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AN EXAMPLE OF A NON-BOREL LOCALLY-CONNECTED FINITE-DIMENSIONAL TOPOLOGICAL GROUP

According to a classical theorem of Gleason and Montgomery, every finite-dimensional locally path-connected topological group is a Lie group. In the paper for every natural number n we construct a locally connected subgroup $G \subset \mathbb{R}^{n+1}$ of dimension n, which is not locally compact. This answers a question posed by S. Maillot on MathOverflow and shows that the local path-connectedness in the result of Gleason and Montgomery can not be weakened to the local connectedness.

Key words and phrases: topological group, Lie group.

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By a classical result of A. Gleason [3] and D. Montgomery [6], every locally path-connected finite-dimensional topological group G is locally compact and hence is a Lie group. Generalizing this result of A. Gleason and D. Montgomery, T. Banakh and L. Zdomskyy [1] proved that a topological group G is a Lie group if G is compactly finite-dimensional and locally continuum-connected. In [5] Sylvain Maillot asked if the locally path-connectedness in the result of A. Gleason and D. Montgomery can be replaced by the local connectedness. In this paper we construct a counterexample to this question of S. Maillot.

We recall that a subset B of a Polish space X is called a *Bernstein set* in X if both B and $X \setminus B$ meet every uncountable closed subset F of X. Bernstein sets in Polish space can be easily constructed by transfinite induction, see [4, 8.24].

Theorem 1. *For every $n \geq 2$ the Euclidean space $\mathbb{R}^n$ contains a dense additive subgroup $G \subset \mathbb{R}^n$ such that*

1) *G is a Bernstein set in $\mathbb{R}^n$;*
2) *G is locally connected;*
3) *G has dimension $\dim(G) = n - 1$;*
4) *G is not Borel and hence not locally compact.*

Proof. Let $(F_\alpha)_{\alpha<\mathfrak{c}}$ be an enumeration of all uncountable closed subsets of $\mathbb{R}^n$ by ordinal $< \mathfrak{c}$. Fix any point $p \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. By transfinite induction, for every ordinal $\alpha < \mathfrak{c}$ we shall choose a point $z_\alpha \in F_\alpha$ such that the subgroup $G_\alpha \subset \mathbb{R}^n$ generated by the set $\{z_\beta\}_{\beta<\alpha}$ does not contain

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the point p. Assume that for some ordinal $\alpha < \mathfrak{c}$ we have chosen points z_β, $\beta < \alpha$, so that the subgroup $G_{<\alpha}$ generated by the set $\{z_\beta\}_{\beta<\alpha}$ does not contain p. Consider the set

$$Z = \left\{ \frac{1}{n}(p - g) : n \in \mathbb{Z} \setminus \{0\},\ g \in G_{<\alpha} \right\}$$

and observe that it has cardinality

$$|Z| \le \omega \cdot |G_{<\alpha}| \le \omega + |\alpha| < \mathfrak{c}.$$

Since the uncountable closed subset F_α of $\mathbb{R}^n$ has cardinality $|F_\alpha| = \mathfrak{c}$ (see [4, 6.5]), there is a point $z_\alpha \in F_\alpha \setminus Z$. For this point we get $p \neq nz_\alpha + g$ for any $n \in \mathbb{Z} \setminus \{0\}$, and $g \in G_{<\alpha}$. Consequently, the subgroup

$$G_\alpha = \{nz_\alpha + g : n \in \mathbb{Z},\ g \in G_{<\alpha}\}$$

generated by the set $\{z_\beta\}_{\beta\le\alpha}$ does not contain the point p. This completes the inductive step.

After completing the inductive construction, consider the subgroup G generated by the set $\{a_\alpha\}_{\alpha<\mathfrak{c}}$ and observe that $p \notin G$ and G meets every uncountable closed subset F of $\mathbb{R}^n$. Moreover, since G meets the closed uncountable set $F - p$, the coset $p + G \subset \mathbb{R}^n \setminus G$ meets F. So, both the subgroup G and its complement $\mathbb{R}^n \setminus G$ meet each uncountable closed subset of $\mathbb{R}^n$, which means that G is a Bernstein set in $\mathbb{R}^n$. The following proposition implies that the group G has properties (2)–(4). □

Proposition 1. *Let $n \ge 2$. Every Bernstein subset B of $\mathbb{R}^n$ has the following properties:*

1) *B is not Borel;*
2) *B is connected and locally connected;*
3) *B has dimension $\dim(B) = n - 1$.*

Proof. 1. By [4, 8.24], the Bernstein set B is not Borel (more precisely, B does not have the Baire property in $\mathbb{R}^n$).

2. To prove that B is connected and locally connected, it suffices to prove that for every open subset $U \subset \mathbb{R}^n$ homeomorphic to $\mathbb{R}^n$ the intersection $U \cap G$ is connected. Assuming the opposite, we could find two non-empty open disjoint sets $U_1, U_2 \subset U$ such that $U \cap B = (U_1 \cap B) \cup (U_2 \cap B)$. Consider the complement $F = U \setminus (U_1 \cup U_2) \subset U \setminus B$ and observe that F is closed in U and hence of type F_σ in $\mathbb{R}^n$. If F is uncountable, then F contains an uncountable closed subset of $\mathbb{R}^n$ and hence meets the set B, which is not the case. So, the closed subset F of U is at most countable and separates the space $U \cong \mathbb{R}^n$, which contradicts Theorem 1.8.14 of [2].

3. Since the subset B has empty interior in $\mathbb{R}^n$, we can apply Theorem 1.8.11 of [2] and conclude that $\dim(B) < n$. On the other hand, Lemma 1.8.16 [2] guarantees that B has dimension $\dim(B) \ge n - 1$ (since B meets every non-trivial compact connected subset of $\mathbb{R}^n$). So, $\dim(B) = n - 1$. □

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Згідно з класичною теоремою Ґлісона-Монтґомері, довільна скінченно-вимірна локально лінійно зв'язна топологічна група є групою Лі. У статті для кожного натурального числа n побудовано локально зв'язну, але не локально компактну адитивну підгрупу $G \subset \mathbb{R}^{n+1}$ топологічного виміру n. Цей приклад дає відповідь на проблему С. Мейло, поставлену на MathOverflow, та показує, що локально лінійну зв'язність у теоремі Ґлісона-Монтґомері не можна послабити до локальної зв'язності.

Ключові слова і фрази: топологічна група, група Лі.

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PROPERTIES OF POWER SERIES OF ANALYTIC IN A BIDISC FUNCTIONS OF BOUNDED L-INDEX IN JOINT VARIABLES

We generalized some criteria of boundedness of **L**-index in joint variables for analytic in a bidisc functions, where $\mathbf{L}(z) = (l_1(z_1, z_2), l_2(z_1, z_2))$, $l_j : \mathbb{D}^2 \to \mathbb{R}_+$ is a continuous function, $j \in \{1, 2\}$, $\mathbb{D}^2$ is a bidisc $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$. We obtained propositions, which describe a behaviour of power series expansion on a skeleton of a bidisc. The power series expansion is estimated by a dominating homogeneous polynomial with a degree that does not exceed some number, depending only from radii of a bidisc. Replacing universal quantifier by existential quantifier for radii of a bidisc, we also proved sufficient conditions of boundedness of **L**-index in joint variables for analytic functions, which are weaker than necessary conditions.

Key words and phrases: analytic function, bidisc, bounded **L**-index in joint variables, maximum modulus, partial derivative, dominating polynomial, power series.

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1 Introduction

Recently, we introduced a concept of boundedness of **L**-index in joint variables for analytic in a bidisc functions [4]–[6]. There were obtained criteria which describes a local behaviour of partial derivatives, give estimate maximum modulus on a skeleton of bidisc and was proved an analog of Hayman's Theorem.

In a fact, inequality (1) in a definition of function of bounded **L**-index in joint variables (see below) contains coefficients of power series expansion at a point $z = (z_1, z_2)$. M. T. Bordulyak and M. M. Sheremeta [9] considered entire functions and obtained a proposition which describe a behavior of homogeneous polynomials with power series coefficients for functions of bounded **L**-index in joint variables in the case $\mathbf{L}(z) = (l_1(z_1), \ldots, l_n(z_n))$. Recently, we generalized [5] their result for entire functions and $\mathbf{L}(z) = (l_1(z), \ldots, l_n(z))$, where $z \in \mathbb{C}^n$. Replacing universal quantifier by existential quantifier, there was proved also new theorem which provides weaker sufficient conditions of boundedness of **L**-index in joint variables.

This leads to such a natural question: *Is there a counterpart of the mentioned Bordulyak–Sheremeta's criterion for functions that are analytic in an arbitrary polydisc domain?* Our answer to the question is affirmative. In particular, it is proved in Theorems 1 and 2 of this paper for a bidisc.

In this paper, we will prove a necessity of Bordulyak–Sheremeta's criterion for analytic in a bidisc functions and $\mathbf{L}(z) = (l_1(z_1, z_2), l_2(z_1, z_2))$. As sufficiency for analytic in $\mathbb{D}^2$ functions, we will deduce an analog of weaker sufficient conditions of boundedness of **L**-index in joint variables from [5].

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2 MAIN DEFINITIONS AND NOTATIONS

We consider two-dimensional complex space $\mathbb{C}^2$. This helps to distinguish main methods of investigation. We need some standard notations. Denote $\mathbb{R}_+ = [0,+\infty)$, $\mathbf{0} = (0,0) \in \mathbb{R}^2_+$, $\mathbf{1} = (1,1) \in \mathbb{R}^2_+$, $R = (r_1,r_2) \in \mathbb{R}^2_+$, $z = (z_1,z_2) \in \mathbb{C}^2$. For $A = (a_1,a_2) \in \mathbb{R}^2$, $B = (b_1,b_2) \in \mathbb{R}^2$ we will use formal notations without violation of the existence of these expressions

$$AB = (a_1b_1,a_2b_2),\ \ A/B = (a_1/b_1,a_2/b_2),\ b_1 \neq 0,\ b_2 \neq 0,\ \ A^B = a_1^{b_1}a_2^{b_2},\ b \in \mathbb{Z}^2_+,$$

and the notation $A < B$ means that $a_j < b_j$, $j \in \{1,2\}$; the relation $A \leq B$ is defined similarly. For $K = (k_1,k_2) \in \mathbb{Z}^2_+$ denote $\|K\| = k_1 + k_2$, $K! = k_1!k_2!$.

The bidisc $\{z \in \mathbb{C}^2 :\ |z_j - z_j^0| < r_j,\ j = 1,2\}$ is denoted by $\mathbb{D}^2(z^0,R)$, its skeleton $\{z \in \mathbb{C}^2 :\ |z_j - z_j^0| = r_j,\ j = 1,2\}$ is denoted by $\mathbb{T}^2(z^0,R)$, and the closed bidisc $\{z \in \mathbb{C}^2 :\ |z_j - z_j^0| \leq r_j,\ j = 1,2\}$ is denoted by $\mathbb{D}^2[z^0,R]$, $\mathbb{D}^2 = \mathbb{D}^2(\mathbf{0},\mathbf{1})$, $\mathbb{D} = \{z \in \mathbb{C} :\ |z| < 1\}$. For $p,q \in \mathbb{Z}_+$ and partial derivative of analytic in $\mathbb{D}^2$ function $F(z)$ we will use the notation

$$F^{(p,q)}(z) = F^{(p,q)}(z_1,z_2) := \frac{\partial^{p+q}F(z_1,z_2)}{\partial z_1^p \partial z_2^q}.$$

Let $\mathbf{L}(z) = (l_1(z),l_2(z))$, where $l_j(z) : \mathbb{D}^2 \to \mathbb{R}_+$ is a continuous function such that for all $z \in \mathbb{D}^2$: $l_j(z) > \beta/(1-|z_j|)$, $j \in \{1,2\}$, where $\beta > 1$ is a some constant, $\boldsymbol{\beta} := (\beta,\beta)$. S.N. Strochyk, M.M. Sheremeta, V.O. Kushnir [14], [20] imposed a similar condition for a function $l : \mathbb{D} \to \mathbb{R}_+$ and $l : G \to \mathbb{R}_+$, where G is arbitrary domain in $\mathbb{C}$.

An analytic function $F\colon \mathbb{D}^2 \to \mathbb{C}$ is called a function of *bounded* $\mathbf{L}$*-index (in joint variables)*, if there exists $n_0 \in \mathbb{Z}_+$ such that for all $z = (z_1,z_2) \in \mathbb{D}^2$ and for all $(p_1,p_2) \in \mathbb{Z}^2_+$

$$\frac{1}{p_1!p_2!}\frac{|F^{(p_1,p_2)}(z)|}{l_1^{p_1}(z)l_2^{p_2}(z)} \leq \max\left\{\frac{1}{k_1!k_2!}\frac{|F^{(k_1,k_2)}(z)|}{l_1^{k_1}(z)l_2^{k_2}(z)} : 0 \leq k_1 + k_2 \leq n_0\right\}. \tag{1}$$

The least such integer n_0 is called the $\mathbf{L}$*-index in joint variables of the function* $F(z)$ and is denoted by $N(F,\mathbf{L},\mathbb{D}^2) = n_0$. This is an analog of definition of entire function of bounded $\mathbf{L}$-index or bounded index ($\mathbf{L} \equiv 1$) in joint variables in $\mathbb{C}^2$ (see [3], [9, 10], [16, 17, 18]) and a definition of analytic in a domain function of bounded index [12]. Note that a primary definition of entire in $\mathbb{C}$ function of bounded index was supposed by B. Lepson [15]. Other approach (so-called L-index in a direction) is considered in [7, 8].

By $Q(\mathbb{D}^2)$ we denote the class of functions $\mathbf{L}$, which satisfy the condition for all $r_j \in [0,\beta]$, $j \in \{1,2\}$

$$0 < \lambda_{1,j}(R) \leq \lambda_{2,j}(R) < \infty,$$

where

$$\lambda_{1,j}(R) = \inf_{z^0\in\mathbb{D}^2} \inf\left\{\frac{l_j(z)}{l_j(z^0)} : z \in \mathbb{D}^2\left[z^0, R/\mathbf{L}(z^0)\right]\right\},$$

$$\lambda_{2,j}(R) = \sup_{z^0\in\mathbb{D}^2} \sup\left\{\frac{l_j(z)}{l_j(z^0)} : z \in \mathbb{D}^2\left[z^0, R/\mathbf{L}(z^0)\right]\right\}.$$

It is easy to prove that the function $L_1(z_1,z_2) = (\beta'/(1-|z_1|),\beta'/(1-|z_2|))$ belongs to $Q(\mathbb{D}^2)$, where $\beta' > \beta$. Other possible methods to construct these functions are considered in [1].

Let $z^0 \in \mathbb{D}^2$. We develop an analytic in $\mathbb{D}^2$ function $F(z)$ in the power series written in a diagonal form

$$F(z) = \sum_{k_1+k_2=0}^{\infty} p_{k_1+k_2}((z_1-z_1^0),(z_2-z_2^0)) = \sum_{k=0}^{\infty}\sum_{j_1+j_2=k} b_{j_1,j_2}(z_1-z_1^0)^{j_1}(z_2-z_2^0)^{j_2}, \qquad (2)$$

where p_k are homogeneous polynomials of degree k. The polynomial $p_{k_0}, k_0 \in \mathbb{Z}_+$, is called a dominating polynomial in the power series expansion (2) on $\mathbb{T}^2(z^0, R)$ if for every $z \in \mathbb{T}^2(z^0, R)$ the next inequality holds:

$$\left|\sum_{k_1+k_2\neq k^0} p_{k_1+k_2}((z_1-z_1^0),(z_2-z_2^0))\right| \le \frac{1}{2}\max\{|b_{j_1,j_2}|r_1^{j_1}r_2^{j_2}\colon j_1+j_2=k^0\},$$

where $b_{j_1,j_2} = \frac{F^{(j_1,j_2)}(z^0)}{j_1!j_2!}$.

3 Some property of power expansion of analytic in a bidisc function of bounded L-index in joint variables

Theorem 1. *Let $\beta > 1$, $\mathbf{L} \in Q(\mathbb{D}^2)$. If an analytic function F in $\mathbb{D}^2$ has bounded $\mathbf{L}$-index in joint variables then there exists $p \in \mathbb{Z}_+$ that for all $d \in (0;\beta]$ there exists $\eta(d) \in (0;d)$ such that for each $z^0 \in \mathbb{D}^2$ and some $r = r(d,z^0) \in (\eta(d), d)$, $k^0 = k^0(d,z^0) \le p$ the polynomial p_{k^0} is the dominating polynomial in the series* (2) *on $\mathbb{T}^2(z^0, \frac{R}{\mathbf{L}(z^0)})$ with $R = (r,r)$.*

Proof. Let F be of bounded $\mathbf{L}$-index in joint variables with $N = N(F,\mathbf{L},\mathbb{D}^2) < +\infty$ and n_0 be $\mathbf{L}$-index in joint variables at a point $z^0 \in \mathbb{D}^2$. Then for each $z^0 \in \mathbb{D}^2$ $n_0 \le N$. We put

$$a^*_{j_1,j_2} = \frac{|b_{j_1,j_2}|}{l_1^{j_1}(z^0)l_2^{j_2}(z^0)} = \frac{|F^{(j_1,j_2)}(z^0)|}{j_1!j_2!l_1^{j_1}(z^0)l_2^{j_2}(z^0)},\ a_k = \max\{a^*_{j_1,j_2}\colon j_1+j_2=k\},$$
$$c = 2((N+1)^3 + 6(N+3)!).$$

Let $d \in (0;\beta]$ be an arbitrary number. We put $r_m = \frac{d}{(d+1)c^m}, m \in \mathbb{Z}_+$ and denote

$$\mu_m = \max\{a_k r_m^k\colon k \in \mathbb{Z}_+\},\ s_m = \min\{k\colon a_k r_m^k = \mu_m\}.$$

Since z^0 is a fixed point the inequality $a^*_{k_1,k_2} \le \max\{a^*_{j_1,j_2}\colon j_1+j_2 \le n_0\}$ is valid for all $(k_2,k_2) \in \mathbb{Z}^2_+$. Then $a_k \le a_{n_0}$ for all $k \in \mathbb{Z}_+$. Hence, for all $k > n_0$ in view of $r_0 < 1$ we have $a_k r_0^k < a_{n_0} r_0^{n_0}$. This implies $s_0 \le n_0$. Since $cr_m = r_{m-1}$, we obtain that for each $k > s_{m-1}$

$$a_{s_{m-1}} r_m^{s_{m-1}} = a_{s_{m-1}} r_{m-1}^{s_{m-1}} c^{-s_{m-1}} \ge a_k r_{m-1}^k c^{-s_{m-1}} = a_k r_m^k c^{k-s_{m-1}} \ge c a_k r_m^k. \qquad (3)$$

From (3) it follows that $s_m \le s_{m-1}$ for all $m \in \mathbb{N}$. Thus, we can rewrite

$$\mu_0 = \max\{a_k r_0^k\colon k \le n_0\},\ \mu_m = \max\{a_k r_m^k\colon k \le s_{m-1}\}.$$

We denote

$$\mu_0^* = \max\{a_k r_0^k\colon s_0 \neq k \le n_0\},\quad \mu_m^* = \max\{a_k r_m^k\colon s_m \neq k \le s_{m-1}\},$$
$$s_0^* = \min\{k\colon k \neq s_0, a_k r_0^k = \mu_0^*\},\quad s_m^* = \min\{k\colon k \neq s_m, a_k r_m^k = \mu_m^*\}, m \in \mathbb{N}$$

and we will show that there exists $m_0 \in \mathbb{Z}_+$ such, that

$$\frac{\mu^*_{m_0}}{\mu_{m_0}} \leq \frac{1}{c}. \tag{4}$$

Suppose that for all $m \in \mathbb{Z}_+$ the next inequality holds

$$\frac{\mu^*_m}{\mu_m} > \frac{1}{c}. \tag{5}$$

If $s^*_m < s_m$ ($s^*_m \neq s_m$ in view of definition) then we have

$$a_{s^*_m} r^{s^*_m}_{m+1} = \frac{a_{s^*_m} r^{s^*_m}_m}{c^{s^*_m}} = \frac{\mu^*_m}{c^{s^*_m}} > \frac{\mu_m}{c^{s^*_m+1}} = \frac{a_{s_m} r^{s_m}_m}{c^{s^*_m+1}} = \frac{a_{s_m} r^{s_m}_{m+1}}{c^{s^*_m+1-s_m}} \geq a_{s_m} r^{s_m}_{m+1},$$

and for all $k > s^*_m, k \neq s_m$, similarly,

$$a_{s^*_m} r^{s^*_m}_{m+1} = \frac{a_{s^*_m} r^{s^*_m}_m}{c^{s^*_m}} \geq \frac{a_k r^k_m}{c^{s^*_m}} \geq \frac{a_k r^k_m}{c^{k-1}} = \frac{c a_k r^k_m}{c^k} = c a_k r^k_{m+1},$$

i.e. $a_{s^*_m} r^{s^*_m}_{m+1} > a_k r^k_{m+1}$ for all $k > s^*_m$. Hence,

$$s_{m+1} \leq s^*_m \leq s_m - 1. \tag{6}$$

On the contrary, if $s_m < s^*_m \leq s_{m-1}$ then the equality $s_{m+1} = s_m$ may hold. But in this case the inequalities $s^*_{m+1} \leq s_m$ and $s^*_m \neq s_{m+1}$ imply that $s^*_{m+1} < s_{m+1}$, $s^*_{m+1} \neq s_{m+1}$. Instead of (6) we have the inequality $s_{m+2} \leq s^*_{m+1} \leq s_{m+1} - 1 = s_m - 1$. Hence, if for all $m \in \mathbb{Z}_+$ estimate (5) is true then for all $m \in \mathbb{Z}_+$ either inequality $s_{m+1} \leq s_m - 1$ or $s_{m+2} \leq s_m - 1$ holds, i.e. $s_{m+2} \leq s_m - 1$, because $s_{m+2} \leq s_{m+1}$. It implies that

$$s_m \leq s_{m-2} - 1 \leq \ldots \leq s_{m-2\lfloor \frac{m}{2} \rfloor} - \left\lfloor \frac{m}{2} \right\rfloor \leq s_0 - \left\lfloor \frac{m}{2} \right\rfloor \leq n_0 - \left\lfloor \frac{m}{2} \right\rfloor \leq N - \left\lfloor \frac{m}{2} \right\rfloor,$$

i.e. $s_m < 0$ if only $m > 2N + 1$, which is impossible. Therefore, there exists $m_0 \leq 2N + 1$ such that (4) holds. We put $r = r_{m_0}$, $\eta(d) = \frac{d}{(d+1)c^{2(N+1)}}$, $p = N$ and $k_0 = s_{m_0}$. Then for all $j_1 + j_2 \neq k_0 = s_{m_0}$ on $\mathbb{T}^2(z^0, \frac{r}{\mathbf{L}(z^0)})$ in view (4) we have

$$|b_{j_1,j_2}||z_1 - z^0_1|^{j_1}|z_2 - z^0_2|^{j_2} = a^*_{j_1,j_2} r^{j_1+j_2} \leq a_{j_1+j_2} r^{j_1+j_2} \leq \mu^*_{m_0} \leq \frac{1}{c}\mu_{m_0} \leq \frac{1}{c} a_{s_{m_0}} r^{s_{m_0}}_{m_0} = \frac{1}{c} a_{k_0} r^{k_0}.$$

Thus, on $\mathbb{T}^2(z^0, \frac{r}{\mathbf{L}(z^0)})$ we obtain

$$\begin{aligned} \left| \sum_{j_1+j_2 \neq k_0} b_{j_1,j_2}(z_1 - z^0_1)^{j_1}(z_2 - z^0_2)^{j_2} \right| &\leq \sum_{j_1+j_2 \neq k_0} a^*_{j_1,j_2} r^{j_1+j_2} \leq \sum_{k=0, k \neq k_0}^{\infty} a_k (k+1)^2 r^k \\ &= \sum_{k=0,\ k \neq s_{m_0}}^{s_{m_0-1}} a_k (k+1)^2 r^k + \sum_{k=s_{m_0-1}+1}^{\infty} a_k (k+1)^2 r^k. \end{aligned} \tag{7}$$

We will estimate two sums in (7). From (4) it follows that $\mu^*_{m_0} \leq \frac{1}{c}\mu_{m_0}$ or

$$\max\{a_k r^k_{m_0} : k \neq s_{m_0},\ k \leq s_{m_0-1}\} \leq \frac{1}{c} \max\{a_k r^k_{m_0} : k \neq s_{m_0}, k \leq s_{m_0-1}\},$$

i. e. $a_k r^k \leq \frac{1}{c} a_{k_0} r^{k_0}$. Then

$$\sum_{k=0,\, k\neq s_{m_0}}^{s_{m_0-1}} a_k (k+1)^2 r^k \leq \frac{a_{k_0} r^{k_0}}{c} \sum_{k=0}^{N} (k+1)^2 \leq \frac{a_{k_0} r^{k_0}}{c} (N+1)^3. \tag{8}$$

For each k the inequality $a_k r^k_{m_0-1} \leq \mu_{m_0-1}$ holds and, hence,

$$a_k r^k_{m_0} = \frac{a_k r^k_{m_0-1}}{c^k} \leq \frac{\mu_{m_0-1}}{c^k}. \tag{9}$$

Using (9) and (4) we deduce

$$\begin{aligned}
\sum_{k=s_{m_0-1}+1}^{\infty} a_k (k+1)^2 r^k &\leq \mu_{m_0-1} \sum_{k=s_{m_0-1}+1}^{\infty} (k+1)^2 \frac{1}{c^k} = a_{s_{m_0-1}} r^{s_{m_0-1}}_{m_0-1} \sum_{k=s_{m_0-1}+1}^{\infty} (k+1)^2 \frac{1}{c^k} \\
&= a_{s_{m_0-1}} \frac{r^{s_{m_0-1}}_{m_0-1}}{c^{s_{m_0-1}}} c^{s_{m_0-1}} \sum_{k=s_{m_0-1}+1}^{\infty} (k+1)^2 \frac{1}{c^k} \leq a_{s_{m_0-1}} r^{s_{m_0-1}}_{m_0} c^{s_{m_0-1}} \sum_{k=s_{m_0-1}+1}^{\infty} (k+1)(k+2) \frac{1}{c^k} \\
&\leq \frac{a_{s_{m_0}} r^{s_{m_0}}}{c} c^{s_{m_0-1}} \left(\sum_{k=s_{m_0-1}+1}^{\infty} x^{k+2} \right)^{(2)} \Bigg|_{x=\frac{1}{c}} = \frac{a_{k_0} r^{k_0}}{c} c^{s_{m_0-1}} \left(\frac{x^{s_{m_0-1}+3}}{1-x} \right)^{(2)} \Bigg|_{x=\frac{1}{c}} \\
&= \frac{a_{k_0} r^{k_0}}{c} c^{s_{m_0-1}} \Bigg(\frac{(s_{m_0-1}+3)(s_{m_0-1}+2) x^{s_{m_0-1}+1}}{1-x} + \frac{2(s_{m_0-1}+3) x^{s_{m_0-1}+2}}{(1-x)^2} \\
&\quad + \frac{2x^{s_{m_0-1}+3}}{(1-x)^3} \Bigg) \Bigg|_{x=\frac{1}{c}} \leq \frac{a_{k_0} r^{k_0}}{c} c^{s_{m_0-1}} 2(s_{m_0-1}+3)(s_{m_0-1}+2) \sum_{j=0}^{2} \frac{x^{s_{m_0-1}+1+j}}{(1-x)^{1+j}} \Bigg|_{x=\frac{1}{c}} \\
&\leq \frac{a_{k_0} r^{k_0}}{c} 2(N+3)! \sum_{j=0}^{2} \frac{1}{(c-1)^{1+j}} \leq \frac{a_{k_0} r^{k_0}}{c} 6(N+3)!,
\end{aligned} \tag{10}$$

because $c \geq 2$. Hence, from (8) and (10) we obtain

$$\begin{aligned}
\left| \sum_{j_1+j_2\neq k_0} b_{j_1,j_2} (z_1 - z_1^0)^{j_1} (z_2 - z_2^0)^{j_2} \right| &\leq \frac{a_{k_0} r^{k_0}}{c} (N+1)^3 + 6 \frac{a_{k_0} r^{k_0}}{c} (N+3)! \\
&= \frac{a_{k_0} r^{k_0}}{c} ((N+1)^3 + 6(N+3)!) = \frac{1}{2} a_{k_0} r^{k_0}.
\end{aligned}$$

Therefore, the polynomial p_{k^0} is the dominating polynomial in the series (2) on the skeleton $\mathbb{T}^2\big(z^0, \frac{R}{\mathbf{L}(z^0)}\big)$. □

Theorem 2. *Let $\beta > 1$, $\mathbf{L} \in Q(\mathbb{D}^2)$. If there exist $p \in \mathbb{Z}_+$, $d \in (0;1]$, $\eta \in (0;d)$ such that for each $z^0 \in \mathbb{D}^2$ and some $R = (r_1, r_2)$ with $r_j = r_j(d, z^0) \in (\eta, d)$, $j \in \{1,2\}$, and certain $k^0 = k^0(d, z^0) \leq p$ the polynomial p_{k^0} is the dominating polynomial in the series* (2) *on $\mathbb{T}^2(z^0, R/\mathbf{L}(z^0))$ then the analytic in $\mathbb{D}^2$ function F has bounded $\mathbf{L}$-index in joint variables.*

Proof. Suppose that there exist $p \in \mathbb{Z}_+$, $d \leq 1$ and $\eta \in (0,d)$ such that for each $z^0 \in \mathbb{D}^2$ and some $R = (r_1, r_2)$ with $r_j = r_j(d, z^0) \in (\eta, d)$, $j \in \{1,2\}$, and $k_0 = k_0(d, z^0) \leq p$ the polynomial p_{k^0} is the dominating polynomial in the series (2) on $\mathbb{T}^2(z^0, R/\mathbf{L}(z^0))$. Let us to denote $r_0 = max\{r_1, r_2\}$. Then

$$\left| \sum_{j_1+j_2\neq k_0} b_{j_1,j_2} (z_1 - z_1^0)^{j_1} (z_2 - z_2^0)^{j_2} \right| = \left| F(z) - \sum_{j_1+j_2=k_0} b_{j_1,j_2} (z_1 - z_1^0)^{j_1} (z_2 - z_2^0)^{j_2} \right| \leq \frac{a_{k_0} r_0^{k_0}}{2}. \tag{11}$$

Using (11) and Cauchy's inequality we have:

$$|b_{j_1,j_2}(z_1-z_1^0)^{j_1}(z_2-z_2^0)^{j_2}| = a^*_{j_1,j_2} r_1^{j_1} r_2^{j_2} \le \frac{a_{k_0} r_0^{k_0}}{2}$$

for all $j_1, j_2 \in \mathbb{Z}_+$, i.e. for all $k_1 + k_2 = k \neq k_0$

$$a_k r_1^{k_1} r_2^{k_2} \le \frac{a_{k_0} r_0^{k_0}}{2}. \tag{12}$$

Suppose that F is not a function of bounded $\mathbf{L}$-index in joint variables.

Let $\mathbf{L} \in Q(\mathbb{D}^2)$. It is known [6] that an analytic function F in $\mathbb{D}^2$ has bounded $\mathbf{L}$-index in joint variables if and only if there exist $p \in \mathbb{Z}_+$ and $c \in \mathbb{R}_+$ such that for each $z = (z_1, z_2) \in \mathbb{D}^2$ the next inequality holds

$$\max\left\{\frac{|F^{(j_1,j_2)}(z)|}{l_1^{j_1}(z) l_2^{j_2}(z)} \colon j_1 + j_2 = p+1\right\} \le c \max\left\{\frac{|F^{(k_1,k_2)}(z)|}{l_1^{k_1}(z) l_2^{k_2}(z)} \colon k_1 + k_2 \le p\right\}.$$

This statement and its generalizations [19, 13, 9, 2, 6] are analogs of known Hayman's Theorem [11] in theory of functions of bounded index. Then by the Hayman Theorem for all $p_1 \in \mathbb{Z}_+$ and $c \ge 1$ there exists $z^0 \in \mathbb{D}^2$ such that the next inequality holds:

$$\max\left\{\frac{|F^{(j_1,j_2)}(z^0)|}{l_1^{j_1}(z^0) l_2^{j_2}(z^0)} \colon j_1 + j_2 = p_1+1\right\} > c \max\left\{\frac{|F^{(k_1,k_2)}(z^0)|}{l_1^{k_1}(z^0) l_2^{k_2}(z^0)} \colon k_1 + k_2 \le p_1\right\}.$$

We put $p_1 = p$ and $c = \left(\frac{(p+1)!}{\eta^{p+1}}\right)^2$. Then for this $z^0(p_1, c)$ we obtain:

$$\max\left\{\frac{|F^{(j_1,j_2)}(z^0)|}{j_1! j_2! l_1^{j_1}(z^0) l_2^{j_2}(z^0)} \colon j_1 + j_2 = p+1\right\} > \frac{1}{\eta^{p+1}} \max\left\{\frac{|F^{(k_1,k_2)}(z^0)|}{k_1! k_2! l_1^{k_1}(z^0) l_2^{k_2}(z^0)} \colon k_1 + k_2 \le p\right\},$$

i.e. $a_{p+1} > \frac{a_{k_0}}{\eta^{p+1}}$ and, hence, $a_{p+1} r_0^{p+1} > \frac{a_{k_0} r_0^{p+1}}{\eta^{p+1}} \ge a_{k_0} r_0^{k_0}$. This is a contradiction with (12). Therefore, F is of bounded $\mathbf{L}$-index in joint variables. □

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Нами узагальнено деякі критерії обмеженості **L**-індексу за сукупністю змінних для аналітичних у бікрузі функцій, де $\mathbf{L}(z) = (l_1(z_1, z_2), l_2(z_1, z_2))$, $l_j : \mathbb{D}^2 \to \mathbb{R}_+$ — неперервна функція, $j \in \{1, 2\}$, $\mathbb{D}^2$ — бікруг $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$. Отримані твердження описують поводження розвинення у степеневий ряд на кістяку бікруга. При цьому сума відповідного степеневого ряду оцінена через домінувальний однорідний многочлен, степінь якого не перевищує деякого числа, залежного тільки від радіусів бікруга. Замінюючи квантор загальності на квантор існування для значень радіусів бікруга, ми також доводимо достатні умови обмеженості **L**-індексу за сукупністю змінних для аналичних функцій, які слабші за необхідні умови.

Ключові слова і фрази: аналітична функція, бікруг, обмежений **L**-індекс за сукупністю змінних, максимум модуля, частинна похідна, головний многочлен, степеневий ряд.

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CONVERGENCE CRITERION FOR BRANCHED CONTINUED FRACTIONS OF THE SPECIAL FORM WITH POSITIVE ELEMENTS

In this paper the problem of convergence of the important type of a multidimensional generalization of continued fractions, the branched continued fractions with independent variables, is considered. This fractions are an efficient apparatus for the approximation of multivariable functions, which are represented by multiple power series. When variables are fixed these fractions are called the branched continued fractions of the special form. Their structure is much simpler then the structure of general branched continued fractions. It has given a possibility to establish the necessary and sufficient conditions of convergence of branched continued fractions of the special form with the positive elements. The received result is the multidimensional analog of Seidel's criterion for the continued fractions. The condition of convergence of investigated fractions is the divergence of series, whose elements are continued fractions. Therefore, the sufficient condition of the convergence of this fraction which has been formulated by the divergence of series composed of partial denominators of this fraction, is established. Using the established criterion and Stieltjes-Vitali Theorem the parabolic theorems of branched continued fractions of the special form with complex elements convergence, is investigated. The sufficient conditions gave a possibility to make the condition of convergence of the branched continued fractions of the special form, whose elements lie in parabolic domains.

Key words and phrases: branced continued fraction of the special form, convergence.

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INTRODUCTION

The convergence problem for continued fractions with positive elements is solved by Seidel's criterion.

Theorem 1 ([9, 12]). *A continued fraction* $b_0 + \mathop{\mathrm{D}}\limits_{n=1}^{\infty} \dfrac{1}{b_n}$ *with positive elements converges if and only if the series* $\sum\limits_{n=1}^{\infty} b_n$ *diverges.*

Convergence criteria for the continued fractions which elements lie in angular [8], parabolic [1, 4, 6] domains was obtained by Seidel's criterion and Stieltjes-Vitaly Theorem.

Necessary, sufficient, necessary and sufficient conditions for convergence of the branched continued fractions (BCF) with N-branches are establised [3, 10, 11]. But, the analog of Seidel's criterion in following statement is not obtained:

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branched continued fraction $b_0+\overset{\infty}{\underset{k=1}{\mathrm{D}}}\sum\limits_{i_k=1}^{N}\frac{1}{b_{i(k)}}$ with positive elements converges if the series $\sum\limits_{k=1}^{\infty}\min\limits_{i(k)} b_{i(k)}$ are divergent.

Establishing the analog of Seidel's criterion for the BCF resulted into construction of different types of BCF, in particular:

$$b_0+\overset{\infty}{\underset{k=1}{\mathrm{D}}}\sum_{i_k=1}^{i_{k-1}}\frac{a_{i(k)}}{b_{i(k)}}=b_0+\sum_{i_1=1}^{N}\cfrac{a_{i(1)}}{b_{i(1)}+\sum\limits_{i_2=1}^{i_1}\cfrac{a_{i(2)}}{b_{i(2)}+\ddots}}, \tag{1}$$

where $a_{i(k)}, b_{i(k)}\in\mathbb{C}$, $i(k)\in\mathcal{I}$, $\mathcal{I}=\{i(k)=i_1i_2\dots i_k\ :1\le i_k\le i_{k-1}\le ...\le i_0;\ k\ge1;\ i_0=N\}$.

This fraction is called the BCF of the special form. There are different convergence ctiteria for this fraction [1, 2, 5].

In the case $b_{i(k)}=1$, and $a_{i(k)}$ are replaced by $a_{i(k)}z_{i_k}$, this fraction is called a multidimensional regular C-fraction with independent variables. This fraction is analog of the BCF for multiple power series. The condition of the correspondence between multiple power series and regular multidimensional C-fraction with independent variables is established in [7].

The analog of Seidel's criterion for the fraction (1) when $a_{i(k)}=1$, $b_{i(k)}>0$, $i(k)\in\mathcal{I}$, and $N=2$ can be found in [6, 11]. The aim of the paper is to establish the analog of Seidel's criterion for arbitrary natural N. Also, using this criterion, the technique of value and elements sets [3, 9] and Stieltjes-Vitaly Theorem [3], to obtain the parabolic convergence region for the following BCF

$$\left(b_0+\overset{\infty}{\underset{k=1}{\mathrm{D}}}\sum_{i_k=1}^{i_{k-1}}\frac{a_{i(k)}}{1}\right)^{-1}, \tag{2}$$

where b_0, $a_{i(k)}$ are complex numbers, $i(k)\in\mathcal{I}$.

1 Main results

In this paper, it will be proved following lemmas for obtaining an analog of Seidel's criterion for the BCF

$$b_0+\overset{\infty}{\underset{k=1}{\mathrm{D}}}\sum_{i_k=1}^{i_{k-1}}\frac{1}{b_{i(k)}}. \tag{3}$$

Lemma 1.1. *Let the BCF* (3) *with positive elements converges and ε be an arbitrary real positive number. Then exists a natural m, depended of ε, such that for each BCF with positive elements*

$$\widehat{b}_0+\overset{\infty}{\underset{k=1}{\mathrm{D}}}\sum_{i_k=1}^{i_{k-1}}\frac{1}{\widehat{b}_{i(k)}}, \tag{4}$$

where $\widehat{b}_{i(k)}=b_{i(k)}$ for all $i(k)\in\mathcal{I}, k<m$, the following estimate holds

$$\left|f_n^{'}-f_k^{'}\right|<\varepsilon$$

for all $n,k\ge m$ and $f_k^{'}$ be a kth approximant of BCF (4).

Proof. If f_k be a kth approximant of BCF (3) and the fraction converges, then for all $\varepsilon > 0$ exists $m \geq 2$: $|f_{m-1} - f_{m-2}| < \varepsilon$.

Since $f_k = f_k^{'}, k = 1, 2, ..., m-1$, using the monotonicity properties of approximants of a BCF with positive elements, we have that for all $\varepsilon > 0$ for all $n, k \in \mathbb{N}, n \geq m, k \geq m$,

$$\left| f_n^{'} - f_k^{'} \right| \leq \left| f_{m-1}^{'} - f_{m-2}^{'} \right| = |f_{m-1} - f_{m-2}| < \varepsilon.$$

□

Lemma 1.2. *Let $\Delta_0, \Delta_{i(k)}$ be absolute errors of b_0 and $b_{i(k)}, i(k) \in \mathcal{I}$, respectively. If $\widehat{b}_0 > 0$, $\widehat{b}_{i(k)} > 0$ are approximants of b_0 and $b_{i(k)}$, respectively, then the absolute value of relative error of f_m, mth approximant of the BCF* (3), *is less or equal to the value*

$$\max_{0 \leq s \leq \left[\frac{m}{2}\right]} \max_{i(2s+1) \in \mathcal{I}} \left\{ \frac{\Delta_{i(2s)}}{b_{i(2s)}}, \frac{\Delta_{i(2s+1)}}{\widehat{b}_{i(2s+1)}} \right\}, \tag{5}$$

where $\Delta_{i_0} = \Delta_0, \Delta_{i(2k+1)} = 0$, if $m = 2k$.

Proof. Let $\delta_\alpha^* = \dfrac{\alpha - \widehat{\alpha}}{\widehat{\alpha}}$, $\delta_\alpha = \dfrac{\widehat{\alpha} - \alpha}{\alpha}$, where $\widehat{\alpha}$ is approximate value of α. If $a > 0, \widehat{a} > 0, b > 0$, $\widehat{b} > 0$, then: $|\delta_{a+b}| \leq \max\left\{|\delta_a|, |\delta_b|\right\}$, $\left|\delta_{a+b}^*\right| \leq \max\left\{|\delta_a^*|, |\delta_b^*|\right\}$, $\left|\delta_{\frac{1}{a}}\right| = |\delta_a^*|$, $|\delta_a^*| = \left|\dfrac{\delta_a}{1+\delta_a}\right|$.

Let $\delta_{i(k)}^{(m)}$ is the relative error of calculation of the BCF $b_{i(k)} + \underset{s=k+1}{\overset{m}{\mathrm{D}}} \sum_{i_s=1}^{i_{s-1}} \dfrac{1}{b_{i(s)}}$. Then the absolute value of relative error of f_m is less or equal to:

$$\max_{i_1} \left\{ |\delta_0|, \left|\delta_{i_1}^{*(m)}\right| \right\} \leq \max_{i_1, i_2} \left\{ |\delta_0|, \left|\delta_{i(1)}^{*}\right|, \left|\delta_{i(2)}^{*(m)}\right| \right\} \leq \max_{i_1, i_2, i_3} \left\{ |\delta_0|, \left|\delta_{i(1)}^{*}\right|, \left|\delta_{i(2)}\right|, \left|\delta_{i(3)}^{*(m)}\right| \right\} \leq$$

$$\leq ... \leq \max_{0 \leq s \leq \left[\frac{m}{2}\right]} \max_{i(2s+1) \in \mathcal{I}} \left\{ \left|\delta_{i(2s)}\right|, \left|\frac{\delta_{i(2s+1)}}{1+\delta_{i(2s+1)}}\right| \right\} = \max_{0 \leq s \leq \left[\frac{m}{2}\right]} \max_{i(2s+1) \in \mathcal{I}} \left\{ \frac{\Delta_{i(2s)}}{b_{i(2s)}}, \frac{\Delta_{i(2s+1)}}{\widehat{b}_{i(2s+1)}} \right\}.$$

□

Let $\mathcal{I}^{(m)} = \left\{i(n) = i_1 i_2 \ldots i_n \; : m \leq i_n \leq i_{n-1} \leq ... \leq i_0;\; n \geq 1;\; i_0 = N\right\}, m = \overline{2, N}$. Let the continued fractions are determined recurrently as follows

$$b_0^{(m)} = b_0^{(m-1)} + \underset{k=1}{\overset{\infty}{\mathrm{D}}} \frac{1}{b_{m[k]}^{(m-1)}},\; b_{i(n)}^{(m)} = b_{i(n)}^{(m-1)} + \underset{k=1}{\overset{\infty}{\mathrm{D}}} \frac{1}{b_{i(n)m[k]}^{(m-1)}},\; m = \overline{1, N}, \tag{6}$$

$m[k] = \underbrace{mm...m}_{k}$, $i(n) \in \mathcal{I}^{(m+1)}$, with the initial conditions $b_0^{(0)} = b_0, b_{i(k)}^{(0)} = b_{i(k)}$, $i(k) \in \mathcal{I}$, where $b_{i(k)}$ are partial denominators of BCF (3).

Theorem 2 (The multidimensional analog of Seidel's criterion)**.** *BCF* (3) *with positive partial denominators converges if and only if for each $m, 1 \leq m \leq N$, and each $i(n), i(n) \in \mathcal{I}^{(m+1)}$, the following series diverge*

$$\sum_{k=1}^{\infty} b_{m[k]}^{(m-1)}, \sum_{k=1}^{\infty} b_{i(n)m[k]}^{(m-1)}, \tag{7}$$

that elements are determined by (6).

