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THE NONLOCAL BOUNDARY PROBLEM WITH PERTURBATIONS OF ANTIPERIODICITY CONDITIONS FOR THE ELIPTIC EQUATION WITH CONSTANT COEFFICIENTS

In this article, we investigate a problem with nonlocal boundary conditions which are perturbations of antiperiodical conditions in bounded m-dimensional parallelepiped using Fourier method. We describe properties of a transformation operator $R : L_2(G) \to L_2(G)$, which gives us a connection between selfadjoint operator L_0 of the problem with antiperiodical conditions and operator L of perturbation of the nonlocal problem $RL_0 = LR$.

Also we construct a commutative group of transformation operators $\Gamma(L_0)$. We show that some abstract nonlocal problem corresponds to any transformation operator $R \in \Gamma(L_0) : L_2(G) \to L_2(G)$ and vice versa. We construct a system $V(L)$ of root functions of operator L, which consists of infinite number of adjoint functions. Also we define conditions under which the system $V(L)$ is total and minimal in the space $L_2(G)$, and conditions under which it is a Riesz basis in the space $L_2(G)$.

In case if $V(L)$ is a Riesz basis in the space $L_2(G)$, we obtain sufficient conditions under which the nonlocal problem has a unique solution in the form of Fourier series by system $V(L)$.

Key words and phrases: differential-operator equation, eigenfunctions, Riesz basis.

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1 INTRODUCTION

Investigation of ordinary differential equations with nonlocal integral conditions begins in works of H. Birkoff, A. Zommerfeld, J. Stone, Ya.D. Tamarkin, W. Feller. Fundamental role in development of nonlocal problems and shift operator theory play works of T. Carleman. The general theory of eliptic boundary problems was formed due to investigations of Y.G. Beid and R.S. Friman, R. Bills, F. Brauder, L. Ehrenpreis, L. Hermander, G. Grub, J.W. Kalkin, Ya.B. Lopatynskiy, M. Malgrange, I.V. Skrypnyk, M. Shekhter, M.I. Vishyk. Nonlocal boundary problems for linear differential equations with partial derivatives in different aspects were investigated by Yu.M. Berezanskiy, A.V. Bitsadze, V.M. Borok, M.L. Gorbachuk, O.O. Dezin, Yu.M. Dybinskiy, M.I. Ionkin, V.S. Ilkiv, P.I. Kalenuyk, A.H. Mamyan, V.A. Mykhailets, B.Yo. Ptashnyk, V.K. Romanko, O.A. Samarskiy, O.L. Skubatchevkiy, S.Ya. Yakubov. Nonlocal elliptic problems were studied in works of A.V. Bitsadze, O.O. Dezin, A.I. Kamynin, S.A. Paneyakha, Ya.A. Roytberg and Z.G. Sheftel, A.A. Samarskiy, L.A. Skybatchevkiy and their followers.

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This paper is denoted to research of nonlocal problems for equations with constant coefficients. The classes of uniqueness and existence of the solution of boundary value problems in unbounded domains (half-space, unbounded strip) for equations with constant coefficients were studied in [7,8,11,12,17,22–26].

Boundary value problems in bounded domains for certain classes of differential equations with constant coefficients have been studied in [1,9,10,13–16,18,20,21,27,28]. The work is a continuation of the studies begun in [2–6].

Let us denote $\mathbb{Z}_0 := \{k : k \geq 0,\ k \in \mathbb{Z}\}$, $\beta = (\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{Z}_0^m$, $|\beta| = \beta_1 + \ldots + \beta_m$,

$$G := \{x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : 0 < x_j < X_j < \infty\ , j = 1, 2, \ldots, m\},$$
$$G_r := \{x^r = (x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_m) \in \mathbb{R}^{m-1} : 0 < x_j < X_j < \infty, j \neq r, j = 1, 2, \ldots, m\}.$$

Let D_j be the operator of differentiation by variable x_j. Denote $D^{2\beta} := D_1^{2\beta_1} D_2^{2\beta_2} \cdot \ldots \cdot D_m^{2\beta_m}$, $W_2^{2n}(G) := \{y \in L_2(G) : D^{2\beta} y \in L_2(G), |\beta| = n\}$,

$$(y, z; W_2^{2n}(G)) := \sum_{j=1}^{m} (D_j^{2n} y, D_j^{2n} z; L_2(G)), \quad |y; W_2^{2n}(G)|^2 := \sum_{j=1}^{m} (D_j^{2n} y, D_j^{2n} y; L_2(G)).$$

