightarrow \hbar_n\left[\varphi_{(j)}^{\otimes(k)}\right] \in L_\chi^2, \qquad (j) \in \mathbb{N}^n$$

is well-defined. In fact, calculating the Fourier transformation by the variable $t \in \mathbb{R}$, we have

$$\begin{aligned}\hbar_1(\varphi_j)(x,y,\tau) &= \frac{\tau e^{ixy/2}(-1)^{j-1}}{\sqrt{2^{j-1}\pi(j-1)!}}\int_{\mathbb{R}} e^{iyt}e^{-(x+t)^2/2}e^{t^2/2}\frac{d^{j-1}}{dt^{j-1}}e^{-t^2}\,dt\\ &= \frac{\tau e^{ixy/2}(-1)^{j-1}(x-iy)^{j-1}}{\sqrt{2^{j-1}(j-1)!}}e^{(-x^2-2ixy-y^2)/4}\end{aligned}$$

for any φ_j. For all (k) such that $k_1+\cdots+k_n=n$ it follows

$$\begin{aligned}\left|\hbar_n(\varphi_{(j)}^{\otimes(k)})\right| &= \prod_{l=1}^{n}|\hbar_1(\varphi_{j_l})(x,y,\tau)|^{k_l}\\ &= e^{-\frac{n}{4}(x^2+y^2)}\prod_{l=1}^{n}\left(\frac{(x^2+y^2)^{j_l-1}}{2^{j_l-1}(j_l-1)!}\right)^{k_l/2}.\end{aligned}$$

Since,

$$\begin{aligned}\int_0^\infty e^{-nu}\prod_{l=1}^{n}\left(\frac{u^{j_l-1}}{(j_l-1)!}\right)^{k_l}du &= \prod_{l=1}^{n}\frac{m!}{(j_l-1)!^{k_l}}\int_0^\infty e^{-nu}\frac{u^m}{m!}\,du\\ &= \prod_{l=1}^{n}\frac{m!}{(j_l-1)!^{k_l}}\frac{1}{n^m}\int_0^\infty e^{-un}\frac{(un)^m}{m!}\,du \le \frac{1}{n}\end{aligned}$$

with $m=\sum_{l=1}^{n}(j_l-1)k_l$ and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(\frac{x^2+y^2}{2}\right)dx\,dy = 4\int_0^\infty\int_0^{\pi/2} f(u)\,du\,d\vartheta = 2\pi\int_0^\infty f(u)\,du,$$

where $x^2=2u\cos^2\vartheta$ and $y^2=2u\sin^2\vartheta$, we obtain that each such function $\hbar_n[\varphi_{(j)}^{\otimes(k)}]$ belongs to L^2_χ and the following estimation holds

$$\int_{\mathbb{H}}\left|\hbar_n[\varphi_{(j)}^{\otimes(k)}]\right|^2 dx\,dy\,d\tau \le \frac{2\pi}{n}. \tag{3.69}$$

Any element $\psi_n \in \odot_{\mathfrak{h}}^n L^2_{\mathbb{R}}$ with $\|\psi_n\|_{\odot_{\mathfrak{h}}^n L^2_{\mathbb{R}}} \le 1$ may be presented in the form of its Fourier decomposition

$$\psi_n = \sum_{(k),(j)}\alpha_{(j)}^{(k)}\varphi_{(j)}^{\otimes(k)}\sqrt{\frac{n!}{(k)!}},\qquad k_1+\cdots+k_n=n,\qquad \sum_{(k),(j)}|\alpha_{(j)}^{(k)}|^2\le 1.$$

Applying the inequality (3.69), we have

$$\begin{aligned}\left\|\sum_{(k),(j)}\alpha_{(j)}^{(k)}\hbar_n[\varphi_{(j)}^{\otimes(k)}]\sqrt{\frac{n!}{(k)!}}\right\|_{L^2_\chi} &\le \sum_{(k),(j)}|\alpha_{(j)}^{(k)}|\sqrt{\frac{n!}{(k)!}}\left\|\hbar_n[\varphi_{(j)}^{\otimes(k)}]\right\|_{L^2_\chi}\\ &\le \sum_{(k),(j)}|\alpha_{(j)}^{(k)}|\sqrt{\frac{n!}{(k)!}\frac{2\pi}{n}}.\end{aligned}$$

It follows that

$$\|\hbar_n[\psi_n]\|^2_{L^2_{\mathsf{X}}} \leq 2\pi(n-1)!\|\psi_n\|^2_{\odot^n_{\mathfrak{h}} L^2_{\mathbb{R}}} \quad \text{or} \quad \|\hbar_n\| \leq \sqrt{2\pi(n-1)!}.$$

If $(\jmath) = (1, \jmath_2, \dots, \jmath_n)$ and $(k) = (n, 0, \dots, 0)$ we have that $\varphi^{\otimes(k)}_{(\jmath)} = \varphi_1^{\otimes n}$ and

$$\begin{aligned}\int_{\mathbb{H}} \left|\hbar_n\left[\varphi_1^{\otimes n}\right]\right|^2 dx\,dy\,d\tau &= \int_{\mathbb{H}} \left|\hbar_1\left[\varphi_1\right](x,y,\tau)\right|^{2n} dx\,dy\,d\tau \\ &= \int_{\mathbb{R}^2} \left|e^{-(x^2+2ixy+y^2)/4}\right|^{2n} dx\,dy = \frac{2\pi}{n}.\end{aligned}$$

Since $1 = \|\varphi_1^{\otimes n}\|_{\odot^n_{\mathfrak{h}} L^2_{\mathbb{R}}} = \sqrt{\dfrac{n}{2\pi}} \left\|\hbar_n\left[\varphi_1^{\otimes n}\right]\right\|_{L^2_{\mathsf{X}}}$, we have that $\varphi_1^{\otimes n} \notin \ker \hbar_n$. This yields

$$\|\hbar_n\| = \sup_{\|\psi_n\|_{\odot^n_{\mathfrak{h}} L^2_{\mathbb{R}}} \leq 1} \|\hbar_n[\psi_n]\|_{L^2_{\mathsf{X}}} \geq \left\|\hbar_n\left[\varphi_1^{\otimes n}\right]\right\|_{L^2_{\mathsf{X}}} = \sqrt{\frac{2\pi}{n}}.$$

Hence,

$$\lim_{n\to\infty} \sqrt[n]{\aleph_n^2} \leq \lim_{n\to\infty} \sqrt[n]{\frac{n}{2\pi}} = 1$$

and the Cauchy type kernel has the form

$$\begin{aligned}\widehat{\mathfrak{C}}\left(\xi, U_{x,y,\tau}\hbar\right) &= 1 + \sum_{n=1}^{\infty} \aleph_n^2 \left\langle \xi \mid U_{x,y,\tau}\hbar\right\rangle^n_{L^2_{\mathbb{R}}} \\ &= 1 + \sum_{n=1}^{\infty} \aleph_n^2 \left(\frac{\tau e^{ixy/2}}{\sqrt[4]{\pi}} \int_{\mathbb{R}} \xi(t) e^{iyt-(t+x)^2/2}\,dt\right)^n\end{aligned}$$

which is a L^2_{X}-valued analytic function by the variable $\xi \in \mathsf{B}$, where $\mathsf{B} \subset L^2_{\mathbb{R}}$ is a nonempty open subset in the unit ball of $L^2_{\mathbb{R}}$. Thus, for any $f \in \mathcal{H}^2_{\mathsf{X}}$ and $\xi \in \mathsf{B}$ we have

$$\widehat{\mathfrak{C}}[f](\xi) = \int_{\mathbb{H}} \widehat{\mathfrak{C}}\left(\xi, U_{x,y,\tau}\hbar\right)(f \circ U_{x,y,\tau})(\hbar)\,dx\,dy\,d\tau,$$

$$\lim_{r\to 1} \int_{\mathbb{H}} \left|\widehat{\mathfrak{C}}_r[f](x,y,\tau) - (f \circ U_{x,y,\tau})(\hbar)\right|^2 dx\,dy\,d\tau = 0,$$

where the functions

$$\widehat{\mathfrak{C}}[f]_r \colon \mathbb{H} \ni (x,y,\tau) \longmapsto \widehat{\mathfrak{C}}[f](rU_{x,y,\tau}\lambda)$$

with $r \in [0,1)$ belong to $\mathcal{H}^2_{\mathsf{X}}$ for any $\lambda \in \mathsf{S}$.

Notes and Remarks. If Ω is a domain in a complex Banach space then the Hardy space $\mathcal{H}^\infty(\Omega)$ which is the uniform algebra of bounded analytic functions on Ω is a standard object of infinite-dimensional complex analysis and was investigated by many authors (see for example [**6, 7, 26, 30, 32, 44**] and others). Integral representations of Hardy spaces $\mathcal{H}^p$ $(p \geq 1)$ with infinitely many variables were an object of research in [**30, 80, 83**].

The Hardy type spaces $\mathcal{H}^2(d\varsigma)$ associated with compact groups were investigated in [**61**]. A more general case of Hardy type spaces associated with locally compact groups were introduced in [**60**]. Section 3.2 is a special case of the results from [**60, 65**]. Theorem 3.13 are proved in [**65**]. Ibid Theorems 3.13, 3.18, and 3.40, establishing an antilinear isomorphism between unitary Hardy spaces and symmetric Fock spaces, are proved. A classic theory of Hardy spaces in the unit ball of $\mathbb{C}^d$ may be find in [**88**, Section 1.4.10]. The Segal-Bargmann space $\mathcal{H}^2_\mu(\mathbb{C}^d)$ been studied in [**12, 14**].

We also analyze the case of unitary Hardy type spaces, which are associated with the the infinite-dimensional unitary matrix group $U(\infty)$ following to [**65**]. Notice that considered here the infinite-dimensional unitary group $U(\infty)$ is one of the basic examples of big groups whose irreducible representations depend on infinitely many parameters. General principles of harmonic analysis on this group developed by G. Olshanski [**79**]. The space of virtual unitary matrices $\mathfrak{U}$ was studied by Y. Neretin [**78**] and G. Olshanski [**79**]. The notion of a virtual space relates to D. Pickrell's space of virtual Grassmannian [**82**] and to S. Kerov, G. Olshanski, and A. Vershik's space of virtual permutations [**53**]. Various spaces of integrable functions with respect to measures that are invariant under infinite-dimensional groups have been widely applied in stochastic processes [**20**], infinite-dimensional probability [**98, 95**] etc.

The space of analytic functions $\mathcal{H}^2_\chi(\mathsf{B})$ with a Haar measure χ on an abstract locally compact group $\mathcal{G}$ were considered in [**60, 64**], where some of its properties was described. An application theory of these spaces to Heisenberg groups were given in [**63**]. Note that in this section we analyze a more general case when a $\mathcal{G}$-invariant measure χ is defined on a unitary orbit G of a locally compact second countable group $\mathcal{G}$ acting in an infinite-dimensional Hilbert space E.

CHAPTER 4

Reproducing kernel spaces of analytic functions

Effective methods for the study of Hardy type spaces on infinite-dimensional domains is representations of analytic functions by means of reproducing kernel and representing measures. In this chapter we analyze various special cases and examples of such spaces.

4.1. Abstract Hardy spaces

4.1.1. Representing measures and abstract Hardy spaces. Let $\mathcal{Q}$ be a compact Hausdorff space. We denote by $C_r(\mathcal{Q})$ and $C(\mathcal{Q})$ the uniform algebras of all continuous real and complex valued functions on $\mathcal{Q}$ respectively. Let $\mathfrak{M}$ be the class of non-negative Baire measures μ on $\mathcal{Q}$ that is isometrically isomorphic to the dual $C_r(\mathcal{Q})'$. Note that for every Baire measure μ from $\mathfrak{M}$ there exists a unique regular Borel extension. This is an unique regular Borel measure on $\mathcal{Q}$ that agrees with μ on the Baire sets of $\mathcal{Q}$.

Throughout this section a subspace of $C_r(\mathcal{Q})$ or $C(\mathcal{Q})$ means a vector subspace *containing the constant functions*, but not necessarily closed.

Let $\mathcal{V}$ be a subspace of $C_r(\mathcal{Q})$ or $C(\mathcal{Q})$, and $x \in \mathcal{Q}$. We define $\mathfrak{M}_x(\mathcal{V})$ to be the subset of $\mathfrak{M}$ consisting of all measures μ on $\mathcal{Q}$ with

$$\int f\,d\mu = f(x), \qquad f \in \mathcal{V}.$$

Then $\mathfrak{M}_x(\mathcal{V})$ is a convex, weak-star compact and always non-empty set, since it must contain at least the point evaluation functional $\delta_x\colon \mathcal{V} \ni f \longmapsto f(x)$.

Let $\mathcal{Q}_0$ be a subset of $\mathcal{Q}$. It is defined by

$$\widehat{\mu}(\mathcal{Q}_0) = \inf \mu(U),$$

where U runs over all open Baire sets that contains $\mathcal{Q}_0$. We define

$$\imath_{\mathcal{V}}(\mathcal{Q}_0) := \Big\{ y \in \mathcal{Q}\colon f(y) = f(x) \text{ for some } x \in \mathcal{Q}_0 \text{ and all } f \in \mathcal{V} \Big\}.$$

If $\mathcal{V}$ is a subspace of either $C_r(\mathcal{Q})$ or $C(\mathcal{Q})$, the *Choquet boundary* of $\mathcal{V}$, denoted by $\partial\mathcal{V}$, is defined to consist of those points x in $\mathcal{Q}$ which are such that any $\mu \in \mathfrak{M}_x(\mathcal{V})$ satisfies the condition

$$\widehat{\mu}\big(\imath_{\mathcal{V}}(x)\big) = 1.$$

The Choquet boundary is not necessarily closed (or even Borel) subset of $\mathcal{Q}$. From the definition it follows that if the subspace $\mathcal{V}$ separates points of $\mathcal{Q}$ and $x \in \partial\mathcal{V}$, then the point evaluation functional δ_x has a unique representing measure that is the atomic measure concentrated at $x \in \mathcal{Q}$.

If $\mathcal{V}$ is a subspace of $C(\mathcal{Q})$, we will denote by $\mathcal{V}_r$ the subspace of $C_r(\mathcal{Q})$, consisting of real parts of the functions in $\mathcal{V}$. It is known **[16]** that

$$\partial\mathcal{V}_r = \partial\mathcal{V} \quad \text{and} \quad \mathfrak{M}_x(\mathcal{V}_r) = \mathfrak{M}_x(\mathcal{V}), \quad \forall x \in \mathcal{Q}.$$

If $\mathcal{V}$ is a subspace of $C_r(\mathcal{Q})$ or $C(\mathcal{Q})$ and μ and ν are in $\mathfrak{M}$, we write $\nu \prec \mu$ if

$$\int f d\nu = \int f\, d\mu \quad \text{and} \quad \int f^2\, d\nu \leq \int f^2\, d\mu, \qquad \forall f \in \mathcal{V}_r.$$

We say that a measure μ is *maximal* for a measure ν, if $\nu \prec \mu$ imply $\nu = \mu$.

Let $\mathcal{V}$ be a subspace of $C_r(\mathcal{Q})$ or $C(\mathcal{Q})$. From Zorn's Lemma it follows that for each ν in $\mathfrak{M}$ there exists a maximal measure $\mu \in \mathfrak{M}$ such that $\nu \prec \mu$ **[16]**.

Let $\mathcal{A}$ be a uniform closed subalgebra of $C(\mathcal{Q})$ that separating point of $\mathcal{Q}$ and contains the identically unit functions 1.

A subset $\mathcal{Q}_0$ of Ω is said to be a *boundary for* $\mathcal{A}$ if for each $f \in \mathcal{A}$ there is some $y \in \mathcal{Q}_0$ with

$$|f(y)| = \|f\|.$$

PROPOSITION 4.1. Let $\mathcal{A}$ be a uniform closed subalgebra of $C(\mathcal{Q})$, which separates point of $\mathcal{Q}$ and contains the constant functions 1. Then the Choquet boundary $\partial\mathcal{A}$ of $\mathcal{A}$ is a boundary and if $\mathcal{Q}$ is metrizable, then $\partial\mathcal{A}$ contained in any boundary of $\mathcal{A}$.

THEOREM 4.2. *Let $\mathcal{A}$ be as in Proposition 4.1 and $\mathfrak{S}$ the σ-ring generated by $\partial\mathcal{A}$ and the Baire sets of $\mathcal{Q}$. Then each linear functional ϕ from the dual $\mathcal{A}'$ has a representation of the form*

$$\phi(f) = \int_{\partial\mathcal{A}} f\, d\mu, \qquad f \in \mathcal{A},$$

where a corresponding measure μ on $\mathfrak{S}$ is such that $\mu(\mathcal{Q}_0) = 0$ for every $\mathcal{Q}_0 \in \mathfrak{S}$ disjoint from $\partial\mathcal{A}$. Furthermore, μ can be chosen to be non-negative if and only if

$$\phi(1) = \|\phi\|.$$

Let $\mathcal{A}$ be an abstract complex commutative uniform Banach algebra with identity 1 and let $M(\mathcal{A})$ be the space of its maximal ideals (complex continuous homomorphisms) endowed with the Gelfand topology. Using the Gelfand transform

$$\mathcal{A} \ni f \longmapsto \widehat{f} \in C\big(M(\mathcal{A})\big),$$

we can consider $\mathcal{A}$ as a closed subalgebra of the algebra $C\big(M(\mathcal{A})\big)$ of all continuous functions on the compact Hausdorff space $M(\mathcal{A})$ and

$$\|f\|_{\mathcal{A}} = \sup_{\phi \in M(\mathcal{A})} \big|\widehat{f}(\phi)\big|, \qquad \widehat{f}(\phi) := \phi(f).$$

Since, $\phi(1) = 1 = \|\phi\|$ for every $\phi \in M(\mathcal{A})$, we get the following.