Proof. Necessity. Let the fraction (3) is convergent, then the following sth tail of this fraction converges:

$$r_{i(s)} = b_{i(s)} + \mathop{\mathrm{D}}_{k=s+1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{1}{b_{i(k)}}, i(s) \in \mathcal{I}.$$

The proof of this fact is analogous to the proof of the Theorem 2.1 [3]. In particular, if $i_s = 1$, then the following continued fractions are convergent

$$r_1 = b_1 + \mathop{\mathrm{D}}_{k=2}^{\infty} \frac{1}{b_{1[k]}},\ r_{i(n)1} = b_{i(n)1} + \mathop{\mathrm{D}}_{k=2}^{\infty} \frac{1}{b_{i(n)1[k]}},\ i(n) \in \mathcal{I}^{(2)}. \tag{8}$$

According to Seidel's criterion, the series $\sum\limits_{k=1}^{\infty} b_{1[k]}, \sum\limits_{k=1}^{\infty} b_{i(n)1[k]}, i(n) \in \mathcal{I}^{(2)}$ diverge. Let $b_0^{(1)} = b_0 + \dfrac{1}{r_1}, b_{i(n)}^{(1)} = b_{i(n)} + \dfrac{1}{r_{i(n)1}}, i(n) \in \mathcal{I}^{(2)}$. Consider the BCF of the special form with $(N-1)$-branches:

$$b_0^{(1)} + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=2}^{i_{k-1}} \frac{1}{b_{i(k)}^{(1)}}. \tag{9}$$

We shall show that the convergence of BCF (9) follows from convergence of the fraction (3). Let f_n be the nth approximant of the BCF (3). The approximants of the BCF (9), $\widetilde{f}_n$, are the figured approximants of the fraction (3).

$$\widetilde{f}_n = b_0 + \mathop{\mathrm{D}}_{k=1}^{n} \sum_{i_k=1}^{i_{k-1}} \frac{1}{\widetilde{b}_{i(k)}}, \widetilde{b}_{i(k)} = \begin{cases} b_{i(k)}, & \text{if } k < n \text{ or } k = n, i_n \neq 1; \\ b_{i(n)} + \mathop{\mathrm{D}}\limits_{p=1}^{\infty} \dfrac{1}{b_{i(n)1[p]}}, & \text{if } k = n, i_n = 1. \end{cases}$$

Applying the method suggested in [3], we can show that the following relation for difference $f_n - \widetilde{f}_n$ is valid:

$$f_n - \widetilde{f}_n = (-1)^n \sum_{i_1=1}^{N} \sum_{i_2=1}^{i_1} \dots \sum_{i_n=1}^{i_{n-1}} \frac{b_{i(n)} - \widetilde{b}_{i(n)}}{\prod\limits_{p=1}^{n} \widetilde{Q}_{i(p)}^{(n)} Q_{i(p)}^{(n)}},$$

where

$$Q_{i(n)}^{(n)} = b_{i(n)},\ Q_{i(s)}^{(n)} = b_{i(s)} + \mathop{\mathrm{D}}_{r=s+1}^{n} \sum_{i_r=1}^{i_{r-1}} \frac{1}{b_{i(r)}},\ \widetilde{Q}_{i(n)}^{(n)} = \widetilde{b}_{i(n)},\ \widetilde{Q}_{i(s)}^{(n)} = \widetilde{b}_{i(s)} + \mathop{\mathrm{D}}_{r=s+1}^{n} \sum_{i_r=1}^{i_{r-1}} \frac{1}{\widetilde{b}_{i(r)}},$$

$n = 1, 2, \ldots;\ s = \overline{1, n-1};\ i(n) \in \mathcal{I};\ i(p) \in \mathcal{I}$. Obviously $b_{i(n)} - \widetilde{b}_{i(n)} = 0$, if $i_n \neq 1$, and $b_{i(n)} - \widetilde{b}_{i(n)} \leq 0$, if $i_n = 1$. Thus, $(-1)^{n+1} \left(f_n - \widetilde{f}_n \right) > 0$, that is $f_{2r} < \widetilde{f}_{2r} < \widetilde{f}_{2r+1} < f_{2r+1}$.

That is to say, the convergence of the fraction (9) follows from the convergence of the fractoin (3). Analogically as for BCF (3), we conclude that series $\sum\limits_{k=1}^{\infty} b_{2[k]}^{(1)}, \sum\limits_{k=1}^{\infty} b_{i(n)2[k]}^{(1)}, i(n) \in \mathcal{I}^{(3)}$, diverge, and from the convergence of the fraction (9) follows that the fraction $b_0 + \mathop{\mathrm{D}}\limits_{k=1}^{\infty} \sum\limits_{i_k=3}^{i_{k-1}} \dfrac{1}{b_{i(k)}}$, $i(k) \in \mathcal{I}^{(3)}$ converges.

Using the same arguments by $(N-2)$ times, we conclude that the series $\sum\limits_{k=1}^{\infty} b_{m[k]}^{(m-1)}$, $\sum\limits_{k=1}^{\infty} b_{i(n)m[k]}^{(m-1)}$ are divergence for each $m : 1 \leq m \leq N-1, i(n) \in \mathcal{I}^{(m+1)}$, also the continued fraction $b_0^{(N-1)} + \mathop{\mathrm{D}}\limits_{k=1}^{\infty} \frac{1}{b_{i(k)}^{(N-1)}}$, $i(k) \in \mathcal{I}^{(N)}$ is convergent. It's equivalent by Seidel's criterion to the divergence of the series $\sum\limits_{k=1}^{\infty} b_{N[k]}^{(N-1)}$. Thus, series (7) diverge.

Sufficiency. By mathematical induction on N, we prove the fact that from diverdgence of the series (7) follows the convergence of the BCF (3).

$N = 1$, the continued fraction with positive elements $b_0 + \mathop{\mathrm{D}}\limits_{k=1}^{\infty} \frac{1}{b_{1[k]}}$ converges by Seidel's criterion, if the series $\sum\limits_{i=1}^{\infty} b_{1[k]}$ is divergent.

$N = 2$, the BCF with positive elements $b_0 + \mathop{\mathrm{D}}\limits_{k=1}^{\infty} \sum\limits_{i_k=1}^{i_{k-1}} \frac{1}{b_{i(k)}}, i(k) \in \mathcal{I}, i_0 = 2$, converges by the Theorem 2.8 [11] if series $\sum\limits_{k=1}^{\infty} b_{1[k]}$, $\sum\limits_{k=1}^{\infty} b_{i(n)1[k]}$, $\sum\limits_{k=1}^{\infty} b_{1[k]}^{(1)}$ diverge.

We suppose that for all $N, N < p$, from the divergence of series (7) follows the convergence of the BCF (3). Consider the convergence of the BCF (3) in the case $N = p$.

$$b_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{1}{b_{i(k)}}, i(k) \in \mathcal{I}, i_0 = p. \tag{10}$$

If $\sum\limits_{k=1}^{\infty} b_{1[k]} = \infty$, $\sum\limits_{k=1}^{\infty} b_{i(n)1[k]} = \infty, i(n) \in \mathcal{I}^{(2)}$, then continued fractions

$$b_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \frac{1}{b_{1[k]}}, \tag{11}$$

$$b_{i(n)} + \mathop{\mathrm{D}}_{k=1}^{\infty} \frac{1}{b_{i(n)1[k]}}, i(n) \in \mathcal{I}^{(2)}, \tag{12}$$

converge to the values $b_0^{(1)}$ and $b_{i(n)}^{(1)}$, respectively. We replace, the continued fractions (11) and (12) by it's values, and obtaine BCF of the special form with $(p-1)$-branches

$$b_0^{(1)} + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=2}^{i_{k-1}} \frac{1}{b_{i(k)}^{(1)}}, i(k) \in \mathcal{I}^{(2)}, i_0 = p. \tag{13}$$

Since, the series (7) diverge, for each m, $2 \leq m \leq N$, the fraction (13) converges by the hypotesis of induction. We shall show that the fraction (10) is convergent. Consider the difference between the nth approximant of BCF (10) and (13).

Let $b_0^{(1,n)}$, $b_{i(n)}^{(1,n)}$ be the nth approximant of continued fractions (11) and (12) respectively. Then the nth approximant of BCF (10) may be written as

$$f_n = b_0^{(1,n)} + \mathop{\mathrm{D}}_{k=1}^{n} \sum_{i_k=2}^{i_{k-1}} \frac{1}{b_{i(k)}^{(1,n-k)}}, \ i(k) \in \mathcal{I}^{(2)}.$$

It's the BCF with $(p-1)$-branches. The nth approximant of BCF (13) may be written as

$$\widehat{f}_n = b_0^{(1)} + \mathop{\mathrm{D}}_{k=1}^{n} \sum_{i_k=2}^{i_{k-1}} \frac{1}{b_{i(k)}^{(1)}},\ i(k) \in \mathcal{I}^{(2)}.$$

According to the Lemma 1, form the convergence of the fraction (13) follows that for all $\varepsilon > 0$ exists $m \in \mathbb{N}$ such that for all $n, k \in \mathbb{N}, n \geq 2m+2$ takes place $\left|\widehat{f}_n - g_n\right| < \varepsilon$, where

$$g_n = b_0^{(1)} + \sum_{i_1=2}^{p} \frac{1}{b_{i(1)}^{(1)}} + \sum_{i_2=2}^{i_1} \frac{1}{b_{i(2)}^{(1)}} + \ldots + \sum_{i_{2m+1}=2}^{i_{2m}} \frac{1}{b_{i(2m+1)}^{(1)}} + \sum_{i_{2m+2}=2}^{i_{2m+1}} \frac{1}{b_{i(2m+2)}^{(1,n-2m-2)}} + \ldots + \sum_{i_n=2}^{i_{n-1}} \frac{1}{b_{i(n)}^{(1,0)}}.$$

Next we estimate the value $\left|f_n - \widehat{f}_n\right|$: $\left|f_n - \widehat{f}_n\right| \leq |f_n - g_n| + \left|g_n - \widehat{f}_n\right|$. Using the Lemma 2, we estimate the first term in the right of inequality:

$$|f_n - g_n| \leq \underset{0\leq s\leq m}{max}\ \underset{i(2s+1)}{max} \left\{ \frac{\left|b_{i(2s)}^{(1,n-2s)} - b_{i(2s)}^{(1)}\right|}{b_{i(2s)}^{(1)}}, \frac{\left|b_{i(2s+1)}^{(1,n-2s-1)} - b_{i(2s+1)}^{(1)}\right|}{b_{i(2s+1)}^{(1,n-2s-1)}} \right\} \cdot g_n.$$

Since the continued fractions (11) converge, we may choose n, $n \geq 2m+2$, such that for all $i(2s+1) \in \mathcal{I}^{(2)}$, $\left|b_{i(2s)}^{(1,n-2s)} - b_{i(2s)}^{(1)}\right| < \dfrac{\varepsilon}{2A}$, $\left|b_{i(2s+1)}^{(1,n-2s-1)} - b_{i(2s+1)}^{(1)}\right| < \dfrac{\varepsilon}{2A}$, where $A = b_0 + \sum\limits_{i_1=1}^{p} \dfrac{1}{b_{i_1}}$.

Thus, $\left|f_n - \widehat{f}_n\right| < \varepsilon$. From the convergence of the fraction (13) follows the convergence of the fraction (10). $\square$

Since the elements of series (7) are difficult to calculate by the relation (6), it's conviniently to use the following sufficient condition for convergence.

Theorem 3. *BCF (3) is divergent, if for each m, $1 \leq m \leq N$, and each, $i(n)$, $i(n) \in \mathcal{I}^{(m+1)}$, the following series are divergent*

$$\sum_{k=1}^{\infty} b_{m[k]}, \sum_{k=1}^{\infty} b_{i(n)m[k]}. \tag{14}$$

The divergence of the series (14) is suffisient for the divergence of the series (7). We shall use the Theorem 3, to obtain the parabolic convergence domain for the BCF (2).

Lemma 1.3. *Let $\left\{V_{i(k)}\right\}$ be the sequense of half-planes*

$$V_{i(k)} = V_{i_k} = \left\{ z \in \mathbb{C} : Re\left(ze^{-i\gamma}\right) > -\frac{1}{2i_{k-1}} cos\gamma \right\}, k = 1, 2, 3, \ldots,\ 1 \leq i_k \leq i_{k-1},\ i_0 = N,$$

and

$$E_{i(k)} = E_{i_k} = \left\{ z \in \mathbb{C} : |z| - Re\left(ze^{-2i\gamma}\right) < \frac{1}{2i_{k-1}} cos^2\gamma \right\},$$

where $-\dfrac{\pi}{2} < \gamma < \dfrac{\pi}{2}$.

Then $\left\{V_{i(k)}\right\}$ and $\left\{E_{i(k)}\right\}$ are the sequenses of value sets and element sets of the BCF (2).

The proof of this Lemma is analogous to the proof of the corresponding Theorem 1.5 [3] for the BCF with N-branches.

Theorem 4. *Let the elements of the BCF* (2) *lie in the parabolic domains* $a_{i(k)} \in \mathcal{P}_{i(k)}$, $i(k) \in \mathcal{I}$, *where*

$$\mathcal{P}_{i(k)}(\varepsilon) = \mathcal{P}_{i_k}(\varepsilon) = \left\{ z \in \mathbb{C} : |z| - Rez < \frac{1-\varepsilon}{2i_{k-1}} \right\}, \tag{15}$$

ε *be an arbitrary small real number,* $0 < \varepsilon < 1$.

Then

1) there exist a finite limits of even and odd approximants of the BCF (2);

2) BCF (2) *converges if* $\sum\limits_{k=1}^{\infty} b_{m[k]} = \infty$, $\sum\limits_{k=1}^{\infty} b_{i(n)m[k]} = \infty$ *for each* m, $1 \le m \le N$, *and each,* $i(n)$, $i(n) \in \mathcal{I}^{(m+1)}$, *where* $b_{i(k)}$ *is definitely determined by the relations*

$$\left|a_{i(k)}\right| = \left(b_{i(k)} b_{i(k-1)}\right)^{-1}, \qquad i(k) \in \mathcal{I}, \qquad b_{i(0)} = b_0 = 1;$$

3) the value region of this fraction is the following circle

$$\mathcal{K} = \{ z \in \mathbb{C} : |z - 1| \le 1 \}.$$

Proof. Let $a_{i(k)} = \left|a_{i(k)}\right| e^{i\alpha_{i(k)}}$, where $\alpha_{i(k)}$ be an argument of number $a_{i(k)}$, $-\pi < \alpha_{i(k)} \le \pi$, if $a_{i(k)} \ne 0$.

We determine the function

$$a_{i(k)}(z) = \begin{cases} 0, & \text{if } a_{i(k)} = 0, \\ \left|a_{i(k)}\right| e^{iz\alpha_{i(k)}}, & \text{if } a_{i(k)} \ne 0 \end{cases}$$

in domain $\Omega_\delta = \{ z \in \mathbb{C} : |Imz| < \delta,\ |Rez| < 1 + \delta \}$, where δ is an arbitrary real number, such that $(1+\delta)^2 e^{\pi\delta} < (1-\varepsilon)^{-1}$.

We shall show that $a_{i(k)}(z) \in \mathcal{P}_{i(k)}(0)$, $i(k) \in \mathcal{I}$, if $z \in \Omega_\delta$.

If $\alpha_{i(k)} = 0$, then $a_{i(k)}(z) \in \mathcal{P}_{i(k)}(0)$. Let $\alpha_{i(k)} \ne 0$ and $z = x + iy$. From $a_{i(k)} \in \mathcal{P}_{i(k)}(\varepsilon)$, we obtain

$$\left|a_{i(k)}(z)\right| - Re a_{i(k)}(z) < \frac{1-\varepsilon}{2i_{k-1}} e^{\pi\delta} \frac{1 - \cos\alpha_{i(k)} x}{1 - \cos\alpha_{i(k)}}. \tag{16}$$

If we determine the extrema for the function $\mathcal{M}\left(\alpha_{i(k)}, x\right) = \dfrac{1 - \cos\alpha_{i(k)} x}{1 - \cos\alpha_{i(k)}}$, where $-\pi < \alpha_{i(k)} \le \pi$, $\alpha_{i(k)} \ne 0$, $|x| \le 1 + \delta$, we obtain $\sup\left(\mathcal{M}\left(\alpha_{i(k)}, x\right)\right) = (1+\delta)^2$. Thus, $\left|a_{i(k)}(z)\right| - Re a_{i(k)}(z) < \dfrac{1}{2i_{k-1}}$, that is $a_{i(k)}(z) \in \mathcal{P}_{i(k)}(0)$, $i(k) \in \mathcal{I}$.

Consider the functional BCF

$$\left(1 + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)}(z)}{1}\right)^{-1}, \ i(k) \in \mathcal{I}. \tag{17}$$

According to the Lemma 3, where $\gamma = 0$, we obtain that the value set of the reciprocal of the fraction (17) is the half-plane $Rez > \dfrac{1}{2}$. Therefore, all approximants of the BCF (17) depend on the domain $\mathcal{K} = \{ z \in \mathbb{C} : |z - 1| \le 1 \}$.

Thus, any nth approximant of the (17), $f_n(z)$, is the holomorphic function in domain Ω_δ. We use the Theorem 2.13 (Stieltjes-Vitali Therem [3]) for sequence $\{f_n(z)\}$, where in particular $a=-1$, $b=-2$, and $\Delta=\{z\in\mathbb{C}\,:\,Rez=0,\ |Imz|<\delta\}$.

If $z\in\Delta$, then we write the BCF (17) in the form

$$\left(1+\mathop{D}_{k=1}^{n}\sum_{i_k=1}^{i_{k-1}}\frac{\widetilde{a}_{i(k)}}{1}\right)^{-1},\ i(k)\in\mathcal{I},\tag{18}$$

where

$$\widetilde{a}_{i(k)}=\begin{cases}0, & \text{if } a_{i(k)}=0,\\ \left|a_{i(k)}\right|e^{-y\alpha_{i(k)}}, & \text{if } a_{i(k)}\neq 0.\end{cases}$$

By equivalence transformstion, we can write the fraction (18), into the form

$$\left(1+\mathop{D}_{k=1}^{\infty}\sum_{i_k=1}^{i_{k-1}}\frac{1}{b_{i(k)}e^{\alpha_{i(k)}y}}\right)^{-1},\ i(k)\in\mathcal{I},\tag{19}$$

where $b_{i(k)}$ is determined by relations $\left|a_{i(k)}\right|=\left(b_{i(k-1)}b_{i(k)}\right)^{-1}$, $b_{i_0}=1$, $i(k)\in\mathcal{I}$.

The divergence of the series $\sum\limits_{k=1}^{\infty}b_{m[k]}$, $\sum\limits_{k=1}^{\infty}b_{i(n)m[k]}$ for each m, $1\le m\le N$, and each $i(n)$, $i(n)\in\mathcal{I}^{(m+1)}$, is equivalent to the divergence of the series $\sum\limits_{k=1}^{\infty}b_{m[k]}e^{\alpha_{m[k]}y}$, $\sum\limits_{k=1}^{\infty}b_{i(n)m[k]}e^{\alpha_{i(n)m[k]}y}$. The convergence of the BCF (19) follows from the Theorem 2. Thus, the fraction (18) converges.

Therefore, according to Stieltjes-Vitali Therem, the BCF (17) converges on every compact subset of Ω_δ. In particular, it converges when $z=1$. This is equivalent to the convergence of the BCF (2).

Using the monotonicity properties of approximants of a BCF with positive elements, we find that finite limits of even and odd approximants of the BCF (2) always exist. $\square$

Analogously we can prove the following statement.

Theorem 5. *Let the elements of the BCF* (2) *lie in the parabolic domains* $a_{i(k)}\in\mathcal{P}_{i(k)}$, $i(k)\in\mathcal{I}$, *where*

$$\mathcal{P}_{i(k)}(\gamma)=\mathcal{P}_{i_k}(\gamma)=\left\{z\in\mathbb{C}:|z|-Re\left(ze^{-2i\gamma}\right)<\frac{1-\varepsilon}{2i_{k-1}}cos^2\gamma\right\},\tag{20}$$

ε is an arbitrary small real number, $0<\varepsilon<1$.

Then

1) there exist a finite limits of even and odd approximants of BCF (2)*;*

2) BCF (2) *converges if* $\sum\limits_{k=1}^{\infty}b_{m[k]}=\infty$, $\sum\limits_{k=1}^{\infty}b_{i(n)m[k]}=\infty$ *for each* m, $1\le m\le N$, *and each* $i(n)$, $i(n)\in\mathcal{I}^{(m+1)}$, *where* $b_{i(k)}$ *is definitely determined by the relations* $\left|a_{i(k)}\right|=\left(b_{i(k)}b_{i(k-1)}\right)^{-1}$, $i(k)\in\mathcal{I}$, $b_{i(0)}=b_0=1$*;*

3) the value region of this fraction is the following circle

$$\mathcal{K}(\gamma)=\left\{z\in\mathbb{C}:\left|z-\frac{e^{-i\gamma}}{2(1-\frac{1}{2}cos\gamma)}\right|\le\frac{1}{2(1-\frac{1}{2}cos\gamma)}\right\}.$$

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Боднар Д.І., Біланик І.Б. *Критерій збіжності гіллястих ланцюгових дробів спеціального вигляду з додатними елементами* // Карпатські матем. публ. — 2017. — Т.9, №1. — С. 13–21.

Досліджується питання збіжності важливого класу багатовимірних узагальнень неперервних дробів — гіллястих ланцюгових дробів (ГЛД) з нерівнозначними змінними. Ці дроби є ефективними при наближенні функцій, заданих кратними степеневими рядами. При фіксованих значеннях змінних вони отримали назву гіллястих ланцюгових дробів спеціального вигляду. Значно простіша структура порівняно із загальними гіллястими ланцюговими дробами дала можливість встановити необхідну і достатню умову їх збіжності у випадку додатних елементів. Отриманий результат є багатовимірним узагальненням критерію збіжності Зейделя для неперервних дробів. Умовою збіжності досліджуваних ГЛД є розбіжність рядів елементами яких є неперервні дроби. Тому доводиться достатня ефективна ознака збіжності, що формулюється через розбіжність рядів складених з частинних знаменників даного ГЛД. Використовуючи встановлену достатню ознаку збіжності та теорему Стілтьєса-Віталі, досліджено параболічні області збіжності для ГЛД спеціального вигляду з комплексними елементами. Встановлена достатня ознака дала можливість послабити умови збіжності ГЛД, елементи котрих лежать в параболічних областях.

Ключові слова і фрази: гіллясті ланцюгові дроби спеціального вигляду, збіжність.

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TOPOLOGY ON THE SPECTRUM OF THE ALGEBRA OF ENTIRE SYMMETRIC FUNCTIONS OF BOUNDED TYPE ON THE COMPLEX L_∞

It is known that the so-called elementary symmetric polynomials $R_n(x) = \int_{[0,1]}(x(t))^n\,dt$ form an algebraic basis in the algebra of all symmetric continuous polynomials on the complex Banach space L_∞, which is dense in the Fréchet algebra $H_{bs}(L_\infty)$ of all entire symmetric functions of bounded type on L_∞. Consequently, every continuous homomorphism $\varphi : H_{bs}(L_\infty) \to \mathbb{C}$ is uniquely determined by the sequence $\{\varphi(R_n)\}_{n=1}^{\infty}$. By the continuity of the homomorphism φ, the sequence $\{\sqrt[n]{|\varphi(R_n)|}\}_{n=1}^{\infty}$ is bounded. On the other hand, for every sequence $\{\xi_n\}_{n=1}^{\infty} \subset \mathbb{C}$, such that the sequence $\{\sqrt[n]{|\xi_n|}\}_{n=1}^{\infty}$ is bounded, there exists $x_\xi \in L_\infty$ such that $R_n(x_\xi) = \xi_n$ for every $n \in \mathbb{N}$. Therefore, for the point-evaluation functional δ_{x_ξ} we have $\delta_{x_\xi}(R_n) = \xi_n$ for every $n \in \mathbb{N}$. Thus, every continuous complex-valued homomorphism of $H_{bs}(L_\infty)$ is a point-evaluation functional at some point of L_∞. Note that such a point is not unique. We can consider an equivalence relation on L_∞, defined by $x \sim y \Leftrightarrow \delta_x = \delta_y$. The spectrum (the set of all continuous complex-valued homomorphisms) M_{bs} of the algebra $H_{bs}(L_\infty)$ is one-to-one with the quotient set $L_\infty/_\sim$. Consequently, M_{bs} can be endowed with the quotient topology. On the other hand, it is naturally to identify M_{bs} with the set of all sequences $\{\xi_n\}_{n=1}^{\infty} \subset \mathbb{C}$ such that the sequence $\{\sqrt[n]{|\xi_n|}\}_{n=1}^{\infty}$ is bounded.

We show that the quotient topology is Hausdorff and that M_{bs} with the operation of coordinatewise addition of sequences forms an abelian topological group.

Key words and phrases: symmetric function, topology on the spectrum.

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INTRODUCTION

Algebras of symmetric functions on the spaces of Lebesgue-measurable functions were studied by a number of authors [1], [4], [5], [6], [7] (see also a survey [2]). In [3] the spectrum of the algebra $H_{bs}(L_\infty)$ of entire symmetric functions of bounded type on L_∞ (see definition below) is described. In this paper the topology on the spectrum of $H_{bs}(L_\infty)$ is investigated.

Let L_∞ be the complex Banach space of all Lebesgue measurable essentially bounded complex-valued functions x on $[0, 1]$ with norm

$$\|x\|_\infty = \operatorname{ess\,sup}_{t\in[0,1]}|x(t)|.$$

Let Ξ be the set of all measurable bijections of $[0, 1]$ that preserve the measure. A function $f : L_\infty \to \mathbb{C}$ is called symmetric if for every $x \in L_\infty$ and for every $\sigma \in \Xi$

$$f(x \circ \sigma) = f(x).$$

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Let $H_{bs}(L_\infty)$ be the Fréchet algebra of all entire symmetric functions $f : L_\infty \to \mathbb{C}$ which are bounded on bounded sets endowed with the topology of uniform convergence on bounded sets. By [3, Theorem 4.3], polynomials $R_n : L_\infty \to \mathbb{C}$, $R_n(x) = \int_{[0,1]}(x(t))^n\,dt$ for $n \in \mathbb{N}$, form an algebraic basis in the algebra of all symmetric continuous polynomials on L_∞. Since every $f \in H_{bs}(L_\infty)$ can be described by its Taylor series of continuous symmetric homogeneous polynomials, it follows that f can be uniquely represented as

$$f(x) = f(0) + \sum_{n=1}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1,\ldots,k_n} R_1^{k_1}(x)\cdots R_n^{k_n}(x).$$

Consequently, for every non-trivial continuous homomorphism $\varphi : H_{bs} \to \mathbb{C}$, taking into account $\varphi(1) = 1$, we have

$$\varphi(f) = f(0) + \sum_{n=1}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1,\ldots,k_n} \varphi(R_1)^{k_1}\cdots \varphi(R_n)^{k_n}.$$

Therefore φ is completely determined by the sequence of its values on R_n :

$$(\varphi(R_1), \varphi(R_2), \ldots).$$

By the continuity of φ, the sequence $\{\sqrt[n]{|\varphi(R_n)|}\}_{n=1}^{\infty}$ is bounded. On the other hand we have following statement.

Theorem 1 ([3]). *For every sequence $\xi = \{\xi_n\}_{n=1}^{\infty} \subset \mathbb{C}$ such that $\sup_{n\in\mathbb{N}} \sqrt[n]{|\xi_n|} < +\infty$, there exists $x_\xi \in L_\infty$ such that $R_n(x_\xi) = \xi_n$ for every $n \in \mathbb{N}$ and $\|x_\xi\|_\infty \leq \frac{2}{M}\sup_{n\in\mathbb{N}} \sqrt[n]{|\xi_n|}$, where*

$$M = \prod_{n=1}^{\infty} \cos\left(\frac{\pi}{2}\frac{1}{n+1}\right). \tag{1}$$

Hence, for every sequence $\xi = \{\xi_n\}_{n=1}^{\infty}$ such that $\sup_{n\in\mathbb{N}} \sqrt[n]{|\xi_n|} < +\infty$, there exists the point-evaluation functional $\varphi = \delta_{x_\xi}$ such that $\varphi(R_n) = \xi_n$ for every $n \in \mathbb{N}$. Since every such a functional is a continuous homomorphism, it follows that the spectrum (the set of all continuous complex-valued homomorphisms) of the algebra $H_{bs}(L_\infty)$, which we denote by M_{bs}, can be identified with the set of all sequences $\xi = \{\xi_n\}_{n=1}^{\infty} \subset \mathbb{C}$ such that $\{\sqrt[n]{|\xi_n|}\}_{n=1}^{\infty}$ is bounded.

There are different approaches to the topologization of the spectra of algebras. The most common approach is to endow the spectrum by the so-called Gelfand topology (the weakest topology, in which all the functions $\widehat{f} : M_{bs} \to \mathbb{C}$, $\widehat{f}(\varphi) = \varphi(f)$, where $f \in H_{bs}(L_\infty)$, are continuous). We consider another natural topology on M_{bs}. Let $\nu : L_\infty \to M_{bs}$ be defined by

$$\nu(x) = (R_1(x), R_2(x), \ldots).$$

Let τ_∞ be the topology on L_∞, generated by $\|\cdot\|_\infty$. Let us define an equivalence relation on L_∞ by $x \sim y \Leftrightarrow \nu(x) = \nu(y)$. Let τ be the quotient topology on M_{bs} :

$$\tau = \{\nu(V) : V \in \tau_\infty\}.$$

Note that ν is a continuous open mapping. Therefore, τ contains the Gelfand topology.

In this work we show that $(M_{bs}, +, \tau)$ is an abelian topological group, where "+" is the operation of coordinate-wise addition.

1 THE MAIN RESULT

Let us denote $B(x,r)$ the open ball with center at $x \in L_\infty$ and radius $r > 0$ in L_∞.

Theorem 2. *(M_{bs}, τ) is a Hausdorff topological space.*

Proof. Let $a = (a_1, a_2, \ldots), b = (b_1, b_2, \ldots) \in M_{bs}$ such that $a \neq b$. Let $m = \min\{j \in \mathbb{N} : a_j \neq b_j\}$. By Theorem 1, there exist $x_a, x_b \in L_\infty$ such that $\nu(x_a) = a$ and $\nu(x_b) = b$. Let

$$\varepsilon = \min\left\{1, \frac{|a_m - b_m|}{3m} \min\left\{\frac{1}{(\|x_a\|_\infty + 1)^{m-1}}, \frac{1}{(\|x_b\|_\infty + 1)^{m-1}}\right\}\right\}.$$

Note that $V_1 = \nu(B(x_a, \varepsilon))$ and $V_2 = \nu(B(x_b, \varepsilon))$ are neighborhoods of a and b respectively. Let us prove that V_1 and V_2 are disjoint. Let $y \in B(x_a, \varepsilon)$ and $z \in B(x_b, \varepsilon)$. Let us show that $R_m(y) \neq R_m(z)$. Note that

$$|a_m - b_m| = |R_m(x_a) - R_m(x_b)| \leq |R_m(x_a) - R_m(y)| + |R_m(y) - R_m(z)| + |R_m(z) - R_m(x_b)|. \quad (2)$$

Since $\|y - x_a\|_\infty < \varepsilon$,

$$\begin{aligned}
|R_m(x_a) - R_m(y)| &\leq \int_{[0,1]} |(x_a(t))^m - (y(t))^m| \, dt \\
&= \int_{[0,1]} |x_a(t) - y(t)| \left|(x_a(t))^{m-1} + (x_a(t))^{m-2}(y(t)) + \ldots + (x_a(t))(y(t))^{m-2} + (y(t))^{m-1}\right| dt \\
&\leq \varepsilon \int_{[0,1]} \left(|x_a(t)|^{m-1} + |x_a(t)|^{m-2}|y(t)| + \ldots + |x_a(t)||y(t)||^{m-2} + |y(t)|^{m-1}\right) dt \\
&\leq \varepsilon \int_{[0,1]} \left(\|x_a\|_\infty^{m-1} + \|x_a\|_\infty^{m-2}\|y\|_\infty + \ldots + \|x_a\|_\infty\|y\|_\infty^{m-2} + \|y\|_\infty^{m-1}\right) dt \\
&\leq \varepsilon \int_{[0,1]} \left(\|x_a\|_\infty^{m-1} + \|x_a\|_\infty^{m-2}(\|x_a\|_\infty + \varepsilon) + \ldots + \|x_a\|_\infty(\|x_a\|_\infty + \varepsilon)^{m-2} + (\|x_a\|_\infty + \varepsilon)^{m-1}\right) dt \\
&\leq \varepsilon m(\|x_a\|_\infty + \varepsilon)^{m-1} \leq \varepsilon m(\|x_a\|_\infty + 1)^{m-1}.
\end{aligned}$$

Since $\varepsilon \leq \frac{|a_m - b_m|}{3m(\|x_a\|_\infty + 1)^{m-1}}$, it follows that $|R_m(x_a) - R_m(y)| \leq \dfrac{1}{3}|a_m - b_m|$. Analogously, we obtain $|R_m(z) - R_m(x_b)| \leq \frac{1}{3}|a_m - b_m|$. Therefore, by (2),

$$|a_m - b_m| \leq \frac{2}{3}|a_m - b_m| + |R_m(y) - R_m(z)|.$$

Hence,

$$|R_m(y) - R_m(z)| \geq \frac{1}{3}|a_m - b_m| > 0.$$

Therefore, $R_m(y) \neq R_m(z)$, and, consequently, $\nu(y) \neq \nu(z)$. Hence, V_1 and V_2 are disjoint. ☐

The operation of coordinate-wise addition $+ : M_{bs}^2 \to M_{bs}$ is defined by

$$a + b = (a_1 + b_1, a_2 + b_2, \ldots)$$

for $a = (a_1, a_2, \ldots), b = (b_1, b_2, \ldots) \in M_{bs}$. Note that $(M_{bs}, +)$ is an abelian group.

Theorem 3. *The operation of coordinate-wise addition $+ : M_{bs}^2 \to M_{bs}$ is continuous with respect to the topology τ.*

Proof. Let $a, b \in M_{bs}$. Let us show that for every neighborhood U of the point $a+b$ there exist neighborhoods V_a and V_b of points a and b respectively, such that $a' + b' \in U$ for every $a' \in V_a$ and $b' \in V_b$.

By Theorem 1, there exist functions $x_{4a}, x_{4b} \in L_\infty$ such that $\nu(x_{4a}) = (4a_1, 4a_2, \ldots)$ and $\nu(x_{4b}) = (4b_1, 4b_2, \ldots)$. Let

$$x_a(t) = \begin{cases} x_{4a}(4t), & \text{if } t \in [0, \frac{1}{4}], \\ 0, & \text{if } t \in (\frac{1}{4}, 1] \end{cases}$$

and

$$x_b(t) = \begin{cases} x_{4b}(4t-2), & \text{if } t \in [\frac{1}{2}, \frac{3}{4}], \\ 0, & \text{if } t \in [0, \frac{1}{2}) \cup (\frac{3}{4}, 1]. \end{cases}$$

Then $\nu(x_a) = a$ and $\nu(x_b) = b$. Note that $\nu(x_a + x_b) = \nu(x_a) + \nu(x_b)$. Hence, $\nu(x_a + x_b) = a + b$. Therefore, $x_a + x_b \in \nu^{-1}(U)$. Since the set $\nu^{-1}(U)$ is open in L_∞, it follows that there exists $\varepsilon > 0$ such that $B(x_a + x_b, \varepsilon) \subset \nu^{-1}(U)$. Let

$$r = \frac{\varepsilon}{2} \frac{M}{M+8},$$

where M is defined by (1). Let $V_a = \nu(B(x_a, r))$ and $V_b = \nu(B(x_b, r))$. Let us show that $a' + b' \in U$ for every $a' \in V_a$ and $b' \in V_b$. Let $y \in B(x_a, r)$ and $z \in B(x_b, r)$ such that $\nu(y) = a'$ and $\nu(z) = b'$. Let

$$y_1(t) = \begin{cases} y(t), & \text{if } t \in [0, \frac{1}{4}], \\ 0, & \text{if } t \in (\frac{1}{4}, 1], \end{cases} \qquad y_2(t) = \begin{cases} 0, & \text{if } t \in [0, \frac{1}{4}], \\ y(t), & \text{if } t \in (\frac{1}{4}, 1], \end{cases}$$

$$z_1(t) = \begin{cases} z(t), & \text{if } t \in [\frac{1}{2}, \frac{3}{4}], \\ 0, & \text{if } t \in [0, \frac{1}{2}) \cup (\frac{3}{4}, 1], \end{cases} \quad z_2(t) = \begin{cases} 0, & \text{if } t \in [\frac{1}{2}, \frac{3}{4}], \\ z(t), & \text{if } t \in [0, \frac{1}{2}) \cup (\frac{3}{4}, 1]. \end{cases}$$

Since $x_a(t) = 0$ for $t \in (\frac{1}{2}, 1]$ and $x_b(t) = 0$ for $t \in [0, \frac{1}{2}) \cup (\frac{3}{4}, 1]$, it follows that

$$\|y - x_a\|_\infty = \max\{\|y_1 - x_a\|_\infty, \|y_2\|_\infty\} \qquad \text{and} \qquad \|z - x_b\|_\infty = \max\{\|z_1 - x_b\|_\infty, \|z_2\|_\infty\}.$$

Since $y \in B(x_a, r)$ and $z \in B(x_b, r)$, it follows that $\|y - x_a\|_\infty < r$ and $\|z - x_b\|_\infty < r$. Consequently,

$$\|y_1 - x_a\|_\infty < r, \quad \|y_2\|_\infty < r, \quad \|z_1 - x_b\|_\infty < r \quad \text{and} \quad \|z_2\|_\infty < r.$$

By Theorem 1, for sequences $\xi = 4\nu(y_2)$ and $\eta = 4\nu(z_2)$ there exist functions $u_\xi, v_\eta \in L_\infty$ such that $\nu(u_\xi) = \xi$, $\nu(u_\eta) = \eta$, $\|u_\xi\|_\infty \leq \frac{2c}{M}$ and $\|v_\eta\|_\infty \leq \frac{2d}{M}$, where $c = \sup_{n \in \mathbb{N}} \sqrt[n]{|\xi_n|}$ and $d = \sup_{n \in \mathbb{N}} \sqrt[n]{|\eta_n|}$. Note that

$$|\xi_n| = |4R_n(y_2)| \leq 4\|y_2\|_\infty^n < 4r^n \qquad \text{and} \qquad |\eta_n| = |4R_n(z_2)| \leq 4\|z_2\|_\infty^n < 4r^n.$$

Therefore, $c, d \leq \sup_{n \in \mathbb{N}} \sqrt[n]{4} r \leq 4r$. Consequently, $\|u_\xi\|_\infty < \frac{8r}{M}$ and $\|v_\eta\|_\infty < \frac{8r}{M}$. Let

$$\widetilde{u}(t) = \begin{cases} 0, & \text{if } t \in [0, \frac{1}{4}] \cup [\frac{1}{2}, 1], \\ u_\xi(4t-1), & \text{if } t \in (\frac{1}{4}, \frac{1}{2}) \end{cases}$$

and

$$\widetilde{v}(t) = \begin{cases} 0, & \text{if } t \in [0, \frac{3}{4}], \\ v_\eta(4t-3), & \text{if } t \in (\frac{3}{4}, 1]. \end{cases}$$

Then
$$\nu(\widetilde{u}) = \nu(y_2) \qquad \text{and} \qquad \nu(\widetilde{v}) = \nu(z_2). \tag{3}$$
Note that $\|\widetilde{u}\|_\infty = \|u_\xi\|_\infty$ and $\|\widetilde{v}\|_\infty = \|v_\eta\|_\infty$. Let $\widetilde{y} = y_1 + \widetilde{u}$ and $\widetilde{z} = z_1 + \widetilde{v}$. Note that
$$\|\widetilde{y} - x_a\|_\infty = \max\{\|y_1 - x_a\|_\infty, \|\widetilde{u}\|_\infty\} \le \|y_1 - x_a\|_\infty + \|\widetilde{u}\|_\infty < r + \frac{8r}{M} = r\frac{M+8}{M} = \frac{\varepsilon}{2}.$$
Analogously, $\|\widetilde{z} - x_b\|_\infty < \frac{\varepsilon}{2}$. Therefore,
$$\|\widetilde{y} + \widetilde{z} - (x_a + x_b)\|_\infty \le \|\widetilde{y} - x_a\|_\infty + \|\widetilde{z} - x_b\|_\infty < \varepsilon.$$
Hence, $\widetilde{y} + \widetilde{z} \in B(x_a + x_b, \varepsilon)$. Therefore, $\nu(\widetilde{y} + \widetilde{z}) \in U$. Note that
$$\nu(\widetilde{y} + \widetilde{z}) = \nu(\widetilde{y}) + \nu(\widetilde{z}).$$
By (3),
$$\nu(\widetilde{y}) = \nu(y_1) + \nu(\widetilde{u}) = \nu(y_1) + \nu(y_2) = \nu(y) = a'$$
and
$$\nu(\widetilde{z}) = \nu(z_1) + \nu(\widetilde{v}) = \nu(z_1) + \nu(z_2) = \nu(z) = b'.$$
Therefore, $\nu(\widetilde{y} + \widetilde{z}) = a' + b'$. Hence, $a' + b' \in U$. ☐

Theorem 4. *The group's inverse operation $\xi \mapsto -\xi$ on $(M_{bs}, +)$ is continuous with respect to the topology τ.*

Proof. Let us prove that the inverse operation is continuous at the identity element $(0,0,\ldots)$ of M_{bs}. Let U be a neighborhood of $(0,0,\ldots)$. Then $\nu^{-1}(U)$ contains $0 \in L_\infty$. Since $\nu^{-1}(U)$ is open, it follows that there exists $\varepsilon > 0$ such that $B(0,\varepsilon) \subset \nu^{-1}(U)$. Let $0 < r < \frac{1}{2}M\varepsilon$, where M is defined by (1), and $V = \nu(B(0,r))$. Note that V is a neighborhood of $(0,0,\ldots)$. Let us show that $-\xi \in U$ for every $\xi \in V$. Let $\xi = (\xi_1, \xi_2, \ldots) \in V$. Then there exists $y_\xi \in B(0,r)$ such that $\nu(y_\xi) = \xi$. Note that
$$|\xi_n| = |R_n(y_\xi)| \le \|y_\xi\|_\infty < r^n$$
for every $n \in \mathbb{N}$. Therefore,
$$\sup_{n\in\mathbb{N}} \sqrt[n]{|\xi_n|} \le r.$$
By Theorem 1, there exists $x_{-\xi} \in L_\infty$ such that $\nu(x_{-\xi}) = -\xi$ and
$$\|x_{-\xi}\|_\infty < \frac{2}{M} \sup_{n\in\mathbb{N}} \sqrt[n]{|-\xi_n|}.$$
Since
$$\sup_{n\in\mathbb{N}} \sqrt[n]{|-\xi_n|} = \sup_{n\in\mathbb{N}} \sqrt[n]{|\xi_n|} \le r$$
and $r < \frac{1}{2}M\varepsilon$, it follows that $\|x_{-\xi}\|_\infty < \varepsilon$, i.e. $x_{-\xi} \in B(0,\varepsilon)$. Therefore, $x_{-\xi} \in \nu^{-1}(U)$ and, consequently, $\nu(x_{-\xi}) \in U$, i.e. $-\xi \in U$. Hence, for every neighborhood U of $(0,0,\ldots)$ there exists neighborhood V of $(0,0,\ldots)$ such that $-\xi \in U$ for every $\xi \in V$. In other words, the inverse operation is continuous at $(0,0,\ldots)$.

For $\eta \in M_{bs}$ let $f_\eta : M_{bs} \to M_{bs}$ be defined by $f_\eta : \xi \mapsto \xi + \eta$. By Theorem 3, f_η is a continuous function for every $\eta \in M_{bs}$. Let ζ be an arbitrary element of M_{bs}. By the continuity of the inverse operation at $(0,0,\ldots)$ and by the continuity of functions $f_{-\zeta}$ and f_ζ at ζ and $(0,0,\ldots)$ respectively, the inverse operation is continuous at ζ as a composition of continuous functions. Hence, the inverse operation is continuous at every point of M_{bs}. ☐

Corollary 1. *$(M_{bs}, +, \tau)$ is an abelian topological group.*

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Василишин Т.В. *Топологія на спектрі алгебри цілих симетричних функцій обмеженого типу на комплексному просторі L_∞* // Карпатські матем. публ. — 2017. — Т.9, №1. — С. 22–27.

Відомо, що так звані елементарні симетричні поліноми $R_n(x) = \int_{[0,1]}(x(t))^n\,dt$ утворюють алгебраїчний базис алгебри усіх симетричних неперервних поліномів на комплексному банаховому просторі L_∞, яка є скрізь щільною в алгебрі Фреше $H_{bs}(L_\infty)$ усіх цілих симетричних функцій обмеженого типу на L_∞. Як наслідок, кожен неперервний гомоморфізм $\varphi : H_{bs}(L_\infty) \to \mathbb{C}$ однозначно визначається послідовністю $\{\varphi(R_n)\}_{n=1}^\infty$. За неперервністю гомоморфізму φ, послідовність $\{\sqrt[n]{|\varphi(R_n)|}\}_{n=1}^\infty$ є обмеженою. З іншого боку, для кожної послідовності $\{\xi_n\}_{n=1}^\infty \subset \mathbb{C}$, такої, що послідовність $\{\sqrt[n]{|\xi_n|}\}_{n=1}^\infty$ є обмеженою, існує $x_\xi \in L_\infty$ така, що $R_n(x_\xi) = \xi_n$ для кожного $n \in \mathbb{N}$. Тому для функціонала обчислення значення в точці δ_{x_ξ} буде $\delta_{x_\xi}(R_n) = \xi_n$ для кожного $n \in \mathbb{N}$. Отже, кожен неперервний комплекснозначний гомоморфізм алгебри $H_{bs}(L_\infty)$ збігається із функціоналом обчислення значення в деякій точці простору L_∞. Зауважимо, що така точка не є єдиною. Розглянемо відношення еквівалентності на L_∞, визначене правилом $x \sim y \Leftrightarrow \delta_x = \delta_y$. Тоді спектр (множина усіх неперервних комплекснозначних гомоморфізмів) M_{bs} алгебри $H_{bs}(L_\infty)$ є у взаємно однозначній відповідності із фактор-множиною $L_\infty/_\sim$. Відповідно, на M_{bs} можна розглянути фактор-топологію. З іншого боку, природно ототожнити M_{bs} із множиною усіх послідовностей $\{\xi_n\}_{n=1}^\infty \subset \mathbb{C}$ таких, що послідовність $\{\sqrt[n]{|\xi_n|}\}_{n=1}^\infty$ є обмеженою.

У роботі показано, що фактор-топологія є гаусдорфовою і що M_{bs} з операцією покоординатного додавання послідовностей утворює абелеву топологічну групу.

Ключові слова і фрази: симетрична функція, топологія на спектрі.