Also we will use the following notations. Let E_j be the identical transformation in the space $L_2(0, X_j)$; E be the identical transformation in the space $L_2(G)$; I_j be an operator of involution in the space $L_2(0, X_j)$, $I_j z(x) := z(X_j - x), z(x) \in L_2(0, X_j)$; p_j be an orthoprojector in the space $L_2(0, X_j)$; $p_j z(x) := \frac{1}{2}(z(x) + (-1)^j z(X_j - x)), z(x) \in L_2(0, X_j)$; $L_{2,r}(0, X_j) := \{z(x_j) \in L_2(0, X_j) : z(x_j) := p_r z(x)\}$, $r = 0, 1$; $W_{2n}^*\left(0, X_j\right)$ be the space of linear continuous functionals on $W_2^{2n}(0, X_j)$; $W_{2n,s}^*(0, X_j) := \{l \in W_{2n}^*(0, X_j) : l(e^{ihx} - (-1)^s e^{ih(X_j - t)}) = 0,\ h \in \mathbb{R},\ x_j \in (0, X_j)\}$, $s = 0, 1$; $Q_m := \{Q := (q_1, q_2, \ldots, q_m) \in \mathbb{Z}^m, q_r \in \{0, 1\},\ r = 1, 2, \ldots, m\}$; $p_q := \prod_{r=1}^{m} p_{q_r}$ be an orthoprojector in the space $L_2\,(G)$; $L_{2,Q}(G) := \{y \in L_2(G) : y := p_Q y\}$.

Let us consider boundary problem

$$L(D)y := \sum_{|\beta| \leq n} (-1)^{|\beta|} a_\beta D^{2\beta} y = f, \quad x \in G, \tag{1}$$
$$\ell_{s,j} y := D_j^{2s-2} y|_{x_j=0} + D_j^{2s-2} y|_{x_j=X_j} = 0, \quad j = 1, 2, \ldots, m, \tag{2}$$
$$\ell_{n+s,j} y := D_j^{2s-1} y|_{x_j=0} + D_j^{2s-1} y|_{x_j=X_j} + l_{s,j}^1 y = 0, \quad j = 1, 2, \ldots, m, \tag{3}$$

where

$$\ell_{s,j}^1 y := \sum_{r=0}^{1} \sum_{q=0}^{m_{s,j}} b_{q,r,s,j} D_j^q y_{x_j = r X_j}, \tag{4}$$

$b_{q,r,s,j} \in \mathbb{R},\ q = 0, 1, \ldots, m_{s,j},\ r = 0, 1,\ s = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m.$

Let us denote by $L : L_2(G) \to L_2(G)$ the operator of problem (1)–(4), $Ly := L(D)y$, $y \in D(L), D(L) := \{y \in W_2^{2n}(G) : \ell_{s,j} y = 0,\ s = 1, 2, \ldots, 2n,\ j = 1, 2, \ldots, m\}$.

Definition 1. *We will denote by function $y \in D(L)$ a solution of the problem* (1)–(4) *that satisfies* $\|Ly - f; L_2(G)\| = 0$.

2 General results

Let us consider the following assumptions

$$P_1 : b_{q,1,s,j} = (-1)^q b_{q,0,s,j};$$
$$P_2 : m_{s,j} \leq 2s-1, \quad s = 1,2,\ldots,n;$$
$$P_3 : |\lambda_k| \geq C_1|k|^{2n} > 0,\ 0 < C_1 < \infty,\ k \in \mathbb{N}^m;$$
$$P_4 : p_1X_1 + p_2X_2 + \ldots = p_mX_m \neq 0,\ p_j \in \mathbb{Z},\ j = 1,2,\ldots,m.$$

Theorem 1. *Let P_1 holds. Then for arbitrary numbers $a_\beta \in \mathbb{R}, |\beta| \leq n$, the operator L has eigenvalues*

$$\lambda_k := \sum_{|\beta|\leq n} a_\beta \prod_{j=1}^{m} \rho_{k,j}^{2\beta_j}, \tag{5}$$

$\rho_{k,j} := (2k_j - 1)\pi X_j^{-1}, j = 1,2,\ldots,m, k = (k_1,\ k_2,\ldots,k_m) \in \mathbb{N}^m$, and a complete and minimal system $V(L)$ of root functions exists in the space $L_2(G)$.

Theorem 2. *Let P_1–P_2 hold. Then the operator L has a system $V(L)$ of root functions, which is a Riesz basis for the space $L_2(G)$.*

Theorem 3. *Let the assumptions P_1–P_2 take place. Then for any function $f \in L_2(G)$ there exists a unique solution of the problem* (1)–(4).