COROLLARY 4.3. For every $\phi \in M(\mathcal{A})$ there is a probability measure μ (so-called, *representing*) on Choquet boundary $\partial\mathcal{A} \subset M(\mathcal{A})$ such that

$$\phi(f) = \int_{\partial\mathcal{A}} \widehat{f}\, d\mu, \qquad f \in \mathcal{A}.$$

Note, for example that the disk algebra $\mathcal{A}(D)$, where D is the closed unit disk of $\mathbb{C}$, admits a unique maximal measure which represents the atomic measure δ_0 concentrated at the origin. This measure is a probability Lebesgue measure on the unit circle [**44**, p. 38].

A point $x \in M(\mathcal{A})$ is called the *peak point* if there is a function $f \in \mathcal{A}$ such that

$$\widehat{f}(x) = 1 \quad \text{and} \quad |\widehat{f}(y)| < 1$$

for all $y \in M(\mathcal{A})$ such that $y \neq x$. A closed subset Z of $M(\mathcal{A})$ is called the *peak set* if there exists a function $f \in \mathcal{A}$ such that

$$\widehat{f}(x) = 1 \quad \text{for} \quad x \in Z \quad \text{and} \quad |\widehat{f}(y)| < 1 \quad \text{for} \quad y \in M(\mathcal{A}) \setminus Z.$$

An intersection of peak sets is called the *p-set*. A point $x \in M(\mathcal{A})$ is called the *p-point* if $\{x\}$ is a p-set.

Let $\mathfrak{P}$ be the set of p-points of $\mathcal{A}$. It is well-known that Choquet boundary of $\mathcal{A}$ coincides with the set of p-points of $\mathcal{A}$, i.e.,

$$\partial\mathcal{A} = \mathfrak{P}. \tag{4.1}$$

Let μ be representing on $\partial\mathcal{A}$ of the complex homomorphism $\phi \in M(\mathcal{A})$. According to [**43**] the *abstract Hardy space* $\mathcal{H}^p(\mu)$ is the closure of $\mathcal{A}$ in the space $L^p_\mu(\partial\mathcal{A})$ for $1 \le p < \infty$ and the *abstract Hardy space* $\mathcal{H}^\infty(\mu)$ is the weak-star closure of $\mathcal{A}$ in the Banach algebra $L^\infty_\mu(\partial\mathcal{A})$.

The functions in $\mathcal{H}^\infty(\mu)$ can be regarded as continuous functions on the maximal ideal space $M = M\left(L^\infty_\mu(\partial\mathcal{A})\right)$ of the Banach algebra $L^\infty_\mu(\partial\mathcal{A})$.

Since $\mathcal{H}^\infty(\mu)$ is closed under uniform convergence, $\mathcal{H}^\infty(\mu)$ becomes a uniform algebra on the quotient space obtained by identifying all points of M which are identified by all functions in $\mathcal{H}^\infty(\mu)$.

Each measure μ on $\partial\mathcal{A}$ determines a unique measure $\widetilde{\mu}$ on M that is representing measure of the given homomorphism ϕ which is well defined on $\mathcal{H}^\infty(\mu)$. It is easy to check that we can identify spaces $L^p_\mu(\partial\mathcal{A})$ and $L^p_{\widetilde{\mu}}(\partial\mathcal{A})$ for every $p \ge 1$. In particular, we obtain the following.

PROPOSITION 4.4. For every $1 \le p \le \infty$,

$$\mathcal{H}^p(\widetilde{\mu}) = \mathcal{H}^p(\mu)$$

via the natural identification.

In what follows we will in particular concentrate on cases of abstract Hardy spaces $\mathcal{H}^2(\mu)$ for uniform Banach algebras $\mathcal{A}$ of analytic complex functions on open balls of infinite dimensional complex Banach spaces.

4.1.2. Abstract Hardy spaces for algebras $\mathcal{A}_a(B)$. For a given Banach space X with the unit open ball B let $\mathcal{A}_w(B)$ be a subalgebra of $\mathcal{H}^\infty_{uc}(B)$, consisting of weakly uniformly continuous functions on B and let $\mathcal{A}_a(B)$ be the subalgebra of $\mathcal{A}_w(B)$ generated by approximable polynomials $\mathcal{P}_a(X)$. In other words, $\mathcal{A}_a(B)$ is the completion of $\mathcal{A}_1(X)$ with respect to the norm of uniform convergence on B.

If X' has the approximation property, then

$$\mathcal{A}_w(B) = \mathcal{A}_a(B).$$

If $X = c_0$, then by the Littlewood-Bogdanowicz-Pełczyński theorem [**81**],

$$\mathcal{A}_a(B) = \mathcal{H}^\infty_{uc}(B).$$

An another space, for which these algebras coincide, is the Tsirelson space [**6**].

Note that every bounded analytic function on B can be extended to a bounded analytic function $\widehat{f}$ on open unit ball $B_{X''}$ in the second dual X'', where $\widehat{f}$ is the Aron-Berner extension of f (see [**36**] for details). Moreover, since B is weak-star dense in $B_{X''}$ and all functions in $\mathcal{A}_a(B)$ are continuous on $B \subset B_{X''}$ with respect to the weak-star topology on X'', the operator of Aron-Berner extension

$$\mathcal{A}_a(B) \ni f \longmapsto \widehat{f} \in C\left(\bar{B}_{X''}\right)$$

coincides with the Gelfand transform on $\mathcal{A}_a(B)$ [**7**].

From results of Chapter 2 easily follows that the spectrum $M_a := M(\mathcal{A}_a(B))$ of $\mathcal{A}_a(B)$ consists of point evaluation functionals on the unit closed ball $\bar{B}_{X''}$ of the second dual X''. Hence,

$$M_a = \bar{B}_{X''}, \tag{4.2}$$

if $\bar{B}_{X''}$ is endowed with the weak-star topology of X''.

A probability measure μ, defined on the spectrum M_a, is said to be *norming* on $\mathcal{A}_a(B)$, if

$$\|f\|_\mu := \left(\int \left|\widehat{f}\right|^2 d\mu\right)^{1/2}, \qquad f \in \mathcal{A}_a(B)$$

is a norm over $\mathcal{A}_a(B)$. Clearly, each norming measure μ defines on $\mathcal{A}_a(B)$ the scalar product

$$\langle f_1 \mid f_2\rangle_\mu := \int \widehat{f_1}\overline{\widehat{f_1}}\, d\mu, \qquad f_1, f_2 \in \mathcal{A}_a(B).$$

PROPOSITION 4.5. Let μ be a norming measure on $\mathcal{A}_a(B)$. Then

$$\|f\|_\mu \le \|f\|_{\mathcal{A}_a(B)}, \qquad f \in \mathcal{A}_a(B).$$

PROOF. Since the probability measure μ is norming on $\mathcal{A}_a(B)$, the natural embedding of $\mathcal{A}_a(B)$ into the Lebesgue space $L^2_\mu\left(\bar{B}_{X''}\right)$ is injective. Moreover,

$$\|f\|^2_\mu = \int \left|\widehat{f}\right|^2 d\mu \le \sup_{x\in B_{X''}} \left|\widehat{f}(x)\right|^2 \mu\left(B_{X''}\right) = \|f\|^2_{\mathcal{A}_a(B)},$$

as required. □

Via the identity (4.2), the space of all complex continuous functions on the space of maximal ideals M_a coincides with the space $C_{w^*}(\bar{B}_{X''})$ of all weakly-star continuous functions on $\bar{B}_{X''}$. By the Stone-Weierstrass Theorem the system of finite sums $h = \sum \widehat{f}_n \overline{\widehat{g}}_m$ is dense in $C_{w^*}(\bar{B}_{X''})$, where f_n, g_m are homogeneous polynomials in $\mathcal{A}_a(B)$ and $\overline{\widehat{g}}_m$ is the complex conjugate of $\widehat{g}_m$.

A measure μ on $\bar{B}_{X''}$ is called *circular* or *scalar invariant* if it is invariant with respect to the scalar group

$$\bar{B}_{X''} \ni x \longmapsto e^{i\vartheta} x \in \bar{B}_{X''}, \qquad \vartheta \in (-\pi, \pi]$$

that is

$$\int h\left(e^{i\vartheta} x\right) d\mu(x) = \int h(x)\, d\mu(x), \qquad x \in \bar{B}_{X''}$$

for every $h \in C_{w^*}(\bar{B}_{X''})$.

We denote by $\mathcal{H}'_n$ with $n \in \mathbb{N}$ the completion in $L^2_\mu(\bar{B}_{X''})$ of all approximable n-homogeneous polynomials $\mathcal{P}_a({}^nX)$.

PROPOSITION 4.6. A norming measure μ on $\bar{B}_{X''}$ is circular if and only if $\mathcal{H}'_m$ is orthogonal to $\mathcal{H}'_n$ for $m \neq n$. Moreover, in this case the following decomposition formula holds

$$\int h(x)\, d\mu(x) = \frac{1}{2\pi} \int d\mu(x) \int_{-\pi}^{\pi} h\left(e^{i\vartheta} x\right) d\vartheta, \qquad h \in C_{w^*}(\bar{B}_{X''}) \tag{4.3}$$

and μ is necessary representing for origin evaluation complex homomorphism δ_0 over the algebra $\mathcal{A}_a(B)$.

PROOF. Suppose that $\mathcal{H}'_m$ is orthogonal to $\mathcal{H}'_n$ for $m \neq n$. Let h be a finite sums $\sum \widehat{f}_n \overline{\widehat{g}}_m$. By the orthogonality $\widehat{f}_n \perp \widehat{g}_m$ we have for every $\vartheta \in [0, 2\pi)$,

$$\begin{aligned}
\int h(x)\, d\mu(x) &= \sum_{n,m \in \mathbb{Z}_+} \int \widehat{f}_n(x) \overline{\widehat{g}}_m(x)\, d\mu(x) = \sum_{n \in \mathbb{Z}_+} \int \widehat{f}_n(x) \overline{\widehat{g}}_n(x)\, d\mu(x) \\
&= \sum_{n \in \mathbb{Z}_+} \int \widehat{f}_n(x) \overline{\widehat{g}}_n(x) e^{i(n-n)\vartheta}\, d\mu(x) \\
&= \sum_{n \in \mathbb{Z}_+} \int \widehat{f}_n\left(e^{i\vartheta} x\right) \overline{\widehat{g}}_n\left(e^{i\vartheta} x\right) d\mu(x) \\
&= \sum_{n,m \in \mathbb{Z}_+} \int \widehat{f}_n\left(e^{i\vartheta} x\right) \overline{\widehat{g}}_m\left(e^{i\vartheta} x\right) d\mu(x) = \int h\left(e^{i\vartheta} x\right) d\mu(x).
\end{aligned}$$

In general, approaching any $h \in C_{w^*}(\bar{B}_{X''})$ by the finite sums $\sum \widehat{f}_n \overline{\widehat{g}}_m$ and using the continuity of integration of functions in $C_{w^*}(\bar{B}_{X''})$ we obtain the equality

$$\int h(x)\, d\mu(x) = \int h\left(e^{i\vartheta} x\right) d\mu(x)$$

for any $h \in C_{w^*}(\bar{B}_{X''})$. So, the measure μ is circular. For every $h \in C_{w^*}(\bar{B}_{X''})$ the function $(\vartheta, x) \longmapsto h(e^{i\vartheta} x)$ is continuous on $(-\pi, \pi] \times \bar{B}_{X''}$. By the Fubini Theorem we have

$$\int d\mu(x) \int_{-\pi}^{\pi} h\left(e^{i\vartheta}x\right) d\vartheta = \int_{-\pi}^{\pi} d\vartheta \int h\left(e^{i\vartheta}x\right) d\mu(x).$$

However, the second integral of the right hand does not depend of $\vartheta \in (-\pi,\pi]$ and $h \circ e^{i\vartheta} \in C_{w^*}(\bar{B}_{X''})$ for every $h \in C_{w^*}(\bar{B}_{X''})$. Taking into account the equality $\int_{-\pi}^{\pi} d\vartheta = 2\pi$ we get (4.3). If $f \in \mathcal{A}_a(B)$, then $t \longmapsto f(tx)$ is an analytic function on the unit disk $D \in \mathbb{C}$. Thus,

$$\int \widehat{f}(x)\, d\mu(x) = \frac{1}{2\pi} \int d\mu(x) \int_{-\pi}^{\pi} \widehat{f}\left(e^{i\vartheta}x\right) d\vartheta = f(0).$$

So, the measure μ is representing over $\mathcal{A}_a(B)$.

Conversely, let the measure μ be circular. Suppose that $f_n \in \mathcal{P}(^nX)$ and $g_m \in \mathcal{P}(^mX)$. Then

$$\int \widehat{f}_n(x)\overline{\widehat{g}}_m(x)\, d\mu = \int \widehat{f}_n\left(e^{i\vartheta}x\right)\overline{\widehat{g}}_m\left(e^{i\vartheta}x\right) d\mu$$
$$= \frac{1}{2\pi} \int \widehat{f}_n(x)\overline{\widehat{g}}_m(x)\, d\mu(x) \int_{-\pi}^{\pi} e^{i(n-m)\vartheta}\, d\vartheta = 0$$

if $m \neq n$. So, $\mathcal{H}'_m$ is orthogonal to $\mathcal{H}'_n$. □

Via the equality (4.2), the Choquet boundary $\partial\mathcal{A}_a(B)$ of the algebra $\mathcal{A}_a(B)$ is a subset of $\bar{B}_{X''}$. Note that Arenson **[3]** for more general case has shown the equality

$$\partial\mathcal{A}_a(B) = \operatorname{ext}_{\mathbb{C}}\bar{B}_{X''}, \tag{4.4}$$

where $\operatorname{ext}_{\mathbb{C}}\bar{B}_{X''}$ means the set of *complex-extreme points* of $\bar{B}_{X''}$, that is, all points $x \in \bar{B}_{X''}$ not belonging to the relative interior of any set $\mathbb{C}(x,y) \cap \bar{B}_{X''}$ with $y \neq x$. Here $\mathbb{C}(x,y) := \{(1-\lambda)x + \lambda y \colon \lambda \in \mathbb{C}\}$ for any $x, y \in X''$.

THEOREM 4.7. *Let X be a separable Banach space. Then there exists a probability measure μ, which is defined on the Choquet boundary $\partial\mathcal{A}_a(B)$ of the uniform algebra $\mathcal{A}_a(B)$ and represents its origin evaluation functional δ_0, and which is norming on $\mathcal{A}_a(B)$.*

PROOF. Let $(y_n)_{n\in\mathbb{Z}_+}$ be a dense sequence in the unit sphere S_X of X. Then (y_n) is weak-star dense in $B_{X''}$. For every y_n we consider the unit circle $e^{i\vartheta}y_n$ with $-\pi < \vartheta \leq \pi$. Put $\chi_n = 2^{-n}\chi$, where χ is the normalized Lebesgue measure on this unit circle. Let U be a Borel subset of $B_{X''} \subset M_a$. We set

$$\nu(U) = \sum_{n\in\mathbb{Z}_+} \chi_n\Big(U \cap \{e^{i\vartheta}y_n \colon 0 \leq \vartheta < 2\pi\}\Big).$$

Let f be a nonzero function in $\mathcal{A}_a(B)$ and $\widehat{f}$ be its Aron-Berner extension. Since $\widehat{f}$ is weakly-star continuous on $B_{X''}$, there is a weakly-star open subset $O \subset X''$ such that $|\widehat{f}(x)|^2 > 0$ for every $x \in O \cap B_{X''}$. So,

$$\int |\widehat{f}|^2\, d\nu \geq \int_{O\cap B_{X''}} |\widehat{f}|^2\, d\nu > 0, \qquad f \in \mathcal{A}_a(B)$$

On the other hand, since

$$\int \hat{f}\, d\nu = \sum_{n\in\mathbb{Z}_+} \frac{1}{2^n} f(0) = f(0), \qquad f \in \mathcal{A}_a(B),$$

the measure ν is representing for δ_0 over $\mathcal{A}_a(B)$. Moreover, the relation $\nu \succ \delta_0$ is satisfied. But, there is a maximal representing measure μ for δ_0 over $\mathcal{A}_a(B)$ with respect to the order $\nu \succ \delta_0$. Via Corollary 4.3 such maximal measure $\mu \succ \delta_0$ is concentrated on the Choquet boundary $\partial\mathcal{A}_a(B)$ and it is norming on the algebra $\mathcal{A}_a(B)$, because

$$\int \left|\hat{f}\right|^2 d\mu \geq \int \left|\hat{f}\right|^2 d\nu, \qquad f \in \mathcal{A}_a(B)$$

by the maximality. Thus, the statement is proved. □

DEFINITION 4.8. We define *the abstract Hardy space* $\mathcal{H}^2_a(\mu)$ to be the closure in the Lebesgue space $L^2_\mu\left(\bar{B}_{X''}\right)$ with respect to a probability measure μ on the Choquet boundary $\partial\mathcal{A}_a(B)$, norming on $\mathcal{A}_a(B)$, and which represents the origin evaluation functional δ_0 over $\mathcal{A}_a(B)$.