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SUPEREXTENSIONS OF THREE-ELEMENT SEMIGROUPS

A family $\mathcal{A}$ of non-empty subsets of a set X is called an *upfamily* if for each set $A \in \mathcal{A}$ any set $B \supset A$ belongs to $\mathcal{A}$. An upfamily $\mathcal{L}$ of subsets of X is said to be *linked* if $A \cap B \neq \varnothing$ for all $A, B \in \mathcal{L}$. A linked upfamily $\mathcal{M}$ of subsets of X is *maximal linked* if $\mathcal{M}$ coincides with each linked upfamily $\mathcal{L}$ on X that contains $\mathcal{M}$. The *superextension* $\lambda(X)$ consists of all maximal linked upfamilies on X. Any associative binary operation $* : X \times X \to X$ can be extended to an associative binary operation $\circ : \lambda(X) \times \lambda(X) \to \lambda(X)$ by the formula $\mathcal{L} \circ \mathcal{M} = \left\langle \bigcup_{a \in L} a * M_a : L \in \mathcal{L},\ \{M_a\}_{a \in L} \subset \mathcal{M} \right\rangle$ for maximal linked upfamilies $\mathcal{L}, \mathcal{M} \in \lambda(X)$. In the paper we describe superextensions of all three-element semigroups up to isomorphism.

Key words and phrases: semigroup, maximal linked upfamily, superextension, projective retraction, commutative.

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INTRODUCTION

In this paper we investigate the algebraic structure of the superextension $\lambda(S)$ of a three-element semigroup S. The thorough study of various extensions of semigroups was started in [11] and continued in [1–7, 12–16]. The largest among these extensions is the semigroup $v(S)$ of all upfamilies on S. A family $\mathcal{A}$ of non-empty subsets of a set X is called an *upfamily* if for each set $A \in \mathcal{A}$ any subset $B \supset A$ belongs to $\mathcal{A}$. Each family $\mathcal{B}$ of non-empty subsets of X generates the upfamily $\langle B \subset X : B \in \mathcal{B} \rangle = \{A \subset X : \exists B \in \mathcal{B} (B \subset A)\}$. An upfamily $\mathcal{F}$ that is closed under taking finite intersections is called a *filter*. A filter $\mathcal{U}$ is called an *ultrafilter* if $\mathcal{U} = \mathcal{F}$ for any filter $\mathcal{F}$ containing $\mathcal{U}$. The family $\beta(X)$ of all ultrafilters on a set X is called the Stone-Čech compactification of X, see [17], [20]. An ultrafilter $\{x\}$, generated by a singleton $\{x\}$, $x \in X$, is called *principal*. Each point $x \in X$ is identified with the principal ultrafilter $\langle \{x\} \rangle$ generated by the singleton $\{x\}$, and hence we consider $X \subset \beta(X) \subset v(X)$. It was shown in [11] that any associative binary operation $* : S \times S \to S$ can be extended to an associative binary operation $\circ : v(S) \times v(S) \to v(S)$ by the formula

$$\mathcal{L} \circ \mathcal{M} = \left\langle \bigcup_{a \in L} a * M_a : L \in \mathcal{L},\ \ \{M_a\}_{a \in L} \subset \mathcal{M} \right\rangle$$

for upfamilies $\mathcal{L}, \mathcal{M} \in v(S)$. In this case the Stone-Čech compactification $\beta(S)$ is a subsemigroup of the semigroup $v(S)$.

The semigroup $v(S)$ contains many other important extensions of S. In particular, it contains the semigroup $\lambda(S)$ of maximal linked upfamilies. The space $\lambda(S)$ is well-known in

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General and Categorial Topology as the *superextension* of S, see [19]- [21]. An upfamily $\mathcal{L}$ of subsets of S is *linked* if $A \cap B \neq \varnothing$ for all $A, B \in \mathcal{L}$. The family of all linked upfamilies on S is denoted by $N_2(S)$. It is a subsemigroup of $v(S)$. The superextension $\lambda(S)$ consists of all maximal elements of $N_2(S)$, see [10], [11].

Each map $f : X \to Y$ induces the map

$$\lambda f : \lambda(X) \to \lambda(Y), \quad \lambda f : \mathcal{M} \mapsto \big\langle f(M) \subset Y : M \in \mathcal{M} \big\rangle \text{ (see [10]).}$$

A non-empty subset I of a semigroup S is called *an ideal* if $IS \cup SI \subset I$. A semigroup S is called *simple* if S is the unique ideal of S. An element z of a semigroup S is called a *zero* (resp. a *left zero*, a *right zero*) in S if $az = za = z$ (resp. $za = z$, $az = z$) for any $a \in S$. A semigroup S is said to be a *left (right) zeros semigroup* if $ab = a$ ($ab = b$) for any $a, b \in S$. A semigroup S is called a *null semigroup* if there exists an element $c \in S$ such that $xy = c$ for any $x, y \in S$. By O_n, LO_n and RO_n we denote a null semigroup, a left zero semigroups and a right zero semigroup of order n respectively. Following the algebraic tradition, we denote by C_n the cyclic group of order n.

Let S be a semigroup and $e \notin S$. The binary operation defined on S can be extended to $S \cup \{e\}$ putting $es = se = s$ for all $s \in S \cup \{e\}$. The notation S^{+1} denotes a monoid $S \cup \{e\}$ obtained from S by adjoining an extra identity e (regardless of whether S is or is not a monoid). Analogous to the above construction, for every semigroup S one can define S^{+0}, a semigroup with attached an extra zero to S.

Let us recall that a *semilattice* is a commutative idempotent semigroup. Idempotent semigroups are called *bands*. So, in a band each element x is an *idempotent*, which means that $xx = x$. By L_n we denote the linear semilattice $\{0, 1, \dots, n\}$ of order n, endowed with the operation of minimum. A semigroup S is called *Clifford* if it is a union of groups.

A semigroup $\langle a \rangle = \{a^n\}_{n \in \mathbb{N}}$ generated by a single element a is called *monogenic* or *cyclic*. If a monogenic semigroup is infinite, then it is isomorphic to the additive semigroup $\mathbb{N}$. A finite monogenic semigroup $S = \langle a \rangle$ also has very simple structure (see [8], [18]). There are positive integer numbers r and m called the *index* and the *period* of S such that

- $S = \{a, a^2, \dots, a^{m+r-1}\}$ and $m + r - 1 = |S|$;
- for any $i, j \in \omega$ the equality $a^{r+i} = a^{r+j}$ holds if and only if $i \equiv j \mod m$;
- $C_m = \{a^r, a^{r+1}, \dots, a^{m+r-1}\}$ is a cyclic and maximal subgroup of S with the neutral element $e = a^n \in C_m$, where m divides n.

We denote by $C_{r,m}$ a finite monogenic semigroup of index r and period m.

An *isomorphism* between S and S' is one-to-one function $\varphi : S \to S'$ such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in S$. If there exist an isomorphism between S and S', then S and S' are said to be *isomorphic*, denoted $S \cong S'$. An *antiisomorphism* between S and S' is one-to-one function $\varphi : S \to S'$ such that $\varphi(xy) = \varphi(y)\varphi(x)$ for all $x, y \in S$. If there exist an antiisomorphism between S and S', then S and S' are said to be *antiisomorphic*, denoted $S \cong_a S'$. If $(S, *)$ is a semigroup, then $(S, \circ)$, where $x \circ y = y * x$, is a semigroup as well. The semigroups $(S, *)$ and $(S, \circ)$ are called *dual*. It is easy to see that dual semigroups are antiisomorphic.

There are exactly five pairwise non-isomorphic semigroups having two elements: C_2, L_2, O_2, LO_2, RO_2. The superextension $\lambda(S)$ of two-element semigroups S consists of two principal ultrafilters and therefore $\lambda(S) \cong S$.

In this paper we concentrate on describing the structure of the superextensions $\lambda(S)$ of three-element semigroups S. Among 19683 different operations on a three-element set $S = \{a, b, c\}$ there are exactly 113 operations which are associative, see [9]. In other words, there exist exactly 113 three-element semigroups, and many of these are isomorphic so that there are essentially only 24 pairwise non-isomorphic semigroups of order 3.

1 PROJECTIVE RETRACTIONS AND SUPEREXTENSIONS

In this section we will apply some properties of proretract semigroups to study the structure of the superextensions of semigroups.

A subset R of a set X is called a *retract* if there exists a *retraction* of X onto R, that is a map of X onto R which leaves each element of R fixed. A retraction $r : S \to T$ of a semigroup S onto a subsemigroup T of S is called a *projective retraction* if $xy = r(x)r(y)$ for any $x, y \in S$. A semigroup S is said to be a *proretract-semigroup* provided that there exists a projective retraction $r : S \to T$ of S onto some proper subsemigroup T of S. In this case T will be called a *projective retract* of S under a projective retraction r, and S will be called a *proretract extension* of T under a projective retraction r. If $r : S \to T$ is a projective retraction of a semigroup S onto a subsemigroup T of S, then r is a homomorphism and T is an ideal of S.

If a semigroup S is simple, then it is not a proretract-semigroup. In particular, groups, left zero and right zero semigroups are not proretract-semigroups.

Proposition 1. *A finite monogenic semigroup $C_{r,m}$ of index r and period m is a proretract-semigroup if and only if $r = 2$.*

Proof. Let $C_{r,m} = \{a, a^2, \ldots, a^r, \ldots, a^{r+m-1} \mid a^{r+m} = a^m\}$. If $r = 1$, then $C_{r,m}$ is simple and thus it is not a proretract-semigroup.

Let $r = 2$. Consider the map $\varphi : C_{2,m} \to C_m = \{a^2, \ldots, a^{m+1}\}$, $\varphi(s) = es$, where e is the identity of the maximal subgroup C_m of $C_{2,m}$. Then $st \in C_m$ and $st = eset = \varphi(s)\varphi(t)$ for any $s, t \in C_{2,m}$. Consequently, φ is a projective retraction.

Let $r > 2$. Suppose that $\varphi : C_{r,m} \to I$ is a projective retraction onto some proper ideal I of S. Then $aa = \varphi(a)\varphi(a)$. In monogenic semigroups of index $r > 2$ the equality $a^2 = \varphi(a)^2$ is possible only in the case $\varphi(a) = a$. Since φ is a homomorphism, then φ leaves each element of $C_{r,m}$ fixed. Therefore, $I = C_{r,m}$, a contradiction. $\square$

Let us note that for a subsemigroup T of a semigroup S the homomorphism $i : \lambda(T) \to \lambda(S)$, $i : \mathcal{A} \to \langle \mathcal{A} \rangle_S$ is injective, and thus we can identify the semigroup $\lambda(T)$ with the subsemigroup $i(\lambda(T)) \subset \lambda(S)$. Therefore, for each family $\mathcal{B}$ of non-empty subsets of T we identify the upfamilies

$$\langle \mathcal{B} \rangle_T = \{A \in T \mid \exists B \in \mathcal{B}(B \subset A)\} \in \lambda(T) \quad \text{and} \quad \langle \mathcal{B} \rangle_S = \{A \in S \mid \exists B \in \mathcal{B}(B \subset A)\} \in \lambda(S).$$

In the following proposition we show that proretract-semigroup property is preserved by superextensions.

Proposition 2. *If $r : S \to T$ is a projective retraction of a semigroup S onto a subsemigroup T of S, then $\lambda r : \lambda(S) \to \lambda(T)$ is a projective retraction of the superextension $\lambda(S)$ onto $\lambda(T)$.*

Proof. Let $\mathcal{L}, \mathcal{M} \in \lambda(S)$. Then

$$\begin{aligned}\lambda r(\mathcal{L}) \circ \lambda r(\mathcal{M}) &= \Big\langle \bigcup_{a \in r(L)} a * r(M)_a : r(L) \in \lambda r(\mathcal{L}),\ \ \{r(M)_a\}_{a \in r(L)} \subset \lambda r(\mathcal{M}) \Big\rangle \\ &= \Big\langle \bigcup_{a \in L} r(a) * r(M)_a : L \in \mathcal{L},\ \ \{r(M)_a\}_{a \in L} \subset \lambda r(\mathcal{M}) \Big\rangle \\ &= \Big\langle \bigcup_{a \in L} a * M_a : L \in \mathcal{L},\ \ \{M_a\}_{a \in L} \subset \mathcal{M} \Big\rangle = \mathcal{L} \circ \mathcal{M}.\end{aligned}$$

□

Corollary 1. *If S is a proretract-semigroup, then $\lambda(S)$ is a proretract-semigroup as well.*

In the next section we show that there exists a semigroup S that is not a proretract-semigroup, but the superextension $\lambda(S)$ is a proretract-semigroup.

Theorem 1. *If S is a null semigroup, then $\lambda(S)$ is a null semigroup as well.*

Proof. Let S be a null semigroup. So there exists $c \in S$ such that $xy = c$ for all $x, y \in S$. Then the map $r : S \to \{c\}$, $r(s) = c$ for any $s \in S$, is a projective retraction. According to Proposition 2 the map $\lambda r : \lambda(S) \to \lambda\{c\} = \{\langle\{c\}\rangle\}$ is a projective retraction as well. Therefore,

$$\mathcal{L} \circ \mathcal{M} = \lambda r(\mathcal{L}) \circ \lambda r(\mathcal{M}) = \langle\{c\}\rangle \circ \langle\{c\}\rangle = \langle\{c\}\rangle$$

for any $\mathcal{L}, \mathcal{M} \in \lambda(S)$. Consequently $\lambda(S)$ is a null semigroup. □

A semigroup S is said to be an *almost null semigroup* if there exist the distinct elements $a, c \in S$ such that $aa = a$ and $xy = c$ for any $(x, y) \in S \times S \setminus \{(a, a)\}$.

Theorem 2. *If S is an almost null semigroup, then $\lambda(S)$ is an almost null semigroup as well.*

Proof. Let S be an almost null semigroup, so there exist the elements $a, c \in S$, $c \neq a$, such that $aa = a$ and $xy = c$ for any $(x, y) \in S \times S \setminus \{(a, a)\}$. Then the map $r : S \to \{a, c\}$, $r(a) = a$ and $r(s) = c$ for any $s \neq a$, is a projective retraction. According to Proposition 2 the map $\lambda r : \lambda(S) \to \lambda\{a, c\}$ is a projective retraction as well. It is easy to see that the semigroup $\lambda\{a, c\} = \{\langle\{a\}\rangle, \langle\{c\}\rangle\} \cong \{a, c\}$ is isomorphic to the semilattice $L_2 = \{0, 1\}$ with operation of minimum.

It is obvious that $\langle\{a\}\rangle \circ \langle\{a\}\rangle = \langle\{a\}\rangle$. If $\mathcal{A} \neq \langle\{a\}\rangle$, then there exists $A \in \mathcal{A}$ such that $a \notin A$ and therefore $r(A) = c$. This implies that $\lambda r(\mathcal{A}) = \{\langle\{c\}\rangle\}$. If $(\mathcal{L}, \mathcal{M}) \in \lambda(S) \times \lambda(S) \setminus \{(\langle\{a\}\rangle, \langle\{a\}\rangle)\}$, then $\lambda r(\mathcal{L}) = \langle\{c\}\rangle$ or $\lambda r(\mathcal{M}) = \langle\{c\}\rangle$. Therefore, $\mathcal{L} \circ \mathcal{M} = \lambda r(\mathcal{L}) \circ \lambda r(\mathcal{M}) = \langle\{c\}\rangle$. Consequently, $\lambda(S)$ is an almost null semigroup. □

Theorem 3. *If S is a left (right) zero semigroup, then $\lambda(S)$ is a left (right) zero semigroup as well.*

Proof. Let S be a left zero semigroup. Then

$$\mathcal{L} \circ \mathcal{M} = \Big\langle \bigcup_{a \in L} a * M_a : L \in \mathcal{L},\ \ \{M_a\}_{a \in L} \subset \mathcal{M} \Big\rangle = \Big\langle \bigcup_{a \in L} \{a\} : L \in \mathcal{L} \Big\rangle = \mathcal{L}$$

for any $\mathcal{L}, \mathcal{M} \in \lambda(S)$. Thus $\lambda(S)$ is a left zero semigroup as well.

For a right zero semigroup the proof is similar. □

2 SUPEREXTENSIONS OF COMMUTATIVE SEMIGROUPS OF ORDER 3

In this section we describe the structure of superextensions of commutative three-element semigroups. Among 24 pairwise non-isomorphic semigroups of order 3 there are 12 commutative semigroups.

For a semigroup $S = \{a,b,c\}$ the semigroup $\lambda(S)$ contains the three principal ultrafilters $\langle\{a\}\rangle, \langle\{b\}\rangle, \langle\{c\}\rangle$ and the maximal linked upfamily $\triangle = \langle\{a,b\},\{a,c\},\{b,c\}\rangle$. Since semigroups S and $\{\langle\{a\}\rangle, \langle\{b\}\rangle, \langle\{c\}\rangle\}$ are isomorphic, then we can assume that $\lambda(S) = S \cup \{\triangle\}$.

In the sequel we will describe the structure of superextensions of three-element semigroups $S = \{a,b,c\}$ defined by Cayley tables using the formula

$$\mathcal{L} \circ \mathcal{M} = \Big\langle \bigcup_{a \in L} a * M_a : L \in \mathcal{L},\ \ \{M_a\}_{a\in L} \subset \mathcal{M} \Big\rangle$$

of product of maximal linked upfamilies $\mathcal{L}, \mathcal{M} \in \lambda(S)$.

The superextension $\lambda(C_3)$ (described by the following Cayley table) of the cyclic group C_3 is isomorphic to $(C_3)^{+0}$ and therefore $\lambda(C_3)$ is a commutative Clifford semigroup. The thorough study of superextensions of groups was started in [7] and continued in [1–3].

$\cdot$	a	b	c	$\triangle$
a	a	b	c	$\triangle$
b	b	c	a	$\triangle$
c	c	a	b	$\triangle$
$\triangle$	$\triangle$	$\triangle$	$\triangle$	$\triangle$

The superextensions of monogenic semigroups were studied in [13]. The cyclic semigroup $C_{2,2}$ is a proretract extension of cyclic subgroup $\{b,c\} \cong C_2$ under retraction $\varphi : \{a,b,c\} \to \{b,c\}$ with $\varphi(a) = c$. The superextension $\lambda(C_{2,2})$ is also a proretract extension of $\lambda\{b,c\} \cong \{b,c\}$ according to Proposition 2. The monogenic semigroup $C_{3,1}$ is not a proretract-semigroup by Proposition 1, but its superextension $\lambda(C_{3,1})$ is a proretract extension of $C_{3,1}$ under retraction $r : \lambda(C_{3,1}) \to C_{3,1}$ with $r(\triangle) = c$, and, therefore, $\lambda(C_{3,1})$ is a proretract-semigroup. Here are the Cayley tables of $\lambda(C_{2,2})$ and $\lambda(C_{3,1})$ respectively:

$\cdot$	a	b	c	$\triangle$
a	b	c	b	b
b	c	b	c	c
c	b	c	b	b
$\triangle$	b	c	b	b

$\cdot$	a	b	c	$\triangle$
a	b	c	c	c
b	c	c	c	c
c	c	c	c	c
$\triangle$	c	c	c	c

The following Cayley tables for the semigroups $\lambda((C_2)^{+0})$ and $\lambda((C_2)^{+1})$, where $C_2 \cong \{a,b\}$, imply that

$$\lambda((C_2)^{+0}) \cong \{a,b,\triangle\}^{+0} \cong ((C_2)^{+0})^{+0}$$

and

$$\lambda((C_2)^{+1}) \cong \{a,b,\triangle\}^{+1} \cong ((C_2)^{+1})^{+1} :$$

$\cdot$	a	b	c	$\triangle$
a	a	b	c	$\triangle$
b	b	a	c	$\triangle$
c	c	c	c	c
$\triangle$	$\triangle$	$\triangle$	c	$\triangle$

$\cdot$	a	b	c	$\triangle$
a	a	b	a	a
b	b	a	b	b
c	a	b	c	$\triangle$
$\triangle$	a	b	$\triangle$	$\triangle$

The superextensions of a null semigroup and an almost null semigroup are a null semigroup and an almost null semigroup as well according to Theorems 1 and 2:

$\cdot$	a	b	c	$\triangle$
a	c	c	c	c
b	c	c	c	c
c	c	c	c	c
$\triangle$	c	c	c	c

$\cdot$	a	b	c	$\triangle$
a	a	c	c	c
b	c	c	c	c
c	c	c	c	c
$\triangle$	c	c	c	c

The following Cayley tables for the semigroups $\lambda((O_2)^{+0})$ and $\lambda((O_2)^{+1})$ imply that

$$\lambda((O_2)^{+0}) \cong \{a,b,\triangle\}^{+0} \cong (O_3)^{+0} \qquad \text{and} \qquad \lambda((O_2)^{+1}) \cong \{a,b,\triangle\}^{+1} \cong (O_3)^{+1}.$$

The semigroups $(O_2)^{+0}$ and $\lambda((O_2)^{+0})$ are proretract extensions of the subsemigroup $\{b,c\} \cong L_2$.

$\cdot$	a	b	c	$\triangle$
a	b	b	c	b
b	b	b	c	b
c	c	c	c	c
$\triangle$	b	b	c	b

$\cdot$	a	b	c	$\triangle$
a	b	b	a	b
b	b	b	b	b
c	a	b	c	$\triangle$
$\triangle$	b	b	$\triangle$	b

The superextensions of semilattices were studied in [4]. The following Cayley tables imply that $\lambda(L_3) \cong L_4$ is a linear semilattice, but the superextension of the non-linear semilattice is its proretract extension and it is not even a Clifford semigroup:

$\cdot$	a	b	c	$\triangle$
a	a	b	c	$\triangle$
b	b	b	c	b
c	c	c	c	c
$\triangle$	$\triangle$	b	c	$\triangle$

$\cdot$	a	b	c	$\triangle$
a	a	c	c	c
b	c	b	c	c
c	c	c	c	c
$\triangle$	c	c	c	c

The structure of the superextension of the last commutative semigroup is shown in the following table. This semigroup and its superextension are proretract extensions of the subgroup $\{a,c\} \cong C_2$.

$\cdot$	a	b	c	$\triangle$
a	c	a	a	a
b	a	c	c	c
c	a	c	c	c
$\triangle$	a	c	c	c

3 SUPEREXTENSIONS OF NON-COMMUTATIVE SEMIGROUPS OF ORDER 3

There are 12 pairwise non-isomorphic non-commutative three-element semigroups. Non-commutative semigroups are divided into the pairs of dual semigroups that are antiisomorphic.

The superextension of a left (right) zero semigroup is a left (right) zero semigroup as well according to Theorem 3. Therefore $\lambda(LO_3) \cong LO_4$ and $\lambda(RO_3) \cong RO_4$.

$\cdot$	a	b	c	$\triangle$
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c
$\triangle$	$\triangle$	$\triangle$	$\triangle$	$\triangle$

$\cdot$	a	b	c	$\triangle$
a	a	b	c	$\triangle$
b	a	b	c	$\triangle$
c	a	b	c	$\triangle$
$\triangle$	a	b	c	$\triangle$

The following Cayley tables for the semigroups $\lambda((LO_2)^{+0})$ and $\lambda((RO_2)^{+0})$ imply that

$$\lambda((LO_2)^{+0}) \cong \{a, b, \triangle\}^{+0} \cong (LO_3)^{+0}$$

and

$$\lambda((RO_2)^{+0}) \cong \{a, b, \triangle\}^{+0} \cong (RO_3)^{+0} :$$

$\cdot$	a	b	c	$\triangle$
a	a	a	c	a
b	b	b	c	b
c	c	c	c	c
$\triangle$	$\triangle$	$\triangle$	c	$\triangle$

$\cdot$	a	b	c	$\triangle$
a	a	b	c	$\triangle$
b	a	b	c	$\triangle$
c	c	c	c	c
$\triangle$	a	b	c	$\triangle$

The following Cayley tables for the semigroups $\lambda((LO_2)^{+1})$ and $\lambda((RO_2)^{+1})$ imply that

$$\lambda((LO_2)^{+1}) \cong \{a, b, \triangle\}^{+1} \cong (\{a, b\}^{+1})^{+1} \cong ((LO_2)^{+1})^{+1}$$

and

$$\lambda((RO_2)^{+1}) \cong \{a, b, \triangle\}^{+1} \cong (\{a, b\}^{+1})^{+1} \cong ((RO_2)^{+1})^{+1} :$$

$\cdot$	a	b	c	$\triangle$
a	a	a	a	a
b	b	b	b	b
c	a	b	c	$\triangle$
$\triangle$	a	b	$\triangle$	$\triangle$

$\cdot$	a	b	c	$\triangle$
a	a	b	a	a
b	a	b	b	b
c	a	b	c	$\triangle$
$\triangle$	a	b	$\triangle$	$\triangle$

The following three-element semigroups and its superextensions are proretract extensions of its subsemigroups, which are isomorphic to LO_2 and RO_2 respectively:

$\cdot$	a	b	c	$\triangle$
a	c	c	c	c
b	b	b	b	b
c	c	c	c	c
$\triangle$	c	c	c	c

$\cdot$	a	b	c	$\triangle$
a	c	b	c	c
b	c	b	c	c
c	c	b	c	c
$\triangle$	c	b	c	c

Other two pairs of non-Clifford non-commutative dual superextensions of three-element semigroups are given by the following Cayley tables:

$\cdot$	a	b	c	$\triangle$
a	c	c	c	c
b	a	b	c	$\triangle$
c	c	c	c	c
$\triangle$	c	c	c	c

$\cdot$	a	b	c	$\triangle$
a	c	a	c	c
b	c	b	c	c
c	c	c	c	c
$\triangle$	c	$\triangle$	c	c

$\cdot$	a	b	c	$\triangle$
a	a	a	a	a
b	b	b	b	b
c	a	a	c	a
$\triangle$	a	a	$\triangle$	a

$\cdot$	a	b	c	$\triangle$
a	a	b	a	a
b	a	b	a	a
c	a	b	c	$\triangle$
$\triangle$	a	b	a	a

The last two three-element semigroups are the examples of non-commutative bands whose superextensions are not Clifford semigroups.

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Сім'я $\mathcal{A}$ непорожніх підмножин множини X називається *монотонною*, якщо для кожної множини $A \in \mathcal{A}$ довільна множина $B \supset A$ належить $\mathcal{A}$. Монотонна сім'я $\mathcal{L}$ підмножин множини X називається *зчепленою*, якщо $A \cap B \neq \varnothing$ для всіх $A, B \in \mathcal{L}$. Зчеплена монотонна сім'я $\mathcal{M}$ підмножин множини X є *максимальною зчепленою*, якщо $\mathcal{M}$ збігається з кожною зчепленою монотонною сім'єю $\mathcal{L}$ на X, яка містить $\mathcal{M}$. *Суперрозширення* $\lambda(X)$ складається з усіх максимальних зчеплених монотонних сімей на X. Кожна асоціативна бінарна операція $* : X \times X \to X$ продовжується до асоціативної бінарної операції $\circ : \lambda(X) \times \lambda(X) \to \lambda(X)$ за формулою $\mathcal{L} \circ \mathcal{M} = \left\langle \bigcup_{a\in L} a * M_a : L \in \mathcal{L},\ \{M_a\}_{a\in L} \subset \mathcal{M} \right\rangle$ для максимальних зчеплених монотонних сімей $\mathcal{L}, \mathcal{M} \in \lambda(X)$. У цій статті описуються суперрозширення всіх трьохелементних напівгруп з точністю до ізоморфізму.

Ключові слова і фрази: напівгрупа, максимальна зчеплена система, суперрозширення, проективна ретракція, комутативність.

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POINTS OF NARROWNESS AND UNIFORMLY NARROW OPERATORS

It is known that the sum of every two narrow operators on L_1 is narrow, however the same is false for L_p with $1 < p < \infty$. The present paper continues numerous investigations of the kind. Firstly, we study narrowness of a linear and orthogonally additive operators on Köthe function spaces and Riesz spaces at a fixed point. Theorem 1 asserts that, for every Köthe Banach space E on a finite atomless measure space there exist continuous linear operators $S, T : E \to E$ which are narrow at some fixed point but the sum $S + T$ is not narrow at the same point. Secondly, we introduce and study uniformly narrow pairs of operators $S, T : E \to X$, that is, for every $e \in E$ and every $\varepsilon > 0$ there exists a decomposition $e = e' + e''$ to disjoint elements such that $\|S(e') - S(e'')\| < \varepsilon$ and $\|T(e') - T(e'')\| < \varepsilon$. The standard tool in the literature to prove the narrowness of the sum of two narrow operators $S + T$ is to show that the pair S, T is uniformly narrow. We study the question of whether every pair of narrow operators with narrow sum is uniformly narrow. Having no counterexample, we prove several theorems showing that the answer is affirmative for some partial cases.

Key words and phrases: narrow operator, orthogonally additive operator, Köthe Banach space.

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INTRODUCTION

The class of narrow operators includes some other classes of "small" operators defined on atomless function spaces and Riesz spaces, such as weakly compact, Dunford-Pettis, absolutely summing etc. It was introduced and studied in [11] for function spaces and in [7] for Riesz spaces, however some results on these operators appeared in 80-th years of XXth century. The importance of narrow operators is explained by different geometric implications of their properties, see survey [13] and textbook [14]. Then the notion was naturally generalized to (nonlinear) orthogonally additive operators in [12]. An operator (linear or, more general, orthogonally additive) $T : E \to X$ from an atomless function space or atomless Riesz space E to a topological vector space X is said to be *narrow* if for every $e \in E$ and every neighborhood V of zero in X there exists a decomposition to disjoint summands $e = e' + e''$ such that $T(e') - T(e'') \in V$. Although it would be natural to consider narrowness at a fixed point $e \in E$, no investigation before [12] (2014) took this point into account. However in [12] the authors considered narrowness of an operator T at a fixed point $e \in E$ only for technical reasons to prove the main result.

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One of the most interesting facts concerning narrow operators is that, for some pairs of spaces (E, F) the sum $S + T$ of every two narrow operators $S, T : E \to X$ is narrow, but for other pairs the same is not true. For instance, the sum of every two narrow operators on L_1 is narrow, however every operator on L_p with $1 < p < \infty$ is a sum of two narrow operators. A number of published papers of different authors devoted to the questions of narrowness of a sum of two narrow operators (see, e.g. [2, 7, 8, 11]). A very different situation appears for narrowness at a fixed point. Theorem 1 asserts that for every Köthe Banach space E on a finite atomless measure space there exist continuous linear operators $S, T : E \to E$ which are narrow at some fixed point but the sum $S + T$ is not narrow at the same point.

A very natural proof that the sum $S + T$ of two narrow operators $S, T : E \to X$ is narrow is reduced to the proof that, for every $e \in E$ and every $\varepsilon > 0$ there exists a partition $e = e' \sqcup e''$ (common for both S and T) such that $\|Se' - Se''\| < \varepsilon/2$ and $\|Te' - Te''\| < \varepsilon/2$. This naturally leads us to a new notion of uniformly narrow pair of operators and to the question of whether every pair of narrow operators with narrow sum is uniformly narrow. Having no counterexample, in Section 2 we prove several theorems showing that the answer is affirmative for some partial cases.

Now we give a brief preliminaries on the notions used below. An *F-space* is a complete metric linear space X over a scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with an invariant metric ρ (i.e., $\rho(x, y) = \rho(x + z, y + z)$ for each $x, y, z \in X$). We set $\|x\| = \rho(x, 0)$, and so, $\rho(x, y) = \|x - y\|$ and call the defined map $\|\cdot\| : X \times X \to [0, +\infty)$ the F-norm of the F-space X. A very important class of F-spaces is the class of Banach spaces. Let (Ω, Σ, μ) be a finite measure space. An F-space E of equivalence classes of measurable functions on Ω is called a *Köthe F-space* if the following conditions hold: (K_i) if $y \in E$ and $|x| \le |y|$ then $x \in E$ and $\|x\| \le \|y\|$; (K_{ii}) $\mathbf{1}_\Omega \in E$. If, moreover, E is a Banach space and (K_{iii}) $E \subseteq L_1(\mu)$ then E is called a *Köthe Banach space*.

By $\mathcal{L}(X, Y)$ we denote the set of all continuous linear operators acting from X to Y.

Let E be a Riesz space (in particular, a Köthe F-space) and X a vector space. A map $T : E \to X$ is called an *orthogonally additive operator* if $T(x + y) = T(x) + T(y)$ for all $x, y \in E$ with $x \perp y$ (for Köthe F-space it means that x and y have disjoint supports). If, moreover, X is a Riesz space then an order bounded orthogonally additive operator $T : E \to X$ is called an *abstract Uryson operator*. We refer the reader to [4, 5, 6, 10] and the bibliography therein for examples and some usual facts on orthogonally additive operators. An element y of a Riesz space E is called a *fragment* (in another terminology, a *component*) of an element $x \in E$, provided $y \perp (x - y)$. The notation $y \sqsubseteq x$ means that y is a fragment of x. A net $(x_\alpha)_{\alpha \in \Lambda}$ in E *order converges* to an element $x \in E$ (notation $x_\alpha \xrightarrow{\mathrm{o}} x$) if there exists a net $(u_\alpha)_{\alpha \in \Lambda}$ in E such that $u_\alpha \downarrow 0$ and $|x_\beta - x| \le u_\beta$ for all $\beta \in \Lambda$. The equality $x = \bigsqcup_{i=1}^n x_i$ means that $x = \sum_{i=1}^n x_i$ and $x_i \perp x_j$ if $i \neq j$. Note that in this case one has that $x_i \sqsubseteq x$ for all i. If E is a Riesz space and $e \in E^+$ then by $\mathfrak{F}_e$ we denote the set of all fragments of e. We say that a net $(x_\alpha)_{\alpha \in \Lambda}$ in E *up-laterally converges* to an element $x \in E$ (notation $x_\alpha \xrightarrow{\ell\uparrow} x$) if $x_\alpha \xrightarrow{\mathrm{o}} x$ and $x_\alpha \sqsubseteq x_\beta$ as $\alpha < \beta$. A function $f : E \to F$ between Riesz spaces is said to be up-laterally continuous if for every net $(x_\alpha)_{\alpha \in \Lambda}$ in E and every $x \in E$ the condition $x_\alpha \xrightarrow{\ell\uparrow} x$ implies $f(x_\alpha) \xrightarrow{\ell\uparrow} f(x)$ in F.

An element e of a Riesz space E is called a *projection element* if the band B_e generated by e is a projection band. A Riesz space E is said to have the *principal projection property* if every element of E is a projection element. For instance, every Dedekind σ-complete Riesz space has the principal projection property. An element $u \neq 0$ of a Riesz space E is called an *atom* whenever $0 \le x \le |u|$, $0 \le y \le |u|$ and $x \wedge y = 0$ imply that either $x = 0$ or $y = 0$. Evidently, if $u \in E$ is an atom then $\mathfrak{F}_u = \{0, u\}$. A Riesz space without a nonzero atom is said to be *atomless*.

1 POINTS OF NARROWNESS

Below we give main definitions of narrow operators adapted to the idea to consider narrowness at a fixed point.

Definition 1.1 (of a narrow map). *Let E be a Riesz space and X be a topological vector space. A function $f : E \to X$ is said to be:*

- ***narrow at a point*** *$e \in E$ if for every neighborhood of zero U in X there exists a decomposition $e = e_1 \sqcup e_2$ such that $f(e_1) - f(e_2) \in U$. The set of all points of E at which f is narrow is denoted by $\mathcal{N}(f)$;*
- ***narrow*** *if $\mathcal{N}(f) = E$.*

Observe that, for linear maps the definition is equivalent to the following one. A linear operator $T : E \to X$ is said to be *narrow at a point* $e \in E$ if for every neighborhood of zero U in X there exists $f \in E$ such that $|f| = |e|$ and $Tf \in U$.

Definition 1.2 (of a strictly narrow map). *Let E be a Riesz space and X be a set. A function $f : E \to X$ is said to be*

- ***strictly narrow at a point*** *$e \in E$ if there exists a decomposition $e = e_1 \sqcup e_2$ such that $f(e_1) = f(e_2)$. The set of all points of E at which f is strictly narrow is denoted by $\mathcal{N}^s(f)$;*
- ***strictly narrow*** *if $\mathcal{N}^s(f) = E$.*

Likewise, if X is a linear space, a linear operator $T : E \to X$ is strictly narrow at a point $e \in E$ if and only if there exists $f \in E$ such that $|f| = |e|$ and $Tf = 0$.

Definition 1.3 (of an order narrow map). *Let E, X be Riesz spaces. A function $f : E \to X$ is said to be:*

- ***order narrow at a point*** *$e \in E$ if there is a net of decompositions $e = e'_\lambda \sqcup e''_\lambda$, $\lambda \in \Lambda$ such that $(f(e'_\lambda) - f(e''_\lambda)) \xrightarrow{\text{o}} 0$ in X. The set of all points of E at which f is order narrow is denoted by $\mathcal{N}^o(f)$;*
- order narrow *if $\mathcal{N}^o(f) = E$.*

Similarly, a linear operator $T : E \to X$ is order narrow at a point $e \in E$ if and only if there exists a net $f_\alpha \in E$ with $|f_\alpha| = |e|$ for all indices α such that $Tf_\alpha \xrightarrow{\text{o}} 0$.

Observe that a narrow (in any sense) function sends any atom to zero. So, to avoid triviality one may consider atomless Köthe F-spaces and atomless Riesz spaces to be the domain spaces of narrow maps. Another simple observation is that 0 is a point of narrowness of any map in any sense of narrowness.

Obviously, if X is a topological vector space then every strictly narrow (at a point, on a set) function is narrow. So, $\mathcal{N}^s(f) \subseteq \mathcal{N}(f)$ for any map $f : E \to X$. Similarly, if X is a Riesz space then every strictly narrow (at a point, on a set) function is order narrow. So, $\mathcal{N}^s(f) \subseteq \mathcal{N}^o(f)$

for any map $f : E \to X$. If one considers a compact linear operator T with zero kernel acting from a Köthe F-space E to an F-space X then $\mathcal{N}^s(T) = \{0\}$, however $\mathcal{N}(T) = E$, because every compact operator is narrow [14, Proposition 2.1]. If, moreover, X is an order continuous Banach lattice then $\mathcal{N}^o(T) = E$ as well, because in this case every narrow operator is order narrow [14, Proposition 10.9].

The connections between narrowness and order narrowness of a map is not so obvious, however it can be easily deduced from the arguments of [7]. Recall that a Banach lattice E is said to be *order continuous* if for each net (x_α) in E the condition $x_\alpha \downarrow 0$ implies that $\|x_\alpha\| \to 0$. Note that in this case the weaker condition $x_\alpha \xrightarrow{\text{o}} 0$ also implies that $\|x_\alpha\| \to 0$.

Proposition 1.1. *Let E be a Riesz space and X a Banach lattice. Then*

(1) *every narrow at a point $e \in E$ map $f : E \to X$ is order narrow at e;*

(2) *if, moreover, X is order continuous then every order narrow at a point $e \in E$ map $f : E \to X$ is narrow at e;*

(3) *there exists an order narrow positive operator $T \in \mathcal{L}(L_\infty)$ that is not narrow.*

Proof. (1) For each $n \in \mathbb{N}$ we choose a decomposition $e = e'_n \sqcup e''_n$ with $\|f(e'_n) - f(x''_n)\| < 2^{-n}$ and set $u_n = \sum_{k \geq n} |f(e'_k) - f(x''_k)|$ (the series obviously satisfies Cauchy's condition and hence converges). To show that $(f(e'_n) - f(e''_n)) \xrightarrow{\text{o}} 0$ is a standard technical exercise.

(2) Let f be order narrow at e. We choose a net of decompositions $e = e'_\lambda \sqcup e''_\lambda$, $\lambda \in \Lambda$ such that $(f(e'_\lambda) - f(e''_\lambda)) \xrightarrow{\text{o}} 0$. By the definition of an order continuous Banach lattice, $\|f(e'_\lambda) - f(e''_\lambda)\| \to 0$, and thus, f is narrow at e.

(3) See Example 3.3 of [7]. $\square$

The following two propositions are simple exercises.

Proposition 1.2. *Let E be a Riesz space and X a topological vector space.*

1. *For a linear operator $T : E \to X$ the following assertions are equivalent:*

   (i) *T is narrow;*

   (ii) *$E^+ \subseteq \mathcal{N}(T)$.*

2. *For an orthogonally additive operator $T : E \to X$ the following are equivalent:*

   (i) *T is narrow;*

   (ii) *$E^+ \cup E^- \subseteq \mathcal{N}(T)$.*

Similar statements are true for strictly narrow and order narrow operators.

Remark that the condition $E^+ \subseteq \mathcal{N}(T)$ for an orthogonally additive operator T does not imply that T is narrow, as the following simple example shows: $Tx = x^-$ for all $x \in E$.

Proposition 1.3. *Let E be a Riesz space and X a topological vector space.*

1. *Assume $T : E \to X$ is a linear operator.*

   (a) *If $e, f \in E$, $e \in \mathcal{N}(T)$ and $|f| = |e|$ then $f \in \mathcal{N}(T)$.*

(b) If $e_1, e_2 \in \mathcal{N}(T)$, $e_1 \perp e_2$ and $a, b \in \mathbb{R}$ then $ae_1 + be_2 \in \mathcal{N}(T)$.

2. *Assume $T : E \to X$ is an orthogonally additive operator. If $e_1, e_2 \in \mathcal{N}(T)$ and $e_1 \perp e_2$ then $e_1 + e_2 \in \mathcal{N}(T)$.*

Similar statements are true for strictly narrow and order narrow operators.

Proposition 1.4. *Let E be a Köthe F-space on a finite atomless measure space (Ω, Σ, μ), X a topological vector space, $T : E \to X$ a uniformly continuous orthogonally additive operator. Then the set of narrowness $\mathcal{N}(T)$ is closed in E.*

Proof. Let e belong to the F-norm closure of $\mathcal{N}(T)$. We show that T is narrow at e. Let V be any neighborhood of zero in X. Choose a neighborhood of zero V_1 in X so that $V_1 + V_1 + V_1 \subseteq V$ and $\delta > 0$ so that if $x, y \in E$ with $\|x - y\| < \delta$ then $T(x) - T(y) \in V_1$. Now choose $e_1 \in \mathcal{N}(T)$ so that $\|e_1 - e\| < \delta$ and choose a decomposition $e_1 = e_1' \sqcup e_1''$ so that $T(e_1') - T(e_1'') \in V_1$. Set $\Omega' = \operatorname{supp} e_1'$, $\Omega'' = \Omega \setminus \Omega'$, $e' = e \cdot \mathbf{1}_{\Omega'}$ and $e'' = e \cdot \mathbf{1}_{\Omega''}$. Then $e = e' \sqcup e''$. We show that $Te' - Te'' \in V$. Indeed, observe that

$$\|e' - e_1'\| = \big\|e \cdot \mathbf{1}_{\Omega'} - e_1 \cdot \mathbf{1}_{\Omega'}\big\| \le \|e - e_1\| < \delta$$

and analogously $\|e'' - e_1''\| < \delta$. Then $Te' - Te_1' \in V_1$ and $Te'' - Te_1'' \in V_1$. Hence,

$$Te' - Te'' = (Te' - Te_1') + (Te_1' - Te_1'') + (Te_1'' - Te'') \in V_1 + V_1 + V_1 \subseteq V. \qquad \square$$

Next we provide an example of a linear operator the set of narrowness of which coincides with the set of all functions with constant modulus.

Example 1. *Let (Ω, Σ, μ) be an atomless probability space (that is, a measure space with $\mu(\Omega) = 1$), $1 \le p < \infty$. Let $\Omega = A \sqcup B$ be any partition to measurable sets A, B. Then for the operator $T \in \mathcal{L}\big(L_p(\mu)\big)$ given by*

$$Tx = x - \Big(\int_\Omega rx \, d\mu\Big) r, \quad \text{where } r = \mathbf{1}_A - \mathbf{1}_B, \ \ x \in L_p(\mu)$$

one has $\mathcal{N}^s(T) = \mathcal{N}(T) = \{e \in E : |e(\omega)| = \lambda \text{ a.e. on } \Omega, \ \lambda \in \mathbb{R}\}$.

Proof. The inclusion $\{e \in E : |e(\omega)| = \lambda \text{ a.e. on } \Omega, \ \lambda \in \mathbb{R}\} \subseteq \mathcal{N}^s(T)$ follows from the observation that $T(\lambda r) = 0$ and $|\lambda r| = |e|$ for any element $e \in E$ with $|e(\omega)| = \lambda$ a.e. on Ω. To show that T is not narrow at each point $e \in E$ with $|e| \ne \lambda r$, $\lambda \in \mathbb{R}$, consider any element of the form $x = e \cdot \mathbf{1}_C - e \cdot \mathbf{1}_D$, where $\Omega = C \sqcup D$ (i.e., an arbitrary element $x \in E$ with $|x| = |e|$). Set $F_1 = A \cap C$, $F_2 = A \cap D$, $F_3 = B \cap C$ and $F_4 = B \cap D$. Then

$$\alpha \overset{\text{def}}{=} \int_\Omega rx \, d\mu = \int_{F_1} e \, d\mu - \int_{F_2} e \, d\mu - \int_{F_3} e \, d\mu + \int_{F_4} e \, d\mu,$$

which implies $|\alpha| \le \int_\Omega |e| \, d\mu = \|e\|_{L_1(\mu)}$.

Hence,

$$\|Tx\| = \|x - \alpha r\| \ge \|x\| - |\alpha| \|r\| = \|e\| - |\alpha| \ge \|e\|_{L_p(\mu)} - \|e\|_{L_1(\mu)}. \tag{1}$$

If we assume that T is narrow at e then by (1), $\|e\|_{L_p(\mu)} - \|e\|_{L_1(\mu)} = 0$ which yields that $|e|$ is a constant. $\square$

The following theorem provides an example of narrow at a fixed point operators on an arbitrary Köthe Banach space with nonnarrow sum at the same point.