3 Self-adjoint problem for ordinary differential equations of even order

Let us denote by $A_{0,j}$ the operator generated in $L_2\left(0, X_j\right)$ by the next boundary problem $-z^{(2)}(x_j) = g(x_j),\ x_j \in (0, X_j), z(0) + z(X_j) = 0,\ z^{(1)}(0) + z^{(1)}(X_j) = 0,\ j = 1,2,\ldots,m$, $A_{0,j}y(x_j) := -y^{(2)}(x_j),\ y \in D(A_{0,j})$ and $D(A_{0,j}) := \{y \in W_2^2(0, X_j) : y^{(r)}(0) + y^{(r)}(X_j) = 0,\ r = 0,1\}$, $\tau_{1,k_j}(x_j) := \frac{2}{\sqrt{2X_j}} \cos \rho_{k,j}x_j,\ k_j = 1,2,\ldots,\ \tau_{0,k_j}(x_j) := \frac{2}{\sqrt{2X_j}} \sin \rho_{k,j}x_j,\ k_j = 1,2,\ldots$, $\tau_j := \{\tau_{r,k_j}(x_j) \in L_2(0, X_j),\ r = 0,1,\ k = 1,2,\ldots\}$ is an orthonormal basis of the space $L_2(0, X_j),\ j = 1,2,\ldots,m$.

Lemma 1. *The operator $A_{0,j}$ has the point spectrum*

$$\sigma(A_{0,j}) := \{\mu_{k,j} \in \mathbb{R} : \mu_{k,j} = \rho_{k,j}^2,\ k = 1,2,\ldots\}$$

and a system of eigenfunctions T_j. Sets $L_{2,s}(0, X_j)$ are invariant for the operator $A_{0,j},\ s = 0,1$.

Proof. By substitution we obtain that $\tau_{r,k_j}(x_j) \in D(A_{0,j})$ and $A_{0,j}\tau_{r,k_j}(x_j) = \mu_{k,j}\tau_{r,k_j}(x_j),\ r = 0,1, k = 1,2,\ldots$.

Therefore operator $A_{0,j}$ has a system of eigenfunctions T_j, which corresponds to the set of eigenvalues $\sigma\left(A_{0,j}\right)$.

Let us notice that subset of eigenfunctions $T_{j,r} := \{\tau_{r,k_j}(x_j) \in L_2(0, X_j),\ k = 1,2,\ldots\}$, $A_{0,j}$ is an orthonormal basis in the space $L_{2,r}(0, X_j),\ r = 0,1.$ □

Let us consider for equation (1) the following problem with boundary conditions

$$\ell_{0,s,j}y := D_j^{2s-2}\, y|_{x_j=0} + D_j^{2s-2}\, y|_{x_j=X_j} = 0,\ s = 1,2,\ldots,n,\ j = 1,2,\ldots,m, \tag{6}$$

$$\ell_{0,n+s,j}y := D_j^{2s-1}\, y|_{x_j=0} + D_j^{2s-1}\, y|_{x_j=X_j} = 0,\ s = 1,2,\ldots,n,\ j = 1,2,..,m. \tag{7}$$

Let $L_0 : W_2^{2n}(G) \to W_2^{2n}(G)$ be the operator of problem (1), (6), (7). Also we denote by $L_0 y := L(D)y,\ y \in D\,(L_0)\,;\ D(L_0) := \{y \in W_2^{2n}(G) : \ell_{0,s,j}y = 0,\ s = 1,2,\ldots,2n,\ j = 1,2,\ldots,m\}$; $V(L_0) := \{v_{0,r,k}(x) \in L_2(G) : v_{0,r,k}(x) := \prod\limits_{j=1}^{m} \tau_{r_j,k_j}(x_j),\ r_j \in \{0,1\},\ j = 1,2,\ldots,m,\ k \in \mathbb{N}^m\}$ the orthonormal basis of the space $L_2(G)$; $L_{0,Q}$ the restriction of the operator L_0 to the space $L_{2,Q}(G)$ and

$$V_Q := \{v_{0,q,k}(x) \in L_2(G) : v_{0,k}(x) := \prod_{j=1}^{m} \tau_{q_j,j}(x_j),\ k_j = 2k_j - j_r,\ k \in \mathbb{N}^m\},\ Q \in Q_m.$$

Lemma 2. *The operator L_0 has eigenvalues* (5) *and a system of eigenfunctions $V(L_0)$.*

Proof. By a substitution it is easy to check that $v_{0,r,k}(x) \in D(L_0)$, $L_0 v_{0,r,k}(x) = \lambda_k v_{0,r,k}(x)$, $k \in \mathbb{N}^m$.