Suppose that X is a Banach complex space with the separable dual X'.

Let here E is defined to be the completion of X' in the abstract Hardy space $\mathcal{H}^2_a(\mu)$ endowed with the induced norm $\|\cdot\|_\mu$ and the scalar product $\langle\cdot\mid\cdot\rangle_\mu$. As usually, E' denotes its dual. It is clear that the weakly-star dense embedding of E into X'' holds. By Proposition 4.5,

$$\|x\|_\mu \leq \|x\|_{X'} \quad \text{or} \quad \|y\|_{X''} \leq \|y\|_\mu$$

for all $x \in X'$ and $y \in \mathsf{E}'$. It means that the embeddings

$$X' \hookrightarrow \mathsf{E} = \mathsf{E}' \hookrightarrow X''$$

are norm continuous, and every open set of E with respect to the norm $\|\cdot\|_{X''}$ is also open relatively of $\|\cdot\|_\mu$.

Let $(\mathfrak{e}_j)_{j\in\mathbb{N}}$ be an orthonormal basis in E. Since X' is a dense subspace of E, we can suppose that

$$\mathfrak{e}_j \in X' \quad \text{for every} \quad j \in \mathbb{N}.$$

Thus, the weakly continuous linear extension of $\mathfrak{e}_j \in X'$,

$$\mathfrak{e}_j \colon X'' \ni x \longmapsto \mathfrak{e}_j(x),$$

is also continuous over E. As a consequence, the restriction $\mathfrak{e}_j\,|_{\mathsf{E}}$ belongs to E via the Riesz theorem. These restrictions uniquely define the linear functionals

$$\mathfrak{e}'_j := \langle\cdot\mid\mathfrak{e}_j\rangle_\mu \in \mathsf{E}' \subset X'', \qquad j \in \mathbb{N},$$

which form an orthonormal basis in E' of continuous linear functionals over E.

Consider the system of polynomials

$$\mathfrak{E} = \bigcup_{n\in\mathbb{Z}_+} \left\{\mathfrak{e}^{(k)}_{\{i\}} \colon \{i\} \in \mathbb{N}^n, (k) \in \mathbb{Z}^n_+, |(k)| = n\right\}$$

with all multi-indexes $(k) = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$ and $\{j\} = (j_1, \dots, j_n) \in \mathbb{N}^n$ such that $j_1 < \dots < j_n$, where

$$\mathfrak{e}_{\{j\}}^{(k)}(x) := \mathfrak{e}_{j_1}^{k_1}(x) \dots \mathfrak{e}_{j_n}^{k_n}(x), \qquad x \in X''.$$

Denote by $C_{w^*}^0(X'')$ the dense subspace of $C_{w^*}(\bar{B}_{X''})$, which consists of all finite sums $h = \sum_{n,m \in \mathbb{Z}_+} \widehat{f}_n \bar{\widehat{g}}_m$, where homogeneous polynomials $f_n \in \mathcal{P}_a({}^nX)$ and $g_m \in \mathcal{P}_a({}^mX)$ are considered as expanded on X''.

Let $l \in X' \subset \mathsf{E}$. Then $l' := \langle \cdot \mid l \rangle_\mu \in \mathsf{E}' \subset X''$. We say that a measure μ on $\bar{B}_{X''}$ is *circular with respect to* $l \in X'$ if

$$\int h(x - l(x)l' + e^{\mathrm{i}\vartheta} l(x)l')\, d\mu(x) = \int h(x)\, d\mu(x)$$

for every $h \in C_{w^*}^0(X'')$ and $\vartheta \in (-\pi, \pi]$.

Repeating our previous arguments with Fubini's theorem we have that if μ is circular with respect to $l \in X'$, then

$$\int h(x)\, d\mu(x) = \frac{1}{2\pi} \int d\mu(x) \int_{-\pi}^{\pi} h(x - l(x)l' + e^{\mathrm{i}\vartheta} l(x)l')\, d\vartheta.$$

THEOREM 4.9. *The system of polynomials $\mathfrak{E}$ forms an orthogonal basis in the abstract Hardy space $\mathcal{H}_a^2(\mu)$ if and only if the measure μ is circular with respect to the basis functional $\mathfrak{e}_j \in X'$ for every $j \in \mathbb{N}$.*

PROOF. Suppose that μ is circular with respect to $\mathfrak{e}_j \in X'$ for every j. Show that polynomials $\mathfrak{E}$ are orthogonal. Suppose that $\{i\} \neq \{j\}$. Then there exists an index $j \in \mathbb{N}$ such that

$$\mathfrak{e}_{\{i\}}^{(k)} = \mathfrak{e}_{i_1}^{k_1} \dots \mathfrak{e}_j^{m} \dots \mathfrak{e}_{i_s}^{k_s} \quad \text{and} \quad \mathfrak{e}_{\{j\}}^{(q)} = \mathfrak{e}_{j_1}^{q_1} \dots \mathfrak{e}_j^{n} \dots \mathfrak{e}_{j_l}^{q_l}$$

for some $m \neq n$. Thus,

$$\mathfrak{e}_{\{i\}}^{(k)}(x - \mathfrak{e}_j(x)\mathfrak{e}_j' + e^{\mathrm{i}\vartheta}\mathfrak{e}_j(x)\mathfrak{e}_j') = x_{i_1}^{k_1} \dots (e^{\mathrm{i}m\vartheta} x_j^m) \dots x_{i_s}^{k_s} = e^{\mathrm{i}m\vartheta} \mathfrak{e}_{\{i\}}^{(k)}(x)$$

and by the same reason

$$\mathfrak{e}_{\{j\}}^{(q)}(x - \mathfrak{e}_j'(x)\mathfrak{e}_j + e^{\mathrm{i}\vartheta}\mathfrak{e}_j'(x)\mathfrak{e}_j) = e^{\mathrm{i}n\vartheta} \mathfrak{e}_{\{j\}}^{(q)}(x), \qquad x \in X'',$$

where $x_j := \mathfrak{e}_j'(x)$. Since $m \neq n$, we have

$$\int \mathfrak{e}_{\{i\}}^{(k)}(x) \overline{\mathfrak{e}_{\{j\}}^{(q)}}(x)\, d\mu(x) = \frac{1}{2\pi} \int \mathfrak{e}_{\{i\}}^{(k)}(x) \overline{\mathfrak{e}_{\{j\}}^{(q)}}(x)\, d\mu(x) \int_{-\pi}^{\pi} e^{\mathrm{i}(m-n)\vartheta}\, d\vartheta = 0.$$

Hence, $\mathfrak{e}_{\{i\}}^{(k)} \perp \mathfrak{e}_{\{j\}}^{(q)}$ in the space $\mathcal{H}_a^2(\mu)$.

Now we prove that $\mathfrak{E}$ is a basis in $\mathcal{H}_a^2(\mu)$. Since $\mathfrak{e}_j \in X'$ for every $j \in \mathbb{N}$, it follows that $\mathfrak{e}_{\{j\}}^{(k)} \in \mathcal{A}_a(B)$ and so $\mathfrak{e}_{\{j\}}^{(k)} \in \mathcal{H}_a^2(\mu)$ for all polynomials $\mathfrak{e}_{\{j\}}^{(k)} \in \mathfrak{E}$. Let $\mathcal{H}_{\mathfrak{E}}$ be a closed linear span of all polynomials $\mathfrak{E}$ in $\mathcal{H}_a^2(\mu)$.

Each function $f \in \mathcal{H}_a^2(\mu)$ can be approximated by functions from $\mathcal{A}_a(B)$ in the norm of $\mathcal{H}_a^2(\mu)$. In turn, each function from $\mathcal{A}_a(B)$ can be approximated by polynomials of finite type in the uniform norm, so in the norm of $\mathcal{H}_a^2(\mu)$ as well. Therefore, if $\mathcal{H}_\mathfrak{E} \neq \mathcal{H}_a^2(\mu)$ then there exists a nonzero polynomial of the form

$$P = l_1 l_2 \dots l_m,$$

where $l_1, \dots, l_m$ is a finite sequence linear functionals in X', such that $P \perp \mathcal{H}_\mathfrak{E}$ in $\mathcal{H}_a^2(\mu)$. Via Proposition 1.4 the corresponding lineal functional f_P belongs to the dual space $(\odot_\pi^m \mathsf{E}')'$ and possesses the orthogonal property $f_P \perp \odot_\pi^m \mathsf{E}'$, since the generalizing of $\mathcal{H}_\mathfrak{E}$ system

$$\left\{\mathfrak{e}_{\{i\}}^{(k)}:\ \{i\} \in \mathbb{N}^m,\ (k) \in \mathbb{Z}_+^m,\ |(k)| = m\right\}$$

is total in $\odot_\pi^m \mathsf{E}'$. It follows that $f_P = 0$ and so $P = 0$, that gives a contradiction. Hence, $\mathcal{H}_\mathfrak{E} = \mathcal{H}_a^2(\mu)$.

Conversely, suppose that the system $\mathfrak{E}$ forms an orthogonal basis in $\mathcal{H}_a^2(\mu)$. Then every function $h \in C_{w^*}^0(X'')$ can be presented in a form

$$h = \sum_{\{i\},\{j\},(k),(q)} \lambda_{\{i\}\{j\}}^{(k)(q)} \mathfrak{e}_{\{i\}}^{(k)} \overline{\mathfrak{e}_{\{j\}}^{(q)}}, \qquad \lambda_{\{i\}\{j\}}^{(k)(q)} \in \mathbb{C},$$

where $\mathfrak{e}_{\{i\}}^{(k)}, \mathfrak{e}_{\{j\}}^{(q)} \in \mathfrak{E}$. It follows that

$$\begin{aligned}
\int h(x)\,d\mu(x) &= \sum_{\{i\},\{j\},(k),(q)} \int \lambda_{\{i\}\{j\}}^{(k)(q)} \mathfrak{e}_{\{i\}}^{(k)}(x) \overline{\mathfrak{e}_{\{j\}}^{(q)}}(x)\,d\mu(x) \\
&= \sum_{\{i\},(k)} \int \lambda_{\{i\}\{i\}}^{(k)(k)} \mathfrak{e}_{\{i\}}^{(k)}(x) \overline{\mathfrak{e}_{\{i\}}^{(k)}}(x)\,d\mu(x) \\
&= \sum_{\{i\},(k)} \int \lambda_{\{i\}\{i\}}^{(k)(k)} \mathfrak{e}_{\{i\}}^{(k)}(x) \overline{\mathfrak{e}_{\{i\}}^{(k)}}(x) \exp\left[\mathrm{i}(m-m)\vartheta\right] d\mu(x) \\
&= \sum_{\{i\},(k)} \int \lambda_{\{i\}\{i\}}^{(k)(k)} \left|\mathfrak{e}_{\{i\}}^{(k)}(x - \mathfrak{e}\mathfrak{e}_j'(x)e_j + e^{\mathrm{i}\vartheta}\mathfrak{e}_j'(x)\mathfrak{e}_j)\right|^2 d\mu(x) \\
&= \sum_{\{i\},\{j\},(k),(q)} \int \lambda_{\{i\}\{j\}}^{(k)(q)} \mathfrak{e}_{\{i\}}^{(k)}(x - \mathfrak{e}_j'(x)\mathfrak{e}_j + e^{\mathrm{i}\vartheta}\mathfrak{e}_j'(x)\mathfrak{e}_j) \times \\
&\qquad \times \overline{\mathfrak{e}_{\{j\}}^{(q)}}(x - \mathfrak{e}_j'(x)\mathfrak{e}_j + e^{\mathrm{i}\vartheta}\mathfrak{e}_j'(x)\mathfrak{e}_j)\,d\mu(x) \\
&= \int h(x - \mathfrak{e}_j'(x)\mathfrak{e}_j + e^{\mathrm{i}\vartheta}\mathfrak{e}_j'(x)\mathfrak{e}_j)\,d\mu(x)
\end{aligned}$$

for every $h \in C_{w^*}^0(X'')$. Hence, μ is circular with respect to $\mathfrak{e}_j'$ with any $j \in \mathbb{N}$. □

Note that some non-circular representing measures on the unit ball of $\mathbb{C}^2$ were discussed in [**89**].

Now we consider question: *Does exist an open subset Q in* E *such that every* $f \in \mathcal{H}_a^2(\mu)$ *can be expressed as an analytic function on* Q*?* We do not know the answer in the general case. Let us observe that each function in $\mathcal{A}_a(B)$ has the Aron-Berner extension to an analytic function on $B_{X''}$. On the other hand, the natural domain for linear functionals from $\mathcal{H}^2_a(\mu)$ is E. So, if an open set $\mathcal{Q}$ exists, it must be a subset of $B_{X''} \cap \mathsf{E}$.

For every $x = \sum_{i\in\mathbb{N}} x_i \mathfrak{e}_i \in \mathsf{E}$ with $x_i \in \mathbb{C}$ we consider a formal power series

$$\eta(x) = \sum_{n\in\mathbb{Z}_+} \sum_{\substack{(k)\in\mathbb{Z}^n_+\\ |(k)|=n}} \sum_{\{i\}\in\mathbb{N}^n} c^{(k)}_{\{i\}} \bar{x}^{(k)}_{\{i\}} \mathfrak{e}^{(k)}_{\{i\}}, \tag{4.5}$$

where is denoted

$$c^{(k)}_{\{i\}} := \frac{1}{\|e^{(k)}_{\{i\}}\|^2_\mu}, \qquad x^{(k)}_{\{i\}} := x^{k_1}_{i_1} \dots x^{k_n}_{i_n}.$$

THEOREM 4.10. *Let X be a complex Banach space with separable dual and μ be a norming representing measure for $\mathcal{A}_a(B)$ such that the system $\mathfrak{E}$ form an orthogonal basis in $\mathcal{H}^2_a(\mu)$ for some $(\mathfrak{e}_j)_{j\in\mathbb{N}}$ in X'. Then the following statements are equivalent.*

(i) *There exists an open subset $\mathcal{Q} \subset \mathsf{E}$ such that series (4.5) is convergent in $\mathcal{H}^2_a(\mu)$ for every $x \in \mathcal{Q}$.*

(ii) *$\mathcal{H}^2_a(\mu)$ is a reproducing kernel space with the reproducing kernel*
$$K(x,z) = \langle \eta(x) \mid \eta(z)\rangle_{\mathsf{E}},$$
which is defined on $\mathcal{Q} \times \mathcal{Q}$ for some open subset $\mathcal{Q} \subset \mathsf{E}$.

(iii) *For every x in an open subset $\mathcal{Q} \subset \mathsf{E}$ the linear functional $x \longmapsto f(x)$ is continuous on $\mathcal{H}^2_a(\mu)$ and each element $f \in \mathcal{H}^2_a(\mu)$ is an analytic function on $\mathcal{Q}$.*

PROOF. Suppose that (4.5) converges on an open subset $\mathcal{Q}$ of E. Then

$$\langle \mathfrak{e}^{(k)}_{\{i\}} \mid \eta(x)\rangle_{\mathsf{E}} = c^{(k)}_{\{i\}} x^{(k)}_{\{i\}} \left\| e^{(k)}_{\{i\}} \right\|^2_\mu = x^{(k)}_{\{i\}} = \mathfrak{e}^{(k)}_{\{i\}}(x), \qquad x \in \mathsf{E}.$$

So, $\langle f \mid \eta(x)\rangle_{\mathsf{E}} = f(x)$ if $f \in \mathcal{A}_a(B)$ and we can put

$$f(x) = \langle f \mid \eta(x)\rangle_{\mathsf{E}}$$

for every $f \in \mathcal{H}^2_a(\mu)$ and $x \in \mathcal{Q}$. Since $\eta(x)$ is an element in $\mathcal{H}^2_a(\mu)$ for any fixed $x \in \mathcal{Q}$, it follows that the linear functional

$$\langle \cdot \mid \eta(x)\rangle_{\mathsf{E}} : f \longmapsto f(x)$$

is continuous on $\mathcal{H}^2_a(\mu)$ for every $x \in \mathcal{Q}$. By [**90**, p. 34],

$$K(x,z) = \langle \eta(z) \mid \eta(x)\rangle_{\mathsf{E}}$$

is a reproducing kernel of $\mathcal{H}^2_a(\mu)$, which is defined on $\mathcal{Q} \times \mathcal{Q}$. So, (i) implies (ii).

According to [**90**, p. 40], the map $K(x,z)$ is continuous on $\mathcal{Q}\times\mathcal{Q}$. So, $x \longmapsto \eta'(x)$ is continuous on $\mathcal{Q}$ as well. On the other hand, from (4.5) we have that the linear functional

$$\eta'(x) := \langle \cdot \mid \eta(x)\rangle_{\mathsf{E}}, \qquad x \in \mathcal{Q}$$

on $\mathcal{H}^2_a(\mu)$ can be expressed by a convergent power series

$$\eta'(x) = \sum_{n\in\mathbb{Z}_+} \sum_{\substack{(k)\in\mathbb{Z}^n_+\\ |(k)|=n}} \sum_{\{i\}\in\mathbb{N}^n} c^{(k)}_{\{i\}} x^{(k)}_{\{i\}} \mathfrak{e}^{(k)}_{\{i\}}, \qquad x \in \mathcal{Q}.$$

Hence η' is G-analytic on $\mathcal{Q}$ [**31**, p. 201]. The both G-analyticity an continuity of η' implies that η' is an analytic map on $\mathcal{Q}$ (see [**31**, p. 198]). Hence, for every $f \in \mathcal{H}_a^2(\mu)$,

$$\mathcal{Q} \ni x \longmapsto f(x)$$

is a composition of the analytic map $\mathcal{Q} \ni x \longmapsto \eta'(x)$ and the linear functional $\eta'(x)(f) = f(x)$. So, it must be analytic and (ii) implies (iii).