Theorem 1. *Let E be a Köthe Banach space on a finite atomless measure space (Ω, Σ, μ). Then there are continuous linear operators $T_1, T_2 \in \mathcal{L}(E)$ each of which is strictly narrow at the point $\mathbf{1} = \mathbf{1}_\Omega$, however the sum $T_1 + T_2$ is not narrow at $\mathbf{1}$.*

Proof. Assume for simplicity of the notation that $\mu(\Omega) = 1$ and $\|\mathbf{1}\| = 1$. Decompose $\Omega = A_1 \sqcup A_2 \sqcup A_3 \sqcup A_4$ with measure $\mu(A_i) = 1/4$ each. Set $r_1 = \mathbf{1}_{A_1} + \mathbf{1}_{A_2} - \mathbf{1}_{A_3} - \mathbf{1}_{A_4}$ and $r_2 = \mathbf{1}_{A_1} - \mathbf{1}_{A_2} + \mathbf{1}_{A_3} - \mathbf{1}_{A_4}$. Define operators $T_1, T_2 \in \mathcal{L}(E)$ by setting

$$T_i x = x - \Big(\int_\Omega r_i x \, d\mu\Big) \, r_i, \quad x \in E, \quad i = 1, 2.$$

It is immediately that T_i are strictly narrow at $\mathbf{1}$, because $T_i r_i = 0$, $i = 1, 2$. We show that $T_1 + T_2$ is not narrow at $\mathbf{1}$. Let $r \in E$ be any element of the form $r = \mathbf{1}_A - \mathbf{1}_B$, where $A, B \in \Sigma$ with $\Omega = A \sqcup B$. We set $D_k = A \cap A_k$ and $F_k = B \cap A_k$ for $k = 1, 2, 3, 4$. Then set

$$\lambda_i = \int_\Omega r r_i \, d\mu, \quad i = 1, 2.$$

Taking into account that $\mu(D_k) + \mu(F_k) = 1/4$ for all k, we obtain

$$\begin{aligned} \lambda_1 &= \mu(D_1) + \mu(D_2) - \mu(D_3) - \mu(D_4) - \mu(F_1) - \mu(F_2) + \mu(F_3) + \mu(F_4) \\ &= 2\mu(D_1) + 2\mu(D_2) - 2\mu(D_3) - 2\mu(D_4) \end{aligned} \tag{2}$$

and analogously

$$\lambda_2 = 2\mu(D_1) - 2\mu(D_2) + 2\mu(D_3) - 2\mu(D_4). \tag{3}$$

Since $|\lambda_i| \le 1$ for $i = 1, 2$ and E is a Köthe Banach space,

$$\begin{aligned} \|(T_1 + T_2)\, r\| = \|2r - \lambda_1 r_1 - \lambda_2 r_2\| &= \big\|(2 - \lambda_1 - \lambda_2)\mathbf{1}_{D_1} + (2 - \lambda_1 + \lambda_2)\mathbf{1}_{D_2} \\ &\quad + (2 + \lambda_1 - \lambda_2)\mathbf{1}_{D_3} + (2 + \lambda_1 + \lambda_2)\mathbf{1}_{D_4} + (-2 - \lambda_1 - \lambda_2)\mathbf{1}_{F_1} \\ &\quad + (-2 - \lambda_1 + \lambda_2)\mathbf{1}_{F_2} + (-2 + \lambda_1 - \lambda_2)\mathbf{1}_{F_3} + (-2 + \lambda_1 + \lambda_2)\mathbf{1}_{F_4}\big\| \\ \ge \max\Big\{ &(2 - \lambda_1 - \lambda_2)\|\mathbf{1}_{D_1}\|, (2 - \lambda_1 + \lambda_2)\|\mathbf{1}_{D_2}\|, (2 + \lambda_1 - \lambda_2)\|\mathbf{1}_{D_3}\|, \\ &(2 + \lambda_1 + \lambda_2)\|\mathbf{1}_{D_4}\|, (2 + \lambda_1 + \lambda_2)\|\mathbf{1}_{F_1}\|, (2 + \lambda_1 - \lambda_2)\|\mathbf{1}_{F_2}\|, \\ &(2 - \lambda_1 + \lambda_2)\|\mathbf{1}_{F_3}\|, (2 - \lambda_1 - \lambda_2)\|\mathbf{1}_{F_4}\|\Big\}. \end{aligned}$$

Since $\mathbf{1} = \mathbf{1}_{D_1} + \mathbf{1}_{D_2} + \mathbf{1}_{D_3} + \mathbf{1}_{D_4} + \mathbf{1}_{F_1} + \mathbf{1}_{F_2} + \mathbf{1}_{F_3} + \mathbf{1}_{F_4}$, one of the summands has norm at least $1/8$. Of course, it is a matter of similar cases, which one. Say, $\|\mathbf{1}_{D_1}\| \ge 1/8$. Then

$$\|(T_1 + T_2)\, r\| \ge (2 - \lambda_1 - \lambda_2)\|\mathbf{1}_{D_1}\| \ge (2 - \lambda_1 - \lambda_2)/8.$$

Fix any $\varepsilon > 0$ and assume that r is chosen so that $\|(T_1 + T_2)\, r\| < \varepsilon$. Then by the above,

$$2 - \lambda_1 - \lambda_2 < 8\varepsilon. \tag{4}$$

We claim that $\lambda_i > 1 - 8\varepsilon$ for $i = 1, 2$. Indeed, if $\lambda_1 \le 1 - 8\varepsilon$ then $2 - \lambda_1 - \lambda_2 \ge 1 - \lambda_1 \ge 8\varepsilon$, which contradicts (4). Analogously, $\lambda_2 > 1 - 8\varepsilon$. Then by (2),

$$\mu(D_1) + \mu(D_2) - \mu(D_3) - \mu(D_4) = \frac{\lambda_1}{2} \ge \frac{1}{2} - 4\varepsilon \tag{5}$$

and by (3),

$$\mu(D_1) - \mu(D_2) + \mu(D_3) - \mu(D_4) = \frac{\lambda_2}{2} \geq \frac{1}{2} - 4\varepsilon. \tag{6}$$

Averaging (5) and (6), one gets $\frac{1}{4} \geq \mu(D_1) \geq \mu(D_1) - \mu(D_4) \geq \frac{1}{2} - 4\varepsilon$, which implies $\varepsilon \geq 1/16$. Thus, $T_1 + T_2$ is not narrow at $\mathbf{1}$. ◻

The following statement characterizes the set of strict narrowness of linear maps.

Proposition 1.5. *Let E be a Riesz space, X a linear space and $T : E \to X$ a linear operator. Then*

$$\mathcal{N}^s(T) = \Big\{x \in E : \ (\exists e \in \ker T)\, |x| = |e|\Big\}.$$

Proof. Let $x \in \mathcal{N}^s(T)$. Choose a decomposition $x = x' \sqcup x''$ so that $T(x') = T(x'')$. Then for $e = x' - x''$ one has that $|e| = |x|$ and $e \in \ker T$.

Assume $e \in \ker T$ $x \in E$ and $|x| = |e|$. Then

$$e = (x^+ \wedge e^+) \sqcup (x^- \wedge e^+) \sqcup \big(-(x^+ \wedge e^-)\big) \sqcup \big(-(x^- \wedge e^-)\big) \tag{7}$$

and

$$x = (x^+ \wedge e^+) \sqcup \big(-(x^- \wedge e^+)\big) \sqcup (x^+ \wedge e^-) \sqcup \big(-(x^- \wedge e^-)\big). \tag{8}$$

Then setting $x' = (x^+ \wedge e^+) - (x^- \wedge e^-)$ and $x'' = -(x^- \wedge e^+) + (x^+ \wedge e^-)$, we obtain $x = x' \sqcup x''$ and by (7) and (8),

$$0 = Te = T(x^+ \wedge e^+) + T(x^- \wedge e^+) - T(x^+ \wedge e^-) - T(x^- \wedge e^-) = Tx' - Tx''.$$ ◻

In particular, $\mathcal{N}^s(T)$ need not be a linear subspace of E. For instance, if $\ker T$ is the set of all constant functions then $\mathcal{N}^s(T)$ equals the set of all functions with constant modulus.

Remark that Proposition 1.5 is not longer true for orthogonally additive operators due to the obvious example $Tx = x^-$ for which $\mathcal{N}^s(T) = E^+$. To provide more examples for orthogonally additive operators we recall some necessary information from [9]. Given any two elements x, y of a Riesz space E, by xy we denote the greatest lower bound of the two-element set $\{x, y\}$ in E with respect to the lateral order $u \sqsubseteq v$ on E, if it exists. If E is a Riesz space of functions then

$$xy(t) = \begin{cases} x(t), & \text{if } x(t) = y(t); \\ 0, & \text{if } x(t) \neq y(t). \end{cases}$$

A Riesz space is said to have the intersection property if every two-point subset $\{x, y\}$ of E has the lateral infimum xy. In particular, the principal projection property implies the intersection property [9].

Example 2. *Let E be a Riesz space with the intersection property and $e \in E$. Then the function $T : E \to E$ given by $Tx = ex$ is an orthogonally additive operator with $\mathcal{N}^s(T) = \{0\} \cup (E \setminus \mathfrak{F}_e)$.*

Example 3. *Let E be a Riesz space with the intersection property and $e \in E$. Then the function $T : E \to E$ given by $Tx = x - ex$ is an orthogonally additive operator with $\mathcal{N}^s(T) = \mathfrak{F}_e$.*

The following example [7, Example 4.2] shows that, a continuous linear functional on an atomless Banach lattice may have the only zero point of narrowness.

Example 4. *There is a continuous linear functional $f \in L_\infty^*$ for which $\mathcal{N}(f) = \mathcal{N}^o(f) = \{0\}$.*

Proof. Denote by $\mathcal{B}$ the Boolean algebra of Borel subsets of $[0,1]$ equals up to measure null sets. Let $\mathcal{U}$ be any ultrafilter on $\mathcal{B}$. Then the linear functional $f : E \to \mathbb{R}$ defined by

$$f(x) = \lim_{A \in \mathcal{U}} \frac{1}{\mu(A)} \int_A x \, d\mu$$

is obviously bounded. However it is not narrow in any sense at every nonzero point. Indeed, for each $x \in L_\infty \setminus \{0\}$ of the form $x = \mathbf{1}_A - \mathbf{1}_B$ where $[0,1] = A \sqcup B$ one has $f(x) = \pm 1$ depending on whether $A \in \mathcal{U}$ or $B \in \mathcal{U}$. □

2 UNIFORMLY NARROW PAIRS OF OPERATORS

Below we define a uniformly narrow pair of operators; even though one can consider an arbitrary uniformly narrow set of operators.

Definition 2.1. *Let E be a Riesz space and X be an F-space. We say that an orthogonally additive operators $S, T : E \to X$ are **uniformly narrow** if for every $e \in E$ and every $\varepsilon > 0$ there exists a partition $e = e' \sqcup e''$ such that $\|Se' - Se''\| < \varepsilon$ and $\|Te' - Te''\| < \varepsilon$.*

As was noted in the introduction, a simple argument shows that, if orthogonally additive operators $S, T : E \to X$ are uniformly narrow then the sum $S + T$ is narrow. The following question naturally arises.

Problem 1. *Let E be a Riesz space and X be an F-space. Are the following assertions equivalent for every pair of narrow linear (orthogonally additive operators) $S, T : E \to X$?*

- (i) *$S + T$ is narrow;*
- (ii) *S, T are uniformly narrow.*

Although we do not know any example of spaces with negative answer to Problem 1, we present below an affirmative solution for some partial cases. We refer the reader to [1] for further standard terminology concerning operators on Riesz spaces.

We say that a Banach space X has the *contains its square* if there are a subspace Y of X and a decomposition $Y = X_1 \oplus X_2$ onto subspaces X_1, X_2 isomorphic to X.

Theorem 2. *Let E be a Riesz space and X be a Banach space containing its square. Let the sum of every two narrow linear bounded operators from E to X is narrow. Then every pair $S, T : E \to X$ of narrow linear bounded operators is uniformly narrow.*

Proof. Let Y be a subspace of X, $Y = X_1 \oplus X_2$ with subspaces X_1, X_2 isomorphic to X. Let $\tau_i : X \to X_i$ be isomorphisms, $i = 1, 2$. Let $S, T : E \to X$ be narrow linear operators. Then the linear operators $S', T' : E \to Y \subseteq X$ defined by setting $S' = \tau_1 \circ S$ and $T' = \tau_2 \circ T$ are narrow as compositions of a narrow operator from the right by a bounded operator from the left. By the assumption, the operator $A = S' + T'$ is narrow. Denote by P the projection of Y onto X_1 parallel to X_2 and by Q the projection of Y onto X_2 parallel to X_1. Observe that $P \circ A = S'$ and $Q \circ A = T'$. Given any $e \in E^+$ and $\varepsilon > 0$, we choose a decomposition $e = e' \sqcup e''$ such that

$$\|Ae' - Ae''\| < \frac{\varepsilon}{\|\tau^{-1}\| \max\{\|P\|, \|Q\|\}}.$$

Then

$$\|Se' - Se''\| \le \|\tau^{-1}\|\|\tau(Se' - Se'')\| = \|\tau^{-1}\|\|S'e' - S'e''\|$$
$$= \|\tau^{-1}\|\|P(Ae' - Ae'')\| \le \|\tau^{-1}\|\|P\|\|Ae' - Ae''\| < \varepsilon.$$

Analogously, $\|Te' - Te''\| < \varepsilon$. □

For example, the assumptions of Theorem 2 are valid for $E = F = L_1$ (see [2] or [14, Theorem 7.46] for the fact that a sum of every two narrow operators on L_1 is narrow).

We say that a Banach lattice X *regularly contains its square* if there are a subspace Y of X and a decomposition $Y = X_1 \oplus X_2$ onto subspaces X_1, X_2 isomorphic to X by means of regular isomorphisms $\tau_i : X \to X_i$, $i = 1, 2$.

Theorem 3. *Let E be a Riesz space and X be a Banach lattice regularly containing its square. Let the sum of every two narrow regular linear operators from E to X is narrow. Then every pair $S, T : E \to X$ of narrow regular linear operators is uniformly narrow.*

Proof. Let Y be a subspace of X, $Y = X_1 \oplus X_2$ with subspaces X_1, X_2 isomorphic to X by means of regular isomorphisms $\tau_i : X \to X_i$, $i = 1, 2$. Let $S, T : E \to X$ be narrow regular linear operators. Then the linear operators $S', T' : E \to Y \subseteq X$ defined by setting $S' = \tau_1 \circ S$ and $T' = \tau_2 \circ T$ are narrow regular as compositions of a narrow regular operator from the right by a bounded regular operator from the left. By the assumption, the operator $A = S' + T'$ is narrow. Starting from this point, the proof is the same as that of Theorem 2. □

Corollary 2.1. *Let E, F be order continuous Banach lattices with E atomless and F regularly containing its square. Then every pair of narrow regular operator $S, T : E \to F$ is uniformly narrow.*

Proof. Accordingly to Theorem 11.8 of [7] (see also [14, Theorem 10.41]), the set of all narrow regular linear operators is a band in the Riesz space of all regular linear operators from E to F. In particular, the sum of every two narrow regular linear operators from E to X is narrow. By Theorem 3, every pair of narrow regular operator $S, T : E \to F$ is uniformly narrow. □

Now we pass to orthogonally additive operators. Let E and F be Riesz spaces. An orthogonally additive operator $T : E \to F$ is called:

- *positive* provided $Tx \ge 0$ holds in F for all $x \in E$;
- *order bounded* it T maps order bounded sets in E to order bounded sets in F.

Observe that if $T : E \to F$ is a positive orthogonally additive operator and $x \in E$ is such that $T(x) \ne 0$ then $T(-x) \ne -T(x)$ (otherwise both $T(x) \ge 0$ and $T(-x) \ge 0$ would imply $T(x) = 0$). Thus, this positivity turns out to be more restrictive than the usual one for linear operators because the only linear operator which is positive in the above sense is zero.

A positive orthogonally additive operator need not be order bounded. Indeed, every function $T : \mathbb{R} \to \mathbb{R}$ with $T(0) = 0$ is an orthogonally additive operator, and obviously, not each of them is order bounded.

Banach lattices E and F are said to be *Riesz isomorphic* if there exists a *Riesz isomorphism* $\tau : E \to F$, that is, an isomorphism between Banach spaces such that both τ and τ^{-1} are order preserving operators.

We say that a Banach lattice X *contains its Riesz square* if there are a subspace Y of X and a decomposition $Y = X_1 \oplus X_2$ onto subspaces X_1, X_2 Riesz isomorphic to X and, moreover, the corresponding projections of Y onto X_i parallel to X_{3-i} are order continuous. For example, the Banach lattice $L_p[0,1]$ with $1 \le p \le \infty$ obviously contains its Riesz square.

Theorem 4. *Let E be an atomless Riesz space and F be an order continuous Banach lattice containing its Riesz square. Let the sum of every two narrow up-laterally continuous abstract Uryson operators from E to X is narrow. Then every pair $S, T : E \to X$ of narrow up-laterally continuous abstract Uryson operators is uniformly narrow.*

Proof. By [12, Lemma 2.7], under the assumptions on E and F, an abstract Uryson operator $B : E \to F$ is narrow if and only if B is order narrow. Let Y be a subspace of X, $Y = X_1 \oplus X_2$ and $\tau_i : X \to X_i$ be Riesz isomorphisms, $i = 1, 2$. Let $S, T : E \to X$ be narrow up-laterally continuous abstract Uryson operators. Then the maps $S', T' : E \to Y \subseteq X$ defined by setting $S' = \tau_1 \circ S$ and $T' = \tau_2 \circ T$ are narrow up-laterally continuous abstract Uryson operators as compositions of such an operator from the right by a bounded regular operator from the left. By the theorem assumptions, the operator $A = S' + T'$ is narrow and so, is order narrow. Denote by P the projection of Y onto X_1 parallel to X_2 and by Q the projection of Y onto X_2 parallel to X_1. Observe that $P \circ A = S'$ and $Q \circ A = T'$. Given any $e \in E^+$ and $\varepsilon > 0$, we choose a net of decompositions $e = e'_\alpha \sqcup e''_\alpha$ with $(Ae'_\alpha - Ae''_\alpha) \xrightarrow{\rm o} 0$. Since the operators τ^{-1} and P are order continuous,

$$Se'_\alpha - Se''_\alpha = \tau^{-1}(S'e'_\alpha - S'e''_\alpha) = \tau^{-1}P(Ae'_\alpha - Ae''_\alpha) \xrightarrow{\rm o} 0.$$

By the order continuity of F, $\|Se'_\alpha - Se''_\alpha\| \to 0$. Analogously, $\|Te'_\alpha - Te''_\alpha\| \to 0$. We choose α so that $\|Se'_\alpha - Se''_\alpha\| < \varepsilon$ and $\|Te'_\alpha - Te''_\alpha\| < \varepsilon$. ◻

As a consequence of [12, Theorem 8.2], we obtain the following assertion.

Corollary 2.2. *Let E be an atomless Riesz space with the principal projection property and F be an order continuous Banach lattice containing its Riesz square. Then every pair $S, T : E \to X$ of narrow up-laterally continuous abstract Uryson operators is uniformly narrow.*

Proof. By [12, Lemma 2.7], under the assumptions on E and F, an abstract Uryson operator $B : E \to F$ is narrow if and only if B is order narrow. So, by [12, Theorem 8.2], the sum of every two narrow up-laterally continuous abstract Uryson operators from E to X is narrow. Then apply Theorem 4. ◻

Recall that an operator $T \in \mathcal{L}(E, X)$ from a Köthe Banach space E on a finite atomless measure space (Ω, Σ, μ) to a Banach space X is called *hereditarily narrow* if for every $A \in \Sigma$, $\mu(A) > 0$ and every atomless sub-σ-algebra $\mathcal{F}$ of $\Sigma(A)$ the restriction of T to $E(\mathcal{F})$ is narrow (here $\Sigma(A) = \{B \in \Sigma : B \subseteq A\}$ and $E(\mathcal{F}) = \{x \in E(A) : x \text{ is } \mathcal{F} - \text{measurable}\}$). We refer the reader to [14, Section 11.1] for more information on hereditarily narrow operators.

Proposition 2.1. *Let E be a Köthe Banach space on $[0,1]$ with an absolutely continuous norm and X be a Banach space. If $S \in \mathcal{L}(E, X)$ is a hereditarily narrow operator and $T \in \mathcal{L}(E, X)$ is a narrow operator then the pair S, T is uniformly narrow.*

The proof of Proposition 2.1 just repeats the proof of [14, Proposition 11.2] (see also [3]).

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Відомо, що сума довільних двох вузьких операторів на L_1 є вузькою, проте для просторів L_p з $1 < p < \infty$ аналогічне твердження хибне. Дана стаття продовжує численні дослідження на цю тему. По-перше, ми вивчаємо вузькість лінійних та ортогонально адитивних операторів на функціональних просторах Кете і векторних ґратках у фіксованій точці. Теорема 1 стверджує, що для кожного банахового простору Кете на просторі зі скінченною безатомною мірою існують лінійні неперервні оператори $S, T : E \to E$, які є вузькими у деякій фіксованій точці, проте сума $S + T$ не є вузькою у цій же самій точці. По-друге, ми уводимо і досліджуємо одностайно вузькі пари операторів $S, T : E \to X$, тобто, для кожного $e \in E$ та кожного $\varepsilon > 0$ існує розклад $e = e' + e''$ на диз'юнктні елементи такий, що $\|S(e') - S(e'')\| < \varepsilon$ та $\|T(e') - T(e'')\| < \varepsilon$. Стандартний метод в літературі доведення вузькості суми двох вузьких операторів $S + T$ полягає в тому, щоби показати, що пара S, T є одностайно вузькою. Ми вивчаємо питання, чи кожна пара вузьких операторів з вузькою сумою є одностайно вузькою. Не маючи жодного контрприкладу, ми доводимо кілька теорем, які надають позитивну відповідь для деяких часткових випадків.

Ключові слова і фрази: вузький оператор, ортогонально адитивний оператор, банахів простір Кете.

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THE STRUCTURE OF SOLUTIONS OF THE MATRIX LINEAR UNILATERAL POLYNOMIAL EQUATION WITH TWO VARIABLES

We investigate the structure of solutions of the matrix linear polynomial equation $A(\lambda)X(\lambda)+B(\lambda)Y(\lambda)=C(\lambda)$, in particular, possible degrees of the solutions. The solving of this equation is reduced to the solving of the equivalent matrix polynomial equation with matrix coefficients in triangular forms with invariant factors on the main diagonals, to which the matrices $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ are reduced by means of semiscalar equivalent transformations. On the basis of it, we have pointed out the bounds of the degrees of the matrix polynomial equation solutions. Necessary and sufficient conditions for the uniqueness of a solution with a minimal degree are established. An effective method for constructing minimal degree solutions of the equations is suggested. In this article, unlike well-known results about the estimations of the degrees of the solutions of the matrix polynomial equations in which both matrix coefficients are regular or at least one of them is regular, we have considered the case when the matrix polynomial equation has arbitrary matrix coefficients $A(\lambda)$ and $B(\lambda)$.

Key words and phrases: matrix polynomial equation, solution of equation, semiscalar equivalence of polynomial matrices.

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INTRODUCTION

Let $\mathcal{F}$ be a field and $\mathcal{F}[\lambda]$ be a polynomial ring over $\mathcal{F}$. The matrix linear polynomial equations

$$A(\lambda)X(\lambda)+B(\lambda)Y(\lambda)=C(\lambda), \tag{1}$$

$$A(\lambda)X(\lambda)+Y(\lambda)B(\lambda)=C(\lambda), \tag{2}$$

where $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ are known, $X(\lambda)$ and $Y(\lambda)$ are unknown $m\times m$ matrices over ring $\mathcal{F}[\lambda]$, find application in the dynamical systems theory, the optimal control theory and in other areas [6, 7, 12–14].

It is clear, that if equations (1) and (2) are solvable, then they have solutions of unlimited on top degrees. Therefore, when we describe the solutions of such equations, it is important to establish their minimal degrees. Some estimations of the degrees of the solutions of the matrix polynomial equation (2) are known in [1, 5, 9]. In [1], it has been established that if in the matrix polynomial equation (2) both matrices $A(\lambda), B(\lambda)$ are regular, then there exists a solution $X(\lambda)$, $Y(\lambda)$, such that

$$\deg X(\lambda)<\deg B(\lambda),\quad \deg Y(\lambda)<\deg A(\lambda) \tag{3}$$

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and it is unique if and only if

$$\deg C(\lambda) \leq \deg A(\lambda) + \deg B(\lambda) - 1 \quad \text{and} \quad (\det A(\lambda), \det B(\lambda)) = 1.$$

In [5], this result has been extended for the matrix equation (2) if at least one of the matrices $A(\lambda)$ or $B(\lambda)$ is regular. We don't know similar estimates of the degrees of the solutions of the matrix polynomial equation (1).

In [2, 8], the matrix linear unilateral and bilateral equations in the form (1) and (2) over other domains have been studied.

In [3], we have obtained some bounds of the degrees of the solutions of the matrix polynomial equation (1) with singular matrix coefficients. In this paper, we have continued studing the structure of solutions of this matrix polynomial equation. The triple of matrices $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ can be simultaneously reduced to triangular forms $T^A(\lambda)$, $T^B(\lambda)$ and $T^C(\lambda)$ with invariant factors on main diagonals by means of semiscalar equivalence transformations [10, 11]. Following this, the bounds of the degrees of the solutions of the matrix polynomial equation (1) have been pointed out. Necessary and sufficient conditions for the uniqueness of a solution with a minimal degree have been established. There is also suggested an effective method for constructing minimal degree solutions of such matrix polynomial equations.

1 Preliminary results

We denote the ring of $m \times m$ matrices over $\mathcal{F}[\lambda]$ by $M(m, \mathcal{F}[\lambda])$, groups of invertible matrices over $\mathcal{F}$ and $\mathcal{F}[\lambda]$ by $GL(m, \mathcal{F})$ and $GL(m, \mathcal{F}[\lambda])$, respectively.

It is well known, that every matrix $A(\lambda) \in M(m, \mathcal{F}[\lambda])$, $\operatorname{rank} A = r$, is equivalent to the Smith normal form $S^A(\lambda)$, that is,

$$S^A(\lambda) = U(\lambda)A(\lambda)V(\lambda) = \operatorname{diag}(\mu_1^A(\lambda), \ldots, \mu_r^A(\lambda), 0, \ldots, 0),$$

where $U(\lambda)$, $V(\lambda) \in GL(m, \mathcal{F}[\lambda])$, $\mu_i^A(\lambda) \mid \mu_{i+1}^A(\lambda)$, $i = 1, \ldots, r-1$. The polynomials $\mu_i^A(\lambda)$ are called the invariant factors of matrix $A(\lambda)$.

Definition 1 ([10, 11]). *Collection of polynomial matrices*

$$A_1(\lambda), \ \ldots, \ A_k(\lambda)$$

is called semiscalar equivalent to the collection of polynomial matrices

$$B_1(\lambda), \ \ldots, \ B_k(\lambda),$$

where $A_i(\lambda)$, $B_i(\lambda) \in M(m, \mathcal{F}[\lambda])$, *if there exist matrices* $Q \in GL(m, \mathcal{F})$ *and* $R_i(\lambda) \in GL(m, \mathcal{F}[\lambda])$ *such that* $B_i(\lambda) = QA_i(\lambda)R_i(\lambda)$, $i = 1, \ldots, k$.

Theorem 1 ([10, 11]). *Collection of nonsingular polynomial matrices*

$$A_1(\lambda), \ \ldots, \ A_k(\lambda), \ \ A_i(\lambda) \in M(m, \mathcal{F}[\lambda]),$$

$i = 1, \ldots, k$, *is semiscalar equivalent to the collection of triangular matrices*

$$T^{A_1}(\lambda), \ \ldots, \ T^{A_k}(\lambda),$$

that is, there exist an upper unitriangular matrix $Q \in GL(m, \mathcal{F})$ *and invertible matrices* $R^{A_i}(\lambda) \in GL(m, \mathcal{F}[\lambda])$ *such that*

$$T^{A_i}(\lambda) = QA_i(\lambda)R^{A_i}(\lambda) = \left\| \begin{array}{cccc} \mu_1^{A_i}(\lambda) & 0 & \cdots & 0 \\ t_{21}^{(i)}(\lambda)\mu_1^{A_i}(\lambda) & \mu_2^{A_i}(\lambda) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ t_{m1}^{(i)}(\lambda)\mu_1^{A_i}(\lambda) & t_{m2}^{(i)}(\lambda)\mu_2^{A_i}(\lambda) & \cdots & \mu_m^{A_i}(\lambda) \end{array} \right\|, \tag{4}$$

where $\deg t_{pq}^{(i)}(\lambda) < \deg \mu_p^{A_i}(\lambda) - \deg \mu_q^{A_i}(\lambda)$*, if* $\deg \mu_p^{A_i}(\lambda) > \deg \mu_q^{A_i}(\lambda)$ *and* $t_{pq}^{(i)}(\lambda) \equiv 0$*, if* $\mu_p^{A_i}(\lambda) = \mu_q^{A_i}(\lambda)$*, for all* $p, q = 1, \ldots, m$*,* $p > q$*;* $i = 1, \ldots, k$*.*

Triangular form $T^{A_i}(\lambda)$ is called **standard form** of polynomial matrix $A_i(\lambda)$ with respect to semiscalar equivalence. Note that the matrix $T^{A_i}(\lambda)$ may be written in the form $T^{A_i}(\lambda) = T_i(\lambda)S^{A_i}(\lambda)$, where $T_i(\lambda)$ is a lower unitriangular matrix, $S^{A_i}(\lambda)$ is the Smith normal form of matrix $A_i(\lambda)$.

It should be noted that this theorem holds if the field $\mathcal{F}$ is infinite or if it is finite but $\sum_{i=1}^{k} s_i < |\mathcal{F}|$, where $|\mathcal{F}|$ is the number of elements of finite field $\mathcal{F}$, $s_i = \deg\det A_i(\lambda)$, $i = 1, \ldots, k$.

2 SOLUTIONS OF MINIMAL DEGREE OF MATRIX POLYNOMIAL EQUATIONS

By Theorem 1, the triple of nonsingular polynomial matrices $A(\lambda)$, $B(\lambda)$, $C(\lambda) \in M(m, \mathcal{F}[\lambda])$ from equation (1) is semiscalar equivalent to the triple of triangular polynomial matrices $T^A(\lambda)$, $T^B(\lambda)$, $T^C(\lambda)$ in standard form, that is,

$$T^A(\lambda) = QA(\lambda)R^A(\lambda),\ T^B(\lambda) = QB(\lambda)R^B(\lambda),\ T^C(\lambda) = QC(\lambda)R^C(\lambda),$$

where $Q \in GL(m, \mathcal{F})$, $R^A(\lambda)$, $R^B(\lambda)$, $R^C(\lambda) \in GL(m, \mathcal{F}[\lambda])$.

Matrices $T^A(\lambda)$, $T^B(\lambda)$ and $T^C(\lambda)$ have the form (4), that is,

$$T^A(\lambda) = \left\| \begin{array}{cccc} \mu_1^{A}(\lambda) & 0 & \cdots & 0 \\ \widetilde{a}_{21}(\lambda)\mu_1^{A}(\lambda) & \mu_2^{A}(\lambda) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \widetilde{a}_{m1}(\lambda)\mu_1^{A}(\lambda) & \widetilde{a}_{m2}(\lambda)\mu_2^{A}(\lambda) & \cdots & \mu_m^{A}(\lambda) \end{array} \right\|,$$

$$T^B(\lambda) = \left\| \begin{array}{cccc} \mu_1^{B}(\lambda) & 0 & \cdots & 0 \\ \widetilde{b}_{21}(\lambda)\mu_1^{B}(\lambda) & \mu_2^{B}(\lambda) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \widetilde{b}_{m1}(\lambda)\mu_1^{B}(\lambda) & \widetilde{b}_{m2}(\lambda)\mu_2^{B}(\lambda) & \cdots & \mu_m^{B}(\lambda) \end{array} \right\|,$$

$$T^C(\lambda) = \left\| \begin{array}{cccc} \mu_1^{C}(\lambda) & 0 & \cdots & 0 \\ \widetilde{c}_{21}(\lambda)\mu_1^{C}(\lambda) & \mu_2^{C}(\lambda) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \widetilde{c}_{m1}(\lambda)\mu_1^{C}(\lambda) & \widetilde{c}_{m2}(\lambda)\mu_2^{C}(\lambda) & \cdots & \mu_m^{C}(\lambda) \end{array} \right\|.$$

Then from equation (1) we obtain the matrix polynomial equation

$$T^A(\lambda)\widetilde{X}(\lambda) + T^B(\lambda)\widetilde{Y}(\lambda) = T^C(\lambda), \tag{5}$$

where $\widetilde{X}(\lambda) = (R^A(\lambda))^{-1}X(\lambda)R^C(\lambda)$, $\widetilde{Y}(\lambda) = (R^B(\lambda))^{-1}Y(\lambda)R^C(\lambda)$.

We will call the equation (5) **associate** to the equation (1).

Lemma 1. *The equation* (1) *is solvable if and only if the equation* (5) *is solvable. Each solution* $X(\lambda)$, $Y(\lambda)$ *of the equation* (1) *corresponds to a solution* $\widetilde{X}(\lambda)$, $\widetilde{Y}(\lambda)$ *of the equation* (5) *and the converse each solution* $\widetilde{X}(\lambda)$, $\widetilde{Y}(\lambda)$ *of the equation* (5) *corresponds to a solution* $X(\lambda)$, $Y(\lambda)$ *of the equation* (1).

Proof. It is well known [6, 13], that the matrix equation (1) is solvable if and only if the left greatest common divisor $D(\lambda)$ of matrices $A(\lambda)$ and $B(\lambda)$ is the left divisor of the matrix $C(\lambda)$. Then the greatest common divisor of triangular forms $T^A(\lambda)$ and $T^B(\lambda)$ is $D_1(\lambda) = QD(\lambda)$. Is it easy to see that if the matrix $D(\lambda)$ is the left divisor of the matrix $C(\lambda)$, then $D_1(\lambda)$ is the divisor of the matrix $T^C(\lambda)$ and conversely.

Furthermore, each solution $\widetilde{X}(\lambda)$, $\widetilde{Y}(\lambda)$ of the equation (5) corresponds to the solution

$$X(\lambda) = R^A(\lambda)\widetilde{X}(\lambda)(R^C(\lambda))^{-1}, \quad Y(\lambda) = R^B(\lambda)\widetilde{Y}(\lambda)(R^C(\lambda))^{-1}$$

of the equation (1) and conversely. □

Thus, the description of solutions of the matrix equation (1) is reduced to the description of solutions of the associated equation (5).

Solutions $X(\lambda)$, $Y(\lambda)$ and $\widetilde{X}(\lambda)$, $\widetilde{Y}(\lambda)$ of the matrix equations (1) and (5) are associate.

We denote the i-th row of matrix A by $row_i(A)$.

Theorem 2. *Let the matrix equation* (5) *be solvable. Then, it has the solution*

$$\widetilde{X}_1(\lambda) = \|\widetilde{x}_{ij}^{(1)}(\lambda)\|_1^m, \quad \widetilde{Y}_1(\lambda) = \|\widetilde{y}_{ij}^{(1)}(\lambda)\|_1^m$$

such that

$$row_i(\widetilde{X}_1(\lambda)) = \mathbf{0} \quad \textit{if} \quad \deg\mu_i^B(\lambda) = 0 \quad (\mu_i^B(\lambda) = 1),\ i = 1, \dots, k, \tag{6}$$

$$\deg row_i(\widetilde{X}_1(\lambda)) < \deg\mu_i^B(\lambda) \quad \textit{if} \quad \deg\mu_i^B(\lambda) \geq 1,\ i = k+1, \dots, m, \tag{7}$$

and the solution $\widetilde{X}_2(\lambda) = \|\widetilde{x}_{ij}^{(2)}(\lambda)\|_1^m$, $\widetilde{Y}_2(\lambda) = \|\widetilde{y}_{ij}^{(2)}(\lambda)\|_1^m$ *such that*

$$row_i(\widetilde{Y}_2(\lambda)) = \mathbf{0} \quad \textit{if} \quad \deg\mu_i^A(\lambda) = 0 \quad (\mu_i^A(\lambda) = 1),\ i = 1, \dots, l, \tag{8}$$

$$\deg row_i(\widetilde{Y}_2(\lambda)) < \deg\mu_i^A(\lambda) \quad \textit{if} \quad \deg\mu_i^A(\lambda) \geq 1,\ i = l+1, \dots, m. \tag{9}$$

Proof. From the matrix equation (5), we obtain the system of linear polynomial equations

$$\sum_{k=1}^{i} \left(\mu_k^A(\lambda)\widetilde{a}_{ik}(\lambda)\widetilde{x}_{kj}(\lambda) + \mu_k^B(\lambda)\widetilde{b}_{ik}(\lambda)\widetilde{y}_{kj}(\lambda) \right) = \mu_j^C(\lambda)\widetilde{c}_{ij}(\lambda), \tag{10}$$

$i,\ j = 1, \dots, m$, where $\widetilde{a}_{ii(\lambda)} = \widetilde{b}_{ii}(\lambda) = \widetilde{c}_{ii}(\lambda) = 1$.

The description of solutions of this system is reduced to the description of solutions of linear polynomial equations in the following form

$$\mu_i^A(\lambda)\widetilde{x}_{ij}(\lambda) + \mu_i^B(\lambda)\widetilde{y}_{ij}(\lambda) = \widehat{c}_{ij}(\lambda), \quad i,\, j = 1, \ldots, m. \tag{11}$$

If the equation (11) is solvable, then it has the solution $\widetilde{x}_{ij}(\lambda) = \widetilde{x}_{ij}^{(1)}(\lambda)$, $\widetilde{y}_{ij}(\lambda) = \widetilde{y}_{ij}^{(1)}(\lambda)$ such that $\deg\widetilde{x}_{ij}^{(1)}(\lambda) < \deg\mu_i^B(\lambda)$ and the solution $\widetilde{x}_{ij}(\lambda) = \widetilde{x}_{ij}^{(2)}(\lambda)$, $\widetilde{y}_{ij}(\lambda) = \widetilde{y}_{ij}^{(2)}(\lambda)$ such that $\deg\widetilde{y}_{ij}^{(2)}(\lambda) < \deg\mu_i^A(\lambda)$ [4,7]. If $\deg\mu_i^B(\lambda) \geq 1$, $i = k+1, \ldots, m$, then for each element in the row $row_i(\widetilde{X}_1(\lambda))$ the condition (7) of the theorem is true. Similarly, if $\deg\mu_i^A(\lambda) \geq 1$, $i = l+1, \ldots, m$, the condition (9) is true.

Among equations of the system (10) there are such polynomial equations

$$\mu_i^A(\lambda)\widetilde{x}_{ii}(\lambda) + \mu_i^B(\lambda)\widetilde{y}_{ii}(\lambda) = \mu_i^C(\lambda). \tag{12}$$

If $\mu_i^A(\lambda) = 1$ and $\mu_i^B(\lambda) = 1$, then this equation has solutions $\widetilde{x}_{ii}(\lambda) = 0$, $\widetilde{y}_{ii}(\lambda) = \mu_i^C(\lambda)$ and $\widetilde{x}_{ii}(\lambda) = \mu_i^C(\lambda)$, $\widetilde{y}_{ii}(\lambda) = 0$. If only one of $\mu_i^A(\lambda) = 1$ or $\mu_i^B(\lambda) = 1$, then this equation has solutions $\widetilde{x}_{ii}(\lambda) = 0$, $\widetilde{y}_{ii}(\lambda) = \frac{\mu_i^C(\lambda)}{\mu_i^B(\lambda)}$ and $\widetilde{x}_{ii}(\lambda) = \frac{\mu_i^C(\lambda)}{\mu_i^A(\lambda)}$, $\widetilde{y}_{ii}(\lambda) = 0$, respectively.

The system (10) also has polynomial equations in the following form

$$\mu_i^A(\lambda)\widetilde{x}_{ij}(\lambda) + \mu_i^B(\lambda)\widetilde{y}_{ij}(\lambda) = 0, \quad i < j,\ i = 1, \ldots, m-1,\ j = 2, \ldots, m. \tag{13}$$

These equations always have a zero solution, that is, $\widetilde{x}_{ii}(\lambda) = 0$, $\widetilde{y}_{ii}(\lambda) = 0$. Thus, the conditions (6) and (8) of the theorem are true. This completes the proof. ☐

From the proof of this theorem, we get a method for constructing solutions of the matrix equation (5). Since, the following inequalities $\deg\mu_i^A(\lambda) \leq \deg\mu_m^A(\lambda)$, $i = 1, \ldots, m-1$, are true for the invariant factors of matrix $A(\lambda)$, then $\deg S^A(\lambda) = \deg\mu_m^A(\lambda)$. Therefore, from Theorem 2 we get the following corollary.

Corollary 1. *Let the matrix equation* (5) *be solvable. Then it has the solution*

$$\widetilde{X}_1(\lambda), \quad \widetilde{Y}_1(\lambda)$$

such that

$$\widetilde{X}_1(\lambda) = \mathbf{0} \quad \textit{if} \quad \deg S^B(\lambda) = 0 \ (B(\lambda) \textit{ is an invertible matrix}),$$
$$\deg\widetilde{X}_1(\lambda) < \deg S^B(\lambda) \quad \textit{if} \quad \deg S^B(\lambda) \geq 1,$$

and the solution

$$\widetilde{X}_2(\lambda), \quad \widetilde{Y}_2(\lambda)$$

such that

$$\widetilde{Y}_2(\lambda) = \mathbf{0} \quad \textit{if} \quad \deg S^A(\lambda) = 0 \ (A(\lambda) \textit{ is an invertible matrix}),$$
$$\deg\widetilde{Y}_2(\lambda) < \deg S^A(\lambda) \quad \textit{if} \quad \deg S^A(\lambda) \geq 1.$$

Theorem 3. *Let*

$$S^A(\lambda) = \operatorname{diag}(\underbrace{1, \ldots, 1}_{k}, \mu_{k+1}^A(\lambda), \ldots, \mu_m^A(\lambda)),\ k \geq 0, \tag{14}$$

and

$$S^B(\lambda) = \operatorname{diag}(\underbrace{1, \ldots, 1}_{l}, \mu_{l+1}^B(\lambda), \ldots, \mu_m^B(\lambda)),\ l \geq 0, \tag{15}$$

be the Smith normal forms of the matrices $A(\lambda)$ and $B(\lambda)$, respectively, and let the matrix equation (5) *be solvable. Without loss of generality, let $k \geq l$.*

(i) If $\deg\mu_i^C(\lambda) \geq \deg\mu_i^A(\lambda) + \deg\mu_i^B(\lambda)$, $\mu_i^A(\lambda) \neq 1$, $\mu_i^B(\lambda) \neq 1$, $i = 1, \ldots, m$, *then the matrix equation* (5) *has the solution*

$$\widetilde{X}(\lambda) = \|\widetilde{x}_{ij}(\lambda)\|_1^m, \quad \widetilde{Y}(\lambda) = \|\widetilde{y}_{ij}(\lambda)\|_1^m$$

such that

$$\deg row_i(\widetilde{X}(\lambda)) < \deg\mu_i^B(\lambda), \deg row_i(\widetilde{Y}(\lambda)) = \deg\mu_i^C(\lambda) - \deg\mu_i^B(\lambda), \tag{16}$$

(ii) if $\deg\mu_i^C(\lambda) = \deg\mu_i^A(\lambda) + \deg\mu_i^B(\lambda)$, $\mu_i^A(\lambda) = 1$ *or* $\mu_i^B(\lambda) = 1$, $i = 1, \ldots, k$, *then the matrix equation* (5) *has solutions* $\widetilde{X}(\lambda)$, $\widetilde{Y}(\lambda)$ *such that*

$$row_i(\widetilde{X}(\lambda)) = \mathbf{0}, \ \deg row_i(\widetilde{Y}(\lambda)) \leq \deg\mu_i^C(\lambda) - \deg\mu_i^B(\lambda), \tag{17}$$

and

$$\deg row_i(\widetilde{X}(\lambda)) \leq \deg\mu_i^C(\lambda) - \deg\mu_i^A(\lambda), \ row_i(\widetilde{Y}(\lambda)) = \mathbf{0}, \tag{18}$$

(iii) if $\deg\mu_i^C(\lambda) < \deg\mu_i^A(\lambda) + \deg\mu_i^B(\lambda)$, $i = k+1, \ldots, m$, *then the matrix equation* (5) *has the solution* $\widetilde{X}(\lambda)$, $\widetilde{Y}(\lambda)$ *such that*

$$\deg row_i(\widetilde{X}(\lambda)) < \deg\mu_i^B(\lambda), \ \deg row_i(\widetilde{Y}(\lambda)) < \deg\mu_i^A(\lambda). \tag{19}$$

Proof. Case (i). In the proof of Theorem 2, it has been shown that the solving of the matrix equation (5) is reduced to the solving of the system of linear polynomial equations (10). This system has equations (12). Then, there exists a solution with the condition $\deg \widetilde{x}_{ii}(\lambda) < \deg\mu_i^B(\lambda)$ of the i-th equation (12) [4,7]. So, $\deg \widetilde{y}_{ij}(\lambda) = \deg\mu_i^C(\lambda) - \deg\mu_i^B(\lambda)$ for a fixed value of i and all values of $j = 1, \ldots, m$. Thus, the matrix equation (5) has the solution $\widetilde{X}(\lambda)$, $\widetilde{Y}(\lambda)$ with the condition (16).

Case (ii). In this case the condition has the form $\deg\mu_i^C(\lambda) = \deg\mu_i^A(\lambda)$ or $\deg\mu_i^C(\lambda) = \deg\mu_i^B(\lambda)$ if $\mu_i^B(\lambda) = 1$ or $\mu_i^A(\lambda) = 1$ for a fixed value of i. If $\mu_i^B(\lambda) = 1$ and $\mu_i^A(\lambda) = 1$ for a fixed value of i, then the condition has the form $\deg\mu_i^C(\lambda) = 0$. In the proof of Theorem 2, it has been shown that the system of linear polynomial equations (11) has equations (12) and (13). In this case, these equations have zero solutions. Thus, the matrix equation (5) has solutions $\widetilde{X}(\lambda)$, $\widetilde{Y}(\lambda)$ with the conditions (17) and (18).