Let now every $f \in \mathcal{H}_a^2(\mu)$ be an analytic function on an open set $\mathcal{Q} \subset \mathsf{E}$ and for every $x \in \mathcal{Q}$ the x-evaluation linear functional $x \longmapsto f(x)$ with $f \in \mathcal{H}_a^2(\mu)$ be well defined and continuous. But this functional coincides with $\eta'(x)$ on the basis functions $\mathfrak{E}$. Hence,

$$f(x) = \langle f \mid \eta(x) \rangle_{\mathsf{E}}$$

for every $x \in \mathcal{Q}$ and so (4.5) is convergent. It follows that (iii) implies (i). □

PROBLEM. Let μ be a norming circular representing measure for $\mathcal{A}_a(B)$ and X' is separable. Does necessary exist an orthonormal basis (e'_j) in E' such that μ is circular with respect to e'_j for every j?

4.1.3. Examples of abstract Hardy spaces.

EXAMPLE 4.11. Let $X = c_0$ with the standard basis $(\mathfrak{e}_i)_{i \in \mathbb{N}}$. Then

$$\mathcal{H}_{uc}^{*}(B_{c_0}) = \mathcal{A}_a(B_{c_0}).$$

Denote by μ a measure on the open in ℓ_∞ unit ball B_{ℓ_∞}, which is the infinity product of one-dimensional Lebesgue probability measures on intervals $[-1, 1]$. It is easy to check that coordinate functionals $(\mathfrak{e}_i)_{i \in \mathbb{N}}$ over ℓ_∞ form an orthogonal basis in E and

$$\left\| \mathfrak{e}_{\{i\}}^{(k)} \right\|_\mu^2 = \int_{[-1,1]^n} |\mathfrak{e}_{i_1}|^{2k_1} \dots |\mathfrak{e}_{i_n}|^{2k_n} \, dt_{i_1} \dots dt_{i_n}$$

$$= \int_{[-1,1]^n} dt_{i_1} \dots dt_{i_n} = 1.$$

So,

$$\eta(x) = \sum_{n \in \mathbb{Z}_+} \sum_{\substack{(k) \in \mathbb{Z}_+^n \\ |(k)| = n}} \sum_{\{i\} \in \mathbb{N}^n} x_{\{i\}}^{(k)} \mathfrak{e}_{\{i\}}^{(k)}, \qquad x \in B_{\ell_\infty}.$$

The space E coincides with the completion of the linear span of $(\mathfrak{e}_i)$ in the ℓ_2-norm and the domain $\mathcal{Q}$ of η is defined by

$$\mathcal{Q} = \left\{ x = \sum_{i \in \mathbb{N}} x_i \mathfrak{e}_i \in \mathsf{E} \colon |x_i| < 1 \right\} = \ell_2 \cap B_{\ell_\infty}.$$

The corresponding space $\mathcal{H}_a^2(\mu)$ consists of all analytic functions on $\mathcal{Q}$ which can be expressed by

$$f(x) = \langle f \mid \eta(x) \rangle_{\mathsf{E}} = \int f \, \overline{\eta(x)} \, d\mu.$$

Evidently that μ is circular and all polynomials $\mathfrak{e}_{\{i\}}^{(k)}$ together with 1 form an orthogonal basis.

Let $\phi \in M(\mathcal{H}^\infty_{uc}(B))$ and (x_α) be a net in X such that $\phi(P) = \lim_\alpha \phi(x_\alpha)$ for every polynomial P. Such net exists according to [6]. A given $t \in \mathbb{C}$ let us consider the net (tx_α). If $|t| \le 1$, then the net (tx_α) determines an element in $M(\mathcal{H}^\infty_{uc}(B))$ which we denote by $t * \phi$, and

$$t * \phi(f) = \sum_{n \in \mathbb{Z}_+} t^n \phi(f_n), \qquad f = \sum_{n \in \mathbb{Z}_+} f_n \in \mathcal{H}^\infty_{uc}(B),$$

where f_n are n-homogeneous polynomials. Let $t = e^{i\vartheta}$ with $-\pi < \vartheta \le \pi$ and χ be the Lebesgue probability measure on the unit circle

$$S^1\phi = e^{i\vartheta} * \phi \subset M(\mathcal{H}^\infty_{uc}(B)).$$

Denote by μ_ϕ the extension of χ to Borel subsets of $M(\mathcal{H}^\infty_{uc}(B))$,

$$\mu_\phi(U) = \chi\left(U \cap S^1\phi\right).$$

Then μ_ϕ represents the point evaluation functional δ_0 and

$$\int g \, d\mu_\phi = \int_{-\pi}^{\pi} g\left(e^{i\vartheta} * \phi\right) \, d\chi(\vartheta)$$

for every continuous function g on $M(\mathcal{H}^\infty_{uc}(B))$. However, μ_ϕ is not norming.

The following example contains some nontrivial representing measure for the space $\mathcal{H}^\infty_{uc}(B_{\ell_p})$ which is trivial on the subalgebra $\mathcal{A}_a(B_{\ell_p})$, where B_{ℓ_p} denotes the open unit ball in ℓ_p. Note that we do not know does exist a norming representing measure on the set of maximal ideals of $\mathcal{H}^\infty_{uc}(B_{\ell_p})$.

EXAMPLE 4.12. (Cf. [2, Examle 3.1]). Let $X = \ell_p$ for some positive integer $p > 1$ with the standard basis $(\mathfrak{e}_i)_{i \in \mathbb{N}}$. For every $n \in \mathbb{N}$, put

$$v_n = \frac{1}{n^{1/p}} (\mathfrak{e}_1 + \cdots + \mathfrak{e}_n).$$

Since $\|v_n\|_{\ell_p} = 1$, the point evaluation functional $\delta_{v_n} : f \longmapsto f(v_n)$ belongs to the set of maximal ideals of $\mathcal{H}^\infty_{uc}(B_{\ell_p})$. By the compactness of $M(\mathcal{H}^\infty_{uc}(B_{\ell_p}))$, there is an accumulating point $\phi \in M(\mathcal{H}^\infty_{uc}(B_{\ell_p}))$ of the sequence (δ_{v_n}). On the other hand, (v_n) is a weakly-zero sequence. Hence,

$$\phi(f) = \lim_{n \to \infty} f(v_n) = f(0) = \delta_0(f)$$

for every weakly continuous function, in particular for every $f \in \mathcal{A}_a(B_{\ell_p})$. Let μ_ϕ be the δ_0-representing measure associated with ϕ, which is introduced above. The support of this measure coincides with the set

$$\left\{e^{i\vartheta} * \phi \colon \vartheta \in (-\pi, \pi]\right\} =$$
$$= \left\{\psi \in M(\mathcal{H}^\infty_{uc}(B_{\ell_p})) : \psi(f) = \lim_{n \to \infty} f(e^{i\vartheta} v_n), \ f \in \mathcal{H}^\infty_{uc}(B_{\ell_p}), \ \vartheta \in (-\pi, \pi]\right\}.$$

Since $(e^{i\vartheta} v_n)$ is a weak-zero sequence for each ϑ, the Gelfand transform of every function $f \in \mathcal{A}_a(B_{\ell_p})$, $\widehat{f}$ vanishes on the support of μ_ϕ. Hence,

$$\int |\widehat{f}|^2 \, d\mu_\phi = f(0)$$

if $f \in \mathcal{A}_a\left(B_{\ell_p}\right)$ and $\mu_\phi = \delta_0$ on $\mathcal{A}_a\left(B_{\ell_p}\right)$.

But the situation is different if a function is not weakly continuous. For example, let

$$Q_p\Big(\sum_{i\in\mathbb{N}} x_i\mathfrak{e}_i\Big) := \sum_{i\in\mathbb{N}} x_i^p \in \mathcal{H}^\infty_{uc}\left(B_{\ell_p}\right).$$

Then we have

$$\phi(Q_p) = \lim_{n\to\infty} \delta_{v_n}(Q_p) = 1 \neq \delta_0(Q_p).$$

Moreover, the following equality holds

$$\int |\widehat{Q}_p|^2\, d\mu_\phi = \int_{-\pi}^{\pi} e^{\mathbf{i}p\vartheta} e^{-\mathbf{i}p\vartheta} \phi(\widehat{Q}_p)\, \overline{\phi(\widehat{Q}_p)}\, d\chi(\vartheta) = 1.$$

4.2. Hilbert-Schmidt analytic functions

Let in this section E means an complex separable Hilbert space with the scalar product $\langle x \mid y\rangle_{\mathsf{E}}$ and the norm $\|x\|_{\mathsf{E}} = \langle x \mid x\rangle_{\mathsf{E}}^{1/2}$, $(x, y \in \mathsf{E})$, in which be fixed an orthonormal basis $\{\mathfrak{e}_j \in \mathsf{E}\colon j \in \mathbb{N}\}$. Denote

$$\mathsf{B} := \{x \in \mathsf{E}\colon \|x\|_{\mathsf{E}} < 1\}, \qquad \mathsf{S} := \{x \in \mathsf{E}\colon \|x\|_{\mathsf{E}} = 1\}.$$

Here, the symmetric Fock space is defined to be the orthogonal sum

$$\mathsf{F} := \bigoplus_{n\in\mathbb{Z}_+} (\odot^n_{\mathfrak{h}}\mathsf{E}) = \mathbb{C} \oplus \mathsf{E} \oplus (\odot^2_{\mathfrak{h}}\mathsf{E}) \oplus (\odot^3_{\mathfrak{h}}\mathsf{E}) \oplus \ldots$$

with the scalar product and the Hilbertian norm, respectively

$$\langle \psi \mid \omega\rangle_{\mathsf{F}} = \sum_{n\in\mathbb{Z}_+} \langle \psi_n \mid \omega_n\rangle_{\otimes^n_{\mathfrak{h}}\mathsf{E}}, \qquad \|\psi\|_{\mathsf{F}} = \langle \psi \mid \psi\rangle_{\mathsf{F}}^{1/2},$$

where $\psi = \bigoplus_n \psi_n$, $\omega = \bigoplus_n \omega_n \in \mathsf{F}$ and $\psi_n, \omega_n \in \odot^n_{\mathfrak{h}}\mathsf{E}$. By Proposition 1.28 the system

$$\begin{aligned} \mathscr{E} &= \left\{\mathscr{E}_n : n \in \mathbb{Z}_+\right\},\\ \mathscr{E}_n &= \left\{\mathfrak{e}^{\otimes(k)}_{\{j\}} \in \odot^n_{\mathfrak{h}}\mathsf{E}\colon \{j\} \in \mathbb{N}^n, (k) \in \mathbb{Z}^n_+, |(k)| = n\right\}, \end{aligned}$$

forms an orthogonal basis in F. Here $\mathfrak{e}^{\otimes(k)}_{\{j\}} = 1$ if $|(k)| = 0$.

PROPOSITION 4.13. For each $x \in \mathsf{E}$ a Fourier decomposition of element $x^{\otimes n}$ in $\odot^n_h\mathsf{E}$ by the orthogonal basis $\mathscr{E}_n$ has the form

$$x^{\otimes n} = \sum_{\substack{(k)\in\mathbb{Z}^n_+\\ |(k)|=n}} \sum_{\{j\}\in\mathbb{N}^n} \frac{n!}{(k)!} \left\langle x^{\otimes n} \mid \mathfrak{e}^{\otimes(k)}_{\{j\}}\right\rangle_{\mathsf{F}} \mathfrak{e}^{\otimes(k)}_{\{j\}}. \tag{4.6}$$

As a F-valued function by the variable $x \in \mathsf{B}$, the series of geometric vectors

$$\omega(x) = \bigoplus_{n\in\mathbb{Z}_+} x^{\otimes n} = 1 \oplus x \oplus x^{\otimes 2} \oplus x^{\otimes 3} \oplus \ldots$$

is an analytic function with the Fourier decomposition

$$\omega(x)=\bigoplus_{n\in\mathbb{Z}_+}\sum_{\substack{(k)\in\mathbb{Z}^n_+\\ |(k)|=n}}\sum_{\{j\}\in\mathbb{N}^n}\frac{n!}{(k)!}\left\langle x^{\otimes n}\mid \mathfrak{e}^{\otimes(k)}_{\{j\}}\right\rangle_{\mathsf{F}}\mathfrak{e}^{\otimes(k)}_{\{j\}}.$$

PROOF. Since $x=\sum_{t\in\mathbb{N}}\mathfrak{e}_t\langle x\mid \mathfrak{e}_t\rangle_{\mathsf{E}}$, we have $\|x\|^2_{\mathsf{E}}=\sum_{t\in\mathbb{N}}\left|\langle x\mid \mathfrak{e}_t\rangle_{\mathsf{E}}\right|^2$. So, a Fourier decomposition has a form of the series

$$x^{\otimes n}=\left(\sum_{t\in\mathbb{N}}\mathfrak{e}_t\langle x\mid \mathfrak{e}_t\rangle_{\mathsf{E}}\right)^{\otimes n}=\sum_{|(k)|=n}\sum_{\{j\}}\frac{n!}{(k)!}\left\langle x^{\otimes n}\mid \mathfrak{e}^{\otimes(k)}_{\{j\}}\right\rangle_{\mathsf{F}}\mathfrak{e}^{\otimes(k)}_{\{j\}},$$

which is convergent in $\odot^n_{\mathfrak{h}}\mathsf{E}$. The last fact follows from the orthogonality of elements $\mathfrak{e}^{\otimes(k)}_{\{j\}}$ in $\odot^n_{\mathfrak{h}}\mathsf{E}$ and the equalities

$$\begin{aligned}\left\|x^{\otimes n}\right\|^2_{\mathsf{F}} &= \sum_{|(k)|=n}\sum_{\{j\}}\frac{n!^2}{(k)!^2}\left|\left\langle x^{\otimes n}\mid \mathfrak{e}^{\otimes(k)}_{\{j\}}\right\rangle_{\mathsf{F}}\right|^2\|\mathfrak{e}^{(k)}_{\{j\}}\|^2_{\mathsf{F}}\\ &= \sum_{|(k)|=n}\sum_{\{j\}}\frac{n!}{(k)!}\left|\left\langle x^{\otimes n}\mid \mathfrak{e}^{\otimes(k)}_{\{j\}}\right\rangle_{\mathsf{F}}\right|^2=\left(\sum_{t\in\mathbb{N}}\left|\langle x\mid \mathfrak{e}_t\rangle_{\mathsf{E}}\right|^2\right)^n.\end{aligned}$$

So, $x^{\otimes n}$ is a continuous n-homogenous $\odot^n_{\mathfrak{h}}\mathsf{E}$-valued polynomials by $x\in\mathsf{E}$.

On the other hand, using the orthogonal property $x^{\otimes n}\perp x^{\otimes m}$ at $n\neq m$ in F, we obtain the following equalities

$$\begin{aligned}\|\omega(x)\|^2_{\mathsf{F}} &= \langle\omega(x)\mid\omega(x)\rangle_{\mathsf{F}}\\ &= \sum_{n\in\mathbb{Z}_+}\left\|x^{\otimes n}\right\|^2_{\mathsf{F}}=\sum_{n\in\mathbb{Z}_+}\|x\|^{2n}_{\mathsf{E}}=\frac{1}{1-\|x\|^2_{\mathsf{E}}}.\end{aligned}$$

Thus, the series $\bigoplus_n x^{\otimes n}$ is absolutely and uniformly convergent in the space F for all $\|x\|\le 1-\varepsilon$ with $\varepsilon\in(0,1)$. The Taylor coefficients of $\omega(x)$ at origin are equal to $x^{\otimes n}$. Then, as is well-know [**50**, Proposition 2.4.2], the F-valued function $\omega(x)$ is analytic by $x\in\mathsf{B}$. Now it remains to substitute instead of coefficients $x^{\otimes n}$ their Fourier decompositions. □

Recall that if all Taylors coefficients of an analytic complex function on B are homogeneous Hilbert-Schmidt polynomials then it means *Hilbert-Schmidt analytic function* (see e.g. [**39**]).