Case (iii). There exists a solution of the equation (11) with the condition $\deg \widetilde{x}_{ij}(\lambda) < \deg\mu_j^B(\lambda)$, $\deg \widetilde{y}_{ij}(\lambda) < \deg\mu_i^A(\lambda)$ if the condition $\deg\mu_i^C(\lambda) < \deg\mu_i^A(\lambda) + \deg\mu_i^B(\lambda)$ is true for a fixed value of i and all values of $j = 1, \ldots, m$ [4,7]. This completes the proof. □

Remark 1. *We should note that in cases (ii) and (iii), opposite propositions hold, that is, their conditions are necessary for the existence of solutions with the conditions* (17)—(19)*.*

Theorem 4. *Let the equation* (5) *be solvable. Then it has solutions*

$$\widetilde{X}(\lambda) = \|\widetilde{x}_{ij}(\lambda)\|_1^m, \ \widetilde{Y}(\lambda) = \|\widetilde{y}_{ij}(\lambda)\|_1^m$$

of lower triangular forms such that

(i) $\deg\widetilde{x}_{ii}(\lambda) < \deg\mu_i^B(\lambda)$, $\deg\widetilde{y}_{ii}(\lambda) < \deg\mu_i^A(\lambda)$

$$\text{if} \quad \deg\mu_i^C(\lambda) < \deg\mu_i^A(\lambda) + \deg\mu_i^B(\lambda), \ i = 1, \ldots, m;$$

(ii) $\deg\widetilde{x}_{ii}(\lambda) < \deg\mu_i^B(\lambda)$, $\deg\widetilde{y}_{ii}(\lambda) = \deg\mu_i^C(\lambda) - \deg\mu_i^B(\lambda)$

$$\text{if} \quad \deg\mu_i^C(\lambda) \geq \deg\mu_i^A(\lambda) + \deg\mu_i^B(\lambda), \;\; i = 1, \dots, m.$$

Proof. We prove this theorem in a similar way to Theorem 2 and Theorem 3. □

We get solutions of the matrix equation (1) from solutions of the matrix equation (5):

$$X(\lambda) = R^A(\lambda)\widetilde{X}(\lambda)(R^C(\lambda))^{-1}, \;\; Y(\lambda) = R^B(\lambda)\widetilde{Y}(\lambda)(R^C(\lambda))^{-1}.$$

3 THE UNIQUENESS OF SOLUTIONS OF MINIMAL DEGREES OF MATRIX POLYNOMIAL EQUATIONS

We will establish the conditions for the uniqueness of solutions of minimal degrees of the matrix equation (5).

Theorem 5. *The matrix equation* (5) *has a unique solution*

$$\widetilde{X}_0^{(1)}(\lambda) = \|\widetilde{x}_{ij}^{(1)}(\lambda)\|_1^m, \; \widetilde{Y}_0^{(1)}(\lambda) = \|\widetilde{y}_{ij}^{(1)}(\lambda)\|_1^m$$

and

$$\widetilde{X}_0^{(2)}(\lambda) = \|\widetilde{x}_{ij}^{(2)}(\lambda)\|_1^m, \; \widetilde{Y}_0^{(2)}(\lambda) = \|\widetilde{y}_{ij}^{(2)}(\lambda)\|_1^m$$

such that

$$row_i(\widetilde{X}_0^{(1)}(\lambda)) = \mathbf{0} \quad \text{if} \quad \deg\mu_i^B(\lambda) = 0, \; i = 1, \dots, k, \tag{20}$$

$$\deg row_i(\widetilde{X}_0^{(1)}(\lambda)) < \deg\mu_i^B(\lambda) \quad \text{if} \quad \deg\mu_i^B(\lambda) \geq 1, \; i = k+1, \dots, m, \tag{21}$$

and

$$row_i(\widetilde{Y}_0^{(2)}(\lambda)) = \mathbf{0} \quad \text{if} \quad \deg\mu_i^A(\lambda) = 0, \; i = 1, \dots, k, \tag{22}$$

$$\deg row_i(\widetilde{Y}_0^{(2)}(\lambda)) < \deg\mu_i^A(\lambda) \quad \text{if} \quad \deg\mu_i^A(\lambda) \geq 1, \; i = k+1, \dots, m, \tag{23}$$

if and only if

$$(\mu_m^A(\lambda), \mu_m^B(\lambda)) = 1.$$

Proof. It is clear that the matrix equation (5) has a unique solution $\widetilde{X}_0^{(1)}(\lambda)$, $\widetilde{Y}_0^{(1)}(\lambda)$ with the condition (21) if and only if each equation (11) has a unique solution $\widetilde{x}_{ij}^{(1)}(\lambda)$, $\widetilde{y}_{ij}^{(1)}(\lambda)$ such that $\deg\widetilde{x}_{ij}^{(1)} < \deg\mu_i^B(\lambda)$. This solution of the equation (11) is unique if and only if $(\mu_i^A(\lambda), \mu_j^B(\lambda)) = 1$ for all $i, j = 1, \dots, m$ [4, 7]. The last condition holds if and only if $(\mu_m^A(\lambda), \mu_m^B(\lambda)) = 1$.

As it has been shown in the proof of Theorem 2, the system (10) has equations (12) and (13). By the condition of the theorem, these equations have a zero solution, which is unique. Thus, the solution $\widetilde{X}_0^{(1)}(\lambda)$, $\widetilde{Y}_0^{(1)}(\lambda)$ with the condition (20) is unique.

Similarly we prove the existence of a unique solution $\widetilde{X}_0^{(2)}(\lambda)$, $\widetilde{Y}_0^{(2)}(\lambda)$ with the conditions (22) and (23). This completes the proof. □

Theorem 6. *Let the matrix equation* (5) *be solvable and let* $S^A(\lambda)$, *and* $S^B(\lambda)$ *be the Smith normal forms* (14) *and* (15) *of the matrices* $A(\lambda)$ *and* $B(\lambda)$, *respectively. Then, there exists a unique solution*

$$\widetilde{X}(\lambda) = \|\widetilde{x}_{ij}(\lambda)\|_1^m,\ \widetilde{Y}(\lambda) = \|\widetilde{y}_{ij}(\lambda)\|_1^m$$

of the matrix equation (5) *with the conditions* (17) *and* (18) *if and only if*

$$\deg\mu_i^C(\lambda) = \deg\mu_i^A(\lambda) + \deg\mu_i^B(\lambda), \quad i = 1, \ldots, k, \quad \text{and} \quad (\mu_m^A(\lambda), \mu_m^B(\lambda)) = 1,$$

and with the condition (19) *if and only if*

$$\deg\mu_i^C(\lambda) < \deg\mu_i^A(\lambda) + \deg\mu_i^B(\lambda), \quad i = k+1, \ldots, m, \quad \text{and} \quad (\mu_m^A(\lambda), \mu_m^B(\lambda)) = 1.$$

Proof. It is clear that a unique solution of the matrix equation (5) exists if and only if a unique solution of the system of linear polynomial equations (10) exists, that is, a unique solution of each linear polynomial equation (11) exists. This system has equations (12). If $\mu_i^A(\lambda) = 1$ and $\mu_i^B(\lambda) = 1$, then by the conditions of the theorem, this equation has solutions $\widetilde{x}_{ii}(\lambda) = 0$, $\widetilde{y}_{ii}(\lambda) = \mu_i^C(\lambda)$ and $\widetilde{x}_{ii}(\lambda) = \mu_i^C(\lambda)$, $\widetilde{y}_{ii}(\lambda) = 0$. If only one of $\mu_i^A(\lambda) = 1$ or $\mu_i^B(\lambda) = 1$, then this equation has solutions

$$\widetilde{x}_{ii}(\lambda) = 0,\ \widetilde{y}_{ii}(\lambda) = \frac{\mu_i^C(\lambda)}{\mu_i^B(\lambda)} \quad \text{and} \quad \widetilde{x}_{ii}(\lambda) = \frac{\mu_i^C(\lambda)}{\mu_i^A(\lambda)},\ \widetilde{y}_{ii}(\lambda) = 0,$$

respectively. The equations (13) always have a zero solution, that is, $\widetilde{x}_{ii}(\lambda) = 0$, $\widetilde{y}_{ii}(\lambda) = 0$. This solution is unique. So, there exists a unique solution with the conditions (17) and (18) of the matrix equation (5).

If $\mu_i^A(\lambda) \neq 1$ and $\mu_i^B(\lambda) \neq 1$, then by the results [4, 7] the solution with the condition (19) of the matrix equation (5) is unique if and only if the solution $\widetilde{x}_{ij}(\lambda)$, $\widetilde{y}_{ij}(\lambda)$ such that $\deg\widetilde{x}_{ij}(\lambda) < \deg\mu_i^B(\lambda)$ and $\deg\widetilde{y}_{ij}(\lambda) < \deg\mu_i^A(\lambda)$ of the equation (11) is unique. There exist such solutions and they are unique if and only if $\deg\mu_i^C(\lambda) < \deg\mu_i^A(\lambda) + \deg\mu_i^B(\lambda)$ and

$$(\mu_i^A(\lambda), \mu_j^B(\lambda)) = 1 \ \ i, j = 1, \ldots, m.$$

The last conditions are true if and only if $(\mu_m^A(\lambda), \mu_m^B(\lambda)) = 1$. This completes the proof. □

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Джалюк Н.С., Петричкович В.М. *Структура розв'язків матричного лінійного однобічного поліноміального рівняння від двох змінних* // Карпатські матем. публ. — 2017. — Т.9, №1. — С. 48–56.

Досліджується структура розв'язків матричного лінійного поліноміального рівняння $A(\lambda)X(\lambda) + B(\lambda)Y(\lambda) = C(\lambda)$, зокрема можливі степені цих розв'язків. Розв'язування цього матричного поліноміального рівняння зводиться до розв'язування еквівалетного матричного поліноміального рівняння з матрицями-коефіцієнтами у трикутних формах з інваріантними множниками на головних діагоналях, до яких зводяться поліноміальні матриці $A(\lambda)$, $B(\lambda)$ і $C(\lambda)$ напівскалярними еквівалентними перетвореннями. На основі цього вказано межі для степенів розв'язків матричних поліноміальних рівнянь. Встановлено необхідні і достатні умови єдиності розв'язку мінімального степеня. Запропоновано ефективний метод побудови розв'язків мінімальних степенів цих рівнянь. На відміну від відомих результатів про оцінки степенів розв'язків матричних поліноміальних рівнянь, в яких обидва або принаймні один із коефіцієнтів є регулярною матрицею, у цій статті розглянуто випадок матричного поліноміального рівняння з довільними коефіцієнтами $A(\lambda)$ і $B(\lambda)$.

Ключові слова і фрази: матричне поліноміальне рівняння, розв'язок рівняння, напівскалярна еквівалентність поліноміальних матриць.

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POINCARÉ SERIES FOR THE ALGEBRAS OF JOINT INVARIANTS AND COVARIANTS OF n QUADRATIC FORMS

We consider one of the fundamental objects of classical invariant theory, namely the Poincaré series for an algebra of invariants of Lie group SL_2. The first two terms of the Laurent series expansion of Poincaré series at the point $z = 1$ give us an important information about the structure of the algebra $\mathcal{I}_d$. It was derived by Hilbert for the algebra $\mathcal{I}_d = \mathbb{C}[V_d]^{SL_2}$ of invariants for binary $d-$form (by V_d we denote the vector space over $\mathbb{C}$ consisting of all binary forms homogeneous of degree d). Springer got this result, using explicit formula for the Poincaré series of this algebra. We consider this problem for the algebra of joint invariants $\mathcal{I}_{2n}=\mathbb{C}[\underbrace{V_2\oplus V_2\oplus \cdots \oplus V_2}_{n \text{ times}}]^{SL_2}$ and the algebra of joint covariants $\mathcal{C}_{2n}=\mathbb{C}[\underbrace{V_2\oplus V_2\oplus \cdots \oplus V_2}_{n \text{ times}} \oplus\mathbb{C}^2]^{SL_2}$ of n quadratic forms. We express the Poincaré series $\mathcal{P}(\mathcal{C}_{2n}, z) = \sum_{j=0}^{\infty} \dim(\mathcal{C}_{2n})_j\, z^j$ and $\mathcal{P}(\mathcal{I}_{2n}, z) = \sum_{j=0}^{\infty} \dim(\mathcal{I}_{2n})_j\, z^j$ of these algebras in terms of Narayana polynomials.

Also, for these algebras we calculate the degrees and asymptotic behavious of the degrees, using their Poincaré series.

Key words and phrases: classical invariant theory, invariants, Poincaré series, combinatorics.

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INTRODUCTION

Let V_2 be the complex vector space of quadratic binary forms endowed with the natural action of the special linear group SL_2. Consider the corresponding action of the group SL_2 on the algebras of polynomial functions $\mathbb{C}[nV_2]$ and $\mathbb{C}[nV_2 \oplus \mathbb{C}^2]$, where $nV_2 := \underbrace{V_2 \oplus V_2 \oplus \cdots \oplus V_2}_{n \text{ times}}$.

Denote by $\mathcal{I}_{2n} = \mathbb{C}[nV_2]^{SL_2}$ and by $\mathcal{C}_{2n} = \mathbb{C}[nV_2 \oplus \mathbb{C}^2]^{SL_2}$ the corresponding algebras of invariant polynomial functions. In the language of classical invariant theory the algebras $\mathcal{I}_{2n}$ and $\mathcal{C}_{2n}$ are called the algebra of joint invariants and the algebra of joint covariants for the n quadratic binary forms respectively.

Let $R = R_0 \oplus R_1 \oplus \cdots$ be a finitely generated graded complex algebra, $R_0 = \mathbb{C}$. Denote by

$$\mathcal{P}(R, z) = \sum_{j=0}^{\infty} \dim R_j z^j$$

its Poincaré series. Letting r be the transcendence degree of the quotient field of R over $\mathbb{C}$, the number

$$\deg(R) := \lim_{z \to 1} (1 - z)^r \mathcal{P}(R, z)$$

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is called the *degree of the algebra* R. The first two terms of the Laurent series expansion of $\mathcal{P}(R,z)$ at the point $z=1$ have the following form

$$\mathcal{P}(R,z)=\frac{\deg(R)}{(1-z)^r}+\frac{\psi(R)}{(1-z)^{r-1}}+\cdots.$$

The numbers $\deg(R),\psi(R)$ are important characteristics of the algebra R. For instance, if R is an algebra of invariants of a finite group G then $\deg(R)^{-1}$ is order of the group G and $2\dfrac{\psi(R)}{\deg(R)}$ is the number of pseudo-reflections in G (see [3]).

Let V_d be the standard $(d+1)-$dimensional complex representation of SL_2. Consider the corresponding algebras of invariants $I_d := C[V_d]^{SL_2}$ and $\mathcal{C}_d=\mathbb{C}[V_d\oplus\mathbb{C}^2]^{SL_2}$ be the corresponding algebra of invariants. Explicit formula for the degree of algebra of invariants for binary $d-$forms $\deg(\mathcal{I}_d)$ was derived by Hilbert in [4] and Springer in [8]. In [2] explicit formula for the degree of algebra of covariants for binary $d-$forms of $\deg(\mathcal{C}_d)$ was derived. For this purpose, in [8] and [2] authors used an explicit formula for the Poincaré series of those algebras.

The formal power series

$$\mathcal{P}(\mathcal{C}_{2n},z)=\sum_{j=0}^{\infty}\dim(\mathcal{C}_{2n})_j\,z^j \quad\text{and}\quad \mathcal{P}(\mathcal{I}_{2n},z)=\sum_{j=0}^{\infty}\dim(\mathcal{I}_{2n})_j\,z^j$$

are called the Poincaré series of the algebras $\mathcal{C}_{2n}$ and $\mathcal{I}_{2n}$. In the paper [1] the following expressions for the Poincaré series of those algebras was derived:

$$\mathcal{PC}_{2n}(z)=\sum_{k=1}^{n}\frac{(-1)^{n-k}}{(n-k)!(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left(\sum_{i=0}^{n-k}\binom{n-k}{i}\frac{(n)_i(n)_{n-k-i}z^{2n-k-i-1}}{(1-z)^{n+i}(1-z^2)^{2n-k-i}}\right),$$
$$\mathcal{PI}_{2n}(z)=\sum_{k=1}^{n}\frac{(-1)^{n-k}}{(n-k)!(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left(\sum_{i=0}^{n-k}\binom{n-k}{i}\frac{(n)_i(n)_{n-k-i}z^{2n-k-i-1}}{(1-z)^{n+i-1}(1-z^2)^{2n-k-i}}\right),$$

where $(n)_m:=n(n+1)\cdots(n+m-1)$, $(n)_0:=1$ denotes the shifted factorial.

In the present paper those formulas are reduced to the following forms:

$$\mathcal{P}(\mathcal{C}_{2n},z)=\frac{W_{n-1}(z^2)}{(1-z)^{3n-1}(1+z)^{2n-1}}\quad\text{and}\quad \mathcal{P}(\mathcal{I}_{2n},z)=\frac{W_{n-1}(z^2)-nzN_{n-1}(z^2)}{(1-z)^{3n-1}(1+z)^{2n-1}},$$

where

$$N_n(z)=\sum_{k=1}^{n}\frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1}z^{k-1}\quad\text{and}\quad W_n(z)=\sum_{k=0}^{n}\binom{n}{k}^2 z^k$$

denotes the *Narayana polynomials* and the *Narayana polynomials of type B* respectively.

Also, the degrees of algebras $\mathcal{I}_{2n}$, $\mathcal{C}_{2n}$ and asymptotic behaviors of the degrees are calculated using the explicit expressions for the Poincaré series.

1 Combinatorial identities

Let us prove several auxiliary combinatorial identities.

Lemma 1. *Let m, n be positive integers. The following identities hold:*

$$(i)\quad \frac{W_{n-1}(z^2)}{(1-z)^a(1-z^2)^{2n-1}} = \sum_{k=1}^{n} \frac{1}{(n-k)!(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{z^{2n-k-1+a}}{(1-z)^{2n-k+a}} \frac{d^{n-k}}{dz^{n-k}} \left(\frac{1}{z^a(1+z)^n} \right) \right),$$

$$(ii)\quad \frac{nzN_{n-1}(z^2)}{(1-z)^a(1-z^2)^{2n-1}} = \sum_{k=1}^{n} \frac{1}{(n-k)!(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{z^{2n-k+a}}{(1-z)^{2n-k+a}} \frac{d^{n-k}}{dz^{n-k}} \left(\frac{1}{z^a(1+z)^n} \right) \right).$$

Proof. We shall prove the relations by induction in a.

For $a = 0$ the statements follow immediately from the next identities (see [5]):

$$\sum_{k=1}^{n} \frac{(-1)^{n-k}(n)_{n-k}}{(k-1)!(n-k)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{z^{2n-k-1}}{(1-z^2)^{2n-k}} \right) = \frac{\sum_{k=0}^{n-1} \binom{n-1}{k}^2 z^{2k}}{(1-z^2)^{2n-1}},$$

$$\sum_{k=1}^{n} \frac{(-1)^{n-k}(n)_{n-k}}{(k-1)!(n-k)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{z^{2n-k}}{(1-z^2)^{2n-k}} \right) = \frac{\sum_{k=0}^{n-2} \binom{n-2}{k}\binom{n}{k+1} z^{2k+1}}{(1-z^2)^{2n-1}}.$$

(i) Assume there is a non-negative m such that

$$\sum_{k=1}^{n} \frac{1}{(n-k)!(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{z^{2n-k-1+m}}{(1-z)^{2n-k+m}} \frac{d^{n-k}}{dz^{n-k}} \left(\frac{1}{z^m(1+z)^n} \right) \right) = \frac{\sum_{i=0}^{n-1} \binom{n-1}{i}^2 z^{2i}}{(1-z)^m(1-z^2)^{2n-1}}.$$

We must prove the formula (i) is true for $a = m+1$:

$$\sum_{k=1}^{n} \frac{1}{(n-k)!(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{z^{2n-k+m}}{(1-z)^{2n-k+m}} \frac{d^{n-k}}{dz^{n-k}} \frac{1}{z^{m+1}(1+z)^n} \right) = \frac{\sum_{i=0}^{n-1} \binom{n-1}{i}^2 z^{2i}}{(1-z)^{m+1}(1-z^2)^{2n-1}}.$$

That is,

$$(1-z) \sum_{k=1}^{n} \frac{1}{(n-k)!(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{z^{2n-k-1+m+1}}{(1-z)^{2n-k+m+1}} \frac{d^{n-k}}{dz^{n-k}} \frac{1}{z^{m+1}(1+z)^n} \right)$$
$$= \sum_{k=1}^{n} \frac{1}{(n-k)!(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{z^{2n-k-1+m}}{(1-z)^{2n-k+m}} \frac{d^{n-k}}{dz^{n-k}} \left(\frac{1}{z^m(1+z)^n} \right) \right).$$

It sufficed to show that (we expanded the functions into the Taylor series about z)

$$\sum_{j=0}^{\min\{k,n-1\}} \sum_{i=0}^{k-j} \binom{n+k-j-1}{k} \binom{n+m+k-i-1}{k-j-i} (-1)^i \binom{n+i-1}{i} \binom{i-m}{j}$$
$$= \sum_{j=0}^{\min\{k,n-1\}} \sum_{i=0}^{k-j} \left(\binom{n+k-j-1}{k} \binom{n+m+k-i}{k-j-i} - \binom{n+k-j-2}{k-1} \binom{n+m+k-i-1}{k-j-i-1} \right)$$
$$\times (-1)^i \binom{n+i-1}{i} \binom{i-m-1}{j}.$$

Using following formulas

$$\binom{n}{m} = \frac{n}{m}\binom{n-1}{m-1}, \binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1},$$

after some algebraic transformations we obtain the last equality.

The proof of (ii) is completely analogous to that of (i). □

2 THE POINCARÉ SERIES OF THE ALGEBRAS OF INVARIANTS AND COVARIANTS

We use the derived above combinatorial identities to express the Poincaré series $\mathcal{P}(\mathcal{I}_{2n}, z)$ and $\mathcal{P}(\mathcal{C}_{2n}, z)$ in terms of Narayana polynomials.

Theorem 1. *The following formulas hold:*

$$(i) \quad \mathcal{P}(\mathcal{C}_{2n}, z) = \frac{W_{n-1}(z^2)}{(1-z)^n(1-z^2)^{2n-1}},$$
$$(ii) \quad \mathcal{P}(\mathcal{I}_{2n}, z) = \frac{W_{n-1}(z^2) - nzN_{n-1}(z^2)}{(1-z)^n(1-z^2)^{2n-1}}.$$

Proof. (i) Note that

$$\begin{aligned}\mathcal{P}(\mathcal{C}_{2n}, z) &= \sum_{k=1}^{n} \frac{(-1)^{n-k}}{(n-k)!(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(\sum_{i=0}^{n-k} \binom{n-k}{i} \frac{(n)_i (n)_{n-k-i} z^{2n-k-i-1}}{(1-z)^{n+i}(1-z^2)^{2n-k-i}} \right) \\ &= \sum_{k=1}^{n} \frac{1}{(n-k)!(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{z^{3n-k-1}}{(1-z)^{3n-k}} \frac{d^{n-k}}{dz^{n-k}} \left(\frac{1}{(z(1+z))^n} \right) \right).\end{aligned}$$

Substituting n for a in Lemma 1 (i), we get

$$\sum_{k=1}^{n} \frac{1}{(n-k)!(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{z^{3n-k-1}}{(1-z)^{3n-k}} \frac{d^{n-k}}{dz^{n-k}} \left(\frac{1}{(z(1+z))^n} \right) \right) = \frac{W_{n-1}(z^2)}{(1-z)^n(1-z^2)^{2n-1}}.$$

(ii)

$$\begin{aligned}\mathcal{P}(\mathcal{I}_{2n}, z) &= \sum_{k=1}^{n} \frac{(-1)^{n-k}}{(n-k)!(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \sum_{i=0}^{n-k} \binom{n-k}{i} \frac{(n)_i (n)_{n-k-i} z^{2n-k-i-1}}{(1-z)^{n+i-1}(1-z^2)^{2n-k-i}} \\ &= \sum_{k=1}^{n} \frac{1}{(n-k)!(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{z^{3n-k-1}}{(1-z)^{3n-k-1}} \frac{d^{n-k}}{dz^{n-k}} \frac{1}{(z(1+z))^n} \right) \\ &= \sum_{k=1}^{n} \frac{1}{(n-k)!(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{z^{3n-k-1}}{(1-z)^{3n-k}} \frac{d^{n-k}}{dz^{n-k}} \frac{1}{(z(1+z))^n} \right) \\ &\quad - \sum_{k=1}^{n} \frac{1}{(n-k)!(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{z^{3n-k}}{(1-z)^{3n-k}} \frac{d^{n-k}}{dz^{n-k}} \frac{1}{(z(1+z))^n} \right).\end{aligned}$$

Substituting n for m in Lemma 1, we get

$$\mathcal{P}(\mathcal{I}_{2n}, z) = \frac{W_{n-1}(z^2) - nzN_{n-1}(z^2)}{(1-z)^n(1-z^2)^{2n-1}}.$$

$\square$

3 THE DEGREES OF THE ALGEBRAS OF INVARIANTS AND COVARIANTS

Let us calculate the degrees of the algebras of joint invariants and covariants of n quadratic binary forms using the formulas for the Poincaré series $\mathcal{P}(\mathcal{I}_{2n}, z)$ and $\mathcal{P}(\mathcal{C}_{2n}, z)$.

Theorem 2. *The following formulas hold*

$$(i) \quad \operatorname{tr}\deg_{\mathbb{C}} \mathcal{C}_{2n} = 3n-1,$$
$$(ii) \quad \operatorname{tr}\deg_{\mathbb{C}} \mathcal{I}_{2n} = 3n-3.$$

Proof. The transcendence degrees over $\mathbb{C}$ for the algebras $\mathcal{I}_{2n}, \mathcal{C}_{2n}$ is equal to order of the pole for $\mathcal{P}(\mathcal{I}_{2n}, z), \mathcal{P}(\mathcal{C}_{2n}, z)$ respectively, see [7]. Since $\frac{W_{n-1}(1)}{2^{2n-1}} \neq 0.$ for all n then $\operatorname{tr}\deg_{\mathbb{C}} \mathcal{C}_{2n} = 3n-1$.

Note that

$$(W_{n-1}(z^2) - nzN_{n-1}(z^2))\,|_{z=1} = \sum_{k=0}^{n-1} \binom{n-1}{k}^2 - n\sum_{k=1}^{n-1} \frac{1}{k}\binom{n-2}{k-1}\binom{n-1}{k-1} = 0,$$
$$(W_{n-1}(z^2) - nzN_{n-1}(z^2))'\,|_{z=1} = 2\sum_{k=1}^{n-1} k\binom{n-1}{k}^2 - n\sum_{k=1}^{n-1} \frac{2k-1}{k}\binom{n-2}{k-1}\binom{n-1}{k-1} = 0,$$
$$(W_{n-1}(z^2) - nzN_{n-1}(z^2))''\,|_{z=1} = \sum_{k=1}^{n-1} 2k(2k-1)\binom{n-1}{k}^2$$
$$- n\sum_{k=2}^{n-1} \frac{1}{k}(2k-1)(2k-2)\binom{n-1}{k-1}\binom{n-2}{k-1}\binom{2n-4}{n-2} \neq 0.$$

Thus, the function $(W_{n-1}(z^2) - nzN_{n-1}(z^2))$ has the pole of order 2 at $z=1$. Let us remember that $\mathcal{P}(\mathcal{I}_{2n}, z) = \dfrac{W_{n-1}(z^2) - nzN_{n-1}(z^2)}{(1-z)^{3n-1}(1+z)^{2n-1}}$. This implies that $\operatorname{tr}\deg_{\mathbb{C}} \mathcal{I}_{2n} = 3n-3$. □

Note that the proof of previous Theorem is direct. Luna's Slice Theorem (see [6]) gives us more general result.

We know explicit forms for the Poincaré series for the algebras of joint invariants and covariants of n linear forms. Thus we can prove the following statement.

Theorem 3. *The degrees of the algebras of joint covariants and invariants of n quadratic binary forms are equal to*

$$(i) \quad \deg(\mathcal{C}_{2n}, z) = \frac{\binom{2n-2}{n-1}}{2^{2n-1}},$$
$$(ii) \quad \deg(\mathcal{I}_{2n}, z) = \frac{\binom{2n-4}{n-2}}{(n-1)2^{2n-1}}.$$

Proof. (i) Using Theorem 1 and Theorem 2, we have:

$$\deg(\mathcal{C}_{2n}) = \lim_{z=1}(1-z)^{3n-1}\mathcal{P}(\mathcal{C}_n, z) = \lim_{z=1}(1-z)^{3n-1}\frac{\sum\limits_{k=0}^{n-1}\binom{n-1}{k}^2 z^{2k}}{(1-z)^n(1-z^2)^{2n-1}} = \frac{\binom{2n-2}{n-1}}{2^{2n-1}}.$$

(ii) Similarly, we have

$$\deg(\mathcal{I}_{2n}) = \lim_{z=1}(1-z)^{3n-3}\mathcal{P}(\mathcal{I}_n, z) = \lim_{z=1}\frac{W_{n-1}(z^2) - nzN_{n-1}(z^2)}{(1-z)^2(1+z)^{2n-1}}$$
$$= \lim_{z=1}\frac{\left(W_{n-1}(z^2) - nzN_{n-1}(z^2)\right)''}{\left((1-z)^2(1+z)^{2n-1}\right)''} = \frac{\binom{2n-4}{n-2}}{(n-1)2^{2n-1}}.$$

□

Note that asymptotically, the Catalan numbers grow as

$$C_n = \frac{1}{n+1}\binom{2n}{n} \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}.$$

It is easy to calculate asymptotic behaviours of the degrees of the algebras $\mathcal{I}_{2n}$ and $\mathcal{C}_{2n}$:

Corollary 1. *Asymptotic behaviours of the degrees of the algebras of joint invariants and covariants of n quadratic binary forms as $n \to \infty$ are follows*

$$\deg(\mathcal{I}_{2n}) \sim \frac{1}{8\sqrt{\pi n^3}} \quad \textit{and} \quad \deg(\mathcal{C}_{2n}) \sim \frac{1}{2\sqrt{\pi n}}.$$

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Ми розглядаємо одну з фундаментальних проблем класичної теорії інваріантів – дослідження ряду Пуанкаре алгебр інваріантів групи Лі SL_2. Відомо, що перші доданки розкладу ряду Пуанкаре в ряд Лорана в околі точки $z = 1$ несуть важливу інформацію про структуру цієї алгебри. Для алгебри $\mathcal{I}_d = \mathbb{C}[V_d]^{SL_2}$ інваріантів однієї бінарної форми вони були обчислені ще Гільбертом (тут V_d– комплексний $d+1-$ вимірний векторний простір бінарних форм степеня d). Пізніше цей же результат отримав Спрінгер, використовуючи явну формулу для ряду Пуанкаре алгебри $\mathcal{I}_d$. Розглядається аналогічна задача для алгебр спільних інваріантів $\mathcal{I}_{2n} = \mathbb{C}[\underbrace{V_2 \oplus V_2 \oplus \cdots \oplus V_2}_{n \text{ times}}]^{SL_2}$ та спільних коваріантів $\mathcal{C}_{2n} = \mathbb{C}[\underbrace{V_2 \oplus V_2 \oplus \cdots \oplus V_2}_{n \text{ times}} \oplus \mathbb{C}^2]^{SL_2}$ n квадратичних форм. Ми виразили ряди Пуанкаре $\mathcal{P}(\mathcal{C}_{2n}, z) = \sum_{j=0}^{\infty} \dim(\mathcal{C}_{2n})_j z^j$ та $\mathcal{P}(\mathcal{I}_{2n}, z) = \sum_{j=0}^{\infty} \dim(\mathcal{I}_{2n})_j z^j$ цих алгебр через поліноми Нараяна. Також ми обчислили степені цих алгебр та асиптотичну поведінку цих степенів, використовуючи ці ряди Пуанкаре.

Ключові слова і фрази: класична теорія інваріантів, інваріанти, ряди Пуанкаре, комбінаторика.

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ON THE GROWTH OF A KLASSS OF DIRICHLET SERIES ABSOLUTELY CONVERGENT IN HALF-PLANE

In terms of generalized orders it is investigated a relation between the growth of a Dirichlet series $F(s) = \sum\limits_{n=1}^{\infty} a_n \exp\{s\lambda_n\}$ with the abscissa of asolute convergence $A \in (-\infty, +\infty)$ and the growth of Dirichlet series $F_j(s) = \sum\limits_{n=1}^{\infty} a_{n,j} \exp\{s\lambda_n\}$, $1 \le j \le 2$, with the same abscissa of absolute convergence if the coefficients a_n are connected with the coefficients $a_{n,j}$ by correlation

$$\beta\left(\frac{\lambda_n}{\ln\left(|a_n|e^{A\lambda_n}\right)}\right) = (1+o(1))\prod_{j=1}^{m}\beta\left(\frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)^{\omega_j}, \quad n\to\infty,$$

where $\omega_j > 0$, $1 \le j \le m$, $\sum\limits_{j=1}^{m}\omega_j = 1$.

Key words and phrases: Dirichlet series, generalized order.

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INTRODUCTION

For an entire function $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ let $\varrho[f]$ be its order and $\sigma[f]$ be its type. Using Hadamard's formulas for the finding of these characteristics, E.G. Calys [1] proved the following theorems.

Theorem A. *Suppose that entire functions $f_1(z) = \sum\limits_{n=0}^{\infty} a_{n,1} z^n$ and $f_2(z) = \sum\limits_{n=0}^{\infty} a_{n,2} z^n$ have finite orders and regular growth (in sence of the equality of order $\varrho[f]$ and lower order $\lambda[f]$) and the sequences $(|a_{n,1}/a_{n+1,1}|)$ and $(|a_{n,2}/a_{n+1,2}|)$ are nondecreasing for $n \ge n_0$. If*

$$\ln(1/|a_n|) = (1+o(1))\sqrt{\ln(1/|a_{n,1}|)\ln(1/|a_{n,2}|)}$$

as $n\to\infty$, then the function f has regular growth and $\varrho[f] = \sqrt{\varrho[f_1]\varrho[f_2]}$.

Theorem B. *Suppose that functions f_1 and f_2 from Theorem A have the same order $\varrho[f_1] = \varrho[f_2] = \varrho \in (0, +\infty)$ and the types $\sigma[f_1] = \sigma_1$, $\sigma[f_2] = \sigma_2$. Also suppose that $a_{n,1} \ne 0$ and $|a_{n,2}| \ge |a_{n,1}|/l(1/|a_{n,1}|)$ for all $n \ge n_0$, where l is slowly varying function. If*

$$|a_n| = (1+o(1))\sqrt{|a_{n,1}||a_{n,2}|}$$

as $n\to\infty$, then the function f has the order $\varrho[f] = \varrho$ and the type $\sigma[f] \le \sqrt{\sigma_1\sigma_1}$.

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In [2] Theorems A and B are generalized on the case of entire Dirichlet series of finite generalized orders by Sheremeta, moreover instead two functions f_1 and f_2 were considered $n \geq 2$ entire Dirichlet series.

Here we will obtain analogues results for Dirichlet series absolutely convergent in a half-plane.

Let $\Lambda = (\lambda_n)$ be an increasing to $+\infty$ sequence of nonnegative numbers and $S(\Lambda, A)$ be a class of Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_n \exp\{sl_n\}, \quad s = \sigma + it \tag{1}$$

with a given sequence (λ_n) of exponents and an abscissa of absolutely convergence $\sigma_a = A \in (-\infty, +\infty)$ and $M(\sigma, F) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$ for $\sigma \in (-\infty, A)$.

By L we denote a class of positive continuous functions α on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0)$ for $x \leq x_0$ and $0 < \alpha(x) \uparrow +\infty$ as $x_0 \leq x \uparrow +\infty$. We say that $\alpha \in L^0$ if $\alpha \in L$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_{si}$, if $\alpha \in L$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x \to +\infty$ for each $c \in (0, +\infty)$, i. e. α is slowly increasing function. Clearly, $L_{si} \subset L^0$.

For $\alpha \in L$ and $\beta \in L$ the values

$$\varrho^A_{\alpha,\beta}[F] = \varlimsup_{\sigma \uparrow A} \frac{\alpha(\ln M(\sigma, F))}{\beta(1/(A - \sigma))}, \quad \lambda^A_{\alpha,\beta}[F] = \varliminf_{\sigma \uparrow A} \frac{\alpha(\ln M(\sigma, F))}{\beta(1/(A - \sigma))}$$

are called [3] generalized order and lower order correspondly of Dirichlet series (1) from the class $S(\Lambda, A)$.

1 ANALOGUES OF THEOREM A.

We need the following lemmas from [3].

Lemma 1.1. *Let $\alpha \in L_{si}$, $\beta \in L_{si}$ and*

$$\frac{x}{\beta^{-1}(c\alpha(x))} \uparrow +\infty, \quad \alpha\left(\frac{x}{\beta^{-1}(c\alpha(x))}\right) = (1 + o(1))\alpha(x) \tag{2}$$

as $x_0 \leq x \to +\infty$ for each $c \in (0, +\infty)$.

If $\alpha(\lambda_n) = o\left(\beta\left(\lambda_n / \ln n\right)\right)$ as $n \to \infty$, then

$$\varrho^A_{\alpha,\beta}[F] = k^A_{\alpha,\beta}[F] =: \varlimsup_{n \to \infty} \frac{\alpha(\lambda_n)}{\beta\left(\lambda_n / \ln\left(|a_n| e^{A\lambda_n}\right)\right)},$$

and if, moreover, $\alpha(\lambda_{n+1}) = (1 + o(1))\alpha(\lambda_n)$ and $\dfrac{\ln |a_{n+1}| - \ln |a_n|}{\lambda_{n+1} - \lambda_n} \nearrow A$ as $n_0 \leq n \to \infty$, then

$$\lambda^A_{\alpha,\beta}[F] = \varkappa^A_{\alpha,\beta}[F] =: \varliminf_{n \to \infty} \frac{\alpha(\lambda_n)}{\beta\left(\lambda_n / \ln\left(|a_n| e^{A\lambda_n}\right)\right)}.$$

Remark 1.1 ([3]). *In order that $\lambda^A_{\alpha,\beta}[F] \geq \varkappa^A_{\alpha,\beta}[F]$, it is sufficient that $\alpha(\lambda_{n+1}) = (1 + o(1))\alpha(\lambda_n)$ as $n \to \infty$.*

Lemma 1.2. *Let $\alpha \in L_{si}$, $\beta \in L_{si}$ and*

$$\frac{x}{\alpha^{-1}(c\beta(x))} \uparrow +\infty, \quad \beta\left(\frac{x}{\alpha^{-1}(c\alpha(x))}\right) = (1+o(1))\beta(x) \tag{3}$$

as $x_0 \le x \to +\infty$ for each $c \in (0, +\infty)$.

If $\alpha(\ln\, n) = o(\beta(\lambda_n))$ as $n \to \infty$, then

$$\varrho^A_{\alpha,\beta}[F] = k^{A*}_{\alpha,\beta}[F] =: \overline{\lim_{n\to\infty}} \frac{\alpha\left(\ln\,\left(|a_n|e^{A\lambda_n}\right)\right)}{\beta(\lambda_n)},$$

and if, moreover, $\beta(\lambda_{n+1}) = (1+o(1))\beta(\lambda_n)$ and $\dfrac{\ln\,|a_{n+1}| - \ln\,|a_n|}{\lambda_{n+1}-\lambda_n} \nearrow A$ as $n_0 \le n \to \infty$, then

$$\lambda^A_{\alpha,\beta}[F] = \varkappa^{A*}_{\alpha,\beta}[F] =: \underline{\lim_{n\to\infty}} \frac{\alpha\left(\ln\,\left(|a_n|e^{A\lambda_n}\right)\right)}{\beta(\lambda_n)}.$$

Remark 1.2 ([3]). *In order that $\lambda^A_{\alpha,\beta}[F] \ge \varkappa^{A*}_{\alpha,\beta}[F]$, it is sufficient that $\beta(\lambda_{n+1}) = (1+o(1))\beta(\lambda_n)$ as $n \to \infty$.*

Suppose that $F_j \in S(\Lambda, A)$, $1 \le j \le m$, and

$$F_j(s) = \sum_{n=1}^{\infty} a_{n,j} \exp\{s\lambda_n\}. \tag{4}$$

Using Lemma 1.1, at first we prove the following analog of Theorem A.

Theorem 1. *Let functions $\alpha \in L_{si}$ and $\beta \in L_{si}$ satisfy conditions* (2)*, $\alpha(\lambda_n) = o\left(\beta\left(\lambda_n/\ln\, n\right)\right)$ and $\alpha(\lambda_{n+1}) = (1+o(1))\alpha(\lambda_n)$ as $n \to \infty$. Suppose that all functions* (4) *have regular $\alpha\beta$-growth (i.e. $\lambda^A_{\alpha,\beta}[F_j] = \varrho^A_{\alpha,\beta}[F_j] < +\infty$) and $\dfrac{\ln\,|a_{n+1,j}| - \ln\,|a_{n,j}|}{\lambda_{n+1}-\lambda_n} \nearrow A$ as $n_0 \le n \to \infty$.*

If $\omega_j > 0$, $1 \le j \le m$, $\sum\limits_{j=1}^{m} \omega_j = 1$ and

$$\beta\left(\frac{\lambda_n}{\ln\,\left(|a_n|e^{A\lambda_n}\right)}\right) = (1+o(1))\prod_{j=1}^{m}\beta\left(\frac{\lambda_n}{\ln\,\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)^{\omega_j}, \quad n \to \infty, \tag{5}$$

then function (1) *has regular $\alpha\beta$-growth and $\varrho^A_{\alpha,\beta}[F] = \prod\limits_{j=1}^{m}(\varrho^A_{\alpha,\beta}[F_j])^{\omega_j}$.*

Proof. Since $\lambda^A_{\alpha,\beta}[F_j] = \varrho^A_{\alpha,\beta}[F_j] = \varrho_j < +\infty$, by Lemma 1.1 we have

$$\lim_{n\to\infty} \frac{\alpha(\lambda_n)}{\beta\left(\lambda_n/\ln\,\left(|a_{n,j}|e^{A\lambda_n}\right)\right)} = \varrho_j.$$

Therefore, from (5) we obtain

$$\lim_{n\to\infty}\frac{1}{\alpha(\lambda_n)}\beta\left(\frac{\lambda_n}{\ln\,\left(|a_n|e^{A\lambda_n}\right)}\right) = \lim_{n\to\infty}\frac{1}{\alpha(\lambda_n)}\prod_{j=1}^{m}\beta\left(\frac{\lambda_n}{\ln\,\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)^{\omega_j}$$

$$= \lim_{n\to\infty}\prod_{j=1}^{m}\left(\frac{1}{\alpha(\lambda_n)}\beta\left(\frac{\lambda_n}{\ln\,\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)\right)^{\omega_j} = \prod_{j=1}^{m}\lim_{n\to\infty}\left(\frac{1}{\alpha(\lambda_n)}\beta\left(\frac{\lambda_n}{\ln\,\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)\right)^{\omega_j}$$

$$= \prod_{j=1}^{m}\left(1/\varrho_j\right)^{\omega_j},$$

that is,

$$\lim_{n\to\infty}\frac{\alpha(\lambda_n)}{\beta\left(\lambda_n/\ln\left(|a_n|e^{A\lambda_n}\right)\right)}=\prod_{j=1}^{m}\varrho_j^{\omega_j}.$$

Using Lemma 1.1 and the Remark 1.1, hence we get $\prod\limits_{j=1}^{m}\varrho_j^{\omega_j}\le\lambda^A_{\alpha,\beta}[F]\le\varrho^A_{\alpha,\beta}[F]=\prod\limits_{j=1}^{m}\varrho_j^{\omega_j}$, that is the function F has regular $\alpha\beta$-growth and $\varrho^A_{\alpha,\beta}[F]=\prod\limits_{j=1}^{m}(\varrho^A_{\alpha,\beta}[F])^{\omega_j}$. Theorem 1 is proved. $\square$

From (2) it follows that the function α increases less rapidly than the function β. It is easy to verify that the functions $\alpha(x)=\ln\ln x$ and $\beta(x)=\ln x$ for $x\ge x_0$ satisfy (2) and the condition $\alpha(\lambda_n)=o\left(\beta\left(\lambda_n/\ln n\right)\right)$ holds as $n\to\infty$, provided $\varlimsup\limits_{n\to\infty}(\ln\ln n)/\ln\lambda_n<1$. Therefore, Theorem 1 implies the following statement.

Corollary 1.1. *Let $\varlimsup\limits_{n\to\infty}(\ln\ln n)/\ln\lambda_n<1$, $\ln\ln\lambda_{n+1}=(1+o(1))\ln\ln\lambda_n$ as $n\to\infty$. Suppose that $\lim\limits_{\sigma\uparrow A}\dfrac{\ln\ln\ln M(\sigma,F_j))}{\ln(1/(A-\sigma))}=\varrho_j$ and $\dfrac{\ln|a_{n+1,j}|-\ln|a_{n,j}|}{\lambda_{n+1}-\lambda_n}\nearrow A$ as $n_0\le n\to\infty$ for all $1\le j\le m$. If*

$$\ln\left(\frac{\lambda_n}{\ln\left(|a_n|e^{A\lambda_n}\right)}\right)=(1+o(1))\prod_{j=1}^{m}\ln^{\omega_j}\left(\frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}\right),\quad\sum_{j=1}^{m}\omega_j=1,$$

as $n\to\infty$ then $\lim\limits_{\sigma\uparrow A}\dfrac{\ln\ln\ln M(\sigma,F))}{\ln(1/(A-\sigma))}=\prod\limits_{j=1}^{m}\varrho_j^{\omega_j}$.

For the proof of the following theorem we will use Lemma 1.2.

Theorem 2. *Let the functions $\alpha\in L_{si}$ and $\beta\in L_{si}$ satisfy the condition* (3)*, $\alpha(\ln n)=o(\beta(\lambda_n))$ and $\beta(\lambda_{n+1})=(1+o(1))\beta(\lambda_n)$ as $n\to\infty$. Suppose that all functions* (4) *have regular $\alpha\beta$-growth and $\dfrac{\ln|a_{n+1,j}|-\ln|a_{n,j}|}{\lambda_{n+1}-\lambda_n}\nearrow A$ as $n_0\le n\to\infty$.*

If $\omega_j>0$, $1\le j\le m$, $\sum\limits_{j=1}^{m}\omega_j=1$ and

$$\alpha\left(\ln\left(|a_n|e^{A\lambda_n}\right)\right)=(1+o(1))\prod_{j=1}^{m}\alpha^{\omega_j}\left(\ln\left(|a_{n,j}|e^{A\lambda_n}\right)\right),\quad n\to\infty,\tag{6}$$

then function (1) *has regular $\alpha\beta$-growth and $\varrho^A_{\alpha,\beta}[F]=\prod\limits_{j=1}^{m}(\varrho^A_{\alpha,\beta}[F_j])^{\omega_j}$.*

Proof. Since $\lambda^A_{\alpha,\beta}[F_j]=\varrho^A_{\alpha,\beta}[F_j]=\varrho_j<+\infty$, by Lemma 1.2 we have

$$\lim_{n\to\infty}\frac{\alpha\left(\ln\left(|a_{n,j}|e^{A\lambda_n}\right)\right)}{\beta(\lambda_n)}=\varrho_j.$$

Therefore, from (6), as in the proof of Theorem 1,

$$\lim_{n\to\infty}\frac{\alpha\left(\ln\left(|a_n|e^{A\lambda_n}\right)\right)}{\beta(\lambda_n)}=\prod_{j=1}^{m}\lim_{n\to\infty}\left(\frac{\alpha\left(\ln\left(|a_{n,j}|e^{A\lambda_n}\right)\right)}{\beta(\lambda_n)}\right)^{\omega_j}=\prod_{j=1}^{m}\varrho_j^{\omega_j},$$

whence, as above, we obtain the regular $\alpha\beta$-growth of the function f and the equality $\varrho^A_{\alpha,\beta}[F]=\prod\limits_{j=1}^{m}(\varrho^A_{\alpha,\beta}[F])^{\omega_j}$. Theorem 2 is proved. $\square$

From (3) it follows that the function β increases less rapidly than the function β. It is easy to verify that the functions $\alpha(x) = \ln x$ and $\beta(x) = \ln\ln x$ for $x \geq x_0$ satisfy (3). Therefore, Theorem 2 implies the following statement.