Applying Proposition 1.27, we can identify the continuous linear form

$$\psi^*_n:=\langle\cdot\mid\psi_n\rangle_{\otimes^n_{\mathfrak{h}}\mathsf{E}},\qquad \psi_n\in\odot^n_{\mathfrak{h}}\mathsf{E},$$

which belongs to the Hermitian dual space $\odot^n_{\mathfrak{h}}\mathsf{E}^*$, with the n-homogeneous Hilbert-Schmidt polynomial

$$\psi^*_n\colon \mathsf{E}\ni x\longmapsto\psi^*_n(x):=\left\langle x^{\otimes n}\mid\psi_n\right\rangle_{\otimes^n_{\mathfrak{h}}\mathsf{E}}.$$

PROPOSITION 4.14. For each $\psi = \bigoplus_{n\in\mathbb{Z}_+}\psi_n \in \mathsf{F}$ with $\psi_n \in \odot_{\mathfrak{h}}^n\mathsf{E}$ assigns the unique Hilbert-Schmidt analytic function on the open ball B, bounded on any balls $\{\varepsilon\mathsf{B}\colon 0<\varepsilon<1\}$,

$$\psi^*(x) := \langle\omega(x)\mid\psi\rangle_{\mathsf{F}} = \sum_{n\in\mathbb{Z}_+}\psi_n^*(x), \qquad x\in\mathsf{B},$$

with the Taylor coefficients at origin

$$\psi_n^*(x) = \sum_{\substack{(k)\in\mathbb{Z}_+^n\\ |(k)|=n}}\sum_{\{j\}\in\mathbb{N}^n}\frac{n!}{(k)!}\mathfrak{e}_{\{j\}}^{*(k)}(x)\big\langle\mathfrak{e}_{\{j\}}^{(k)}\mid\psi_n\big\rangle_{\otimes_{\mathfrak{h}}^n\mathsf{E}}, \qquad x\in\mathsf{E}.$$

PROOF. It is enough to use the previous Proposition 4.13. In fact, any function ψ^* is the composition

$$\psi^*(x) = (\psi^*\circ\omega)(x)$$

of corresponding linear continuous functional $\psi^* = \langle\cdot\mid\psi\rangle_{\mathsf{F}}$ over F and F-valued analytic function by $x\in\mathsf{B}$. Hence, it is also analytic on B (see [50, Proposition 3.1.2]).

For any $x = ra\in\mathsf{B}$ with $\|a\|_{\mathsf{E}} = 1$ and $r\in[0,1)$ we have

$$\psi_n^*(a) = \frac{1}{n!}\frac{d_0^n\psi^*(ra)}{dr^n}\Big|_{r=0}.$$

So, the polynomial ψ_n^* is a Taylor coefficient of the analytic function ψ^*. Now its Fourier decomposition follows from (4.6). □

The Hilbert space of complex Hilbert-Schmidt analytic functions in the open ball B and bounded on subballs $\{\varepsilon\mathsf{B}\colon 0<\varepsilon<1\}$,

$$\mathcal{H}^2 = \{\psi^*\colon\psi\in\mathsf{F}\} \qquad \text{with the Fock norm} \qquad \|\psi^*\|_{\mathcal{H}^2} = \|\psi\|_{\mathsf{F}},$$

we mean *the dual symmetric Fock space*, associated with E. Respectively, the subspace of all n-homogeneous Hilbert-Schmidt polynomials

$$\mathcal{P}_{\mathfrak{h}}(^n\mathsf{E}) := \{\psi_n^*\colon\psi_n\in\odot_{\mathfrak{h}}^n\mathsf{E}\}\hookrightarrow\mathcal{H}^2$$

we equip by the induced norm

$$\|\psi_n^*\|_{\mathcal{H}^2} = \|\psi_n\|_{\otimes_{\mathfrak{h}}^n\mathsf{E}}.$$

Consider in more details the system of homogeneous Hilbert-Schmidt polynomials

$$\mathscr{E}^* := \Big\{\mathscr{E}_n^*\colon n\in\mathbb{Z}_+\Big\},$$
$$\mathscr{E}_n^* := \Big\{\mathfrak{e}_{\{j\}}^{*(k)}\colon \{j\}\in\mathbb{N}^n,\ (k)\in\mathbb{Z}_+^n,\ |(k)| = n\Big\},$$

generated by $\mathscr{E}$, where

$$\mathfrak{e}_{\{j\}}^{*(k)}(x) := \big\langle x^{\otimes n}\mid\mathfrak{e}_{\{j\}}^{\otimes(k)}\big\rangle_{\otimes_{\mathfrak{h}}^n\mathsf{E}} = \langle x\mid\mathfrak{e}_{j_1}\rangle_{\mathsf{E}}^{k_1}\cdots\langle x\mid\mathfrak{e}_{j_n}\rangle_{\mathsf{E}}^{k_n}, \qquad x\in\mathsf{E}.$$

PROPOSITION 4.15. Let $\psi = \bigoplus_{n\in\mathbb{Z}_+}\psi_n \in \mathsf{F}$ with $\psi_n \in \odot_{\mathfrak{h}}^n \mathsf{E}$. The antilinear mapping

$$(4.7)\qquad {}^*\colon \mathsf{F} \ni \psi \longmapsto \psi^* \in \mathcal{H}^2 \qquad \big(\text{resp. } {}^*\colon \odot_{\mathfrak{h}}^n \mathsf{E} \ni \psi_n \longmapsto \psi_n^* \in \mathcal{P}_{\mathfrak{h}}({}^n\mathsf{E})\big)$$

is an isometry onto the space $\mathcal{H}^2$ $\big(\text{resp. onto } \mathcal{P}_{\mathfrak{h}}({}^n\mathsf{E})\big)$ with the scalar product

$$(4.8)\qquad \langle \psi_1^* \mid \psi_2^* \rangle_{\mathcal{H}^2} = \langle \psi_2 \mid \psi_1 \rangle_{\mathsf{F}}, \qquad \psi_1, \psi_2 \in \mathsf{F}$$

and the orthogonal basis $\mathscr{E}^*$ $\big(\text{resp. } \mathscr{E}_n^* \text{ in } \mathcal{P}_{\mathfrak{h}}({}^n\mathsf{E})\big)$. The operators

$$(4.9)\qquad \begin{aligned} \psi^* &\longmapsto \mathcal{C}[\psi^*] := \langle \psi^* \mid \mathcal{C}(x,\cdot) \rangle_{\mathcal{H}^2} \\ \psi_n^* &\longmapsto \mathcal{C}[\psi_n^*] := \langle \psi_n^* \mid x^{*n} \rangle_{\mathcal{H}^2}, \end{aligned}$$

generated by the kernel

$$\begin{aligned} \mathcal{C}(x,y) &:= \langle \omega(x) \mid \omega(y) \rangle_{\mathsf{F}} = \langle \omega(y^*) \mid \omega(x^*) \rangle_{\mathcal{H}^2} \\ &= 1 + y^*(x) + y^{*2}(x) + \dots \end{aligned}$$

for all $x \in \mathsf{B}$ and $y \in \mathsf{B} \cup \mathsf{S}$ (being Hilbert-Schmidt analytic by $x \in \mathsf{B}$ and anti-analytic by $y \in \mathsf{B}$) and by the kernel

$$y^{*n}(x) = \langle y^{*n} \mid x^{*n} \rangle_{\mathcal{H}^2_{\mathfrak{h}}} = \sum_{\substack{(k)\in\mathbb{Z}_+^n \\ |(k)|=n}} \sum_{\{j\}\in\mathbb{N}^n} \frac{n!}{(k)!}\, \mathfrak{e}^{*(k)}_{\{j\}}(x)\, \overline{\mathfrak{e}^{*(k)}_{\{j\}}}(y)$$

for all $x, y \in \mathsf{E}$ are the identity mappings in $\mathcal{H}^2$ and $\mathcal{P}_{\mathfrak{h}}({}^n\mathsf{E})$, respectively.

PROOF. The one-to-one anti-linear mappings (4.7) directly follow from Proposition 1.28. The antilinear isometry $*$ is a consequence of the norm $\|\cdot\|_{\mathcal{H}^2}$ definition. Consequently, the set $\mathfrak{e}^{*(k)}_{\{j\}}$ with $|(k)| = n$ and all $\{j\}$ forms an orthogonal basis in $\mathcal{P}_{\mathfrak{h}}({}^n\mathsf{E})$. Using (4.6), we obtain

$$\begin{aligned} \mathcal{C}(x,y) = \langle \omega(x) \mid \omega(y) \rangle_{\mathsf{F}} &= \sum_{n\in\mathbb{Z}_+} \langle x^{\otimes n} \mid y^{\otimes n} \rangle_{\otimes_{\mathfrak{h}}^n \mathsf{E}} \\ &= \sum_{n\in\mathbb{Z}_+} \sum_{\substack{(k)\in\mathbb{Z}_+^n \\ |(k)|=n}} \sum_{\{j\}\in\mathbb{N}^n} \frac{n!}{(k)!}\, \mathfrak{e}^{*(k)}_{\{j\}}(x)\, \overline{\mathfrak{e}^{*(k)}_{\{j\}}}(y) \end{aligned}$$

where the series is convergent for all $x \in \mathsf{B}$ and $y \in \mathsf{B} \cup \mathsf{S}$, since

$$|\mathcal{C}(x,y)| \le \sum_{n\in\mathbb{Z}_+} |\langle x \mid y \rangle_{\mathsf{E}}|^n = \frac{1}{1 - |\langle x \mid y \rangle_{\mathsf{E}}|}.$$

This implies that $\mathcal{C}(x,y)$ is analytic by $x \in \mathsf{B}$ and anti-analytic by $y \in \mathsf{B}$. Substituting the above expression for $\mathcal{C}(\cdot,\cdot)$ in (3.14) and using the orthogonal property $\psi_n^* \perp x^{*m}$ at $n \neq m$ in $\mathcal{H}^2$, we have

$$\psi^*(x) = \sum_{n\in\mathbb{Z}_+} \langle \psi_n^* \mid x^{*m} \rangle_{\mathcal{H}^2} = \langle \psi^* \mid \mathcal{C}(x,\cdot) \rangle_{\mathcal{H}^2}$$

for all $x \in \mathsf{B}$. In particular,

$$\langle \mathfrak{e}^{*(k)}_{(j)} \mid x^{*n} \rangle_{\mathcal{H}^2} = \frac{n!}{(k)!} \mathfrak{e}^{*(k)}_{(j)}(x) \langle \mathfrak{e}^{*(k)}_{(j)} \mid \mathfrak{e}^{*(k)}_{(j)} \rangle_{\mathcal{H}^2} = \mathfrak{e}^{*(k)}_{(j)}(x)$$

for all $\mathfrak{e}^{\otimes(k)}_{(j)} \in \mathscr{E}_n$, hence $\mathcal{C}(\cdot,\cdot)$ produces the identity mapping in $\mathcal{H}^2$. □

Theorem 4.15 can also be treated as some functional generalization of the Cauchy type integral formula for analytic functions. This can be formulated as follows.

COROLLARY 4.16. Each analytic function $\psi^* \in \mathcal{H}^2$ can be uniquely written in the Cauchy type form

$$\mathcal{C}[\psi^*](x) = \langle \psi^* \mid \mathcal{C}(x,\cdot) \rangle_{\mathcal{H}^2}, \qquad x \in \mathsf{B}.$$

COROLLARY 4.17. Each analytic function $\psi^* \in \mathcal{H}^2$ can be uniquely written in the Poisson type form

$$\mathscr{P}[\psi^*](x) = \langle \psi^* \mid \mathscr{P}(x,\cdot) \rangle_{\mathcal{H}^2}, \qquad x \in \mathsf{B} \tag{4.10}$$

with the kernel

$$\mathscr{P}(x,y) := \frac{1 - \|x\|^2_{\mathsf{E}}}{|1 - \langle x \mid y \rangle_{\mathsf{E}}|^2} > 0, \qquad y \in \mathsf{B} \cup \mathsf{S}.$$

PROOF. If we put

$$\phi^*(y) := \frac{\mathcal{C}(y,x)}{\mathcal{C}(x,x)} \psi^*(x)$$

then $\phi^* \in \mathcal{H}^2$ and $\phi^*(x) = \psi^*(x)$ for all $x \in \mathsf{B}$. Hence, we have

$$\mathscr{P}[\psi^*](x) = \langle \phi^* \mid \mathcal{C}(x,\cdot) \rangle_{\mathcal{H}^2} = \langle \psi^* \mid \mathscr{P}(x,\cdot) \rangle_{\mathcal{H}^2},$$

since

$$\mathscr{P}(x,y) = \frac{1 - \|x\|^2_{\mathsf{E}}}{|1 - \langle x \mid y \rangle_{\mathsf{E}}|^2} = \frac{\mathcal{C}(x,y)\mathcal{C}(y,x)}{\mathcal{C}(x,x)}$$

for all $x \in \mathsf{B}$ and $y \in \mathsf{B} \cup \mathsf{S}$. □

The Hardy class $\mathcal{H}^2(d\varsigma)$ from the Section 3.1 and the Hermitian dual of symmetric Fock space

$$\mathcal{H}^2 = \mathsf{F}^*$$

possesses the same orthogonal basis $\mathscr{E}^*$ (see Theorem 3.6).

PROPOSITION 4.18. Let $f = \sum_{n \in \mathbb{Z}_+} f_n \in \mathsf{F}$ with $f_n \in \odot^n_{\mathfrak{h}} \mathsf{E}$. Every element

$$f^* = \sum_{n \in \mathbb{Z}_+} f^*_n \in \mathcal{H}^2 \qquad \text{with} \qquad f^*_n \in \mathcal{P}_{\mathfrak{h}}({}^n\mathsf{E})$$

generates the analytic function on the Hilbertian open ball B

$$\mathfrak{F}[f^*](x) := f^*\Big(\bigoplus_{n \in \mathbb{Z}_+} x^{\otimes n}\Big) = \sum_{n \in \mathbb{Z}_+} f^*_n(x), \qquad x \in \mathsf{B} \tag{4.11}$$

with the Taylor series at the origin

$$f_n^*(x) = \sum_{\substack{(k)\in\mathbb{Z}_+^n \\ |(k)|=n}} \sum_{\{j\}\in\mathbb{N}^n} \frac{n!}{(k)!} \mathfrak{e}^{*(k)}_{\{j\}}(x) \left\langle \mathfrak{e}^{(k)}_{\{j\}} \mid f_n \right\rangle_{\otimes_{\mathfrak{h}}^n \mathsf{E}}, \qquad x \in \mathsf{E}$$

and the point-evaluation functional

$$\delta_x^{\mathsf{F}} \colon f^* \mapsto \mathfrak{F}[f^*](x)$$

is continuous for every $x \in \mathsf{B}$.

PROOF. In fact, since

$$\mathfrak{F}[f^*](x) = \langle \omega(x) \mid f \rangle_{\mathsf{F}}$$

and $\omega(x) \in \mathsf{F}$ for every $x \in \mathsf{B}$, the functional δ_x^{F} is continuous on the space $\mathcal{H}^2$ endowed with the norm induced by F. □

PROPOSITION 4.19. In the case if

$$n_r = 1 \qquad \text{for all} \qquad r \in \mathbb{N},$$

the following contractive dense embeddings

$$\mathcal{H}^2(d\varsigma) \hookrightarrow \mathcal{H}^2 \quad \text{and} \quad \mathcal{H}^2_n \hookrightarrow \mathcal{P}_{\mathfrak{h}}(^n\mathsf{E}) \tag{4.12}$$

for all $n \in \mathbb{Z}_+$ hold.

PROOF. By Proposition 1.28 the system $\mathscr{E}$ forms an orthogonal bases in the symmetric Fock space F and

$$\left\|\mathfrak{e}^{\otimes(k)}_{\{j\}}\right\|^2_{\mathsf{F}} = \left\|\mathfrak{e}^{\otimes(k)}_{\{j\}}\right\|^2_{\otimes_{\mathfrak{h}}^n \mathsf{E}} = \frac{(k)!}{n!}, \qquad n = |(k)| \quad \text{for all} \quad \{j\} \in \mathbb{N}^n.$$

Now from Theorem 3.7 it follows, that

$$\left\|\mathfrak{e}^{\otimes(k)}_{\{j\}}\right\|^2_{\mathsf{F}} \le 1 = \left\|\mathfrak{e}^{*(k)}_{\{j\}}\right\|^2_{L^2_\varsigma}.$$

Via Theorem 3.6 for every function $f_n^* \in \mathcal{H}^2_n$ there exists a Fourier decomposition

$$f_n^* = \sum_{\substack{(k)\in\mathbb{Z}_+^n \\ |(k)|=n}} \sum_{\{j\}\in\mathbb{N}^n} \alpha^{(k)}_{\{j\}} \mathfrak{e}^{*(k)}_{\{j\}} \qquad \text{in} \qquad \mathcal{H}^2_n$$

with the coefficients $\alpha^{(k)}_{\{j\}} \in \mathbb{C}$. It follows, that

$$\begin{aligned} \|f_n\|^2_{\otimes_{\mathfrak{h}}^n \mathsf{E}} &= \sum_{|(k)|=n} \sum_{\{j\}\in\mathbb{N}^n} \left|\alpha^{(k)}_{\{j\}}\right|^2 \frac{(k)!}{n!} \\ &\le \sum_{|(k)|=n} \sum_{\{j\}\in\mathbb{N}^n} \left|\alpha^{(k)}_{\{j\}}\right|^2 = \|f_n^*\|^2_{L^2_\varsigma}. \end{aligned}$$

Hence, the embedding $\mathcal{H}^2_n \hookrightarrow \mathcal{P}_{\mathfrak{h}}(^n\mathsf{E})$ is contractive for all n. Therefore

$$\|f\|^2_{\mathsf{F}} = \sum_{n\in\mathbb{Z}_+} \|f_n\|^2_{\otimes_{\mathfrak{h}}^n \mathsf{E}} \le \sum_{n\in\mathbb{Z}_+} \|f_n^*\|^2_{L^2_\varsigma} = \|f^*\|^2_{L^2_\varsigma}$$

for all $f^* = \sum_n f_n \in \mathcal{H}^2(d\varsigma)$ with $f_n^* \in \mathcal{H}_n^2$ and the embeddings (4.12) are proved. Since the system $\mathscr{E}_n^*$ forms an orthogonal basis in $\mathcal{H}_n^2$ for all n, the embeddings (4.12) are dense. □

THEOREM 4.20. *Let $x \in \mathsf{B}\bigcap\mathsf{B}_{\mathcal{G}}^{\infty}$ and $f^* \in \mathcal{H}^2\bigcap\mathcal{H}^2(d\varsigma)$. Then*

$$\mathfrak{C}[f^*](x) = \mathfrak{F}[f^*](x)$$

for any integer sequence (n_r).

PROOF. We observe that

$$\mathfrak{C}[f^*](x) = \delta_x(f^*), \qquad \mathfrak{F}[f^*](x) = \delta_x^{\mathsf{F}}(f^*)$$

and both δ_x and δ_x^{F} are continuous if $x \in \mathsf{B}\bigcap\mathsf{B}_{\mathcal{G}}^{\infty}$. So they coincide on the common domain if they are equal each to other on basis functions. But

$$\begin{aligned}\delta_x\left(\mathfrak{e}_{\{j\}}^{*\otimes(k)}\right) &= \mathfrak{e}_{j_1}^{*\otimes k_1}(x)\cdot\mathfrak{e}_{j_2}^{*\otimes k_2}(x)\cdot\ldots\cdot\mathfrak{e}_{j_n}^{*\otimes k_n}(x)\\ &= \delta_x^{\mathsf{F}}\left(\mathfrak{e}_{\{j\}}^{*\otimes(k)}\right)\end{aligned}$$

for all multi-indexes (k) and $\{j\}$. So,

$$\mathfrak{C}[f^*](x) = \mathfrak{F}[f^*](x)$$

for every $f^* \in \mathcal{H}^2\bigcap\mathcal{H}^2(d\varsigma)$ and $x \in \mathsf{B}\bigcap\mathsf{B}_{\mathcal{G}}^{\infty}$. □

The following proposition gives a natural isomorphism between $\mathcal{H}^2$ and $\mathcal{H}^2(d\varsigma)$ for any integer sequence (n_r).

PROPOSITION 4.21. Let J be a linear operator from $\mathcal{H}^2$ to $\mathcal{H}^2(d\varsigma)$ defined on the basis functions by the following way

$$J\left(\mathfrak{e}_{\{j\}}^{*\otimes(k)}\right) = \sqrt{\frac{(k)!}{n!}}\frac{\mathfrak{e}_{\{j\}}^{*\otimes(k)}}{\sqrt{\left\langle \mathfrak{e}_{\{j\}}^{*\otimes(k)} \mid \mathfrak{e}_{\{j\}}^{*\otimes(k)}\right\rangle_{L^2_\varsigma}}}, \qquad n = |(k)|.$$

Then J is an isometrical isomorphism.

PROOF. From the definition of J we have that

$$J\left(\frac{e_{\{j\}}^{*\otimes(k)}}{\|e_{\{j\}}^{\otimes(k)}\|_{\mathsf{F}}}\right) = \frac{e_{\{j\}}^{*\otimes(k)}}{\|e_{\{j\}}^{*\otimes(k)}\|_{L^2_\varsigma}}.$$

That is, J maps one-to-one the orthonormal basis of $\mathcal{H}^2$ onto the orthonormal basis of $\mathcal{H}^2(d\varsigma)$. So J is an isometrical isomorphism. □

Note that if $n_r = 1$ for all $r \in \mathbb{N}$, then

$$J\left(e_{\{j\}}^{*\otimes(k)}\right) = \sqrt{\frac{(k)!}{n!}}\,e_{\{j\}}^{*\otimes(k)}.$$

4.3. Holomorphicity and generalized symmetric Fock spaces

4.3.1. Generalized symmetric Fock spaces. Let E means an complex separable Hilbert space in which is fixed an orthonormal basis $\{\mathfrak{e}_i \in \mathsf{E} \colon i \in \mathbb{N}\}$. For any Fourier expansion $x = \sum_{i\in\mathbb{N}} x_i\mathfrak{e}_i$ with $x_i \in \mathbb{C}$ convergent in E we have that $x_i = \langle x \mid \mathfrak{e}_i\rangle_{\mathsf{E}}$. In the algebraic direct sum

$$\mathbb{C} \dotplus \mathsf{E} \dotplus \odot^2\mathsf{E} \dotplus \cdots \dotplus \odot^n\mathsf{E} \dotplus$$

we consider the system

$$\mathscr{E} = \bigcup_{n\in\mathbb{Z}_+} \left\{\mathfrak{e}_{\{i\}}^{\otimes(k)} \colon \{i\} \in \mathbb{N}^n,\ (k) \in \mathbb{Z}_+^n,\ |(k)| = n\right\},$$

where $\{i\} = (i_1, \ldots, i_n) \in \mathbb{N}^n$ such that $i_1 < \cdots < i_n$ and $\mathfrak{e}_{\{i\}}^{(k)} \equiv 1$ if $|(k)| = 0$. It is also clear that $\mathfrak{e}_{\{i\}}^{\otimes(k)} \in \odot^n\mathsf{E}$ and for all $x \in \mathsf{E}$ we have

$$\mathfrak{e}_{\{i\}}^{*(k)}(x) = \left\langle x^{\otimes n} \mid \mathfrak{e}_{\{i\}}^{\otimes(k)}\right\rangle_{\otimes_h^n\mathsf{E}}, \qquad n = |(k)|.$$

Then we can consider the system

$$\mathscr{E}^* = \bigcup_{n\in\mathbb{Z}_+} \left\{\mathfrak{e}_{\{i\}}^{*(k)} \colon \{i\} \in \mathbb{N}^n,\ (k) \in \mathbb{Z}_+^n,\ |(k)| = n\right\},$$

of Hilbert-Schmidt polynomials over E,

$$\mathfrak{e}_{\{i\}}^{*(k)}(x) = x_{\{i\}}^{(k)}, \qquad x_{\{i\}}^{(k)} := x_{i_1}^{k_1} \ldots x_{i_n}^{k_n},$$

where $\{i\} = (i_1, \ldots, i_n) \in \mathbb{N}^n$ such that $i_1 < \cdots < i_n$ and $\mathfrak{e}_{\{i\}}^{(k)} \equiv 1$ if $|(k)| = 0$.

We say that a Hilbert space F_η with an arbitrary norm $\|\cdot\|_\eta$ is an *(generalized) symmetric Fock space* over E if the system $\mathscr{E}$ forms an orthogonal basis in F_η (not necessarily, orthonormal).

Evidently, the norm $\|\cdot\|_\eta$ is completely defined by its value on the basis vectors $\mathscr{E}$. Hence, setting $\left\|\mathfrak{e}_{\{i\}}^{\otimes(k)}\right\|_\eta$ by arbitrary positive numbers, we can get various symmetric Fock type space over E. Let $\langle\cdot \mid \cdot\rangle_\eta$ be the scalar product in F_η. Put

$$c_{\{i\}}^{(k)} := \left\|\mathfrak{e}_{\{i\}}^{\otimes(k)}\right\|_\eta^{-2}, \qquad c_{\{i\}}^{(k)} = \begin{cases} 1 & : n = |(k)| = 0, \\ 1 & : n = |(k)| = 1 \end{cases} \quad \text{for all } \{i\} = i \in \mathbb{N}.$$

Let us consider a formal power series

$$\eta(x) = \sum_{n\in\mathbb{Z}_+} \sum_{\substack{(k)\in\mathbb{Z}_+^n \\ |(k)|=n}} \sum_{\{i\}\in\mathbb{N}^n} c_{\{i\}}^{(k)} x_{\{i\}}^{(k)} \mathfrak{e}_{\{i\}}^{\otimes(k)}, \qquad x \in \mathsf{E} \tag{4.13}$$

THEOREM 4.22. *Suppose, that there are constants $c > 0$ and $M > 0$ such that for all multi-indexes $\{i\} \in \mathbb{N}^n$ and $(k) \in \mathbb{Z}_+^n$ inequalities*

$$c_{\{i\}}^{(k)} \le cM^{2n}\frac{n!}{(k)!}, \qquad n = |(k)| \in \mathbb{N} \tag{4.14}$$

hold. Then there exists an open zero neighbourhood $\mathcal{Q} \subset \mathsf{E}$ such that:

(i) *the series* (4.13) *is convergent for every* $x \in \mathcal{Q}$ *and* η *is an analytic map from* $\mathcal{Q}$ *into* F_η;

(ii) *for every* $\phi \in \mathsf{F}_\eta$ *the map*

$$f_\phi(x) = \langle \eta(x) \mid \phi \rangle_\eta, \qquad x \in \mathcal{Q}$$

is an analytic function on $\mathcal{Q}$;

(iii) *the function* $\langle \eta(x) \mid \mathfrak{e}^{\otimes(k)}_{\{i\}} \rangle_\eta$ *is an* n*-homogeneous polynomial and*

$$\langle \eta(x) \mid \mathfrak{e}^{\otimes(k)}_{\{i\}} \rangle_\eta = x^{(k)}_{\{i\}}, \qquad x \in \mathsf{E}.$$

PROOF. Evidently, $\eta(0) = 1$. For any fixed $n \in \mathbb{Z}_+$ let

$$\eta_n(x) = \sum_{\substack{(k)\in\mathbb{Z}^n_+ \\ |(k)|=n}} \sum_{\{i\}\in\mathbb{N}^n} c^{(k)}_{\{i\}} x^{(k)}_{\{i\}} \mathfrak{e}^{\otimes(k)}_{\{i\}}, \qquad x \in \mathsf{E}.$$

It is clear that $\eta_n(x)$ is an n-homogeneous Hilbert-Schmidt polynomial from E to F_η. For every $x \in \mathsf{E}$, $\|x\|_\mathsf{E} \le 1$ we can write

$$\begin{aligned}
\|\eta_n(x)\|^2_\eta &= \sum_{\substack{(k)\in\mathbb{Z}^n_+ \\ |(k)|=n}} \sum_{\{i\}\in\mathbb{N}^n} \left\langle c^{(k)}_{\{i\}} x^{(k)}_{\{i\}} \mathfrak{e}^{\otimes(k)}_{\{i\}} \;\middle|\; c^{(k)}_{\{i\}} x^{(k)}_{\{i\}} \mathfrak{e}^{\otimes(k)}_{\{i\}} \right\rangle_\eta \\
&= \sum_{\substack{(k)\in\mathbb{Z}^n_+ \\ |(k)|=n}} \sum_{\{i\}\in\mathbb{N}^n} \left(c^{(k)}_{\{i\}}\right)^2 \left|x^{(k)}_{\{i\}}\right|^2 \left\|\mathfrak{e}^{\otimes(k)}_{\{i\}}\right\|^2_\eta = \sum_{\substack{(k)\in\mathbb{Z}^n_+ \\ |(k)|=n}} \sum_{\{i\}\in\mathbb{N}^n} c^{(k)}_{\{i\}} \left|x^{(k)}_{\{i\}}\right|^2 \\
&\le \sum_{\substack{(k)\in\mathbb{Z}^n_+ \\ |(k)|=n}} \sum_{\{i\}\in\mathbb{N}^n} cM^{2n} \frac{n!}{(k)!} \left|x^{(k)}_{\{i\}}\right|^2 \\
&= cM^{2n} \left(\sum_{i\in\mathbb{N}} |x_i|^2\right)^n = cM^{2n} \|x\|^n_\mathsf{E}.
\end{aligned}$$

Thus, we obtain

$$\sup_{\|x\|\le 1} \|\eta_n(x)\|_\eta \le \sqrt{c} M^n \|x\|^n_\mathsf{E}.$$

Hence, the radius of uniform convergence of η at zero is equal to

$$\varrho_0(\eta) = \left(\limsup_{n\to\infty} \|\eta_n\|^{1/n}_\eta\right)^{-1} \ge \left(\limsup_{n\to\infty} \left(\sqrt{c}M^n\right)^{1/n}\right)^{-1} = \frac{1}{M}.$$

Thus η is an analytic map in an open neighborhood of zero of radius $1/M$. Denote by $\mathcal{Q}$ the domain of analyticity of η in E.

Let $\phi \in \mathsf{F}_\eta$. Then the map f_ϕ is the composition of the analytic maps $\eta(x)$ and ϕ and so it is analytic (see [**50**]). Also, we can see

$$\begin{aligned}
\langle \eta(x) \mid \mathfrak{e}^{\otimes(k)}_{\{i\}} \rangle_\eta &= \left\langle \sum_{n\in\mathbb{Z}_+} \sum_{\substack{(k)\in\mathbb{Z}^n_+ \\ |(k)|=n}} \sum_{\{i\}\in\mathbb{N}^n} c^{(k)}_{\{i\}} x^{(k)}_{\{j\}} \mathfrak{e}^{\otimes(k)}_{\{j\}} \;\middle|\; \mathfrak{e}^{\otimes(k)}_{(i)} \right\rangle_\eta \\
&= c^{(k)}_{\{i\}} x^{(k)}_{\{i\}} \left\|\mathfrak{e}^{\otimes(k)}_{\{i\}}\right\|^2_\eta = x^{(k)}_{\{i\}} = x^{k_1}_{i_1} \ldots x^{k_n}_{i_n}
\end{aligned}$$

for any $x \in \mathcal{Q}$, that it was necessary to prove. □

Let us denote $\mathcal{H}_\eta$ the Hilbert space of analytic function

$$f_\phi = \langle \eta(\cdot) \mid \phi \rangle_\eta, \qquad \phi \in \mathsf{F}_\eta$$

that is Hermitian dual to F_η. We will use the same symbol $\langle \cdot \mid \cdot \rangle_\eta$ for the scal product in $\mathcal{H}_\eta$. Define an involution in F_η respectively $\langle \cdot \mid \cdot \rangle_\eta$, using the compl conjugation in $\mathcal{H}_\eta$. Thus, $\bar{f} \in \mathsf{F}_\eta$ is such that $f = \langle \cdot \mid \bar{f} \rangle_\eta$ for all $f \in \mathcal{H}_\eta$. I particularly, $f(x) = \langle \eta(x) \mid \bar{f} \rangle_\eta$, thus $f_{\bar{f}} = f$. Also for any $g \in \mathsf{F}_\eta$ we mean function from $\mathcal{H}_\eta$ such that $g = \langle \cdot \mid \bar{g} \rangle_\eta$.

Proposition 4.23. A map $K \colon \mathsf{E} \times \mathsf{E} \longrightarrow \mathbb{C}$ defined by

$$K(x,z) = \langle \bar{\eta}(z) \mid \bar{\eta}(x) \rangle_\eta = \langle \eta(x) \mid \eta(z) \rangle_\eta$$

is a reproducing kernel for $\mathcal{H}_\eta$.

The proof immediately follows from Theorem 1.33 for $h(x) := \bar{\eta}(x)$.

Example 4.24. For a fixed positive integer m set

$$\begin{aligned}
\eta(x) &= \sum_{k \in \mathbb{Z}_+} \frac{(m-1+k)!}{(m-1)!k!} x^{\otimes k} = \sum_{k \in \mathbb{Z}_+} \frac{(m-1+k)!}{(m-1)!k!} \Big(\sum_{k \in \mathbb{Z}_+} x_i e_i \Big)^{\otimes k} \\
&= \sum_{n \in \mathbb{Z}_+} \sum_{\substack{(k) \in \mathbb{Z}_+^n \\ |(k)| = n}} \sum_{\{i\} \in \mathbb{N}^n} \frac{(m-1+k)!}{(m-1)!k!} \frac{|(k)|!}{(k)!} e_{\{i\}}^{\otimes (k)} \\
&= \sum_{n \in \mathbb{Z}_+} \sum_{\substack{(k) \in \mathbb{Z}_+^n \\ |(k)| = n}} \sum_{\{i\} \in \mathbb{N}^n} \frac{(|(k)|! + m - 1)!}{(m-1)!(k)!} e_{\{i\}}^{\otimes (k)} x_{\{i\}}^{(k)}.
\end{aligned}$$

It follows that

$$c_{\{i\}}^{(k)} = \frac{(|(k)|! + m - 1)!}{(m-1)!(k)!}.$$

We can show directly that η is an analytic map from the unit open ball $\mathsf{B} \subset$ to F_η for every $m \in \mathbb{N}$. Indeed, since

$$\frac{1}{(1-t)^m} = \frac{1}{(m-1)!} \left(\frac{1}{1-t} \right)^{(m-1)} = \sum_{k \in \mathbb{Z}_+} \frac{(m-1+k)!}{(m-1)!k!} t^k,$$

where t is an independent variable and $\left(\frac{1}{1-t}\right)^{(m-1)}$ is the $(m-1)$th derivation $\frac{1}{1-t}$. So,

$$\|\eta(x)\|_\eta^2 = \sum_{k \in \mathbb{Z}_+} \frac{(m-1+k)!}{(m-1)!k!} \|x\|_{\mathsf{E}}^{2k} = \frac{1}{(1 - \|x\|_{\mathsf{E}}^2)^m}.$$

Thus η is well defined on B and locally bounded. Moreover η is a G-analytic map o B as an absolutely convergent power series on B intersected with an arbitrary finit dimensional subspace. Hence, η is analytic and $\mathcal{H}_\eta$ is a Hilbert space of analyt function on B. Note that $\mathcal{H}_\eta$ coincides with the classical Hardy space on the un ball if (and only if) $\dim \mathsf{E} = m$.