Corollary 1.2. *Let* $\ln\ln n = o(\ln\ln\lambda_n))$ *and* $\ln\ln\lambda_{n+1} = (1+o(1))\ln\ln\lambda_n$ *as* $n \to \infty$. *Suppose that* $\lim\limits_{\sigma\uparrow A} \dfrac{\ln\ln M(\sigma, F_j))}{\ln\ln(1/(A-\sigma))} = \varrho_j$ *and* $\dfrac{\ln|a_{n+1,j}| - \ln|a_{n,j}|}{\lambda_{n+1}-\lambda_n} \nearrow A$ *as* $n_0 \leq n \to \infty$ *for all* $1 \leq j \leq m$. *If*

$$\ln\ln\left(|a_n|e^{A\lambda_n}\right) = (1+o(1))\prod_{j=1}^{m}\ln^{\omega_j}\ln\left(|a_{n,j}|e^{A\lambda_n}\right), \quad \sum_{j=1}^{m}\omega_j = 1,$$

as $n \to \infty$ *then* $\lim\limits_{\sigma\uparrow A} \dfrac{\ln\ln M(\sigma, F))}{\ln\ln(1/(A-\sigma))} = \prod\limits_{j=1}^{m}\varrho_j^{\omega_j}$.

2 ANALOGUES OF THEOREM B.

Suppose, as above, that $\alpha \in L_{si}$ and $\beta \in L_{si}$. In order to get the analogues of Theorem B, except the generalized order $\varrho^A_{\alpha,\beta}[F] \in (0, +\infty)$, it is needed to enter a (generalized) type. A definition of the type depends on what from the functions α or β grows slower.

Suppose at first that the function β increases less rapidly than the function α and define a type by the formula

$$T^{A*}_{\alpha,\beta}[F] = \varlimsup_{\sigma\uparrow A} \frac{\ln M(\sigma, F)}{\alpha^{-1}(\varrho^A_{\alpha,\beta}[F]\beta(1/(A-\sigma)))}.$$

Since $T^{A*}_{\alpha,\beta}[F] = \varrho^A_{\alpha_1,\beta_1}[F]$, where $\alpha_1(x) = x \notin L_{si}$ and $\beta_1(x) = \alpha^{-1}(\varrho^A_{\alpha,\beta}[F]\beta(x))$ for $x \geq x_0$, we can apply none from the lemmas indicated above. However the following statement is true [3].

Lemma 2.1. *Let* $\alpha_1(x) = x$ *for* $x \geq x_0$, $\beta_1 \in L_{si}$ *and*

$$\frac{x}{\beta_1(x)} \uparrow +\infty, \quad \beta_1\left(\frac{x}{\beta_1(x)}\right) = (1+o(1))\beta_1(x), \quad x_0 \leq x \to +\infty.$$

If $\ln n = o(\beta_1(\lambda_n))$ *as* $n \to \infty$ *then* $\varrho^A_{\alpha_1,\beta_1}[F] = \varlimsup\limits_{n\to\infty} \dfrac{\ln\left(|a_n|e^{A\lambda_n}\right)}{\beta_1(\lambda_n)}$.

Since $\beta_1(x) = \alpha^{-1}(\varrho^A_{\alpha,\beta}[F]\beta(x))$ for $x \geq x_0$ then Lemma 2.1 implies the following statement.

Lemma 2.2. *Let* $\alpha \in L_{si}$ *and* $\beta \in L_{si}$ *be such that* $\alpha^{-1}(c\beta(x)) \in L_{si}$ *for each* $c \in (0, +\infty)$ *and*

$$\frac{x}{\alpha^{-1}(c\beta(x))} \uparrow +\infty, \quad \alpha^{-1}\left(c\beta\left(\frac{x}{\alpha^{-1}(c\beta(x)}\right)\right) = (1+o(1))\alpha^{-1}(c\beta(x)) \tag{7}$$

as $x_0 \leq x \to +\infty$ *for each* $c \in (0, +\infty)$. *If* $\ln n = o(\alpha^{-1}(c\beta(\lambda_n))$ *as* $n \to \infty$ *for each* $c \in (0, +\infty)$, *then*

$$T^{A*}_{\alpha,\beta}[F] = \varlimsup_{n\to\infty} \frac{\ln\left(|a_n|e^{A\lambda_n}\right)}{\alpha^{-1}(\varrho^A_{\alpha,\beta}[F]\beta(\lambda_n))}.$$

The following theorem generalizes Theorem B.

Theorem 3. *Let $\beta \in L_{si}$, $\alpha(e^x) \in L^0$, $\alpha^{-1}(c\beta(x)) \in L_{si}$, conditions* (7) *hold and* $\ln n = o(\alpha^{-1}(c\beta(\lambda_n)))$ *as $n \to \infty$ for each $c \in (0, +\infty)$. Suppose that all Dirichlet series* (4) *have the same generalised order $\varrho^A_{\alpha,\beta}[F_j] = \varrho \in (0, +\infty)$ and the types $T^{A*}_{\alpha,\beta}[F_j] \in (0, +\infty)$. Suppose also that $a_{n,1} \neq 0$ for all $n \geq n_0$ and for all $2 \leq j \leq m$*

$$\ln \ln \left(|a_{n,j}|e^{A\lambda_n}\right) \geq (1 + o(1)) \ln \ln \left(|a_{n,1}|e^{A\lambda_n}\right), \quad n \to \infty. \tag{8}$$

If $\omega_j > 0$, $1 \leq j \leq m$, $\sum\limits_{j=1}^{m} \omega_j = 1$ and

$$\ln \left(|a_n|e^{A\lambda_n}\right) = (1 + o(1)) \prod_{j=1}^{m} \left(\ln \left(|a_{n,j}|e^{A\lambda_n}\right)\right)^{\omega_j}, \quad n \to \infty, \tag{9}$$

then Dirichlet series (1) *has the generalized order $\varrho^A_{\alpha,\beta}[F] = \varrho$ and the type*

$$T^{A*}_{\alpha,\beta}[F] \leq \prod_{j=1}^{m} T^{A*}_{\alpha,\beta}[F_j]^{\omega_j}.$$

Proof. Since $\alpha(e^x) \in L^0$, then for each $c \in (0, +\infty)$ we have

$$\alpha(cx) = \alpha(e^{\ln x + \ln c}) = \alpha(e^{(1+o(1)) \ln x}) = (1 + o(1))\alpha(e^{\ln x}) = (1 + o(1))\alpha(x)$$

as $x \to +\infty$, that is $\alpha \in L_{si}$. Hence it follows that $\alpha^{-1}((1 - \eta)x) = o(\alpha^{-1}(x))$ as $x \to +\infty$ for each $\eta \in (0, 1)$, because if $\alpha^{-1}((1 - \eta)x_k) \geq h\alpha^{-1}(x_k))$ for some number $h > 0$ and an increasing to $+\infty$ sequence (x_k) then $(1 - \eta)x_k \geq \alpha(h\alpha^{-1}(x_k)) = (1 + o(1))x_k$ as $k \to \infty$, that is impossible.

Therefore, conditions (7) imply the conditions (3). Indeed, if for some $c \in (0, +\infty)$, $\eta \in (0, 1)$ and an increasing to $+\infty$ sequence (x_k) the inequality

$$\beta\left(x_k/\alpha^{-1}(c\beta(x_k))\right) \leq (1 - \eta)\beta(x_k)$$

is true then $\alpha^{-1}\left(c\beta\left(x_k/\alpha^{-1}(c\beta(x_k))\right)\right) \leq \alpha^{-1}\left(c(1 - \eta)\beta(x_k)\right) = o(\alpha^{-1}\left(c\beta(x_k)\right))$ as $k \to \infty$, that is impossible in view of (7).

Finally, from the condition $\ln n = o(\alpha^{-1}(c\beta(\lambda_n))$ as $n \to \infty$ for each $c \in (0, +\infty)$ we have $\ln n \leq \alpha^{-1}(c\beta(\lambda_n))$ for $n \geq n_0$ and each $c \in (0, +\infty)$, that is $\alpha(\ln n) \leq c\beta(\lambda_n)$ and in view of the arbitrariness of c we obtain $\alpha(\ln n) = o(\beta(\lambda_n))$ as $n \to \infty$.

Thus, from the conditions on the functions α and β and the sequence (λ_n) in Theorem 3 the conditions of Lemma 1.2 follows.

Since all Dirichlet series (4) have the same generalized order $\varrho^A_{\alpha,\beta}[F_j] = \varrho \in (0, +\infty)$, then by Lemma 1.2 for every $\varrho_1 > \varrho$ and all $n \geq n_0(\varrho_1)$ we have $\ln\left(|a_{n,j}|e^{A\lambda_n}\right) \leq \alpha^{-1}(\varrho_1\beta(\lambda_n))$. Therefore, from (9) we obtain

$$\begin{aligned}\varrho^A_{\alpha,\beta}[F] &= \varlimsup_{n\to\infty} \frac{\alpha\left(\ln\left(|a_n|e^{A\lambda_n}\right)\right)}{\beta(\lambda_n)} = \varlimsup_{n\to\infty} \frac{1}{\beta(\lambda_n)} \alpha\left(\prod_{j=1}^{m}\left(\ln\left(|a_{n,j}|e^{A\lambda_n}\right)\right)^{\omega_j}\right)\\ &= \varlimsup_{n\to\infty} \frac{1}{\beta(\lambda_n)} \alpha\left(\exp\left\{\sum_{j=1}^{m} \omega_j \ln\ln\left(|a_{n,j}|e^{A\lambda_n}\right)\right\}\right)\\ &\leq \varlimsup_{n\to\infty} \frac{1}{\beta(\lambda_n)} \alpha\left(\exp\left\{\sum_{j=1}^{m} \omega_j \ln \alpha^{-1}(\varrho_1\beta(\lambda_n))\right\}\right) = \varrho_1,\end{aligned}$$

that is in view of the arbitrariness of ϱ_1 we obtain the inequality $\varrho^A_{\alpha,\beta}[F] \le \varrho$.

On the other hand, in view of the conditions (8) and $\alpha(e^x) \in L^0$ we have

$$\begin{aligned}
\varrho^A_{\alpha,\beta}[F] &= \varlimsup_{n\to\infty} \frac{1}{\beta(\lambda_n)}\alpha\left(\exp\left\{\sum_{j=1}^m \omega_j \ln\ln\left(|a_{n,j}|e^{A\lambda_n}\right)\right\}\right)\\
&= \varlimsup_{n\to\infty} \frac{1}{\beta(\lambda_n)}\alpha\left(\exp\left\{\omega_1 \ln\ln\left(|a_{n,1}|e^{A\lambda_n}\right) + \sum_{j=2}^m \omega_j \ln\ln\left(|a_{n,j}|e^{A\lambda_n}\right)\right\}\right)\\
&\ge \varlimsup_{n\to\infty} \frac{1}{\beta(\lambda_n)}\alpha\left(\exp\left\{\omega_1 \ln\ln\left(|a_{n,1}|e^{A\lambda_n}\right) + \sum_{j=2}^m \omega_j(1+o(1)) \ln\ln\left(|a_{n,1}|e^{A\lambda_n}\right)\right\}\right)\\
&= \varlimsup_{n\to\infty} \frac{1}{\beta(\lambda_n)}\alpha\left(\exp\left\{(1+o(1))\sum_{j=1}^m \omega_j \ln\ln\left(|a_{n,1}|e^{A\lambda_n}\right)\right\}\right)\\
&= \varlimsup_{n\to\infty} \frac{(1+o(1))}{\beta(\lambda_n)}\alpha\left(\exp\left\{\sum_{j=1}^m \omega_j \ln\ln\left(|a_{n,1}|e^{A\lambda_n}\right)\right\}\right)\\
&= \varlimsup_{n\to\infty} \frac{\alpha\left(\ln\left(|a_{n,1}|e^{A\lambda_n}\right)\right)}{\beta(\lambda_n)} = \varrho.
\end{aligned}$$

Thus, $\varrho^A_{\alpha,\beta}[F] = \varrho$ and for $T^{A*}_{\alpha,\beta}[F]$ by Lemma 2.2 from (9) we obtain

$$\begin{aligned}
T^{A*}_{\alpha,\beta}[F] &= \varlimsup_{n\to\infty} \frac{\ln\left(|a_n|e^{A\lambda_n}\right)}{\alpha^{-1}(\varrho^A_{\alpha,\beta}[F]\beta(\lambda_n))} = \varlimsup_{n\to\infty} \frac{1}{\alpha^{-1}(\varrho^A_{\alpha,\beta}[F]\beta(\lambda_n))}\prod_{j=1}^m \left(\ln\left(|a_{n,j}|e^{A\lambda_n}\right)\right)^{\omega_j}\\
&= \varlimsup_{n\to\infty} \prod_{j=1}^m \left(\frac{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}{\alpha^{-1}(\varrho^A_{\alpha,\beta}[F]\beta(\lambda_n))}\right)^{\omega_j} \le \prod_{j=1}^m \varlimsup_{n\to\infty} \left(\frac{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}{\alpha^{-1}(\varrho^A_{\alpha,\beta}[F]\beta(\lambda_n))}\right)^{\omega_j} = \prod_{j=1}^m T^{A*}_{\alpha,\beta}[F_j]^{\omega_j}.
\end{aligned}$$

The proof of Theorem 3 is complete. □

It is easy to verify that the functions $\alpha(x) = \ln x$ and $\beta(x) = \ln\ln x$ for $x \ge x_0$ satisfy the conditions of Theorem 3. Therefore, the following statement is true.

Corollary 2.1. *Let Diriclet series* (4) *be such that for all* $1 \le j \le m$

$$\lim_{\sigma\uparrow A} \frac{\ln\ln M(\sigma, F_j)}{\ln\ln(1/(A-\sigma))} = \varrho, \quad \lim_{\sigma\uparrow A} \frac{\ln M(\sigma, F_j))}{\ln^{\varrho}(1/(A-\sigma))} = T_j,$$

and $\ln n = O(\ln\ln \lambda_n)$ *as* $n \to \infty$. *Then the conditions* (8) *and* (9) *imply*

$$\lim_{\sigma\uparrow A} \frac{\ln\ln M(\sigma, F)}{\ln\ln(1/(A-\sigma))} = \varrho, \quad \lim_{\sigma\uparrow A} \frac{\ln M(\sigma, F)}{\ln^{\varrho}(1/(A-\sigma))} \le \prod_{j=1}^m T_j^{\omega_j}.$$

Since $\varrho^A_{\alpha,\beta}[F] = \varlimsup\limits_{\sigma\uparrow A} \dfrac{\ln\exp\{\alpha(\ln M(\sigma, F)\}}{\ln\exp\{\beta(1/(A-\sigma)\}}$, we define the type also by the formula

$$T^A_{\alpha,\beta}[F] = \varlimsup_{\sigma\uparrow A} \frac{\exp\{\alpha(\ln M(\sigma, F)\}}{\exp\{\varrho^A_{\alpha,\beta}[F]\beta(1/(A-\sigma)\}},$$

and for the finding by the coefficients we use Lemma 1.1. We obtain the following statement.

Lemma 2.3. *Suppose that the function $e^{\alpha(x)}$ and $e^{\beta(x)}$ belongs to L_{si} and*

$$\frac{x}{\beta^{-1}(\ln c+\alpha(x))}\uparrow+\infty,\quad \exp\left\{\alpha\left(\frac{x}{\beta^{-1}(\ln c+\alpha(x))}\right)\right\}=(1+o(1))e^{\alpha(x)} \tag{10}$$

as $x\to+\infty$ for each $c\in(0,+\infty)$. If $\exp\{\alpha(\lambda_n)\}=o\left(\exp\left\{\beta\left(\lambda_n/\ln n\right)\right\}\right)$ as $n\to\infty$ then

$$T^A_{\alpha,\beta}[F]=\varlimsup_{n\to\infty}\frac{\exp\{\alpha(\lambda_n)\}}{\exp\left\{\varrho^A_{\alpha,\beta}[F]\beta\left(\frac{\lambda_n}{\ln\left(|a_n|e^{A\lambda_n}\right)}\right)\right\}}.$$

Theorem 4. *Let the function $e^{\alpha(x)}$ and $e^{\beta(x)}$ belongs to L_{si}, the conditions* (2) *and* (10) *hold and $\alpha(\lambda_n)=o\left(\beta\left(\lambda_n/\ln n\right)\right)$ as $n\to\infty$. Suppose that all Dirichlet series* (4) *have the same generalized order $\varrho^A_{\alpha,\beta}[F_j]=\varrho\in(0,+\infty)$ and the types $T^A_{\alpha,\beta}[F_j]\in(0,+\infty)$. Suppose also that $a_{n,1}\neq 0$ for all $n\geq n_0$ and for all $2\leq j\leq m$*

$$\beta\left(\frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)\leq(1+o(1))\beta\left(\frac{\lambda_n}{\ln\left(|a_{n,1}|e^{A\lambda_n}\right)}\right),\quad n\to\infty. \tag{11}$$

If $\omega_j>0$, $1\leq j\leq m$, $\sum\limits_{j=1}^m\omega_j=1$ and

$$\exp\left\{\beta\left(\frac{\lambda_n}{\ln\left(|a_n|e^{A\lambda_n}\right)}\right)\right\}=(1+o(1))\prod_{j=1}^m\exp\left\{\omega_j\beta\left(\frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)\right\} \tag{12}$$

as $n\to\infty$ then Dirichlet series (1) *has the generalized order $\varrho^A_{\alpha,\beta}[F]=\varrho$ and type*

$$T^A_{\alpha,\beta}[F]\leq\prod_{j=1}^m T^A_{\alpha,\beta}[F_j]^{\omega_j}.$$

Proof. From (12) we have

$$\beta\left(\frac{\lambda_n}{\ln\left(|a_n|e^{A\lambda_n}\right)}\right)=\sum_{j=1}^m\omega_j\beta\left(\frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)+o(1) \tag{13}$$

as $n\to\infty$. Therefore, by Lemma 1.1

$$\frac{1}{\varrho^A_{\alpha,\beta}[F]}=\varliminf_{n\to\infty}\frac{1}{\alpha(\lambda_n)}\beta\left(\frac{\lambda_n}{\ln\left(|a_n|e^{A\lambda_n}\right)}\right)\geq\sum_{j=1}^m\varliminf_{n\to\infty}\frac{\omega_j}{\alpha(\omega_n)}\beta\left(\frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)=\frac{1}{\varrho}.$$

On the other hand, in view of (11) from (13) we obtain

$$\frac{1}{\varrho^A_{\alpha,\beta}[F]}\leq\varliminf_{n\to\infty}\sum_{j=1}^m\frac{\omega_j}{\alpha(\lambda_n)}\beta\left(\frac{\lambda_n}{\ln\left(|a_{n,1}|e^{A\lambda_n}\right)}\right)=\frac{1}{\varrho},$$

that is $\varrho^A_{\alpha,\beta}[F]=\varrho$. From (12) and Lemma 2.3 also it follows that

$$\begin{aligned}\frac{1}{T^A_{\alpha,\beta}[F]}&=\varliminf_{n\to\infty}\frac{1}{\exp\{\alpha(\lambda_n)\}}\exp\left\{\varrho\beta\left(\frac{\lambda_n}{\ln\left(|a_n|e^{A\lambda_n}\right)}\right)\right\}\\&=\varliminf_{n\to\infty}\frac{1}{\exp\{\alpha(\lambda_n)\}}\prod_{j=1}^m\exp\left\{\varrho\omega_j\beta\left(\frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)\right\}\\&\geq\prod_{j=1}^m\varliminf_{n\to\infty}\left(\frac{\exp\left\{\varrho\beta\left(\frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)\right\}}{\exp\{\alpha(\lambda_n)\}}\right)^{\omega_j}=\prod_{j=1}^m\left(\frac{1}{T^A_{\alpha,\beta}[F_j]}\right)^{\omega_j}.\end{aligned}$$

Theorem 4 is proved. $\square$

It is easy to verify that the functions $\alpha(x) = \ln\ln x$ and $\beta(x) = \ln\ln x$ for $x \geq x_0$ satisfy the conditions (2) and (10). The condition $\alpha(\lambda_n) = o\left(\beta\left(\lambda_n / \ln n\right)\right)$ as $n \to \infty$ holds, provided $\varlimsup\limits_{n\to\infty} (\ln\ln n)/\ln\lambda_n < 1$. Therefore, Theorem 4 implies the following statement.

Corollary 2.2. *Let* $\varlimsup\limits_{n\to\infty} (\ln\ln n)/\ln\lambda_n < 1$ *and for all* $1 \leq j \leq m$

$$\varlimsup_{\sigma\uparrow A} \frac{\ln\ln\ln\ln M(\sigma, F_j))}{\ln\ln(1/(A-\sigma))} = \varrho, \quad \varlimsup_{\sigma\uparrow A} \frac{\ln\ln\ln M(\sigma, F_j))}{\ln^{\varrho}(1/(A-\sigma))} = T_j \in (0, +\infty).$$

Suppose that $a_{n,1} \neq 0$ *for all* $n \geq n_0$ *and for all* $2 \leq j \leq m$

$$\ln\ln \frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)} \leq (1+o(1)) \ln\ln \frac{\lambda_n}{\ln\left(|a_{n,1}|e^{A\lambda_n}\right)}, \quad n \to \infty.$$

If $\omega_j > 0, 1 \leq j \leq m, \sum\limits_{j=1}^{m} \omega_j = 1$ *and*

$$\ln \frac{\lambda_n}{\ln\left(|a_n|e^{A\lambda_n}\right)} = (1+o(1)) \prod_{j=1}^{m} \left(\ln \frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)^{\omega_j}$$

as $n \to \infty$ *then*

$$\varlimsup_{\sigma\uparrow A} \frac{\ln\ln\ln\ln M(\sigma, F)}{\ln\ln(1/(A-\sigma))} = \varrho, \quad \varlimsup_{\sigma\uparrow A} \frac{\ln\ln\ln M(\sigma, F)}{\ln^{\varrho}(1/(A-\sigma))} \leq \prod_{j=1}^{m} T_j^{\omega_j}.$$

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У термінах узагальнених порядків досліджено зв'язок між зростанням ряду Діріхле $F(s) = \sum\limits_{n=1}^{\infty} a_n \exp\{s\lambda_n\}$ з абсцисою абсолютної збіжності $A \in (-\infty, +\infty)$ і зростанням рядів Діріхле $F_j(s) = \sum\limits_{n=1}^{\infty} a_{n,j} \exp\{s\lambda_n\}$, $1 \leq j \leq 2$, з такою ж абсцисою абсолютної збіжності, якщо, наприклад, коефіцієнти a_n повязані з коефіцієнтами $a_{n,j}$ співвідношенням

$$\beta\left(\frac{\lambda_n}{\ln\left(|a_n|e^{A\lambda_n}\right)}\right) = (1+o(1)) \prod_{j=1}^{m} \beta\left(\frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)^{\omega_j}, \quad n \to \infty,$$

де $\omega_j > 0, 1 \leq j \leq m, \sum\limits_{j=1}^{m} \omega_j = 1$.

Ключові слова і фрази: ряд Діріхле, узагальнений порядок.

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LITOVCHENKO V.A., UNGURYAN G.M.

PARABOLIC SYSTEMS OF SHILOV-TYPE WITH COEFFICIENTS OF BOUNDED SMOOTHNESS AND NONNEGATIVE GENUS

The Shilov-type parabolic systems are parabolically stable systems for changing its coefficients unlike of parabolic systems by Petrovskii. That's why the modern theory of the Cauchy problem for class by Shilov-type systems is developing abreast how the theory of the systems with constant or time-dependent coefficients alone. Building the theory of the Cauchy problem for systems with variable coefficients is actually today. A new class of linear parabolic systems with partial derivatives to the first order by the time t with variable coefficients that includes a class of the Shilov-type systems with time-dependent coefficients and non-negative genus is considered in this work. A main part of differential expression concerning space variable x of each such system is parabolic (by Shilov) expression. Coefficients of this expression are time-dependent, but coefficients of a group of younger members may depend also a space variable. We built the fundamental solution of the Cauchy problem for systems from this class by the method of sequential approximations. Conditions of minimal smoothness on coefficients of the systems by variable x are founded, the smoothness of solution is investigated and estimates of derivatives of this solution are obtained. These results are important for investigating of the correct solution of the Cauchy problem for this systems in different functional spaces, obtaining forms of description of the solution of this problem and its properties.

Key words and phrases: fundamental matrix of solutions, Cauchy problem, Shilov-type parabolic systems.

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INTRODUCTION

The definition of parabolicity formulated by G.Ye. Shilov [12] generalizes the definition of parabolicity by I.G. Petrovskii [11] and extends considerably the Petrovskii's class of the first-order on time systems by the systems with constant coefficients with order different form the parabolicity factor. The parabolic (by Shilov) systems were investigated, in part, in papers [2, 4, 6, 7] containing the results on description of the classes of uniqueness and correctness of the Cauchy problem, developing the methods of study of fundamental solution, rating the correct solvability of the Cauchy problem at various functional spaces, and ascertaining qualitative properties of solutions for such systems. However, these results concern to the systems with constant or time-dependent coefficients alone. The attempts to derive any results for parabolic (by Shilov) systems with variable coefficients, which are space-dependent ones, were unsuccessful, while it has been shown [5] that such systems are parabolically unstable to changing coefficients.

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Interesting approach to expansion of the Shilov class of parabolic systems has been proposed by Ya.I. Zhitomirskii [13] defining a new class of parabolically stable systems to variations of the lower coefficients. This class adds naturally to the Petrovskii's class of systems with variable coefficients and covers the parabolic (by Shilov) systems. These systems are referred to as the Shilov-type parabolic systems with variable coefficients.

The Shilov-type parabolic systems of the p-th order are of the form

$$\partial_t u(t;x) = \{P_0(t;i\partial_x) + P_1(t,x;i\partial_x)\}u(t;x), \quad t \in (0;T],\ x \in \mathbb{R}^n, \tag{1}$$

where $u := \mathrm{col}(u_1, \ldots, u_m)$, $P_0(t;i\partial_x)$ and $P_1(t,x;i\partial_x)$ are the matrix differential expressions of the orders p and p_1, respectively, with coefficients dependent on time t, and for P_1 on spatial variable x as well. For that, the system

$$\partial_t u(t;x) = P_0(t;i\partial_x)u(t;x), \quad t \in (0;T],\ x \in \mathbb{R}^n, \tag{2}$$

is the parabolical (by Shilov) system with the parabolicity factor h, $0 < h \le p$, kind of μ and of reduced order p_0 (see [4, p.72, p.133]), and p_1 satisfies the following conditions:

$$(A) \qquad 0 \le p_1 < h - n\Big(1 - h\mu/p_0\Big) - (m-1)(p-h), \quad 0 \le \mu;$$
$$(\widehat{A}) \qquad 0 \le p_1 < h - n(1-\mu) - (m-1)(p-h), \quad \mu < 0.$$

For the systems (1) Ya.I. Zhitomirskii has ascertained by the method of sequential approximations correct solvability of the Cauchy problem at the class of smooth bounded initial functions for the case, when the coefficients of the differential expression for P_0 are constant, and the coefficients of the expression P_1 are limited being dependent on x, alone functions, which are differentiable up to some order.

Further elaboration of the Cauchy problem for the Shilov-type parabolic systems with variable coefficients presumed construction of the fundamental solution of the Cauchy problem (FSCP) and comprehensive investigation of it.

For the systems (1) of nonnegative kind μ and the coefficients, which are boundedly continuous on t and infinitely differentiable on x, the FSCP has been derived and its main properties have been studied [8]. These results enable to develop the theory of the Cauchy problem [1, 9, 10] for such systems at spaces S of I.M. Gelfand and G.Ye. Shilov and, in part, to prove correct solvability of the Cauchy problem with generalized initial conditions of kind of the Gevrey's ultra-distributions, to find out the form of classical solutions with generalized boundary values at initial hyperplane, to study the properties of localization and stabilization of the solutions, and to describe the sets of generalized initial functions for which the corresponding solutions are the elements of the L. Swartz space S or any of spaces of I.M. Gelfand and G.Ye. Shilov.

In this paper, we continue the study of the systems (1) for $\mu \ge 0$ with coefficients of bounded smoothness. We determine the conditions of minimal smoothness of the coefficients with respect to the variable x, for which the classical FSCP exists, construct this solution and investigate its main properties. These results are important for further development of the classical theory of the Cauchy problem for parabolic systems and its unification.

1 Auxiliary data

Let T be a fixed number from $(0;+\infty)$, $\mathbb{N}$ be the set of natural numbers; $\mathbb{N}_m := \{1, \ldots, m\}$; $\mathbb{R}^n$ be the real n-dimension space; $\mathbb{R} := \mathbb{R}^1$; $\mathbb{Z}_+^n$ be the set of all n-dimension multi-indices,

$\mathbb{Z}_+ := \mathbb{Z}^1_+$; i – imaginary unit; $(\cdot,\cdot)$ – scalar product at the space $\mathbb{R}^n$; $\|x\| := (x,x)^{1/2}$, $x \in \mathbb{R}^n$; $|x+iy| := (x^2+y^2)^{1/2}$, $\{x,y\} \subset \mathbb{R}$; $|(a_{lj})^m_{l,j=1}| := \max\limits_{\{l,j\}\subset \mathbb{N}_m} |a_{lj}|$; $|z|_+ := |z_1| + \ldots + |z_n|$, $z^l := z_1^{l_1}\ldots z_n^{l_n}$, if $z \in \mathbb{R}^n$, $l \in \mathbb{Z}^n_+$; $\Pi_M := \{(t;x)|\ t \in M, x \in \mathbb{R}^n\}$, $M \subset \mathbb{R}$.

We will consider here only the systems (1) with $\mu \geq 0$, where the differential expressions for P_0 and P_1 are of the form

$$P_0(t; i\partial_x) = \sum_{|k|_+ \leq p} A_{0,k}(t)\partial_x^k,\ \ P_1(t,x; i\partial_x) = \sum_{|k|_+\leq p_1} A_{1,k}(t;x)\partial_x^k,$$

where $A_{0,k}(t) := i^{|k|_+}\left(a^{lj}_{0,k}(t)\right)^m_{l,j=1}$, $A_{1,k}(t;x) := i^{|k|_+}\left(a^{lj}_{1,k}(t;x)\right)^m_{l,j=1}$ are matrix coefficients.

By $G(t,\tau;\cdot)$, $0 \leq \tau < t \leq T$, we denote FSCP of system (2). It is known that $G(t,\tau;\cdot) = F\left[\Theta^t_\tau(\xi)\right](t,\tau;\cdot)$, where $F[\cdot]$ is the Fourier transformation operator, and $\Theta^t_\tau(\cdot)$ is a matriciant of the corresponding Fourier duality of the system. The following statement is proper [1,6].

Proposition 1.1. *For all $T > 0$ there exists $\delta > 0$ and for all $k \in \mathbb{Z}^n_+$ there exists $c > 0$ such that for all $t \in (\tau;T]$, $\tau \in [0;T)$ and $\{x,\xi\} \subset \mathbb{R}^n$ takes place*

$$|\partial_x^k G(t,\tau;x-\xi)| \leq c(t-\tau)^{-\frac{n+|k|_++\gamma}{h}} e^{-\delta\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}, \tag{3}$$

where $\gamma := (m-1)(p-h)$ and $\alpha := \mu/p_0$.

Here, we consider systems (1), which satisfy, in addition to condition (A), the following condition:

(B) the coefficients $a^{lj}_{0,k}(t)$, $a^{lj}_{1,k}(t;x)$ are continuous in the variable t uniformly with respect to x, differentiable with respect to the variable x up to the order α_* inclusively, and bounded together with their derivatives by complex-valued functions in a ball $\Pi_{[0;T]}$.

In [8], FSCP of system (1) was constructed in the form

$$Z(t,x;\tau,\xi) = G(t,\tau;x-\xi) + W(t,x;\tau,\xi),\ \ (t,x;\tau,\xi) \in \Pi^2_T, \tag{4}$$

where $\Pi^2_T := \{(t,x;\tau,\xi)|\ 0 \leq \tau < t \leq T, \{x,\xi\} \subset \mathbb{R}^n\}$ and

$$W(t,x;\tau,\xi) := \int\limits_\tau^t d\beta \int\limits_{\mathbb{R}^n} G(t,\beta;x-y)\Phi(\beta,y;\tau,\xi)dy. \tag{5}$$

Here

$$\Phi(t,x;\tau;\xi) = \sum_{l=1}^\infty K_l(t,x;\tau,\xi), \tag{6}$$

where

$$\begin{aligned} &K_1(t,x;\tau,\xi) := P_1(t,x;i\partial_x)G(\tau,t;x-\xi),\\ &K_l(t,x;\tau,\xi) := \int\limits_\tau^t d\beta \int\limits_{\mathbb{R}^n} K_1(t,x;\beta,y)K_{l-1}(\beta,y;\tau,\xi)dy,\ l>1. \end{aligned} \tag{7}$$

In this case, it was established that condition (A) and the boundedness of the coefficients of system (1) ensure the absolute uniform convergence of the functional series (6) for all $\{x,\xi\} \subset$

$\mathbb{R}^n$, $t \in (\tau; T]$, and $\tau \in [0, T)$. Moreover, its sum Φ and the iterated kernels K_l satisfy the estimates

$$|\Phi(t, x; \tau, \xi)| \leq c_1 (t-\tau)^{\alpha_0 - (1+\alpha n)} e^{-\delta_1 \left(\frac{\|x-\xi\|}{(t-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}}}, \tag{8}$$

$$\begin{aligned} |K_l(t, x; \tau, \xi)| \leq c_0^l & \left(\prod_{j=1}^{l-1} c_{(j\varepsilon)} B(\alpha_0, j\alpha_0) \right) \\ & \times (t-\tau)^{l\alpha_0 - (1+\alpha n)} e^{-\delta(1-(l-1)\varepsilon) \left(\frac{\|x-\xi\|}{(t-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}}}, \ \varepsilon \in (0; 1), \end{aligned} \tag{9}$$

with the estimating constants independent of t, τ, x, and ξ. Here

$$\alpha_0 := 1 + \alpha n - (n + p_1 + \gamma)/h > 0$$

and $B(\cdot, \cdot)$ is the Euler beta-function.

We note that estimates (3) and (8) for $\{x, \xi\} \subset \mathbb{R}^n$ and $0 \leq \tau < t \leq T$ guarantee the absolute convergence of the integral, by which the potential W is determined. Thus, the matrix function $Z(t, x; \tau, \xi)$ is properly determined by formula (4) on the whole set Π_T^2.

Completing this item, we present the following estimates from [3], which will be of importance in what follows:

$$e^{-\delta \left\{ \left(\frac{\|x-y\|}{(t-\beta)^\alpha} \right)^{\frac{1}{1-\alpha}} + \left(\frac{\|y-\xi\|}{(\beta-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}} \right\}} \leq e^{-\delta \left(\frac{\|x-\xi\|}{(t-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}}}; \tag{10}$$

$$\int_{\mathbb{R}^n} e^{-\delta \left\{ \left(\frac{\|x-y\|}{(t-\beta)^\alpha} \right)^{\frac{1}{1-\alpha}} + \left(\frac{\|y-\xi\|}{(\beta-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}} \right\}} \frac{dy}{((t-\beta)(\beta-\tau))^{\alpha n}} \leq \frac{c_\varepsilon e^{-\delta(1-\varepsilon) \left(\frac{\|x-\xi\|}{(t-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}}}}{(t-\tau)^{\alpha n}}, \quad \delta > 0, \tag{11}$$

(here, $\{x, y, \xi\} \subset \mathbb{R}^n$, $\beta \in (\tau; t)$, $0 \leq \tau < t \leq T$, $\varepsilon \in (0; 1)$, and $\delta > 0$, and the quantity c_ε depends only on ε).

2 Properties of FSPC

First, we estimate the derivatives of the iterated kernels K_l.

According to representation (7), the smoothness of the kernel $K_1(t, x; \tau, \xi)$ in the spatial variables x and ξ is determined, respectively, by the smoothness of the coefficients of system (1) and the function $G(t, \tau; x - \xi)$. Therefore, there exist the derivatives $\partial_\xi^r \partial_x^q K_1$ for $\{r, q\} \subset \mathbb{Z}_+^n$, $|q|_+ \leq \alpha_*$, and the following equality holds:

$$\partial_\xi^r \partial_x^q K_1(t, x; \tau, \xi) = \sum_{|k|_+ \leq p_1} \sum_{l=0}^{q} C_q^l \left(\partial_x^l A_{1,k}(t; x) \right) \left(\partial_{(x-\xi)}^{k+r+q-l} G(t, \tau; x-\xi) \right),$$

where C_q^l is a binomial coefficient. From whence, with regard for condition (B) and estimate (3) for $\{r, q\} \subset \mathbb{Z}_+^n$, $|q|_+ \leq \alpha_*$, $(t, x; \tau, \xi) \in \Pi_T^2$, we get

$$|\partial_\xi^r \partial_x^q K_1(t, x; \tau, \xi)| \leq c_{r,q} (t-\tau)^{-\frac{n+p_1+\gamma+|r+q|_+}{h}} e^{-\delta \left(\frac{\|x-\xi\|}{(t-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}}} \tag{12}$$

(here, the estimating constants are independent of t, τ, x, and ξ).

For $l > 1$, we will use the representation

$$\begin{aligned} K_l(t,x;\tau,\xi) &= \int_\tau^{t_1} d\beta \int_{\mathbb{R}^n} K_1(t,x;\beta,\eta+\xi)K_{l-1}(\beta,\eta+\xi;\tau,\xi)d\eta \\ &+ \int_{t_1}^{t} d\beta \int_{\mathbb{R}^n} K_1(t,x;\beta,x-z)K_{l-1}(\beta,x-z;\tau,\xi)dz, \ t_1 := \frac{t+\tau}{2}. \end{aligned} \tag{13}$$

According to it,

$$\begin{aligned} \partial_\xi^r \partial_x^q K_l(t,x;\tau,\xi) &= \sum_{|r_1|_+ \le |r|_+} C_r^{r_1} \int_\tau^{t_1} d\beta \int_{\mathbb{R}^n} \left(\partial_\xi^{r_1}\partial_x^q K_1(t,x;\beta,\eta+\xi)\right) \\ &\times \left(\partial_\xi^{r-r_1} K_{l-1}(\beta,\eta+\xi;\tau,\xi)\right) d\eta + \sum_{|q_1|_+ \le |q|_+} C_q^{q_1} \int_{t_1}^{t} d\beta \int_{\mathbb{R}^n} \left(\partial_x^{q_1} K_1(t,x;\beta,x-z)\right) \\ &\times \left(\partial_\xi^r \partial_x^{q-q_1} K_{l-1}(\beta,x-z;\tau,\xi)\right) dz, \ |q|_+ \le \alpha_*, (t,x;\tau,\xi) \in \Pi_T^2. \end{aligned} \tag{14}$$

Hence, the estimation of $|\partial_\xi^r \partial_x^q K_l(t,x;\tau,\xi)|$ is reduced to that of the expressions $|\partial_\xi^r \partial_x^q K_1(t,x;\tau,\eta+\xi)|$, $|\partial_x^q K_1(t,x;\tau,x-z)|$, $|\partial_\xi^r K_{l-1}(t,\eta+\xi;\tau,\xi)|$, $|\partial_\xi^r \partial_x^q K_{l-1}(t,x-z;\tau,\xi)|$.

In view of the boundedness of $\partial_x^q a_{1,k}^{lj}(t;x)$, $|q|_+ \le \alpha_*$, and estimate (3), for all $\{q,r\} \in \mathbb{Z}_+^n$, $|q|_+ \le \alpha_*$, $\{x,\eta,\xi\} \in \mathbb{R}^n$, $t \in (\tau;T]$, and $\tau \in [0;T)$, we have

$$\begin{aligned} \left|\partial_\xi^r \partial_x^q K_1(t,x;\tau,\eta+\xi)\right| &\le m \sum_{|k|_+ \le p_1} \sum_{|q_1|_+ \le |q|_+} C_q^{q_1} \left|\partial_x^{q_1} A_{1,k}(t;x)\right| \left|\partial_{(x-\eta-\xi)}^{k+r+q-q_1} G(t,\tau;x-\eta-\xi)\right| \\ &\le c_{r,q}(t-\tau)^{-\frac{n+p_1+\gamma+|r+q|_+}{h}} e^{-\delta\left(\frac{\|x-\eta-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}; \end{aligned} \tag{15}$$

$$\begin{aligned} \left|\partial_x^q K_1(t,x;\tau,x-\xi)\right| &= \left|\partial_x^q\Big(\sum_{|k|_+ \le p_1} A_{1,k}(t;x)\partial_x^k G(t,\tau;\xi)\Big)\right| \le m\left|\partial_x^q A_{1,0}(t;x)\right| \left|G(t,\tau;\xi)\right| \\ &\le \widehat{c}_q(t-\tau)^{-\frac{n+\gamma}{h}} e^{-\delta\left(\frac{\|\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}} \le c_q(t-\tau)^{-\frac{n+p_1+\gamma}{h}} e^{-\delta\left(\frac{\|\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}. \end{aligned} \tag{16}$$

We now estimate the expression $\left|\partial_\xi^r K_l(t,\eta+\xi;\tau,\xi)\right|$. Since

$$\partial_\xi^r K_1(t,\eta+\xi;\tau,\xi) = \sum_{|k|_+ \le p_1} \partial_\xi^r A_{1,k}(t;\eta+\xi)\partial_\eta^k G(t,\tau;\eta), \ (t,x;\tau,\xi) \in \Pi_T^2, \tag{17}$$

we have, according to condition (B), that the iterated kernels $K_l(t,\eta+\xi;\tau,\xi)$ are differentiable with respect to the variable ξ only to the order α_*. This fact and (14) imply that $\partial_x^q K_l(t,x;\tau,\xi)$, $|q|_* \le \alpha_*$, is also a function differentiable with respect to ξ only to this order α_*.

Representation (17) and estimate (3) yield

$$\left|\partial_\xi^r K_1(t,\eta+\xi;\tau,\xi)\right| \le c_{1,r}(t-\tau)^{-\frac{n+p_1+\gamma}{h}} e^{-\delta\left(\frac{\|\eta\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}. \tag{18}$$

We note that

$$\partial_\xi^r K_2(t,\eta+\xi;\tau,\xi) = \partial_\xi^r\Bigg(\int_\tau^t d\beta \int_{\mathbb{R}^n} K_1(t,\eta+\xi;\beta,y)K_1(\beta,y;\tau,\xi)dy\Bigg).$$

Let us change the order of integration in the last integral by the formula $y = z+\xi$. In view of estimates (18) and (11) and the equalities

$$\int_\tau^t ((t-\beta)(\beta-\tau))^{\alpha_0-1}d\beta = (t-\tau)^{2\alpha_0-1}B(\alpha_0,\alpha_0) \tag{19}$$

and

$$\partial_\xi^r K_1(t,\eta+\xi;\tau,z+\xi) = \partial_\zeta^r K_1(t,(\eta-z)+\zeta;\tau,\zeta)\Big|_{\zeta=z+\xi},$$

we get

$$\begin{aligned}
&\big|\partial_\xi^r K_2(t,\eta+\xi;\tau,\xi)\big| \\
&\quad\le m\sum_{|r_1|_+\le|r|_+} C_r^{r_1}\int_\tau^t d\beta\int_{\mathbb{R}^n}\Big|\partial_\xi^{r_1}K_1(t,\eta+\xi;\beta,z+\xi)\Big|\,\Big|\partial_\xi^{r-r_1}K_1(\beta,z+\xi;\tau,\xi)\Big|dz \\
&\quad\le m\sum_{|r_1|_+\le|r|_+} C_r^{r_1}c_{1,r_1}c_{1,(r-r_1)}\int_\tau^t \big((t-\beta)(\beta-\tau)\big)^{-\frac{n+p_1+\gamma}{h}}\int_{\mathbb{R}^n} e^{-\delta\Big(\big(\frac{\|\eta-z\|}{(t-\beta)^\alpha}\big)^{\frac{1}{1-\alpha}}+\big(\frac{\|z\|}{(\beta-\tau)^\alpha}\big)^{\frac{1}{1-\alpha}}\Big)}dzd\beta \\
&\quad\le c_{2,r}(\varepsilon)B(\alpha_0,\alpha_0)(t-\tau)^{\alpha_0-\frac{n+p_1+\gamma}{h}}e^{-\delta(1-\varepsilon)\big(\frac{\|\eta\|}{(t-\tau)^\alpha}\big)^{\frac{1}{1-\alpha}}},\ \ \varepsilon\in(0;1).
\end{aligned} \tag{20}$$

By reasoning analogously step by step, we arrive at the inequality

$$\big|\partial_\xi^r K_l(t,\eta+\xi;\tau,\xi)\big| \le c_{l,r}(\varepsilon)\Big(\prod_{j=1}^{l-1}B(\alpha_0,j\alpha_0)\Big)(t-\tau)^{(l-1)\alpha_0-\frac{n+p_1+\gamma}{h}}e^{-\delta(1-(l-1)\varepsilon)\big(\frac{\|\eta\|}{(t-\tau)^\alpha}\big)^{\frac{1}{1-\alpha}}}, \tag{21}$$

which is satisfied for all $\{\eta,\xi\}\subset\mathbb{R}^n$, $|r|_+\le\alpha_*$, $0\le\tau<t\le T$, $\varepsilon\in(0;1)$, and $l\in\mathbb{N}\setminus\{1\}$ and, hence, until the existence of such number l_*, for which

$$\big|\partial_\xi^r K_{l_*}(t,\eta+\xi;\tau,\xi)\big| \le c_{l_*,r}(\varepsilon)\Big(\prod_{j=1}^{l_*-1}B(\alpha_0,j\alpha_0)\Big)e^{-\delta(1-(l_*-1)\varepsilon)\big(\frac{\|\eta\|}{(t-\tau)^\alpha}\big)^{\frac{1}{1-\alpha}}} \tag{22}$$

(here, the quantities $c_{l,r}(\varepsilon)>0$ do not depend on the variables t,τ,η, and ξ, which vary in the above-indicated way).