Theorem 4.22 admits the next generalization. Let

$$\eta(x) = \sum_{n\in\mathbb{Z}_+} \sum_{\substack{(k)\in\mathbb{Z}_+^n \\ |(k)|=n}} \sum_{\{i\}\in\mathbb{N}^n} b_{\{i\}}^{(k)} x_{\{i\}}^{(k)} \mathfrak{e}_{\{i\}}^{\otimes(k)}, \qquad b_{\{i\}}^{(k)} \in \mathbb{C} \tag{4.15}$$

be a formal power series with $x = \sum_{i\in\mathbb{N}} x_i \mathfrak{e}_i \in \mathsf{E}$ and $(x_i) \in \ell_2$. Denote by $\mathcal{N}$ the set of pairs of multi-indexes $(\{i\},(k))$ such that $b_{\{i\}}^{(k)} = 0$. Let

$$\mathsf{V}_{\mathcal{N}} = \operatorname*{span}_{\mathbb{C}} \left\{ \mathfrak{e}_{\{i\}}^{\otimes(k)} : (\{i\},(k)) \in \mathcal{N} \right\}.$$

The linear space $\mathsf{F}/\mathsf{V}_{\mathcal{N}}$ is spanned on the all vectors $\mathfrak{e}_{\{i\}}^{\otimes(k)}$ with $(\{i\},(k)) \notin \mathcal{N}$. We define a norm on $\mathsf{F}/\mathsf{V}_{\mathcal{N}}$ by

$$\left\| \mathfrak{e}_{\{i\}}^{\otimes(k)} \right\|_\eta^2 = \frac{1}{|b_{\{i\}}^{(k)}|}$$

assuming that such $\left(\mathfrak{e}_{\{i\}}^{\otimes(k)}\right)$ forms an orthogonal basis in $(\mathsf{F}/\mathsf{V}_{\mathcal{N}}, \|\cdot\|_\eta)$. Denote by F_η the completion of $\mathsf{F}/\mathsf{V}_{\mathcal{N}}$ by such $\|\cdot\|_\eta$. Notice that power series (4.15) is an analytic map if and only if

$$\mathcal{Q}_{\mathcal{N}} \ni x \longmapsto \sum_{n\in\mathbb{Z}_+} \sum_{\substack{(k)\in\mathbb{Z}_+^n \\ |(k)|=n}} \sum_{\{i\}\in\mathbb{N}^n} \left| b_{\{i\}}^{(k)} \right| x_{\{i\}}^{(k)} \mathfrak{e}_{\{i\}}^{\otimes(k)}$$

is an analytic map in an open domain $\mathcal{Q}_{\mathcal{N}} \subset \mathsf{E}$ and

$$\langle \eta(x) \mid \mathfrak{e}_{\{i\}}^{\otimes(k)} \rangle_\eta = \frac{b_{\{i\}}^{(k)}}{|b_{\{i\}}^{(k)}|} x_{\{i\}}^{(k)}, \qquad x \in \mathcal{Q}_{\mathcal{N}}.$$

4.3.2. Hilbert spaces of entire functions. In this section we consider the case when $\mathcal{H}_\eta = \mathsf{F}_\eta^*$ consists with entire functions on a separable Hilbert complex space E with an orthonormal basis

$$\{\mathfrak{e}_i \in \mathsf{E} : i \in \mathbb{N}\}$$

PROPOSITION 4.25. Suppose that there exists a constant $c > 0$ and a sequence of positive numbers (M_n), $M_n \to 0$ as $n \to \infty$ such that

$$c_{\{i\}}^{(k)} \leq c M_n^{2n} \frac{n!}{k!}$$

for all indexes $(k) \in \mathbb{Z}_+^n$ so that $n = |(k)|$, where

$$c_{\{i\}}^{(k)} = \left\| \mathfrak{e}_{\{i\}}^{\otimes(k)} \right\|_\eta^{-2}$$

and $\mathfrak{e}_{\{i\}}^{\otimes(k)} \in \mathscr{E}$ is an orthogonal basis in F_η. Then

$$\mathcal{H}_\eta = \mathsf{F}_\eta^*$$

is a Hilbert space of entire functions of bounded type on E.

PROOF. Repeating the proof of Theorem 4.22 we can see that the series

$$\eta(x) = \sum_{n\in\mathbb{Z}_+} \sum_{\substack{(k)\in\mathbb{Z}_+^n \\ |(k)|=n}} \sum_{\{i\}\in\mathbb{N}^n} c_{\{i\}}^{(k)} x_{\{i\}}^{(k)} \mathrm{e}_{\{i\}}^{\otimes(k)}$$

is an analytic map with the radius of uniform convergence

$$\varrho_0(\eta) = \limsup_{n\to\infty} \frac{1}{M_n} = \infty.$$

Hence η is an entire map of bounded type from E into F_η. Thus, every function $f \in \mathcal{H}_\eta$ is an entire function of bounded type on E, as a composition of bounded type entire map and linear continuous functional. □

The next proposition gives another test for $\mathcal{H}_\eta$ to be a space of entire functions.

PROPOSITION 4.26. Suppose that

$$\lim_{n\to\infty} \frac{\|\eta_{n+1}\|_\eta}{\|\eta_n\|_\eta} = 0.$$

Then $\mathcal{H}_\eta$ consists with bounded type entire functions.

PROOF. By the ratio test the power series $\sum_{n\in\mathbb{Z}_+} \|\eta_n\| t^n$ is absolutely convergent for every $t \in \mathbb{C}$. Thus, by the Cauchy-Hadamard formula,

$$\limsup_{n\to\infty} \left(\|\eta_n\|^{1/n} \right)^{-1} = \infty$$

and η is hence an entire mapping. □

EXAMPLE 4.27. Let

$$\eta(x) = \sum_{n\in\mathbb{Z}_+} \frac{x^{\otimes n}}{n!}, \qquad x \in \mathsf{E}.$$

Denote by $\mathcal{H}^2(\mathsf{E})$ the corresponding space $\mathcal{H}_\eta$. It is easy to see that $\mathcal{H}^2(\mathsf{E})$ consists with bounded type entire functions on E and

$$\left\| \mathrm{e}_{\{i\}}^{\otimes(k)} \right\|_\eta^2 = (k)!, \qquad n = |(k)|.$$

The reproduction kernel of this space is

$$\begin{aligned} K(z,x) = \langle \eta(x) \mid \eta(z) \rangle_\eta &= \sum_{n\in\mathbb{Z}_+} \frac{\langle x^{\otimes n} \mid z^{\otimes n} \rangle_\eta}{(n!)^2} \\ &= \sum_{n\in\mathbb{Z}_+} \sum_{\substack{(k)\in\mathbb{Z}_+^n \\ |(k)|=n}} \sum_{\{i\}\in\mathbb{N}^n} \frac{1}{n!} \left(\frac{n!}{(k)!} \right)^2 \left\| \mathrm{e}_{\{i\}}^{\otimes(k)} \right\|_\eta^2 x_{\{i\}}^{(k)} z_{\{i\}}^{(k)} \\ &= \sum_{n\in\mathbb{Z}_+} \frac{n!}{(k)!} x_{\{i\}}^{(k)} = \sum_{n\in\mathbb{Z}_+} \frac{\langle x \mid z \rangle_{\mathsf{E}}^n}{n!} = e^{\langle x \mid z \rangle_{\mathsf{E}}}, \end{aligned}$$

and for every function from $\mathcal{H}^2(\mathsf{E})$ there exists a unique $w \in \mathsf{F}_\eta$ such that

$$f_w(x) = \langle \eta(x) \mid w \rangle. \tag{4.16}$$

On the contrary, for any vector $w \in \mathsf{F}_\eta$ we have $f_w \in \mathcal{H}^2(\mathsf{E})$. According to [80] $\mathcal{H}^2(\mathsf{E})$ is an infinite tensor product of

$$\mathcal{H}^2(\mathbb{C}) = \left\{ f \in \mathcal{H}(\mathbb{C}) \colon \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dz < \infty \right\}.$$

4.3.3. More examples of reproducing function. Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$. Denote by Γ_r, $0 < r \leq \infty$ the set of all analytic functions on $r\mathbb{D}$ if $r < \infty$, and on $\mathbb{C}$ if $r = \infty$,

$$r\mathbb{D} \ni t \longmapsto \gamma(t) = \sum_{n \in \mathbb{Z}_+} \gamma_n t^n, \qquad \gamma_n > 0.$$

Let Ω be a stand for both $r\mathbb{D}$ and $\mathbb{C}$. Evidently, Γ_r is an open convex subset of the Fréchet space of all analytic functions $\mathcal{H}^2(\Omega)$ on Ω.

PROPOSITION 4.28. For a given $\gamma \in \Gamma_r$ and a Hilbert space E the function

$$r\mathsf{B} \ni x \longmapsto \eta[\gamma](x) = \sum_{n \in \mathbb{Z}_+} \gamma_n x^n$$

is an analytic map from the ball $r\mathsf{B} \subset \mathsf{E}$ (where $r\mathsf{E} = \mathsf{E}$ if $r = \infty$) to F_η and

$$\|\eta[\gamma](x)\|_\eta^2 = \langle \eta[\gamma](x) \mid \eta[\gamma](x) \rangle_{\eta[\gamma]} = \gamma\left(\|x\|_{\mathsf{E}}^2\right) \tag{4.17}$$

for every $x \in r\mathsf{B}$.

PROOF. By the Cauchy-Hadamard formula

$$\limsup_{n \to \infty} (\gamma_n)^{1/n} \leq 1/r.$$

On the other hand,

$$\eta[\gamma](x) = \sum_{n \in \mathbb{Z}_+} \gamma_n x^n = \sum_{n \in \mathbb{Z}_+} \sum_{\substack{(k) \in \mathbb{Z}_+^n \\ |(k)| = n}} \sum_{\{i\} \in \mathbb{N}^n} \gamma_n \frac{n!}{(k)!} \mathsf{e}_{\{i\}}^{\otimes (k)} x_{\{i\}}^{(k)}. \tag{4.18}$$

We can apply Theorem 4.22 for $M_n = \gamma_n$. Calculating we have

$$\begin{aligned} \|\eta[\gamma](x)\|_\eta^2 &= \langle \eta[\gamma](x) \mid \eta[\gamma](x) \rangle_{\eta[\gamma]} = \sum_{n \in \mathbb{Z}_+} \gamma_n^2 \langle x \mid x \rangle_{\mathsf{E}}^n \\ &= \sum_{n \in \mathbb{Z}_+} \gamma_n^2 \sum_{\substack{(k) \in \mathbb{Z}_+^n \\ |(k)| = n}} \sum_{\{i\} \in \mathbb{N}^n} \left(\frac{n!}{(k)!} \right)^2 \left| x_{\{i\}}^{(k)} \right|^2 \left\| \mathsf{e}_{\{i\}}^{\otimes (k)} \right\|_\eta^2. \end{aligned}$$

Since $\gamma_n \dfrac{n!}{(k)!} = \dfrac{1}{\|\mathsf{e}_{\{i\}}^{(k)}\|_\eta^2}$, we obtain

$$\begin{aligned} \|\eta[\gamma](x)\|_\eta^2 &= \sum_{n \in \mathbb{Z}_+} \gamma_n \sum_{\substack{(k) \in \mathbb{Z}_+^n \\ |(k)| = n}} \sum_{\{i\} \in \mathbb{N}^n} \left(\frac{n!}{(k)!} \right)^2 \left| x_{\{i\}}^{(k)} \right|^2 \\ &= \sum_{n \in \mathbb{Z}_+} \gamma_n \|x\|^{2n} = \gamma(\|x\|^2). \end{aligned}$$

Hence, the equation (4.17) is proved. □

We say that η is generated by γ. Note, that the reproducing function η in Example 4.24 is generated by

$$\gamma(t) = \frac{1}{(1-t)^m},$$

and in Example 4.27 is generated by $\gamma(t) = e^t$.

It is clear that operations of addition and multiplication of functions preserve the set Γ_r. The proof of next technical lemma directly follows from definitions.

LEMMA 4.29. *Let η_1 and η_2 be reproducing functions with the same domain $\mathcal{Q} \subset \mathsf{E}$. Then*

$$\eta = \frac{\eta_1 + \eta_2}{2}$$

is a reproducing function with domain $\mathcal{Q}$ and

$$\left\| e_{\{i\}}^{\otimes(k)} \right\|_\eta^2 = \frac{2}{\|e_{\{i\}}^{\otimes(k)}\|_{\eta_1}^{-2} + \|e_{\{i\}}^{\otimes(k)}\|_{\eta_2}^{-2}}. \tag{4.19}$$

COROLLARY 4.30. Let γ_1 and γ_2 belongs to Γ_r. Then the reproducing function

$$\eta\left[\frac{\gamma_1+\gamma_2}{2}\right]$$

which is defined by $\frac{\gamma_1+\gamma_2}{2}$ is coincides with

$$\frac{\eta[\gamma_1] + \eta[\gamma_2]}{2},$$

where η_1 and η_2 are defined to be $\eta[\gamma_1]$ and $\eta[\gamma_2]$, respectively. Hence, norms of basic vectors of $\mathcal{H}_\eta$ can be computed by (4.19).

Let $\gamma, \tau \in \Gamma_r$, $\gamma(t) = \sum_{n\in\mathbb{Z}_+} t^n \gamma_n$, and $\tau(t) = \sum_{n\in\mathbb{Z}_+} t^n \tau^n$. Then $\xi(t) := \gamma(t)\tau(t) \in \Gamma_r$ and

$$\xi_n = \sum_{k=0}^{n} \gamma_k \tau_{n-k} = \sum_{l+m=n} \gamma_l \tau_m.$$

Let $\{i\} \in \mathbb{N}^N$ and $(k) \in \mathbb{Z}_+^N$. Comparing (4.18) with (4.13) for

$$x = e_{i_1} + \cdots + e_{i_N},$$

where $c_{\{i\}}^{(k)} := \left\| e_{\{i\}}^{\otimes(k)} \right\|_{\eta[\gamma]}^{-2}$, we have for arbitrary $(k) \in \mathbb{Z}_+^N$ such that $|(k)| = l$, $\gamma_l = \dfrac{(k)!}{l!\|e_{\{i\}}^{\otimes(k)}\|_{\eta[\gamma]}^2}$. By the same reasons, for any $(p) \in \mathbb{Z}_+^N$ such that $|(k)| = m$, $\tau_m = \dfrac{(p)!}{m!\|e_{\{i\}}^{(p)}\|_{\eta[\tau]}^2}$. Hence, we have $\xi_n = \displaystyle\sum_{l+m=n} \frac{(k)!(p)!}{l!m!\|e_{\{i\}}^{(k)}\|_{\eta[\gamma]}^2 \|e_{\{i\}}^{(p)}\|_{\eta[\tau]}^2}$. On the other hand, for $(q) \in \mathbb{Z}_+^N$ such that $|(k)| = l + m = n$, and $\xi_n = \dfrac{(q)!}{n!\|\mathfrak{e}_{\{i\}}^{(q)}\|^2_{\eta[\xi]}}$ and so

$$\left\|\mathfrak{e}_{\{i\}}^{(q)}\right\|^2_{\eta[\xi]} = \frac{(q)!}{|(q)|!}\left(\sum_{|(k)|+|(p)|=|(q)|}\frac{(k)!(p)!}{|(k)|!|(p)|!\|\mathfrak{e}_{\{i\}}^{(k)}\|^2_{\eta[\gamma]}\|\mathfrak{e}_{\{i\}}^{(p)}\|^2_{\eta[\tau]}}\right)^{-1}, \tag{4.20}$$

where the sum taken over all multi-indexes $(k), (p) \in \mathbb{Z}_+^N$ such that

$$\mathfrak{e}_{\{i\}}^{\otimes(k)} \odot \mathfrak{e}_{\{i\}}^{\otimes(p)} = \mathfrak{e}_{\{i\}}^{\otimes(q)}.$$

Therefore we have proved the following assertion.

PROPOSITION 4.31. If γ and τ belongs to Γ_r, then $\gamma\tau$ belongs to Γ_r and (4.20) holds.

Let now $\gamma \in \Gamma_r$ and $\tau \in \Gamma_d$. Denote by

$$\gamma \otimes \tau(t,s) := \gamma(t)\tau(s)$$

the tensor product of functions. Set

$$\eta[\gamma \otimes \tau](x,y) := \sum_{k,m\in\mathbb{Z}_+} \gamma_k\tau_m x^{\otimes k} y^{\otimes m},$$

where $x \in \mathsf{E}_1$ and $y \in \mathsf{E}_2$ for some Hilbert spaces E_1 and E_2. Let $(\mathfrak{e}_k)$ be an orthonormal basis in E_1 and $(\mathfrak{a}_k)$ an orthonormal basis in E_2.

PROPOSITION 4.32. The function $\eta[\gamma \otimes \tau]$ is reproducing with the domain $rB_{\mathsf{E}_1} \times tB_{\mathsf{E}_2}$ and

$$\left\|\mathfrak{e}_{\{i\}}^{(k)}\mathfrak{a}_{\{j\}}^{(m)}\right\|_{\eta[\gamma\otimes\tau]} = \left\|\mathfrak{e}_{\{i\}}^{(k)}\right\|_{\eta[\gamma]}\left\|\mathfrak{a}_{\{j\}}^{(m)}\right\|_{\eta[\tau]}.$$

PROOF. Let $x = \mathfrak{e}_{i_1} + \dots + \mathfrak{e}_{i_n}$ and $y = \mathfrak{a}_{j_1} + \dots + \mathfrak{a}_{j_l}$. Then the proof it follows from Theorem 4.22 and the representation

$$\eta[\gamma \otimes \tau](x,y) = \sum_{(k),(m),\{i\},\{j\}} \gamma_{|(k)|}\tau_{|(m)|}\frac{|(k)|!|(m)|!}{(k)!(m)!}\mathfrak{e}_{\{i\}}^{(k)}\mathfrak{a}_{\{j\}}^{(m)}x_{\{i\}}^{(k)}y_{\{j\}}^{(m)},$$

where the sum taken over all multi-indexes $(k) \in \mathbb{Z}_+^n$, $(m) \in \mathbb{Z}_+^l$, $\{i\} \in \mathbb{N}^n$, and $\{j\} \in \mathbb{N}^l$. □

Notes and Remarks. In [**16**] E. Bishop and K. De Leeuw introduced and investigated representing measures on subspaces and subalgebras of algebra $C(Q)$ and proved Theorem 4.2. Abstract Hardy spaces were investigated by T.Gamelin. In particular, Proposition 4.4 is in [**43**]. In Section 4.2 we present some specification of statements from [**56**] (other interpretation of these statements was given in [**83**, Theorem 2.6]). Proposition 4.14 is proved in [**56**]. Notice that an approach to Hilbertian Hardy type classes, being reproducing kernel spaces on infinite-dimensional balls, which generally not having form of a polydisk, using the Bishop-De Leeuw theorem about representing measures, have been proposed in [**59**]. Other examples of reproducing kernel Hilbertian Hardy type spaces were studied in [**55, 57**]. Theorem 4.20 is proved in [**61**].

Bibliography

[1] R. Aron, *An introduction to polynomials on Banach spaces*, Extracta Mathematics, **17**(3), (2002), 303–329.

[2] R. Alencar, R. Aron, P. Galindo and A. Zagorodnyuk, *Algebra of symmetric holomorphic functions on ℓ_p*, Bull. Lond. Math. Soc. **35**, (2003), 55–64.

[3] E.L. Arenson, *Gleason parts and the Choquet boundary of a function algebra on a convex compactum*, J. Math. Sci., **29**(6), (1983), 1832–1834.

[4] R.M. Aron and P.D. Berner, *A Hahn-Banach extension theorem for analytic mappings*, Bull. Soc. Math. France, **106**, (1978), 3–24.

[5] R.M. Aron, D. Carando, T.W. Gamelin, S. Lasalle and M. Maestre, *Cluster Values of Analytic Functions on a Banach space*, Math. Ann., **353**, (2012), 293–303.

[6] R.M. Aron, B.J. Cole and T.W. Gamelin, *Spectra of algebras of analytic functions on a Banach space*, J. Reine Angew. Math., **415**, (1991), 51–93.

[7] R.M. Aron, B.J. Cole and T.W. Gamelin, *Weak-star continuous analytic funtions*, Can. J. Math., **47**, (1995), 673–683.

[8] R.M. Aron, C. Hervés and M. Valdivia, *Weakly continuous mappings on Banach spaces*, J. Funct. Anal., **52**, (1983), 189–204.

[9] R.M. Aron, P. Galindo, D. Garcia and M. Maestre, *Regularity and algebras of analytic function in infinite dimensions*, Trans. Amer. Math. Soc., **348**, (1996), 543–559.

[10] R.M. Aron and J.B. Prolla, *Polynomial approximation of differentiable functions on Banach spaces*, J. Reine Angew. Math., **313**, (1980), 195–216.

[11] N. Aronzajn, *Teory of reproducing kernels*, Trans. Amer. Math. Soc., **68**, (1950), 337–404.

[12] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform*, Comm. Pure Appl. Math., **14**, (1961), 187–214.

[13] V. Bargmann, *Remarks on a Hilbert space of analytic functions*, Proc. Nat. Acad. Sci. U.S.A., **48**, (1962), 199–204.

[14] C.A. Berger and L.A. Coburn, *Toeplitz operators on the Segal-Bargmann space*, Trans. Amer. Math. Soc., **301**, (1987), 813–829.

[15] Yu.M. Berezanski and Yu.G. Kondratiev, *Spectral Methods in Infinite-Dimensional Analysis*, Kluwer Acad. Publ., Dordrecht, 1995.

[16] E. Bishop and K.De Leeuw, *The representations of linear functionals by measures on sets of extreme points*, Ann. Inst. Fourier, **9**, (1959), 305–331.

[17] P. Biström, J.A. Jaramillo and M. Lindström, *Polynomial compactness in Banach spaces*, Rocky Mont. J. Math., **28**, (1998), 1203–1225.

[18] J. Bochnak and J. Siciak, *Polynomials and multilinear mappings in topological vector spaces*, Studia Mathematica, **39**, (1970), 59–76.

[19] J. Bonet and A. Peris. *Hypercyclic operators on non-normable Fréchet space*, J. Funct. Anal., **159**, (1998), 387–395.

[20] A. Borodin and G. Olshanski, *Harmonic analysis on the infinite-dimensional unitary group and determinantal point processes*, Ann. Math., **161**, (2005), 1319–1422.

[21] N. Bourbaki, *Integration II*, Springer, Berlin, 2004.

[22] F. Cabello, Sánchez, R. Garcia and I. Villanueva, *Extension of multilinear operators on Banach spaces*, Extracta Math., **15**, (2000), 291–334.

[23] D. Carando, D. García and M. Maestre, *Homomorphisms and composition operators on algebras of analytic functions of bounded type*, Adv. Math., **197**(2), (2005), 607–629.

[24] D. Carando, D. Garc M. Maestre and P. Sevilla-Peris, *A Riemann manifold structure of the spectra of weighted algebras of holomorphic functions*, Topology, **48**, (2009), 54–65.

[25] D. Carando and P. Sevilla-Peris, *Spectra of weighted algebras of holomorphic functions*, Math. Z., **263**(4), (2009), 887–902.

[26] T.K. Carne, B. Cole and T.W. Gamelin, *A uniform algebra of analytic functions on a Banach space*, Trans. Amer. Math. Soc., **314**, (1989), 639–659.

[27] I. Chernega, P. Galindo and A. Zagorodnyuk, *Some algebras of symmetric analytic functions and their spectra*. Proc. Edinburgh Math. Soc., **55**, (2012), 125–142.

[28] I. Chernega, P. Galindo and A. Zagorodnyuk, *The convolution operation on the spectra of algebras of symmetric analytic functions*, J. Math. Anal. & Appl., **395**(2), (2012), 569–577.

[29] I. Chernega, P. Galindo and A. Zagorodnyuk, *The multiplicative convolution on the spectra of algebras of symmetric analytic functions*, Rev. Mat. Complut. In press. DOI: 10.1007/s13163-013-0128-0

[30] B. Cole and T.W. Gamelin, *Representing measures and Hardy spaces for the infinite polydisk algebra*, Proc. London Math. Soc., **53**, (1986), 112–142.

[31] S.B. Chae, *Holomorphy and calculus in normed spaces*, Pure and Applied Mathematics, A Series of Monographs and Textbooks, vol. 92, Marcel Dekker, Inc., New York, Basel, 1985.

[32] A.M. Davie and T.W. Gamelin, *A theorem on polynomial-star approximation*, Proc. Amer. Math. Soc., **106**, (1989), 351–356.

[33] A. Defant and K. Floret, *Tensor norms and operator ideals*, Mathematics Studies, vol. 176, North-Holland, Amsterdam, 1993.

[34] V. Dimant and R. Gonzalo, *Block diagonal polynomials*, Trans. Amer. Math. Soc., **353**, (2000), 733–747.

[35] S. Dineen, *Complex analysis in locally convex spaces*, Mathematics Studies, vol. 57, North-Holland, Amsterdam, 1981.

[36] S. Dineen, *Complex analysis on infinite dimensional spaces*, Monographs in Mathematics, Springer, 1999.

[37] P. Dixon, *Scalar homomorphisms on algebras of infinetely long polynomials with an application to automatic continuity theory*, J. London Math. Soc., **19**(2), (1979), 488–496.

[38] N. Dunford, *Uniformity in linear spaces*, Trans. Amer. Math. Soc., **44**, (1938), 305–356.

[39] T.A.W. Dwyer, Partial differential equations in Fischer-Fock spaces for the Hilbert- Schmidt holomorphy type, Bull. Amer. Math. Soc., **77**(5), (1971), 725–730.

[40] G.B. Folland, *Harmonic analysis in phase space*, Princeton, 1989.

[41] K. Floret, *Natural norms on symmetric tensor products of normed spaces*, Note di Matematica, **17**, (1997), 153–188.

[42] P. Galindo, D. García, M. Maestre and J. Mujica, *Extension of multilinear mappings on Banach spaces*, Studia. Math., **108**, (1994), 55–76.

[43] T.W. Gamelin, *Uniform algebras*, 2nd ed., Chelsea - New York, 1984.

[44] T.W. Gamelin, *Analytic functions on Banach spaces*, in Complex Function Theory, Ed. Gauthier and Sabidussi, Kluwer Academic Publishers, Amsterdam, (1994), 187–223.

[45] D. García, M.L. Lourenço, M. Maestre and L.A. Moraes, *The spectrum of analytic mappings of bounded type*, J. Math. Anal. & Appl., **245**, (2000), 447–470.

[46] D. García, M.L. Lourenço, L.A. Moraes and O.W. Paques, *The spectra of some algebras of analytic mappings*, Indag. Mathem., **10**, (1999), 393–406.

[47] R. Gonzalo, *Multilinear forms, subsymmetric polynomials, and spreading models on Banach spaces*, J. Math. Anal. App., **202**, (1996), 379–397.

[48] K.-G. Grosse-Erdmann. *Universal families and hypercyclic operators*, Bull. Amer. Math. Soc. (N.S.), **36**, (1999), 345–381.

[49] A. Grotendieck, *Sur certains espaces do fonctions holomorphes*, J. Reine Angew. Math., **192**, (1953), 35–64.

[50] M. Hervé, *Analyticity in infiite dimensional spaces*, de Gruyter Stud. in Math., vol. 10, Berlin, 1989.

[51] E. Hewitt and K.A. Ross, *Abstract harmonic analysis*, vol. 2, Springer, Berlin 1970.
[52] E. Hille and R. Phillips, *Functional Analysis and Semi-groups*, AMS reprint, 2000.
[53] S. Kerov, G. Olshanski and A. Vershik, *Harmonic analysis on the infinite symmetric group deformation of regular representation*, C.R.Acad.Sci. Paris Ser. Math., **316**, (1993), 773–778.
[54] M. Lindström and R.A. Ryan, *Applications of ultraproducts to infinite dimensional holomorphy*, Math. Scand., **71**, (1992), 229–242.
[55] O. Lopushansky and A. Zagorodnyuk, *Groups of isometries of Hardy-type spaces of analytic functionals in unit ball of the space L1*, J. Math. Sci., **107**(1), (2001), 3570–3676.
[56] O. Lopushansky and A. Zagorodnyuk, *Hilbert spaces of analytic functions of infinitely many variables*, Ann. Pol. Math., **81**(2), (2003), 111–122.
[57] O. Lopushansky and A. Zagorodnyuk, *Hilbert spaces of functions of infinitely many variables*, Methods Funct. Anal. & Topology, **10**(2), (2004), 13–20.
[58] O. Lopushansky and S. Sharyn, *Polynomial ultradistributions on cone* $\mathbb{R}^d_+$, Topology, **48**(4), (2009), 80–90.
[59] O. Lopushansky and A. Zagorodnyuk, *Representing measures and infinite-dimensional holomorphy*, J. Math. Anal. & App., **333**(2), (2007), 614–625.
[60] O. Lopushansky, *Hardy spaces for irreducible representations of locally compact groups*, Topology, **48**, (2009), 169–177.
[61] O. Lopushansky and A. Zagorodnyuk, *Hardy type spaces associated with compact unitary groups*, Nonlinear analysis, **74**(2), (2011), 556–572.
[62] O. Lopushansky, *Polynomial ultradistributions: differentiation and Laplace transformation*, Banach Center Publ., **88**, (2010), 195–209.
[63] O. Lopushansky and M. Oleksienko, *A Poisson type formula for Hardy classes on Heisenberg's group*, Carpathian Math. Publ., **2**(1), (2010), 87–95.
[64] O. Lopushansky and M. Oleksienko, *Hardy type spaces on infinite dimensional group orbits*, Methods Funct. Anal. & Topology, **17**(3), (2011), 225–234.
[65] O. Lopushansky, *Hardy-type space associated with an infinite-dimensional unitary matrix group*, Abstract & Appl. An., ID 810735 (2013), 1–7.
[66] G. W. Mackey, *Induced representations of locally compact groups, i,ii*, Ann. Math. **55** (1952), 101–139, & **58**, (1953), 193–221.
[67] R. S. Martin, *Contributions to the theory of functionals*, Ph.D. thesis, University of California, 1932.
[68] S. Mazur and W. Orlicz, *Grundlegende eigenschaften der polynomischen operationen I, II*, Studia Math., **5**, (1935), 50–68, 179–189.
[69] S. Mazur and W. Orlicz, *Sur la divisibilite des polynomes absraits*, C. R. Acad. Sci. Paris., **207**, (1936), 621–623.
[70] L.A. Moraes and A.F. Pereira, *The spectra of algebras of Lorch analytic mappings*, Topology, **48**, (2009), 91–99.
[71] A.D. Michal and A.H. Clifford, *Fonctions analytiques implicites dans des espaces vectoriels abstraits*, C. R. Acad. Sci. Paris., **197**, (1933), 735–737.
[72] A.D. Michal and R.S. Martin, *Some expansions in vector spaces*, J. Math. Pures. Appl., **13**(9), (1934), 69–91.
[73] A.E. Taylor, *Additions to the theory of polynomials in normed linear spaces*, Tohoku Math. Journal, **44**, (1938), 302–318.
[74] A.E. Taylor, *Notes on the history of the uses of analyticity in operator theory*, Amer. Math. Monthly, **78**, (1971), 331–342.
[75] E. Michael, *Locally multiplicatively convex topological algebras*, Mem. Amer. Math. Soc., **11**, Providence, 1952.
[76] J. Mujica, *Complex Analysis in Banach Spaces*, North-Holland, Amsterdam, 1986.
[77] J. Mujica, *Ideals of holomorphic functions on Tsirelson's space*, Archiv der Mathematik, **76**, (2001), 292–298.
[78] Y. Neretin, *Hua type integrals over unitary groups and over projective limits of unitary groups*, Duke Math. J., **114**(2), (2002), 239–266.

[79] G. Olshanski, *The problem of harmonic analysis on the infinite-dimensional unitary group*, J. Funct. Analysis, **205**, (2003), 464–524.

[80] B. Ørted and K.H. Neeb, *Hardy spaces in an infinite dimensional setting*, Proceedings of II international workshop "Lie theory and its application in physics", **6**, (1998), 9–27.

[81] A. Pełczyński, *A property of multilinear operations*, Studia Math., **16**, (1957), 173–182.

[82] D. Pickrell, *Measures on infinite-dimensional Grassmann manifolds*, J. Funct. Analysis, **70**, (1987), 323–356.

[83] D. Pinasco, L^p*-represenable functions on Banach spaces*, J. Math. Anal. Appl., **335**, (2007), 480–497.

[84] D. Pinasco and I. Zalduendo, *Integral representations of holomorphic functions on Banach spaces*, J. Math. Anal. & Appl., **308**, (2005), 159–174.

[85] A. Plichko and A. Zagorodnyuk, *On automatic continuity and three problems of "The Scottish Book" concerning the boundedness of polynomial functionals*, J. Math. Anal.& Appl., **220**, (1998), 477–494.

[86] C.J. Read. *The invariant subspace problem for a class of Banach spaces. II. Hypercyclic operators*, Israel J. Math., **63**, (1988), 1–40.

[87] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. II.* Academic Press, New York, 1975.

[88] W. Rudin,*Function theory in the unit ball of* $\mathbb{C}^n$, Springer, Berlin, 1980.

[89] W. Rudin, *Some representing measures for the ball algebra*, Michigan Math. J., **27**, (1980), 315–320.

[90] S. Saitoh, *Integral Transforms, Reproducig Kernels and Their Applications*, Pitman Research Notes in Math. Ser., vol. 369, Longman, 1997.

[91] S. Thangavelu, Harmonic analysis on the Heisenberg group, Progress in Math. 159, Birkhauser, Boston, 1998.

[92] A. Ülger, *Weakly compact bilinear forms and Arens regularity*, Proc. Amer. Math. Soc., **101**, (1987), 697–704.

[93] T.V. Vasylyshyn, *Description of spectra and derivations of Wiener type algebras of functions generated by (p, q)-polynomials on Banach spaces*, Carpathian Math. Publ., **4**(2), (2012), 212–228.

[94] B.L. van der Waerden, *Modern Algebra*, Ungar, 1964.

[95] Y. Yamasaki, *Projective limit of Haar measures on* $O(n)$, Publ. RIMS, Kyoto Univ., **8**, (1972/73), 141–149.

[96] A. Zagorodnyuk, *Spectra of algebras of entire functions on Banach spaces*, Proc. Amer. Math. Soc., **134**, (2006), 2559–2569.

[97] A. Zagorodnyuk, *Spectra of algebras of analytic functions and polynomials on Banach spaces*, Function spaces, Contemp. Math., **435**, (2007), 381–394

[98] E. Tomas, *On Prohorov's criterion for projective limits*, Operator Theory: Adv. Appl., **168**, (2006), 251–261.