Since

$$\partial_\xi^r\partial_x^q K_l(t,x;\tau,\eta+\xi) = \partial_\zeta^r\partial_x^q K_l(t,x;\tau,\zeta)\Big|_{\zeta=\eta+\xi}$$

and

$$\partial_\xi^r\partial_x^q K_l(t,x-z;\tau,\xi) = \partial_\xi^r\partial_y^q K_l(t,y;\tau,\xi)\Big|_{y=x-z}$$

then the expressions $\partial_\xi^r\partial_x^q K_l(t,x;\tau,\eta+\xi)$, $\partial_\xi^r\partial_x^q K_l(t,x-z;\tau,\xi)$ and $\partial_\xi^r\partial_x^q K_l(t,x;\tau,\xi)$ are of the same type. Therefore, with regard for representation (14) and estimates (15), (16), (21), and (11), we have

$$\begin{aligned}
&\left|\partial_\xi^r\partial_x^q K_2(t,x;\tau,\xi)\right| \le m2^{|r+q|_+}\Bigg(\sum_{|r_1|_+\le|r|_+} c_{r_1,q}c_{1,(r-r_1)}\int\limits_\tau^{t_1}(t-\beta)^{-\frac{n+p_1+\gamma+|r_1+q|_+}{h}}\\
&\times(\beta-\tau)^{-\frac{n+p_1+\gamma}{h}}\int\limits_{\mathbb{R}^n} e^{-\delta\left(\left(\frac{\|x-\eta-\xi\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}}+\left(\frac{\|\eta\|}{(\beta-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}\right)}d\eta d\beta+\sum_{|q_1|_+\le|q|_+} c_{q_1}c_{r,(q-q_1)}\\
&\times\int\limits_{t_1}^{t}(\beta-\tau)^{-\frac{n+p_1+\gamma+|r+q-q_1|_+}{h}}(t-\beta)^{-\frac{n+p_1+\gamma}{h}}\int\limits_{\mathbb{R}^n} e^{-\delta\left(\left(\frac{\|z\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}}+\left(\frac{\|x-z-\xi\|}{(\beta-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}\right)}dzd\beta\Bigg)\\
&\le m2^{|r+q|_+}c_\varepsilon e^{-\delta(1-\varepsilon)\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}(t-\tau)^{-\alpha n}\Bigg(\sum_{|r_1|_+\le|r|_+} c_{r_1,q}c_{1,(r-r_1)}\\
&\times\int\limits_\tau^{t_1}(t-\beta)^{\alpha n-\frac{n+p_1+\gamma+|r_1+q|_+}{h}}(\beta-\tau)^{\alpha_0-1}d\beta+\sum_{|q_1|_+\le|q|_+} c_{q_1}c_{r,(q-q_1)}\\
&\times\int\limits_{t_1}^{t}(t-\beta)^{\alpha_0-1}(\beta-\tau)^{\alpha n-\frac{n+p_1+\gamma+|r+q-q_1|_+}{h}}d\beta\Bigg),\quad |r|_+\le\alpha_*,\, |q|_+\le\alpha_*,\, \varepsilon\in(0;1).
\end{aligned}\tag{23}$$

In view of the estimates

$$\int\limits_\tau^{t_1}(t-\beta)^{\alpha n-\frac{n+p_1+\gamma+|r_1+q|_+}{h}}(\beta-\tau)^{\alpha_0-1}d\beta\le 2^{\frac{|r_1+q|_+}{h}}(t-\tau)^{2\alpha_0-\left(1+\frac{|r_1+q|_+}{h}\right)}B(\alpha_0,\alpha_0)$$

and

$$\int\limits_{t_1}^{t}(t-\beta)^{\alpha_0-1}(\beta-\tau)^{\alpha n-\frac{n+p_1+\gamma+|r+q-q_1|_+}{h}}d\beta\le 2^{\frac{|r+q-q_1|_+}{h}}(t-\tau)^{2\alpha_0-\left(1+\frac{|r+q-q_1|_+}{h}\right)}B(\alpha_0,\alpha_0),$$

we get the inequality

$$\left|\partial_\xi^r\partial_x^q K_2(t,x;\tau,\xi)\right|\le c_{2,\xi}^{r,q}(t-\tau)^{2\alpha_0-\left(1+\alpha n+\frac{|r+q|_+}{h}\right)}B(\alpha_0,\alpha_0)e^{-\delta(1-\varepsilon)\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}.$$

By continuing stepwise the process of estimation, we obtain

$$\left|\partial_\xi^r\partial_x^q K_l(t,x;\tau,\xi)\right|\le c_{l,\varepsilon}^{r,q}(t-\tau)^{l\alpha_0-\left(1+\alpha n+\frac{|r+q|_+}{h}\right)},$$

$$\begin{aligned}
\left|\partial_\xi^r\partial_x^q K_l(t,x;\tau,\xi)\right|&\le c_{l,\varepsilon}^{r,q}(t-\tau)^{l\alpha_0-\left(1+\alpha n+\frac{|r+q|_+}{h}\right)}\\
&\le e^{-\delta(1-(l-1)\varepsilon)\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}\Big(\prod_{j=1}^{l-1}B(\alpha_0,j\alpha_0)\Big),
\end{aligned}\tag{24}$$

for all $|r|_+\le\alpha_*$, $|q|_+\le\alpha_*$, $\{x,\xi\}\subset\mathbb{R}^n$, $0\le\tau<t\le T$, $\varepsilon\in(0;1)$ and $l\in\mathbb{N}\backslash\{1\}$.

Let us pass to the estimation of the expression $\left|\partial_\xi^r \partial_x^q K_l(t,x;\tau,\xi)\right|$, which will be suitable for the establishment of the differentiability of the matrix function Φ with respect to the spatial variables. Directly from (24), we arrive at the existence of a number l^* such that

$$\left|\partial_\xi^r \partial_x^q K_{l^*}(t,x;\tau,\xi)\right| \le c_{l^*,\varepsilon}^{r,q} e^{-\delta(1-(l^*-1)\varepsilon)\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}} \Big(\prod_{j=1}^{l^*-1} B(\alpha_0, j\alpha_0) \Big).$$

Let us set $l_+ := \max\{l_*, l^*\}$, $l_- := \min\{l_*, l^*\}$, where l_* is the corresponding number from (22), $\varepsilon := \frac{1}{r_* l_+}$, $\delta_* := \delta\big(1 - \frac{1}{r_*}\big)$, $r_* > 2$, $T_0 := \max\{1, T\}$, and

$$c_*^0 := \max_{l \in \mathbb{N}_{l_+} \setminus \{1\}} \Big\{ c_{1,r}, c_{l,r}(\varepsilon) \Big(\prod_{j=1}^{l-1} B(\alpha_0, j\alpha_0) \Big), c_{r,q}, c_{l,\varepsilon}^{r,q} \Big(\prod_{j=1}^{l-1} B(\alpha_0, j\alpha_0) \Big) \Big\},$$

$c_* := c_*^0 (T_0)^{l_+ - l_-}$. Then (21) and (24) imply that, for all $\{x, \xi, \eta\} \subset \mathbb{R}^n$, $0 \le \tau < t \le T$, $|r|_+ \le \alpha_*$, and $|q|_+ \le \alpha_*$,

$$\left|\partial_\xi^r \partial_x^q K_{l_+}(t,x;\tau,\xi)\right| \le c_* e^{-\delta_* \left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}, \left|\partial_\xi^r K_{l_+}(t,\eta+\xi;\tau,\xi)\right| \le c_* e^{-\delta_* \left(\frac{\|\eta\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}.$$

In view of this result, estimate (10), the equality

$$\int_{\mathbb{R}^n} e^{-\delta_0 \left(\frac{\|x-y\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}}} \frac{dy}{(t-\beta)^{\alpha n}} = \int_{\mathbb{R}^n} e^{-\delta_0 \|z\|^{\frac{1}{1-\alpha}}} dz =: \widehat{E} < +\infty,$$

representation (14), and inequalities (15) and (16), we obtain

$$\begin{aligned}
&\left|\partial_\xi^r K_{l_+ + 1}(t,\eta+\xi;\tau,\xi)\right| \\
&\quad \le \sum_{|r_1|_+ \le |r|_+} C_r^{r_1} \int_\tau^t d\beta \int_{\mathbb{R}^n} \left|\partial_\xi^{r_1} K_1(t,\eta+\xi;\beta,z+\xi) \partial_\xi^{r-r_1} K_{l_+}(\beta,z+\xi;\tau,\xi)\right| dz \\
&\quad \le m c_*^2 \Big(\sum_{|r_1|_+ \le |r|_+} C_r^{r_1} \Big) \int_\tau^t (t-\beta)^{\alpha_0 - 1} \int_{\mathbb{R}^n} e^{-\delta_* \left(\left(\frac{\|\eta-z\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\|z\|}{(\beta-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}} \right)} e^{-\frac{\delta}{r_*} \left(\frac{\|\eta-z\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}}} \frac{dz}{(t-\beta)^{n\alpha}} d\beta \\
&\quad \le m c_r^0 \widehat{E} c_*^2 B(\alpha_0, 1) (t-\tau)^{\alpha_0} e^{-\delta_* \left(\frac{\|\eta\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}, \qquad c_r^0 := \sum_{|r_1|_+ \le |r|_+} C_r^{r_1};
\end{aligned} \tag{25}$$

$$
\begin{aligned}
&\left|\partial_\xi^r \partial_x^q K_{l_+ +1}(t,x;\tau,\xi)\right| \\
&\quad\le \sum_{|r_1|_+\le|r|_+} C_r^{r_1}\int_\tau^{t_1} d\beta \int_{\mathbb{R}^n} \left|\partial_\xi^{r_1}\partial_x^q K_1(t,x;\beta,\eta+\xi)\partial_\xi^{r-r_1}K_{l_+}(\beta,\eta+\xi;\tau,\xi)\right| d\eta \\
&\quad+ \sum_{|q_1|_+\le|q|_+} C_q^{q_1}\int_{t_1}^{t} d\beta \int_{\mathbb{R}^n} \left|\partial_x^{q_1} K_1(t,x;\beta,x-z)\partial_\xi^{r}\partial_x^{q-q_1}K_{l_+}(\beta,x-z;\tau,\xi)\right| dz \\
&\quad\le mc_*^2\Bigg(\sum_{|r_1|_+\le|r|_+} C_r^{r_1}\int_\tau^{t_1}(t-\beta)^{\alpha_0-\left(1+\frac{|r_1+q|_+}{h}\right)} \\
&\qquad\times\int_{\mathbb{R}^n} e^{-\delta_*\left(\left(\frac{\|x-\eta-\xi\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}}+\left(\frac{\|\eta\|}{(\beta-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}\right)} e^{-\frac{\delta}{r_*}\left(\frac{\|x-\eta-\xi\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}}}\frac{d\eta}{(t-\beta)^{\alpha n}}d\beta \\
&\quad+\sum_{|q_1|_+\le|q|_+} C_q^{q_1}\int_{t_1}^{t}(t-\beta)^{\alpha_0-1}\int_{\mathbb{R}^n} e^{-\delta_*\left(\left(\frac{\|z\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}}+\left(\frac{\|x-z-\xi\|}{(\beta-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}\right)} e^{-\frac{\delta}{r_*}\left(\frac{\|z\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}}}\frac{dz}{(t-\beta)^{\alpha n}}d\beta \\
&\quad\le mc_*^2\widehat{E}e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}\int_\tau^t (t-\beta)^{\alpha_0-1}d\beta\left(\left(\sum_{|r_1|_+\le|r|_+} C_r^{r_1}(t-t_1)^{-\frac{|r_1+q|_+}{h}}\right)+c_q\right) \\
&\quad\le mc_*^2\widehat{E}e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}(t-\tau)^{\alpha_0}B(\alpha_0,1)\left(\left(2^{\frac{|r+q|_+}{h}}\sum_{|r_1|_+\le|r|_+} C_r^{r_1}(t-\tau)^{-\frac{|r_1+q|_+}{h}}\right)+c_q\right) \\
&\quad\le mc_{r,q}^0 c_*^2\widehat{E}(2T_0)^{\frac{|r+q|_+}{h}}B(\alpha_0,1)(t-\tau)^{\alpha_0-\frac{|r+q|_+}{h}}e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}},\ c_{r,q}^0:=c_r+c_q.
\end{aligned}
\tag{26}
$$

Applying the method of induction, we can verify firstly the validity of the estimate

$$
\begin{aligned}
&\left|\partial_\xi^r K_{l_+ +l}(t,\eta+\xi;\tau,\xi)\right| \\
&\qquad\le c_*(mc_r^0 c_*\widehat{E}(t-\tau)^{\alpha_0})^l e^{-\delta_*\left(\frac{\|\eta\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}\left(\prod_{j=1}^{l-1}B(\alpha_0,1+j\alpha_0)\right),
\end{aligned}
\tag{27}
$$

and, hence, the estimate

$$
\begin{aligned}
\left|\partial_\xi^r\partial_x^q K_{l_+ +l}(t,x;\tau,\xi)\right| &\le c_*\left(mc_{r,q}^0 c_*\widehat{E}(2T_0)^{\frac{|r+q|_+}{h}}\right)^l (t-\tau)^{l\alpha_0-\frac{|r+q|_+}{h}} \\
&\quad\times e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}\left(\prod_{j=1}^{l-1}B(\alpha_0,1+j\alpha_0)\right),
\end{aligned}
\tag{28}
$$

for $|r|_+\le\alpha_*$, $|q|_+\le\alpha_*$, $(t,x;\tau,\xi)\in\Pi_T^2$ and $l\in\mathbb{N}\backslash\{1\}$.

The following propositions hold true.

Lemma 2.1. *The matrix function $\Phi(t,x;\tau,\xi)$ on the set Π_T^2 is a function differentiable with respect to each of the spatial variables x and ξ to the order α_* inclusively, and their derivatives satisfy the following estimates:*

$$
\left|\partial_\xi^r\partial_x^q\Phi(t,x;\tau,\xi)\right| \le c_1(t-\tau)^{\alpha_0-(1+\alpha n+\frac{|r+q|_+}{h})}e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}},
\tag{29}
$$

$$\left|\partial_\xi^r \Phi(t,\eta+\xi;\tau,\xi)\right| \le c_2(t-\tau)^{\alpha_0-(1+\alpha n)} e^{-\delta_*\left(\frac{\|\eta\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}, \{\eta,\xi\} \subset \mathbb{R}^n \tag{30}$$

(here, the estimating constants c_1, c_2, and δ_* are independent of t, τ, x, ξ, η).

Proof. In any way, let us fix a point $(x_0;\xi_0)$ from $\mathbb{R}^{2n}$, and consider a ball $\mathbb{K}^\delta_{(x_0;\xi_0)}$ with radius $\delta > 0$, which is centered at the point $(x_0;\xi_0)$, in this space. Then, in view of structure (6) of the function Φ and the differentiability of the iterated kernels K_l with respect to spatial variables on $\mathbb{R}^{2n}$ to the order α_* inclusively, we can conclude that, in order to prove the differentiability of the matrix function Φ at the point $(x_0;\xi_0)$ to the indicated order, it is necessary only to prove the uniform convergence of the formally differentiated series (6) in the variables x and ξ on the set $\mathbb{K}^\delta_{(x_0;\xi_0)}$, $\delta > 0$ (at every fixed t and τ, $0 \le \tau < t \le T$):

$$\sum_{l=1}^{\infty} \partial_\xi^r \partial_x^q K_l(t,x;\tau,\xi), \quad |r|_+ \le \alpha_*,\ |q|_+ \le \alpha_*. \tag{31}$$

Directly from estimates (24) and (28) and the equality

$$\prod_{j=0}^{l-1} B(\alpha_0, 1+j\alpha_0) = \frac{\big(\Gamma(\alpha_0)\big)^l}{\Gamma(1+l\alpha_0)},$$

where $\Gamma(\cdot)$ is the Euler gamma-function, for $\{r,q\} \subset \mathbb{Z}^n_+$, $|r|_+ \le \alpha_*$, $|q|_+ \le \alpha_*$, and $(t,x;\tau,\xi) \in \Pi^2_T$, we have

$$\begin{aligned}
&\Big|\sum_{l=1}^{\infty} \partial_\xi^r \partial_x^q K_l(t,x;\tau,\xi)\Big| \le \sum_{l=1}^{l_+} \left|\partial_\xi^r \partial_x^q K_l(t,x;\tau,\xi)\right| + \sum_{l=l_++1}^{\infty} \left|\partial_\xi^r \partial_x^q K_l(t,x;\tau,\xi)\right| \\
&\le c_*\Bigg(\sum_{l=1}^{l_+} (t-\tau)^{l\alpha_0-(1+\alpha n+\frac{|r+q|_+}{h})} + \sum_{l=1}^{\infty} \big(m c^0_{r,q} c_* \widehat{E}(2T_0)^{\frac{|r+q|_+}{h}}\big)^l (t-\tau)^{l\alpha_0-\frac{|r+q|_+}{h}} \\
&\times \Big(\prod_{j=1}^{l-1} B(\alpha_0, 1+j\alpha_0)\Big)\Bigg) e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}} \le c_1(t-\tau)^{\alpha_0-\left(1+\alpha n+\frac{|r+q|_+}{h}\right)} \times e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}.
\end{aligned} \tag{32}$$

Frow here, we get the uniform convergence of series (31) in x and ξ and, hence, the validity of estimates (29).

Due to the corresponding estimates (21) and (27), we can verify analogously the validity of estimate (30). The lemma is proven. □

Lemma 2.2. *The volumetric potential $W(t,x;\tau,\xi)$ on the set Π^2_T is a function differentiable with respect to each of the spatial variables x and ξ to the orders $\alpha_* + p_1$ and α_* respectively inclusively. In this case,*

$$\begin{aligned}
\partial_\xi^r \partial_x^q W(t,x;\tau,\xi) &= \sum_{l=0}^{r} C_r^l \int_\tau^{t_1} d\beta \int_{\mathbb{R}^n} \partial_\xi^l \partial_x^q G(t,\beta;x-y-\xi)\partial_\xi^{r-l}\Phi(\beta,y+\xi;\tau,\xi)dy \\
&+ \int_{t_1}^{t} d\beta \int_{\mathbb{R}^n} \partial_x^q G(t,\beta;x-y)\partial_\xi^r \Phi(\beta,y;\tau,\xi)dy,\ |q|_+ \le p_1, |r|_+ \le \alpha_*,
\end{aligned} \tag{33}$$

$$
\begin{aligned}
\partial_\xi^r \partial_x^q W(t,x;\tau,\xi) &= \sum_{l=0}^{r} C_r^l \int_\tau^{t_1} d\beta \int_{\mathbb{R}^n} \partial_\xi^l \partial_x^q G(t,\beta;x-y-\xi)\partial_\xi^{r-l}\Phi(\beta,y+\xi;\tau,\xi)dy \\
&+ \int_{t_1}^{t} d\beta \int_{\mathbb{R}^n} \partial_\eta^k G(t,\beta;\eta)\partial_\xi^r \partial_x^{q-k}\Phi(\beta,x-\eta;\tau,\xi)d\eta, \ |r|_+ \le \alpha_*, \\
& |k|_+ = p_1, \ p_1 < |q|_+ \le \alpha_* + p_1.
\end{aligned} \tag{34}
$$

Proof. For $|q|_+ \le p_1$ and $|r|_+ \le \alpha_*$, we use the representation

$$
\begin{aligned}
W(t,x;\tau,\xi) &= \int_\tau^{t_1} d\beta \int_{\mathbb{R}^n} G(t,\beta;x-y-\xi)\Phi(\beta,y+\xi;\tau,\xi)dy \\
&+ \int_{t_1}^{t} d\beta \int_{\mathbb{R}^n} G(t,\beta;x-y)\Phi(\beta,y;\tau,\xi)dy.
\end{aligned}
$$

From here, by the formal differentiation under the sign of integral, we obtain equality (33). Hence, in order to substantiate the validity of equality (33), it is sufficient to prove the uniform convergence of the following integrals in the variables x and ξ on $\mathbb{R}^{2n}$:

$$
\begin{aligned}
\mathcal{I}_1^{r,l,q}(t_1,x;\tau,\xi) &:= \int_\tau^{t_1} d\beta \int_{\mathbb{R}^n} |\partial_\xi^l \partial_x^q G(t,\beta;x-y-\xi)||\partial_\xi^{r-l}\Phi(\beta,y+\xi;\tau,\xi)|dy, \ |l|_+ \le |r|_+; \\
\mathcal{I}_2^{r,q}(t,x;t_1,\xi) &:= \int_{t_1}^{t} d\beta \int_{\mathbb{R}^n} |\partial_x^q G(t,\beta;x-y)||\partial_\xi^r \Phi(\beta,y;\tau,\xi)|dy.
\end{aligned} \tag{35}
$$

This convergence becomes obvious, if we take condition (A) and the following estimates into account for $\{x,\xi\} \subset \mathbb{R}^n$ and $0 \le \tau < t \le T$:

$$
\begin{aligned}
\mathcal{I}_1^{r,l,q}(t_1,x;\tau,\xi) &\le cc_2 \widehat{E} e^{-\delta_* \left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}} (t-t_1)^{-\frac{n+\gamma+|l+q|_+}{h}} \\
&\times \int_\tau^{t_1} (\beta-\tau)^{\alpha_0-1} d\beta, \ |l|_+ \le |r|_+;
\end{aligned} \tag{36}
$$

$$
\mathcal{I}_2^{r,q}(t,x;t_1,\xi) \le cc_1 \widehat{E} e^{-\delta_* \left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}} (t_1-\tau)^{-\frac{n+p_1+\gamma+|r|_+}{h}} \int_{t_1}^{t} (t-\beta)^{\alpha_0-1+\frac{p_1-|q|_+}{h}} d\beta. \tag{37}
$$

These estimates follow directly from (3), (29), and (30).

We now prove the validity of formula (34). For this purpose, we fix any $k \in \mathbb{Z}_+^n$ such that $|k|_+ = p_1$. Then, according to (33) for $p_1 < |q|_+ \le \alpha_* + p_1$ and $|r|_+ \le \alpha_*$, we have

$$
\begin{aligned}
\partial_\xi^r \partial_x^q W(t,x;\tau,\xi) &= \sum_{l=0}^{r} C_r^l \partial_x^{q-k} \int_\tau^{t_1} d\beta \int_{\mathbb{R}^n} \partial_\xi^l \partial_x^k G(t,\beta;x-y-\xi)\partial_\xi^{r-l}\Phi(\beta,y+\xi;\tau,\xi)dy \\
&+ \partial_x^{q-k} \int_{t_1}^{t} d\beta \int_{\mathbb{R}^n} \partial_\eta^k G(t,\beta;\eta)\partial_\xi^r \Phi(\beta,x-\eta;\tau,\xi)d\eta, \ (t,x;\tau,\xi) \in \Pi_T^2.
\end{aligned}
$$

Hence, it remains to substantiate the possibility to introduce the operation ∂_x^{q-k} under the signs of the corresponding integrals. In other words, we should prove the uniform convergence in x and ξ of the following integrals on $\mathbb{R}^{2n}$ for $0 \le \tau < t \le T$:

$$\int_{\tau}^{t_1} d\beta \int_{\mathbb{R}^n} \partial_\xi^l \partial_x^q G(t,\beta; x-y-\xi)\Phi(\beta, y+\xi; \tau,\xi)dy,$$

$$\int_{t_1}^{t} d\beta \int_{\mathbb{R}^n} \partial_\eta^k G(t,\beta;\eta)\partial_\xi^r \partial_x^{q-k}\Phi(\beta, x-\eta;\tau,\xi)d\eta.$$

By reasoning similarly to the case of integrals (35) and using estimates (3), (29), and (30), we get the necessary convergence of the indicated integrals. The lemma is proven. □

The main result can be formulated as the following proposition.

Theorem 1. *Let the system* (1) *satisfy conditions (A) and (B). Then the corresponding function* $Z(t,x;\tau,\xi)$ *defined by equality* (4) *is a function differentiable with respect to each of the spatial variables* x *and* ξ *on the set* Π_T^2 *to the orders* $\alpha_* + p_1$ *and* α_* *respectively inclusively, and exists* $\delta > 0$ *for all* $\{r,q\} \subset \mathbb{Z}_+^n$, $|q|_+ \le \alpha_* + p_1$, $|r|_+ \le \alpha_*$, *exists* $c > 0$ *for all* $(t,x;\tau,\xi) \in \Pi_T^2$:

$$|\partial_\xi^r \partial_x^q Z(t,x;\tau,\xi)| \le c(t-\tau)^{-\frac{n+|r+q|_+ +\gamma}{h}} e^{-\delta\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}; \tag{38}$$

$$|\partial_\xi^k Z(t,x+\xi;\tau,\xi)| \le c_k (t-\tau)^{\beta_k - \frac{n+\gamma}{h}} e^{-\delta_1\left(\frac{\|x\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}, \tag{39}$$

where $|k|_+ \le \alpha_*$, $0 \le \tau < t \le T$, $\{x,\xi\} \subset \mathbb{R}^n$, $\beta_k := \begin{cases} 0, & k=0, \\ \alpha_0, & k \ne 0 \end{cases}$ *(here, the estimating constants are independent of* t, τ, x, *and* ξ*).*

Proof. With regard for structure (4) and the infinite differentiability of the function $G(t,\tau;\xi)$ with respect to the variable ξ, the smoothness of the function $Z(t,x;\tau,\xi)$ in the variables x and ξ becomes obvious directly from the assertion of Lemma 2.

Let $|q|_+ \le p_1$ and $|r|_+ \le \alpha_*$. Then, according to (33), we get

$$|\partial_\xi^r \partial_x^q Z(t,x;\tau,\xi)| \le |\partial_{x-\xi}^{r+q} G(t,\tau;x-\xi)| + \sum_{l=0}^{r} C_r^l \mathcal{I}_1^{r,l,q}(t_1,x;\tau,\xi) + \mathcal{I}_2^{r,q}(t,x;t_1,\xi).$$

From here, by using estimates (3), (36), and (37), we obtain assertion (38).

In a similar way, by using formula (34), we verify the validity of assertion (38) also for $p_1 < |q|_+ \le \alpha_*$ and $|r|_+ \le \alpha_*$.

Then, according to estimates (3) and (30), we have

$$\begin{aligned} Y_k(t,x;\tau,\xi) &:= \Big| \int_{\tau}^{t} d\beta \int_{\mathbb{R}^n} G(t,\beta;x-\zeta)\partial_\xi^k \Phi(\beta,\zeta+\xi;\tau,\xi)d\zeta \Big| \\ &\le cc_2 \int_{\tau}^{t} (t-\beta)^{\alpha_0 + \frac{p_1}{h} - 1}(\beta-\tau)^{\alpha_0 - 1} \int_{\mathbb{R}^n} \exp\Big\{ -\delta_0 \Big\{ \Big(\frac{\|x-\zeta\|}{(t-\beta)^\alpha}\Big)^{\frac{1}{1-\alpha}} \\ &\quad + \Big(\frac{\|\zeta\|}{(\beta-\tau)^\alpha}\Big)^{\frac{1}{1-\alpha}} \Big\}\Big\} \frac{dyd\beta}{((t-\beta)(\beta-\tau)^{\alpha n}}, \end{aligned}$$

$$\delta_0 := \min\{\delta, \delta_*\}, |k|_+ \le \alpha_*.$$

Using estimate (11) and equality (19), we get

$$Y_k(t,x;\tau,\xi) \le c_\varepsilon (t-\tau)^{\alpha_0 - \frac{n+\gamma}{h}} e^{-\delta_0(1-\varepsilon)\left(\frac{\|x\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}, \quad \varepsilon \in (0;1),$$

where $|k|_+ \le \alpha_*$, $0 \le \tau < t \le T$ and $\{x,\xi\} \subset \mathbb{R}^n$. From whence, with regard for inequality (3) and the representation

$$Z(t,x+\xi;\tau,\xi) = G(t,\tau;x) + \int\limits_\tau^t d\beta \int\limits_{\mathbb{R}^n} G(t,\beta;x-\zeta)\Phi(\beta,\zeta+\xi;\beta,\xi)d\zeta,$$

we arrive at estimate (39).

The theorem is proven. □

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На відміну від параболічних за Петровським систем, параболічні за Шиловим системи, взагалі кажучи, є параболічно нестійкими до зміни своїх коефіцієнтів. Саме тому сучасна теорія задачі Коші для систем класу Шилова розвинена на рівні систем із сталими, або залежними лише від часу t коефіцієнтами. Проблема побудови теорії задачі Коші для таких систем із змінними коефіцієнтами досі залишається відкритою. У даній роботі розглянуто новий клас лінійних параболічних систем рівнянь із частинними похідними першого порядку за t із змінними коефіцієнтами, який повністю охоплює клас Шилова систем з коефіцієнтами, залежними від t та невід'ємним родом. Головна частина диференціального виразу стосовно просторової змінної x кожної такої системи є параболічним за Шиловим виразом, коефіцієнти якого залежать від t тоді, як коефіцієнти групи молодших членів можуть залежати ще й від просторової змінної. Методом послідовного наближення побудовано фундаментальний розв'язок задачі Коші для систем із цього класу. З' ясовано умови мінімальної гладкості на коефіцієнти системи за змінною x, за яких існує фундаментальний розв'язок, досліджено його гладкість та одержано оцінки похідних цього розв'язку. Зазначені результати є важливими, зокрема, для встановлення коректної розв'язності задачі Коші для таких систем у різних функціональних просторах, одержанні форм зображення розв'язку цієї задачі та дослідженні його властивостей.

Ключові слова і фрази: фундаментальна матриця розв'язків, задача Коші, параболічні системи типу Шилова.

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BOUNDARY PROBLEM FOR THE SINGULAR HEAT EQUATION

The scheme for solving of a mixed problem with general boundary conditions is proposed for a heat equation

$$a(x)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial T}{\partial x}\right)$$

with coefficient $a(x)$ that is the generalized derivative of a function of bounded variation, $\lambda(x) > 0$, $\lambda^{-1}(x)$ is a bounded and measurable function. The boundary conditions have the form

$$\begin{cases} p_{11}T(0,\tau) + p_{12}T_x^{[1]}(0,\tau) + q_{11}T(l,\tau) + q_{12}T_x^{[1]}(l,\tau) = \psi_1(\tau), \\ p_{21}T(0,\tau) + p_{22}T_x^{[1]}(0,\tau) + q_{21}T(l,\tau) + q_{22}T_x^{[1]}(l,\tau) = \psi_2(\tau), \end{cases}$$

where by $T_x^{[1]}(x,\tau)$ we denote the quasiderivative $\lambda(x)\frac{\partial T}{\partial x}$. A solution of this problem seek by the reduction method in the form of sum of two functions $T(x,\tau) = u(x,\tau) + v(x,\tau)$. This method allows to reduce solving of proposed problem to solving of two problems: a quasistationary boundary problem with initial and boundary conditions for the search of the function $u(x,\tau)$ and a mixed problem with zero boundary conditions for some inhomogeneous equation with an unknown function $v(x,\tau)$. The first of these problems is solved through the introduction of the quasiderivative. Fourier method and expansions in eigenfunctions of some boundary value problem for the second-order quasidifferential equation $(\lambda(x)X'(x))' + \omega a(x)X(x) = 0$ are used for solving of the second problem. The function $v(x,\tau)$ is represented as a series in eigenfunctions of this boundary value problem. The results can be used in the investigation process of heat transfer in a multilayer plate.

Key words and phrases: boundary problem, quasiderivative, eigenfunctions, Fourier method.

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INTRODUCTION

Boundary problems for differential equations of heat conduction with smooth coefficients were studied quite comprehensively in the literature (e.g., see [5]). However, during the modeling of heat transfer processes, the boundary problems with piecewise continuous coefficients or coefficients that have generalized derivatives of discontinuous functions are often appeared. Such problems have already begun to be studied in the works [3, 4].

The present paper deals with solving of a boundary problem for a heat equation with a coefficient that is the generalized derivative of a function of bounded variation. A reduction method [5] is used for solving of this problem. This method allows to reduce solving of this problem to solving of two problems: a quasistationary boundary problem with initial and boundary conditions and a mixed problem with zero boundary conditions for some inhomogeneous equation. Fourier method and expansions in eigenfunctions of some boundary value

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problem for the second-order quasidifferential equation are used for solving of the second of these problems.

Quasidifferential equations are equations that contain terms of the form $(p(x)y^{(m)})^{(n)}$. These equations cannot be reduced to conventional differential equations by n-fold differentiation if the coefficient $p(x)$ is not sufficiently smooth. The introduction of quasiderivatives is used for their research [2].

1 FORMULATION OF THE PROBLEM

Consider the next boundary value problem for a differential heat equation. It is necessary to find a solution $T(x,\tau)$ of the equation

$$a(x)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial T}{\partial x}\right) \tag{1}$$

with boundary conditions

$$\begin{cases} p_{11}T(0,\tau) + p_{12}T_x^{[1]}(0,\tau) + q_{11}T(l,\tau) + q_{12}T_x^{[1]}(l,\tau) = \psi_1(\tau), \\ p_{21}T(0,\tau) + p_{22}T_x^{[1]}(0,\tau) + q_{21}T(l,\tau) + q_{22}T_x^{[1]}(l,\tau) = \psi_2(\tau) \end{cases} \tag{2}$$

and initial condition

$$T(x,0) = \varphi(x), \tag{3}$$

where $a(x) = b'(x)$, $b(x)$ is a right continuous nondecreasing real function of bounded variation on the interval $[0,l]$, $\lambda(x) > 0$, $\lambda^{-1}(x)$ is a bounded and measurable function on the interval $[0,l]$, $\varphi(x)$ is a continuous function on the interval $[0,l]$, $\psi_1(\tau)$ and $\psi_2(\tau)$ are continuously differentiable functions for $\tau \geqslant 0$, p_{ij}, q_{ij} $(i,j = 1,2)$ are real numbers. By $T_x^{[1]}(x,\tau)$ we denote the quasiderivative $\lambda(x)\frac{\partial T}{\partial x}$. The prime in the formula $a(x) = b'(x)$ stands for the generalized differentiation, and hence the function $a(x)$ is a measure, i.e., a zero-order distribution on the space of continuous compactly supported functions [1].

A solution of problem (1)–(3) seek by the reduction method in the form of sum of two functions

$$T(x,\tau) = u(x,\tau) + v(x,\tau). \tag{4}$$

Any of functions u or v can be chosen by a special way, then another one will be determined uniquely.

2 QUASISTATIONARY BOUNDARY PROBLEM FOR $u(x,\tau)$

We define $u(x,\tau)$ as the solution of the boundary problem

$$\frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial u}{\partial x}\right) = 0, \tag{5}$$

$$\begin{cases} p_{11}u(0,\tau) + p_{12}u_x^{[1]}(0,\tau) + q_{11}u(l,\tau) + q_{12}u_x^{[1]}(l,\tau) = \psi_1(\tau), \\ p_{21}u(0,\tau) + p_{22}u_x^{[1]}(0,\tau) + q_{21}u(l,\tau) + q_{22}u_x^{[1]}(l,\tau) = \psi_2(\tau), \end{cases} \tag{6}$$

which is derived from problem (1)–(3) if τ is a parameter. Here the quasiderivative $u_x^{[1]}(x,\tau) \overset{df}{=} \lambda(x)\frac{\partial u}{\partial x}$, then $\frac{\partial u}{\partial x} = \frac{u^{[1]}}{\lambda(x)}$. With the help of the vector $\bar{u} = (u, u^{[1]})^T$ equation (5) is reduced to the system

$$\begin{pmatrix} u \\ u^{[1]} \end{pmatrix}' = \begin{pmatrix} 0 & \frac{1}{\lambda(x)} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ u^{[1]} \end{pmatrix}. \tag{7}$$

Boundary conditions (6) are also represented in the vector form

$$P \cdot \bar{u}(0,\tau) + Q \cdot \bar{u}(l,\tau) = \bar{\Gamma}(\tau), \tag{8}$$

where

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}, \quad Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}, \quad \bar{\Gamma}(\tau) = \begin{pmatrix} \psi_1(\tau) \\ \psi_2(\tau) \end{pmatrix}.$$

By direct verification one can make sure such that the Cauchy matrix $B(x,s)$ of system (7) has the form

$$B(x,s) = \begin{pmatrix} 1 & \sigma(x,s) \\ 0 & 1 \end{pmatrix}, \quad \sigma(x,s) = \int_s^x \frac{dt}{\lambda(t)}.$$

Then $\bar{u}(x,\tau) = B(x,0)\bar{u}_0$, where $\bar{u}_0 = \bar{u}(0,\tau)$. We shall determine $\bar{u}_0$. From boundary conditions (8) we obtain $P \cdot \bar{u}_0 + Q \cdot B(l,0) \cdot \bar{u}_0 = \bar{\Gamma}$ whence $\bar{u}_0 = (P + Q \cdot B(l,0))^{-1} \cdot \bar{\Gamma}$. Therefore,

$$\bar{u}(x,\tau) = B(x,0) \cdot (P + Q \cdot B(l,0))^{-1} \cdot \bar{\Gamma}(\tau). \tag{9}$$

3 MIXED PROBLEM FOR $v(x,\tau)$

We substitute $u(x,\tau)$ and $v(x,\tau)$ into equation (1)

$$a(x)\left(\frac{\partial u}{\partial \tau} + \frac{\partial v}{\partial \tau}\right) = \frac{\partial}{\partial x}\left(\lambda(x)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right)\right).$$

In consequence of (5) we have the equation

$$a(x)\frac{\partial v}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial v}{\partial x}\right) - a(x)\frac{\partial u}{\partial \tau}. \tag{10}$$

According to formula (9) the derivative $\frac{\partial u}{\partial \tau}$ is a continuous function of the variable x on $[0,l]$ and so the last term in equation (10) is correct.

By taking into account formula (4), we define the boundary conditions for v from conditions (2)

$$\begin{aligned} &p_{11}u(0,\tau) + p_{12}u_x^{[1]}(0,\tau) + q_{11}u(l,\tau) + q_{12}u_x^{[1]}(l,\tau) \\ &\qquad + p_{11}v(0,\tau) + p_{12}v_x^{[1]}(0,\tau) + q_{11}v(l,\tau) + q_{12}v_x^{[1]}(l,\tau) = \psi_1(\tau), \\ &p_{21}u(0,\tau) + p_{22}u_x^{[1]}(0,\tau) + q_{21}u(l,\tau) + q_{22}u_x^{[1]}(l,\tau) \\ &\qquad + p_{21}v(0,\tau) + p_{22}v_x^{[1]}(0,\tau) + q_{21}v(l,\tau) + q_{22}v_x^{[1]}(l,\tau) = \psi_2(\tau). \end{aligned}$$

By virtue of (6), we obtain

$$\begin{cases} p_{11}v(0,\tau) + p_{12}v_x^{[1]}(0,\tau) + q_{11}v(l,\tau) + q_{12}v_x^{[1]}(l,\tau) = 0, \\ p_{21}v(0,\tau) + p_{22}v_x^{[1]}(0,\tau) + q_{21}v(l,\tau) + q_{22}v_x^{[1]}(l,\tau) = 0. \end{cases} \tag{11}$$

The initial condition is determined similarly

$$v(x,0) = \varphi(x) - u(x,0) \overset{df}{=} \tilde{\varphi}(x). \tag{12}$$

4 FOURIER METHOD AND EIGENVALUE PROBLEM

We search for non-trivial solutions of the homogeneous differential equation

$$a(x)\frac{\partial v}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial v}{\partial x}\right)$$

with boundary conditions (11) in the form

$$v(x,\tau) = e^{-\omega\tau}X(x), \tag{13}$$

where ω is a parameter, and $X(x)$ is a function. Then

$$-\omega a(x)e^{-\omega\tau}X(x) = (\lambda(x)X'(x))'e^{-\omega\tau}$$

whence we get the quasidifferential equation

$$(\lambda(x)X'(x))' + \omega a(x)X(x) = 0. \tag{14}$$

Substituting formula (13) in boundary conditions (11), we obtain

$$\begin{cases} p_{11}X(0) + p_{12}X^{[1]}(0) + q_{11}X(l) + q_{12}X^{[1]}(l) = 0, \\ p_{21}X(0) + p_{22}X^{[1]}(0) + q_{21}X(l) + q_{22}X^{[1]}(l) = 0. \end{cases} \tag{15}$$

We denote by ω_k the eigenvalues of boundary problem (14), (15). Let $X_k(\omega_k, x)$ be the corresponding eigenfunctions, $k = 1, 2, \ldots, \infty$.

By [6], all eigenvalues ω_k of boundary problem (14), (15) are real, there are a countable number of them, and their set has not a finite limit point. The eigenfunctions $X_k(\omega_k, x)$ that are corresponded to the different eigenvalues are orthogonal in the sense

$$\int_0^l X_m(\omega_m, x)X_n(\omega_n, x)db(x) = 0, \quad \omega_m \neq \omega_n.$$

5 METHOD OF THE EIGENFUNCTIONS

We seek $v(x,\tau)$ in the form of the series

$$v(x,\tau) = \sum_{k=1}^{\infty} t_k(\tau)X_k(\omega_k, x), \tag{16}$$

where $X_k(\omega_k, x)$ are the eigenfunctions of boundary problem (14), (15). We substitute formula (16) into equation (10)

$$a(x)\frac{\partial}{\partial \tau}\left(\sum_{k=1}^{\infty} t_k(\tau)X_k\right) = \frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial}{\partial x}\left(\sum_{k=1}^{\infty} t_k(\tau)X_k\right)\right) - a(x)\frac{\partial u}{\partial \tau}$$

where, under the assumption of uniform convergence of series (16) and series derived from it by differentiation by x or τ, we have

$$a(x)\sum_{k=1}^{\infty} t_k'(\tau)X_k = \sum_{k=1}^{\infty} t_k(\tau)\big(\lambda(x)X_k'\big)' - a(x)\frac{\partial u}{\partial \tau}.$$

As a result of equation (14) there is equality $\left(\lambda(x)X_k'\right)' = -\omega_k a(x) X_k$, then

$$a(x)\sum_{k=1}^{\infty} t_k'(\tau)X_k = -\sum_{k=1}^{\infty} t_k(\tau)\omega_k a(x)X_k - a(x)\frac{\partial u}{\partial \tau}.$$

Therefore,

$$\sum_{k=1}^{\infty}\left(t_k'(\tau)+\omega_k t_k(\tau)\right)X_k = -\frac{\partial u}{\partial \tau}. \tag{17}$$

We expand the known function $\frac{\partial u}{\partial \tau}$ in a series in the eigenfunctions of boundary problem (14), (15):

$$\frac{\partial u}{\partial \tau} = \sum_{k=1}^{\infty} d_k(\tau)X_k(\omega_k, x), \tag{18}$$

where

$$d_k(\tau) = \frac{1}{\|X_k\|}\int_0^l \frac{\partial u}{\partial \tau} X_k(\omega_k, x)db(x), \quad \|X_k\| = \int_0^l X_k^2(\omega_k, x)db(x).$$

By substituting formula (18) into (17), we obtain

$$t_k'(\tau) + \omega_k t_k(\tau) = -d_k(\tau), \quad k = 1, 2, \ldots, \infty. \tag{19}$$

Since formulas (12) and (16), we have

$$v(x,0) = \sum_{k=1}^{\infty} t_k(0)X_k(\omega_k, x) \equiv \tilde{\varphi}(x).$$

We expand the function $\tilde{\varphi}(x)$ in a series in the eigenfunctions

$$\tilde{\varphi}(x) = \sum_{k=1}^{\infty} \varphi_k X_k(\omega_k, x), \quad \varphi_k = \frac{1}{\|X_k\|}\int_0^l \tilde{\varphi}(x)X_k(\omega_k, x)db(x).$$

Consequently,

$$t_k(0) = \varphi_k, \quad k = 1, 2, \ldots, \infty. \tag{20}$$

Then for all positive integer k we have Cauchy problems (19), (20) for ordinary differential equations.

General solutions of linear inhomogeneous equations (19) acquire the formulas

$$t_k(\tau) = \left(C_k - \int_0^{\tau} d_k(s)e^{\omega_k s}ds\right)e^{-\omega_k \tau},$$

where C_k are arbitrary constants. Therefore, by using initial conditions (20), we find for each positive integer k the solution of the corresponding Cauchy problem

$$t_k(\tau) = \varphi_k e^{-\omega_k \tau} - \int_0^{\tau} d_k(s)e^{\omega_k(s-\tau)}ds.$$

Then, by virtue of formula (16), we obtain

$$v(x,\tau) = \sum_{k=1}^{\infty}\left(\varphi_k e^{-\omega_k \tau} - \int_0^{\tau} d_k(s)e^{\omega_k(s-\tau)}ds\right)X_k(\omega_k, x).$$

Thus, by using the reduction method, Fourier method and the expansion in a series in eigenfunctions, we built the solution of the boundary problem for the heat equation with a distribution. The results can be used in the investigation of the process of heat transfer in a multilayer plate.

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Запропоновано схему розв'язування мішаної задачі за загальних крайових умов для рівняння теплопровідності

$$a(x)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial T}{\partial x}\right)$$

з коефіцієнтом $a(x)$, який є узагальненою похідною функції обмеженої варіації, $\lambda(x) > 0$, $\lambda^{-1}(x)$ – обмежена і вимірна функція. Крайові умови мають вигляд

$$\begin{cases} p_{11}T(0,\tau) + p_{12}T_x^{[1]}(0,\tau) + q_{11}T(l,\tau) + q_{12}T_x^{[1]}(l,\tau) = \psi_1(\tau), \\ p_{21}T(0,\tau) + p_{22}T_x^{[1]}(0,\tau) + q_{21}T(l,\tau) + q_{22}T_x^{[1]}(l,\tau) = \psi_2(\tau), \end{cases}$$

де через $T_x^{[1]}(x,\tau)$ позначено квазіпохідну $\lambda(x)\frac{\partial T}{\partial x}$. Розв'язок цієї задачі шукається методом редукції у вигляді суми двох функцій $T(x,\tau) = u(x,\tau) + v(x,\tau)$. Цей метод дає змогу звести розв'язування поставленої задачі до розв'язування двох задач: крайової квазістаціонарної задачі з початковими і крайовими умовами для відшукання функції $u(x,\tau)$ і мішаної задачі з нульовими крайовими умовами для деякого неоднорідного рівняння з невідомою функцією $v(x,\tau)$. Перша з цих задач розв'язується з допомогою введення квазіпохідної. Для розв'язування другої задачі застосовується метод Фур'є і розвинення за власними функціями деякої крайової задачі для квазідиференціального рівняння другого порядку $(\lambda(x)X'(x))' + \omega a(x)X(x) = 0$. Функція $v(x,\tau)$ подається у вигляді ряду за власними функціями цієї крайової задачі. Отримані результати можна використовувати для дослідження процесу теплопередачі в багатошаровій плиті.

Ключові слова і фрази: крайова задача, квазіпохідна, власні функції, метод Фур'є.

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APPROXIMATION OF CAPACITIES WITH ADDITIVE MEASURES

For a space of non-additive regular measures on a metric compactum with the Prokhorov-style metric, it is shown that the problem of approximation of arbitrary measure with an additive measure on a fixed finite subspace reduces to linear optimization problem with parameters dependent on the values of the measure on a finite number of sets.

An algorithm for such an approximation, which is more efficient than the straighforward usage of simplex method, is presented.

Key words and phrases: Prokhorov metric, non-additive measure, approximation, compact metric space.

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INTRODUCTION

Capacities were introduced by Choquet [1] and found numerous applications in different branches of mathematics. Spaces of upper semicontinuous capacities on compacta were systematically studied in [5]. In particular, in the latter paper functoriality of the construction of a space of capacities was proved and Prokhorov-style and Kantorovich-Rubinstein-style metrics on the set of capacities on a metric compactum were introduced. Needs of practice require that a capacity can be approximated with capacities of simpler structure or with some convenient properties.

We follow the terminology and notation of [5] and denote by $\exp X$ the set of all non-empty closed subsets of a compactum X. We call a function $c : \exp X \cup \{\varnothing\} \to I$ a *capacity* on a compactum X if the three following properties hold for all subsets $F, G \underset{\mathrm{cl}}{\subset} X$:

1. $c(\varnothing) = 0$;
2. if $F \subset G$, then $c(F) \leqslant c(G)$ (monotonicity);
3. if $c(F) < a$, then there is an open subset $U \supset F$ such that for all $G \subset U$ the inequality $c(G) < a$ is valid (upper semicontinuity).

If, additionally, $c(X) = 1$ (or $c(X) \le 1$) holds, then the capacity is called *normalized* (resp. *subnormalized*). We denote by $\overline{M}X$, MX, and $\underline{M}X$ the sets of all capacities on X, of all normalized, and of all subnormalized capacities on X respectively.

It was shown in [5] that MX carries a compact Hausdorff topology with the subbase of all sets of the form

$$O_{-}(F, a) = \{c \in MX \mid c(F) < a\}, \text{where} F \underset{\mathrm{cl}}{\subset} X, a \in I,$$

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and

$$O_+(U,a) = \{c \in MX \mid c(U) > a\}$$
$$= \{c \in MX \mid \text{there is a compactum } F \subset U, c(F) > a\}, \text{where } U \underset{\text{op}}{\subset} X, a \in I.$$

The same formulae determine a subbase of a compact Hausdorff topology on $\underline{M}X$ so that $MX \subset \underline{M}X$ is a subspace.

Previously we have considered the following subclasses of MX:

1) $M_\cap X$ is the set of the so-called $\cap$-*capacities* (or necessity measures) with the property: $c(A \cap B) = \min\{c(A), c(B)\}$ for all $A, B \underset{\text{cl}}{\subset} X$.

2) $M_\cup X$ is the set of the so-called $\cup$-*capacities* (or possibility measures) with the property: $c(A \cup B) = \max\{c(A), c(B)\}$ for all $A, B \underset{\text{cl}}{\subset} X$.

3) Class MX_0 of capacities defined *on a closed subspace* $X_0 \subset X$. We regard each capacity c_0 on X_0 as a capacity on X extended with the formula $c(F) = c_0(F \cap X_0)$, $F \underset{\text{cl}}{\subset} X$.

4) Class $M_{Lip}X$ of capacities that are non-expanding w.r.t. the Hausdorff metric on $\exp X$.

Analogous subclasses are defined in $\underline{M}X$ and $\bar{M}X$, with the obvious denotations.

It was proved in [2, 3] that the subsets $M_\cap X$, $M_\cup X$, $M_{Lip}X$, and MX_0 are closed in MX, hence for a compactum X they are compacta as well, similarly for the respective subsets in $\underline{M}X$ and $\bar{M}X$.

We consider the metric on the set $\bar{M}X$ of capacities on a metric compactum (X,d) :

$$\hat{d}(c,c') = \inf\{\varepsilon > 0 \mid c(\bar{O}_\varepsilon(F)) + \varepsilon \geqslant c'(F), c'(\bar{O}_\varepsilon(F)) + \varepsilon \geqslant c(F), \forall F \underset{\text{cl}}{\subset} X\},$$

here $\bar{O}_\varepsilon(F)$ is the closed ε-neighborhood of a subset $F \subset X$. The restrictions of this metric on $\underline{M}X$ and MX are admissible [5].

For an arbitrary capacity c on a metric compactum X, explicit constructions for the closest to c point in the four above subclasses were presented in [3, 4].

Now we consider probably the most important class of *additive* regular measures.

Our goal is to approximate a capacity c on a metric compactum X with an additive measure on a *finite subspace* of X. Such measures are dense in the space $\overline{P}X$ of all finite additive regular measures and have nice representation as linear combinations of Dirac measures.

1 Algorithm for approximation of a capacity with an additive measure on a finite subspace

Consider a capacity c on a metric compactum (X,d) and a finite subspace $X_0 = \{x_1, x_2, \ldots, x_n\} \subset X$. We are going to find the distance between $c \in \bar{M}X$ and the subspace $\overline{P}X_0 \subset \bar{M}X$, in particular to find an additive measure m on X_0 that is (almost) the closest to c with respect to the distance $\hat{d}$.

The inequality $\hat{d}(c,m) \leqslant \varepsilon$ means that there is $0 \leqslant z \leqslant \varepsilon$ satisfying

$$\begin{cases} m(A) \leqslant c(\bar{O}_\varepsilon A) + z, \\ c(A) \leqslant m(\bar{O}_\varepsilon A) + z \end{cases}$$

for all $A \underset{\text{cl}}{\subset} X$. Obviously it is sufficient to verify the first inequality $m(A) \leqslant c^+_\varepsilon(A) + z$, where we denote $c^+_\varepsilon = c(\bar{O}_\varepsilon(A))$, only for all $A \subset X_0$. Similarly, for the second condition we verify

$c(B) \leqslant m(A) + z$ for all $B \subset X$ and $A \subset X_0$ such that $(\bar{O}_\varepsilon B) \cap X_0 \subset A$. This is equivalent to $m(A) \geqslant c_\varepsilon^-(A) - z$ for all $A \subset X_0$, where

$$c_\varepsilon^-(A) = c(X \setminus \bar{O}_\varepsilon(X_0 \setminus A)) = \sup\{c(B) \mid B \underset{\mathrm{cl}}{\subset} X, B \cap \bar{O}_\varepsilon(X_0 \setminus A) = \varnothing\}.$$

Obviously $c_\varepsilon^-(A) \leqslant c_\varepsilon^+(A)$ for all $A \subset X_0$.

All additive measures on X_0 are of the form $m = y_1\delta_{x_1} + y_2\delta_{x_2} + \cdots + y_n\delta_{x_n}$. Thus, to find the least z that satisfies the above conditions for some m, we have to solve the linear programming problem w.r.t. the variables $y_1, y_2, \ldots, y_n, z \geqslant 0$:

$$\begin{cases} y_1, y_2, \ldots, y_n, z \geqslant 0, \\ \sum_{x_i \in A} y_i \leqslant c_\varepsilon^+(A) + z & \text{for all } A \subset X_0, \\ \sum_{x_i \in A} y_i \geqslant c_\varepsilon^-(A) - z & \text{for all } A \subset X_0, \\ z \to \min, \end{cases}$$

which we rewrite as follows:

$$\begin{cases} y_1, y_2, \ldots, y_n, z \geqslant 0, \\ -\sum_{x_i \in A} y_i + z \geqslant -c_\varepsilon^+(A) & \text{for all } A \subset X_0, \\ \quad \sum_{x_i \in A} y_i + z \geqslant \quad c_\varepsilon^-(A) & \text{for all } A \subset X_0, \\ z \to \min. \end{cases}$$

We embed the set $\operatorname{Exp} X_0$ into $\mathbb{R}^n$ by identifying each subset $A \subset X_0$ with the vector containing 1 at all i-th positions such that $x_i \in A$ and 0 at all other positions. E.g., $\varnothing$ is represented by $(0, \ldots, 0)$, and X_0 by $(1, \ldots, 1)$. By $-\operatorname{Exp} X_0$ we denote the set of the opposites to elements of $\operatorname{Exp} X_0 \subset \mathbb{R}^n$. Define a function $c_\varepsilon : \operatorname{Exp} X_0 \cup (-\operatorname{Exp} X_0) \to \mathbb{R}$ by the formula

$$c_\varepsilon(A) = \begin{cases} c_\varepsilon^-(A), & A \in \operatorname{Exp} X_0, \\ -c_\varepsilon^+(-A), & A \in (-\operatorname{Exp} X_0). \end{cases}$$

The common element $\varnothing = (0, \ldots, 0) \in \operatorname{Exp} X_0 \cap (-\operatorname{Exp} X_0)$ leads to no contradiction because $c_\varepsilon^-(\varnothing) = c_\varepsilon^+(\varnothing) = 0$.

We also denote by $(A|1)$ the vector obtained by appending a trailing 1 to the sequence $A = (a_1, a_2, \ldots, a_n) \in \operatorname{Exp} X_0 \cup (-\operatorname{Exp} X_0)$. Then the linear optimization problem can we written as

$$\begin{cases} y_1, y_2, \ldots, y_n, z \geqslant 0, \\ (A|1) \cdot (y_1, y_2, \ldots, y_n, z) \geqslant c_\varepsilon(A) \text{ for all } A \in \operatorname{Exp} X_0 \cup (-\operatorname{Exp} X_0), \\ z \to \min. \end{cases}$$

It has a straightforward geometric interpretation: of all functionals of the form

$$\gamma(t_1, t_2, \ldots, t_n) = y_1 t_1 + y_2 t_2 + \cdots + y_n t_n + z$$

such that $\gamma(A) \geqslant c_\varepsilon(A)$ for all $A \in \operatorname{Exp} X_0 \cup (-\operatorname{Exp} X_0)$, choose one with the minimal z, i.e., with the least value $\gamma(\vec{0})$. Now it is clear that, due to monotonicity of the function c_ε, the restrictions $y_1, y_2, \ldots, y_n \geqslant 0$ can be dropped. Observe also that the restriction $z \geqslant 0$ is equivalent to

$$(\varnothing|1) \cdot (y_1, y_2, \ldots, y_n, z) \geqslant c_\varepsilon(\varnothing),$$

hence can be dropped as well.

Geometric arguments also show that the problem is solved if affinely independent

$$A_1, A_2, \dots, A_{n+1} \in \operatorname{Exp} X_0 \cup (-\operatorname{Exp} X_0)$$

are found such that $\vec{0}$ is in their convex hull (in the sequel we call such $A_1, A_2, \dots, A_{n+1}$ *basic subsets*), and the solutions $y_1, y_2, \dots, y_n, z$ of the system

$$\begin{cases} (A_1|1) \cdot (y_1, y_2, \dots, y_n, z) & = c_\varepsilon(A_1), \\ (A_2|1) \cdot (y_1, y_2, \dots, y_n, z) & = c_\varepsilon(A_2), \\ \dots \\ (A_{n+1}|1) \cdot (y_1, y_2, \dots, y_n, z) & = c_\varepsilon(A_{n+1}) \end{cases}$$

satisfy

$$(A|1) \cdot (y_1, y_2, \dots, y_n, z) \geqslant c_\varepsilon(A)$$

for all $A \in \operatorname{Exp} X_0 \cup (-\operatorname{Exp} X_0)$.

Therefore we propose the following algorithm, which essentially is equivalent to the simplex algorithm, but is better suited for our needs. Choose initial basic subsets, e.g., $A_1 = \{x_1\}$, $A_2 = \{x_2\}$, …, $A_n = \{x_n\}$, $A_{n+1} = -\{x_n\}$, then calculate $y_1, y_2, \dots, y_n, z$ as

$$(y_1, y_2, \dots, y_n, z)^T = \big(M(A_1, A_2, \dots, A_n)\big)^{-1} (c(A_1), c(A_2), \dots, c(A_{n+1}))^T,$$

where $(-)^T$ means transposition, and

$$M(A_1, A_2, \dots, A_n) = \begin{bmatrix} A_1 & | & 1 \\ A_2 & | & 1 \\ \dots & & \dots \\ A_{n+1} & | & 1 \end{bmatrix},$$

i.e., it is the matrix with the rows $(A_1|1)$, $(A_2|1)$, …, $(A_{n+1}|1)$.

We will permanently need the inverse matrix

$$\big(M(A_1, A_2, \dots, A_n)\big)^{-1} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1,n+1} \\ \lambda_{21} & \lambda_{22} & \dots & \lambda_{2,n+1} \\ \dots & \dots & \ddots & \dots \\ \lambda_{n1} & \lambda_{n2} & \dots & \lambda_{n,n+1} \\ \mu_1 & \mu_2 & \dots & \mu_{n+1} \end{bmatrix}.$$

For any $A \in \operatorname{Exp} X_0 \cup (-\operatorname{Exp} X_0)$ the column $\big(M(A_1, A_2, \dots, A_n)\big)^{-1} (A|1)^T$ consists of the coefficients $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$ such that $\alpha_1 + \alpha_2 + \dots + \alpha_{n+1} = 1$ and $\alpha_1 A_1 + \alpha_2 A_2 + \dots + \alpha_{n+1} A_{n+1} = A$ (in the above sense). In particular, $\mu_1 A_1 + \mu_2 A_2 + \dots + \mu_{n+1} A_{n+1} = \varnothing$, and $\lambda_{i1} A_1 + \lambda_{i2} A_2 + \dots + \lambda_{i,n+1} A_{n+1} = \{x_i\}$ for all $1 \leqslant i \leqslant n$.

Now, having $y_1, y_2, \dots, y_n, z$ calculated, compare the differences

$$c_\varepsilon(A) - (A|1)(y_1, y_2, \dots, y_n, z)$$

for all $A \in \operatorname{Exp} X_0 \cup (-\operatorname{Exp} X_0)$. If the basic subsets $A_1, A_2, \dots, A_{n+1}$ provide a solution, then all the differences are not greater than 0. Otherwise find the greatest difference $\Delta =$

$c_\varepsilon(A') - (A'|1)(y_1, y_2, \dots, y_n, z)$, which is positive, and replace with A' a subset A_i such that $\vec{0}$ is in the convex hull of $A_1, A_2, \dots, A_{i-1}, A', A_{i+1}, \dots, A_{n+1}$.

Let $(\alpha_1, \alpha_2, \dots, \alpha_{n+1})^T = \big(M(A_1, A_2, \dots, A_n)\big)^{-1}(A'|1)^T$, hence $A' = \alpha_1 A_1 + \alpha_2 A_2 + \dots + \alpha_{n+1}A_{n+1}$, then

$$A_i = \frac{1}{\alpha_i}A' - \frac{\alpha_1}{\alpha_i}A_1 - \dots - \frac{\alpha_{i-1}}{\alpha_i}A_{i-1} - \frac{\alpha_{i+1}}{\alpha_i}A_{i+1} - \frac{\alpha_{n+1}}{\alpha_i}A_{n+1}.$$

Therefore

$$\begin{aligned}\varnothing = \big(\mu_1 - \mu_i\frac{\alpha_1}{\alpha_i}\big)A_1 + \dots + \big(\mu_{i-1} - \mu_i\frac{\alpha_{i-1}}{\alpha_i}\big)A_{i-1} + \big(\mu_{i+1} - \mu_i\frac{\alpha_{i+1}}{\alpha_i}\big)A_{i+1} \\ + \dots + \big(\mu_{n+1} - \mu_i\frac{\alpha_{n+1}}{\alpha_i}\big)A_{n+1} + \frac{\mu_i}{\alpha_i}A'.\end{aligned}$$

The coefficients in the new decomposition of $\varnothing$ should be nonnegative, hence $\alpha_i > 0$ is required, as well as either $\alpha_j \leqslant 0$ or $\mu_j - \mu_i\dfrac{\alpha_j}{\alpha_i} \geqslant 0$ for all $j \neq i$. If $\alpha_j > 0$, then the latter inequality is equivalent to $\dfrac{\mu_j}{\alpha_j} \geqslant \dfrac{\mu_i}{\alpha_i}$. Hence $\dfrac{\mu_i}{\alpha_i}$ should be the least of $\dfrac{\mu_j}{\alpha_j}$ for $1 \leqslant j \leqslant n+1$ such that $\alpha_j > 0$.

Now we replace A_i with $A'_i = A'$, and the inverse matrix

$$\big(M(A_1, A_2, \dots, A_{i-1}, A'_i, A_{i+1}, \dots, A_n)\big)^{-1} = \begin{bmatrix} \lambda'_{11} & \lambda'_{12} & \dots & \lambda'_{1,n+1} \\ \lambda'_{21} & \lambda'_{22} & \dots & \lambda'_{2,n+1} \\ \dots & \dots & \ddots & \dots \\ \lambda'_{n1} & \lambda'_{n2} & \dots & \lambda'_{n,n+1} \\ \mu'_1 & \mu'_2 & \dots & \mu'_{n+1} \end{bmatrix}$$

is adjusted accordingly:

$$\begin{aligned} \mu'_i &= \frac{\mu_i}{\alpha_i}, \qquad \mu'_j = \mu_j - \alpha_j\frac{\mu_i}{\alpha_i}, \qquad && 1 \leqslant j \leqslant n+1,\ j \neq i, \\ \lambda'_{ki} &= \frac{\lambda_{ki}}{\alpha_i}, \qquad \lambda'_{kj} = \lambda_{kj} - \alpha_j\frac{\lambda_{ki}}{\alpha_i}, \qquad && 1 \leqslant k, j \leqslant n+1,\ j \neq i. \end{aligned}$$

Now look how $y_1, y_2, \dots, y_n, z$ have changed. Taking into account

$$\begin{aligned} z &= \mu_1 c_\varepsilon(A_1) + \dots + \mu_{i-1}c_\varepsilon(A_{i-1}) + \mu_i c_\varepsilon(A_i) \\ &\quad + \mu_{i+1}c_\varepsilon(A_{i+1}) + \dots + \mu_{n+1}A_{n+1}, \\ z' &= \big(\mu_1 - \alpha_1\frac{\mu_i}{\alpha_i}\big)c_\varepsilon(A_1) + \dots + \big(\mu_{i-1} - \alpha_{i-1}\frac{\mu_i}{\alpha_i}\big)c_\varepsilon(A_{i-1}) + \frac{\mu_i}{\alpha_i}c_\varepsilon(A'_i) \\ &\quad + \big(\mu_{i+1} - \alpha_{i+1}\frac{\mu_i}{\alpha_i}\big)c_\varepsilon(A_{i+1}) + \dots + \big(\mu_{n+1} - \alpha_{n+1}\frac{\mu_i}{\alpha_i}\big)c_\varepsilon(A_{n+1}), \end{aligned}$$

obtain

$$z' - z = \frac{\mu_i}{\alpha_i}\big(c_\varepsilon(A'_i) - (\alpha_1 c_\varepsilon(A_1) + \dots + \alpha_{n+1}c_\varepsilon(A_{n+1}))\big) = \frac{\mu_i}{\alpha_i}\cdot\Delta.$$

Similarly

$$y'_k - y_k = \frac{\lambda_{ki}}{\alpha_i}\big(c_\varepsilon(A'_i) - (\alpha_1 c_\varepsilon(A_1) + \dots + \alpha_{n+1}c_\varepsilon(A_{n+1}))\big) = \frac{\lambda_{ki}}{\alpha_i}\cdot\Delta.$$

This simplifies calculation of z' and all y'_k. We iterate the above step until $\Delta = 0$. The final value of z, which we denote $z(\varepsilon)$, is the least z such that

$$\begin{cases} m(A) \leqslant c(\bar{O}_\varepsilon A) + z, \\ c(A) \leqslant m(\bar{O}_\varepsilon A) + z \end{cases}$$

for some $m \in \overline{P}X_0$ and all $A \underset{\text{cl}}{\subset} X$.

Observe that $z(\varepsilon)$ is non-increasing with respect to ε, hence the distance between c and $\overline{P}X_0$ is the least ε such that $z(\varepsilon) \leqslant \varepsilon$. This distance is not greater than $z(0)$, therefore it is easy to bisect the segment $[0, z(0)]$ to find the distance and an approximating additive measure with arbitrary precision.

2 CONCLUDING REMARKS

The proposed algorithm was implemented as a C program and tested on data sets with cardinality of X_0 up to 10.

However, each iteration of the presented algorithm requires previously calculated values of a capacity for all $2^{\text{cardinality of the space}}$ subsets, which is not appropriate even for $\geqslant 40$ points. Hence, to handle subspaces of greater cardinality, we need to cut memory and time requirements using the metric structure and the only reliable property of a capacity, i.e., its monotonicity. This requires deeper investigation combining both topological properties of non-additive measures, e.g., their dimensional characteristics, and computational aspects.

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Для простору неадитивних регулярних мір на метричному компакті з відстанню в стилі Прохорова показано, що задача наближення довільної міри адитивною мірою на фіксованому скінченному підпросторі зводиться до задачі лінійної оптимізації з параметрами, залежними від значень вихідної міри на скінченному числі множин.

Запропоновано алгоритм такого наближення, ефективніший порівняно з прямолінійним застосуванням симплекс-методу.

Ключові слова і фрази: метрика Прохорова, неадитивна міра, апроксимація, компактний метричний простір.

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A NOTE ON THE NECESSITY OF FILTERING MECHANISM FOR POLYNOMIAL OBSERVABILITY OF TIME-DISCRETE WAVE EQUATION

The problem of uniform polynomial observability was recently analyzed. It is shown that, when the continuous model is uniformly polynomially observable, it is sufficient to filter initial data to derive uniform polynomial observability inequalities for suitable time-discretization schemes. In this note, we prove that a filtering mechanism of high frequency modes is necessary to obtain uniform polynomial observability.

More precisely, we give a counterexample which proves that this latter fails without filtering the initial data for time semi-discrete approximations of the wave equation.

Key words and phrases: observability inequality, time discretization, filtering techniques.

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1 Introduction

We consider the following wave equation on interval of length 1

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0, & 0 < x < 1,\ 0 < t < T, \\ u(0,t) = u(1,t) = 0, & 0 < t < T, \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), & 0 < x < 1, \end{cases} \tag{1}$$

where $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$. It is easy to check (see [1]) that this system is well posed in the energy space $H_0^1(0,1) \times L^2(0,1)$. More precisely, for any $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$ there exists a unique solution $u \in C((0,T), H_0^1) \cap C^1((0,T), L^2(0,1))$ of (1).

The energy of solutions of (1) is conserved in time, i.e.,

$$E(t) = \frac{1}{2}\int_0^1 \left(|u_t(x,t)|^2 + |u_x(x,t)|^2\right) dx = E(0) \qquad \text{for all} \quad 0 \leq t \leq T.$$

Define the output function

$$y(t) = u_t(\xi, t), \quad \xi \in (0,1). \tag{2}$$

It was proved in [1] that system (1) is polynomially observable when $\xi \in \mathcal{S}$, where $\mathcal{S}$ is the set of all numbers $\rho \in (0,1)$ such that $\rho \notin \mathbb{Q}$ (the set of rational numbers) and if $[a_0, a_1, \dots, a_n, \dots]$ is the expansion of ρ as a continued fraction, then (a_n) is bounded. More precisely, we have the following assertion.

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Proposition 1. *Let $T > 0$ be fixed. Then for all $\xi \in S$ the solution u of* (1) *satisfies*

$$C_\xi \int_0^T (u_t(\xi,t))^2 dt \geq \|u^0\|^2_{L^2(0,1)} + \|u^1\|^2_{H^{-1}(0,1)}, \tag{3}$$

where $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$, C_ξ is a constant depending only on ξ.

In the remainder of this paper, ξ is fixed and belongs to $\mathcal{S}$. In this paper, we are interested in time discretization of system (1). The analysis of observability properties of numerical approximation schemes for the wave equation has been the object of intensive studies. However most analytical results concern the case of exact observability for discrete systems ([2, 7]). Recently in [3, 4], time semi-discretization of polynomial observability was analyzed. The author shows that a filtering technique allows to restore a uniform (with respect to the parameter of discretization) polynomial observability for the discrete model. But there is no result provided the necessity of this method. Consequently the main goal of our note is to give a counterexample which proves that uniform polynomial observability fails without filtering the initial data for time semi-discrete approximations of the wave equation.

2 Non uniform polynomial observability

We set the time step Δt by $\Delta t = T/(N+1)$, where $N > 0$ is a given integer. Denote by u_k the approximation of the solution u of system (1) at time $t_k = k\Delta t$, for any $k = 0, \dots, N+1$. We then introduce the following trapezoidal time semi-discretization of system (1)

$$\begin{cases} \frac{u_{k+1}+u_{k-1}-2u_k}{(\Delta t)^2} - \frac{\partial^2}{\partial x^2}\left(\frac{u_{k+1}+u_{k-1}}{2}\right) = 0, & k = 1, \dots, N,\ 0 < x < 1, \\ u_k(0) = u_k(1) = 0, & k = 0, \dots, N+1, \\ u_0 = u^0,\ u_1 = u^0 + (\Delta t)u^1, & 0 < x < 1. \end{cases} \tag{4}$$

Here $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$ are the initial data given in system (1). As in the continuous case, we will check an observability inequality for system (4) which can be formulated as follows:

we must find positive constant C such that we have

$$C\Delta t \sum_{k=0}^{N} \left| \frac{u_{k+1}(\xi) - u_k(\xi)}{\Delta t} \right|^2 \geq \|(u_0, u_1)\|^2_{L^2(0,1)\times H^{-1}(0,1)} \tag{5}$$

for all $(u_0, u_1) \in H_0^1(0,1) \times L^2(0,1)$. But there is not the case. Indeed, as in [6], we will choose a particular initial data which don't satisfy (5) uniformly with respect to the discretization parameter. The following theorem provides a quantitative statement of this negative result.

Theorem 1. *For all $T > 0$, there exist a positive constant $C(T, \Delta t)$ and initial data $(u_0, u_1) \in H_0^1(0,1) \times L^2(0,1)$, such that the solution u_k of* (4) *satisfies*

$$C(T,\Delta t)\Delta t \sum_{k=0}^{N} \left| \frac{u_{k+1}(\xi) - u_k(\xi)}{\Delta t} \right|^2 \leq \|(u_0, u_1)\|^2_{L^2(0,1)\times H^{-1}(0,1)}.$$

Proof. We denote by $(\mu_j^2)_{j\geq 1}$ the eigenvalues of the Dirichlet Laplacian and $(\varphi_j)_{j\geq 1}$ the corresponding eigenvectors. Assume that

$$u_0 = \sum_{j=1}^{\infty} a_j \varphi_j, \quad u_1 = \sum_{j=1}^{\infty} (a_j + b_j \Delta t)\varphi_j.$$

Then, by proceeding as in Lemma 2.2 of [6], we easily show that the solution of system (4) is given by

$$u_k = \sum_{j=1}^{\infty} r_j^k \varphi_j, \tag{6}$$

where

$$r_j^k = e^{-iw_j k} \frac{(e^{iw_j} - 1)a_j - \Delta t b_j}{2i \sin(w_j)} + e^{iw_j k} \frac{(1 - e^{iw_j})a_j + \Delta t b_j}{2i \sin(w_j)},$$

and

$$w_j = \arccos\left(\frac{1}{1 + \Delta t^2 \mu_j^2 / 2}\right).$$

If a_j and b_j are chosen so that $(e^{iw_j} - 1)a_j = \Delta t b_j$ for $j = 1, 2, \ldots,$ then

$$u_k = \sum_{j=1}^{\infty} a_j e^{iw_j k} \varphi_j.$$

Now, by using continuous fractions (see [5] and references therein for details) we construct a sequence $(q_m) \subset \mathbb{N}$ such that $q_m \to \infty$ and

$$|\sin(q_m \pi \xi)| \leq \frac{\pi}{q_m} \qquad \text{for all} \quad m \geq 1. \tag{7}$$

Since $q_m \to +\infty$ as $m \to +\infty$, one can choose a $m_0 = m_0(\Delta t)$ such that

$$\frac{1}{(\Delta t)^{\frac{3}{2}}} \leq q_{m_0}, \tag{8}$$

which leads to

$$q_{m_0} \Delta t \to +\infty, \quad \text{as} \ \ \Delta t \to 0. \tag{9}$$

We choose $u_0 = a_{q_{m_0}} \varphi_{q_{m_0}}$, $u_1 = a_{q_{m_0}} e^{iw_{q_{m_0}}} \varphi_{q_{m_0}}$, then $u_k = a_{q_{m_0}} e^{ikw_{q_{m_0}}} \varphi_{q_{m_0}}$, $k \geq 0$. A simple calculations give $\|u_0\|^2_{L^2(0,1)} = a^2_{q_{m_0}}/2$ and $\|u_1\|^2_{H^{-1}(0,1)} = a^2_{q_{m_0}}/2\mu^2_{q_{m_0}}$. On the other hand, one has

$$\left|\frac{u_{k+1}(\xi) - u_k(\xi)}{\Delta t}\right|^2 = \frac{2a^2_{q_{m_0}}}{(\Delta t)^2} \varphi^2_{q_{m_0}}(\xi)(1 - \cos(w_{q_{m_0}})),$$

and then, since $(N+1) = T/\Delta t$,

$$\Delta t \sum_{k=0}^{N} \left|\frac{u_{k+1}(\xi) - u_k(\xi)}{\Delta t}\right|^2 = \frac{2Ta^2_{q_{m_0}} \mu^2_{q_{m_0}} \varphi^2_{q_{m_0}}(\xi)}{2 + (\mu_{q_{m_0}} \Delta t)^2}.$$

Using (7), we get

$$C(T, \Delta t) \Delta t \sum_{k=0}^{N} \left|\frac{u_{k+1}(\xi) - u_k(\xi)}{\Delta t}\right|^2 \leq \|(u_0, u_1)\|^2_{L^2(0,1) \times H^{-1}(0,1)},$$

where $C(T, \Delta t) = (2 + (\mu_{q_{m_0}} \Delta t)^2)/4T\pi^4$. □

The above inequality and (9) claim that (5) fails uniformly with respect to the discretization parameter. Indeed, it is clear that $C(T, \Delta t) \to +\infty$ as $\Delta t \to 0$, and then

$$\frac{\|(u_0, u_1)\|^2_{L^2(0,1) \times H^{-1}(0,1)}}{\Delta t \sum_{k=0}^{N} \left|\frac{u_{k+1}(\xi) - u_k(\xi)}{\Delta t}\right|^2} \to +\infty \ \ \text{as} \ \ \Delta t \to 0.$$

Consequently, filtering the initial data is needed to obtain (5) uniformly with respect to the discretization parameter.

3 FILTERING MECHANISM

We first transform system (4) into a first order time-discrete scheme as in [2]. For simplicity, we denote $A_0 = -\partial^2/\partial x^2$. We have

$$(I + \frac{\Delta t^2}{2} A_0)(u_{k+1} + u_{k-1}) - 2u_k = 0,$$

then

$$(I + \frac{\Delta t^2}{2} A_0)(u_{k+1} + u_{k-1} - 2u_k) = -\Delta t^2 A_0 u_k,$$

which gives

$$(I + \frac{\Delta t^2}{4} A_0)(u_{k+1} + u_{k-1} - 2u_k) = -\Delta t^2 A_0 (\frac{u_{k+1} + u_{k-1} + 2u_k}{4}).$$

Consequently (4) can be rewritten as

$$\frac{u_{k+1} + u_{k-1} - 2u_k}{(\Delta t)^2} + A_1 (\frac{u_{k+1} + u_{k-1} + 2u_k}{4}), \tag{10}$$

with $A_1 = A_0 (I + \frac{\Delta t^2}{4} A_0)^{-1}$. Now using the following change of variables

$$\begin{cases} y^1_{k+1} = \frac{u_{k+1} - u_k}{\Delta t} + i A_1^{1/2} (\frac{u_{k+1} + u_k}{2}), \\ \\ y^2_{k+1} = \frac{u_{k+1} - u_k}{\Delta t} - i A_1^{1/2} (\frac{u_{k+1} + u_k}{2}), \end{cases}$$

we obtain

$$\begin{cases} \frac{y_{k+1} - y_k}{\Delta t} = A (\frac{y_{k+1} + y_k}{2}), \\ \\ y_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \end{cases} \tag{11}$$

with

$$A = \begin{pmatrix} i A_1^{1/2} & 0 \\ 0 & -i A_1^{1/2} \end{pmatrix}, \quad y^{k+1} = \begin{pmatrix} y^1_{k+1} \\ \\ y^2_{k+1} \end{pmatrix}. \tag{12}$$

Note that the spectrum of A is explicitly given by the spectrum of A_0. More precisely, the eigenvalues of A are $i\lambda_j$ with corresponding eigenvectors

$$\varphi_j = \begin{pmatrix} \varphi_j \\ 0 \end{pmatrix}, \quad \varphi_{-j} = \begin{pmatrix} 0 \\ \varphi_j \end{pmatrix}, \quad j \in \mathbb{N}^*,$$

where $\lambda_j = \mu_j / \sqrt{1 + \Delta t^2 \mu_j^2 / 4}$. Moreover we define

$$\mathcal{C}_s = \text{span}\{\varphi_j \text{ such that } \quad \mu_j \leq s\}.$$

We are ready to prove the following uniform boundary polynomial observability of the time discrete wave equation.

Theorem 2. *For any $\delta > 0$, there exists $T_\delta > 0$ such that for any $T > T_\delta$, there exists a positive constant $C = C_{T,\delta}$, independent of Δt, such that for Δt small enough, the solution u_k of* (4) *satisfies*

$$C\Delta t \sum_{k=0}^{N} \left| \frac{u_{k+1}(\xi) - u_k(\xi)}{\Delta t} \right|^2 \geq \|(u_0, u_1)\|^2_{L^2(0,1)\times H^{-1}(0,1)} \qquad \text{for all} \quad (u_0, u_1) \in \mathcal{C}^2_{\delta/\Delta t}. \tag{13}$$

Proof. We have, for all $k \neq l$, $|\lambda_k - \lambda_l| = |f(\mu_k) - f(\mu_l)|$, where f is defined by $f(t) = t/(\sqrt{1 + (t^2\Delta t^2)/4})$. Applying the mean value theorem to the function f, there exists a point c between μ_k and μ_l such that

$$|\lambda_k - \lambda_l| = |f'(c)||\mu_k - \mu_l|.$$

Simple calculations give that $f'(c) = 1/(1 + \frac{\Delta t^2 c^2}{4})^{3/2}$. It is easy to check that $|f'(c)| \geq 1/(1 + \frac{\delta^2}{4})^{3/2}$, and $|\mu_k - \mu_l| \geq \pi$ for all $k \neq l$. Consequently there exists $\gamma > 0$ such that, for all $k \neq l$ $|\lambda_k - \lambda_l| \geq \gamma$. Besides, we have (see [1]) $|\sin(j\pi\xi)| \geq \frac{\nu}{j}$, for all $j \geq 1$, for some $\nu > 0$, and then $|\sin(j\pi\xi)| \geq \frac{\theta}{\lambda_j}$, for all $j \geq 1$, with $\theta = \nu\pi/\sqrt{1 + \frac{\delta^2}{4}}$. Hence, applying Proposition 2.5 of [3], we obtain the desired result. □

Remark 1. *In the last proof, we used Proposition 2.5 of [3] in which we assumed that the damping operator is bounded, but this assumption is not needed in the proof of Proposition 2.5, and the result still correct even if the dissipation is unbounded.*

4 OPEN PROBLEMS

1. In this paper we dealt with the polynomial observability of time discrete wave equation. The question of space semi-discrete polynomial observability for wave equation still open. Another interesting open problem is whether the fully discrete schemes have these properties of observability uniformly with respect to the discretization parameters.

2. At the continuous case, it is well-known that polynomial observability implies polynomial stability for associated dissipative system (see [1]). At the discreet level, the only result excitant, in this context, is [3] which deals with bounded dissipation. However the situation is complicated when the dissipation is unbounded, as for example the case of wave equation with punctual dissipation (which correspond to the associated dissipative system of (1)–(2)), and this issue requires further work.

3. Other question arise when discretizing in time and/or in space semilinear dissipative wave equations. It would be interesting to analyze the uniform (with respect to the steps) decay properties of solutions when the conservative system satisfies a polynomial observability inequality. Actually, this question is also open at the continuous level.

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Хаджеж З. *Про необхідність механізму фільтрації для поліноміального дослідження часово дискретних хвильових рівнянь* // Карпатські матем. публ. — 2017. — Т.9, №1. — С. 98–103.

У статті проаналізовано питання поліноміального дослідження. Показано, що якщо неперевні моделі є рівномірно поліноміально досліджувані, то достатньо відфільтрувати початкові дані для виокремлення поліноміально досліджувальних нерівностей у відповідних часово дискретизованих схемах. У зв'язку з цим ми доводимо, що механізм фільтрування частотних модулів є необхідним для існування рівномірного поліноміального дослідження.

А саме, побудовано контрприклад, який показує, що процедура дослідження пізніше не реалізується без початкового фільтрування даних у напівдискретній апроксимації хвильового рівняння.

Ключові слова і фрази: нерівність спостереження, часова дискретизація, техніки фільтрації.

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Богдан Йосипович Пташник

28.09.1937 — 22.02.2017

22 лютого 2017 року перестало битися серце Богдана Йосиповича Пташника — видатного українського математика, професора, члена-кореспондента НАН України, дійсного члена Наукового товариства імені Шевченка, почесного доктора Прикарпатського національного університету імені Василя Стефаника.

Богдан Йосипович Пташник народився 28 вересня 1937 року в селищі Богородчани Станіславської (нині Івано-Франківської) області. У 1959 році закінчив з відзнакою фізико-математичний факультет Станіславського державного педагогічного інституту, здобувши спеціальність "учитель математики і фізики середньої школи". Вже у студентські роки він робить перші вагомі кроки у науці. Науково-дослідницька робота третьокурсника Богдана Пташника "Вирощування монокристалів цинку" була відзначена грамотою Оргкомітету міжобласного огляду наукових робіт студентів. Математикою майбутній член-кореспондент НАН України почав серйозно займатися з четвертого курсу під керівництвом В.П. Заровного, учня відомого геометра О.С. Смогоржевського. За роботу "Послаблення умов, що забезпечують виконання в абстрактній групі аксіоми паралельності Евкліда", виконану у 1959 році, молодого науковця нагороджено грамотою Міністерства вищої і середньої спеціальної освіти УРСР.

Трудову діяльність Б.Й. Пташник розпочав у серпні 1959 року вчителем математики середньої школи у селі Росільна Богородчанського району Станіславської області. У 1961–1963 рр. працював асистентом кафедри математики Станіславського державного педагогічного інституту. У 1963–1966 рр. навчався в аспірантурі відділу диференціальних рівнянь Інституту математики АН УРСР. Науковим керівником молодого науковця був професор Віталій Якович Скоробогатько. Після закінчення аспірантури у 1966–1969 рр. Б.Й. Пташник працював асистентом кафедри диференціальних рівнянь Львівського державного університету імені Івана Франка. У цьому ж університеті 26 лютого 1968 р. він захистив дисертацію "Задача Валле–Пуссена та деякі крайові задачі для лінійних гіперболічних рівнянь" на здобуття наукового ступеня кандидата фізико-математичних наук за спеціальністю "диференціальні та інтегральні рівняння".

З 1969 року до останнього дня свого життя Б.Й. Пташник працював в установах Академії наук України. У 1969–1972 роках — старший науковий співробітник відділу теорії диференціальних рівнянь Фізико-механічного інституту АН України (м. Львів), з 1973 року — старший науковий співробітник Львівського філіалу відділу математичної фізики Інституту математики АН України (з 1978 року — Інституту прикладних проблем механіки і математики АН України). З 1982 по 1990 рік очолював лабораторію некласичних задач математичної фізики Інституту прикладних проблем механіки і математики АН України. 4 квітня 1989 р. в Інституті математики АН України (м. Київ) захистив дисертацію "Некласичні крайові задачі для диференціальних рівнянь із частинними похідними" на здобуття наукового ступеня доктора фізико-математичних наук за спеціальністю "диференціальні рівняння", а у 1990 році йому присвоєно вчене звання професора. З 1990 року — завідувач відділу математичної фізики та керівник математичного сектора Інституту прикладних проблем механіки і математики ім. Я.С. Підстригача НАН України, а з 2003 року — голова секції теоретичних і прикладних проблем математики при Вченій раді цієї наукової установи.

Професор Б.Й. Пташник — автор понад 200 наукових праць з теорії диференціальних рівнянь із частинними похідними, теорії гіллястих ланцюгових дробів та історії математики, зокрема, трьох монографій. Найбільш вагомі, всесвітньо визнані результати отримані ним у теорії рівнянь із частинними похідними. Під його керівництвом успішно захищено 18 кандидатських і 3 докторські дисертації.

Разом із учнями Б.Й. Пташник розробив оригінальні методи дослідження коректності та побудови розв'язків багатьох некласичних задач для рівнянь і систем рівнянь із частинними похідними, а також для диференціально-операторних рівнянь, зокрема, задач з локальними багатоточковими умовами, з умовами типу умов Діріхле, задач про періодичні та майже періодичні розв'язки, нелокальних крайових та багатоточкових задач. Такі задачі є, взагалі, некоректними, а їх розв'язність у багатьох випадках пов'язана з проблемою малих знаменників. При дослідженні цих задач для гіперболічних, параболічних і безтипних рівнянь та систем рівнянь виникли малі знаменники складної нелінійної структури, оцінювання знизу яких призвело до нових, раніше не розв'язаних, задач метричної теорії чисел. У роботах Б.Й. Пташника встановлено умови існування, єдиності та неперервної залежності від правих частин рівнянь і крайових умов розв'язків наведених задач у різних функціональних просторах, а також побудовано явні формули для розв'язків у вигляді узагальнених рядів Фур'є за системами ортогональних функцій та розроблені алгоритми знаходження наближених розв'язків. На відміну від робіт інших авторів, у роботах професора Б.Й. Пташника не тільки аксіоматично накладаються умови на малі знаменники, що забезпечує розв'язність задачі, але й доводяться твердження метричного характеру про оцінки знизу малих знаменників, з яких випливає однозначна розв'язність задачі для майже всіх (стосовно міри Лебега) векторів, компоненти яких виражаються через параметри області, коефіцієнти рівнянь і крайових умов. Ці дослідження Б.Й. Пташника стали новим етапом розвитку загальної теорії крайових задач, стимулювали розвиток нових аспектів теорії умовно коректних задач і метричної теорії чисел.

Своїми знаннями й досвідом Богдан Йосипович Пташник щедро ділився з молоддю. Упродовж багатьох років він читав спецкурси та керував науковою роботою магістрів, аспірантів і докторантів у Прикарпатському національному університеті імені Василя Стефаника, Національному університеті "Львівська політехніка", у Львівському національному університеті імені Івана Франка, Інституті прикладних проблем механіки та математики ім. Я.С. Підстригача НАН України, був професором Національного університету "Львівська політехніка", керівником Львівського міського семінару з диференціальних рівнянь та загальноінститутського математичного семінару Інституту прикладних проблем механіки та математики ім. Я.С. Підстригача НАН України, членом редколегій провідних математичних журналів "Український математичний журнал", "Математичні методи та фізико-механічні поля", "Математичні студії" та "Карпатські математичні публікації", членом фізико-математичної секції Наукового товариства імені Шевченка, у рамках якої здійснював дослідження з історії математики у Галичині.

Б.Й. Пташник був організатором і натхненником всеукраїнських наукових конференцій "Нові підходи до розв'язування диференціальних рівнянь" (м. Дрогобич) та "Нелінійні проблеми аналізу" (м. Івано-Франківськ). У 2007 р. його обрано почесним доктором Прикарпатського національного університету імені Василя Стефаника — навчального закладу, де він робив свої перші кроки у науці. У 1989 р. Б.Й. Пташника обрано членом, а у 2006 р. — дійсним членом Наукового товариства імені Шевченка. У 2002 р. його обрано академіком Академії наук вищої школи України, у 2003 р. — членом-кореспондентом НАН України.

Багато сил і часу віддавав Б. Й. Пташник науково-організаційній роботі з координації наукових досліджень та підготовки наукових кадрів високої кваліфікації з математики в Західному регіоні України. Починаючи з 1976 року, він був секретарем секції механіки і математики, головою секції математики (1991–2000 рр.), заступником голови секції математики і математичного моделювання (2001–2006 рр.), а з 2007 року — керівником відділення фізико-технічних і математичних наук та керівник секції математики і математичного моделювання Західного наукового центру НАН України та МОН України.

Його життєвий шлях — високий приклад самовідданого служіння рідній землі та науці. Друзі, колеги та учні Богдана Йосиповича Пташника з глибокою вдячністю пам'ятатимуть його яскраву постать, вплив якої на розвиток української науки неможливо переоцінити. Пам'ять про Вченого, Вчителя, Педагога, Українця назавжди залишиться у серцях усіх тих, хто його знав.

П.Б. Василишин, О.Д. Власій, Т.П. Гой, А.В. Загороднюк, Р.А. Заторський, А.І. Казмерчук, М.І. Копач, В.В. Мазуренко, О.В. Махней, Г.П. Малицька, О.Р. Никифорчин, М.М. Осипчук, В.М. Пилипів, І.Я. Савка, М.М. Симотюк, П.В. Філевич, С.В. Шарин.