Therefore, the operator L_0 has a system of eigenfunctions $V(L_0)$ which corresponds to the set of eigenvalues $\sigma(L_0) := \{\lambda_k \in \mathbb{R},\ k \in \mathbb{N}^m\}$. □

4 NONSELFADJOINT PROBLEM OF ORDINARY DIFFERENTIAL EQUATION OF THE SECOND ORDER

Let us consider the following spectral boundary problem

$$-z^{(2)}(x_j) = \mu z(x_j),\ x_j \in \{0, X_j\},\ \mu \in \mathbb{C}, \tag{8}$$

$$z(0) + z(X_j) = 0,\ z^{(1)}(0) + z^{(1)}(X_j) + b(z(1)(0) - z(1)(X_j)) = 0. \tag{9}$$

Let $B_j = B_{j,b}$ be the operator of problem (8), (9). Solutions $\pm\rho$ of the characteristic equation $-\rho^2 = \lambda$ are such that $Re\mu \le 0$.

We define the fundamental system of solutions of the equation (8) by equations

$$z_r(x_j,\rho) := \exp \imath\rho x_j + (-1)^r \exp \imath\rho(X_j - x_j) \in L_{2,r}(0, X_j),\ r = 0,1.$$

The general solution of equation (8) can be represented as the sum

$$z\left(x_j,\rho\right) := c_0 z_0\left(x_j,\rho\right) + c_1 z_1\left(x_j,\rho\right).$$

If we substitute this solution into boundary conditions (9), we obtain an equation which roots define eigenvalues of the operator B_j

$$\Delta(\rho) := \Delta_0(\rho)\Delta_1(\rho) = 0, \tag{10}$$

where $\Delta_0(\rho) = (1 + \exp \imath\rho X_j)$, $\Delta_1(\rho) = \imath\rho(1 + \exp \imath\rho X_j)$.

Equation (10) has two-fold roots $\pm\rho_{k_j}, \rho_{k_j} := (2k-1)\pi X_j^{-1}$, $k = 1,2,\ldots$. Therefore the operator B_j has two-fold eigenvalues $\mu_{k,j} = ((2k_j - 1)\pi X_j^{-1})^2$, $k_j = 1,2,\ldots$. Since $\tau_{2k-1,j}(x_j) \in D(B_j)$ and $B_j \tau_{2k-1,j}(x_j) = \mu_{2k-1,j}\tau_{2k-1,j}(x_j)$, $k_j = 1,2,\ldots,$ we define eigenfunction of the operator B_j by the formula

$$v_{1,k}\left(x_j, B_j\right) = \frac{2}{\sqrt{2X_j}} \cos \rho_{k,j} x_j,\ k = 1,2,\ldots.$$

We define an adjoint function of the operator B_j by the following relation

$$v_{0,k}(x_j, B_j) = (1 + c_j(2x_j - X_j))\frac{2}{\sqrt{2X_j}} \sin \rho_{k_j} x_j,\ c_j \in \mathbb{C}.$$

If we substitute this expression to boundary condition (9) we define $c_j = b$.

So the operator B_j has an adjoint function

$$v_{0,k}(x_j, B_j) = (1 + b(2x_j - X_j))\frac{2}{\sqrt{2X_j}} \sin \rho_{k_j} x_j. \tag{11}$$

Root functions of the operator B_j are defined by equations

$$B_j v_{0,k}(x_j, B_j) = \mu_{k,j} v_{0,k}(x_j, B_j) + \xi_{k,j} v_{1,k}(x_j, B_j),\ \xi_{k,j} = 4b\rho_{k,j},\ k_j = 1, 2, \ldots, \tag{12}$$

$$B_j v_{2k-1}(x_j, AB_j) = \mu_{k,j} v_{2k-1}(x_j, B_j),\ k = 1, 2, \ldots. \tag{13}$$

Since the boundary conditions (9) are regular by Birkhoff, from Shkalikov's theorem [29] we obtain: the system $V(B_j)$ is total and minimal in the space $L_2(0, X_j)$ for all $b \in \mathbb{R}$, $j = 1, 2, \ldots, m$.

Let us prove that $V\left(B_j\right)$ is a Bessel system. Summands in the formula (11) are orthogonal in the space $L_2(0, X_j)$. Therefore for any function $h \in L_2(0, X_j)$ we have

$$|(h, v_{0,k}(x_j, B_j); L_2(0, X_j)|^2 \leq (1 + 2|b|^2 X_j)||(h, \tau_{2k}(x_j); L_2(0, X_j)|^2,$$
$$|(h, v_{1,k}(x_j, B_j; L_2(0, X_j)|^2 = |(h, \tau_{2k-1}(x_j); L_2(0, X_j)|^2.$$

If we consider the sum for $k = 1, 2, \ldots,$ we have inequality

$$\sum_{r=0}^{1} \sum_{k=1}^{\infty} |(h, v_{r,k}(x_j, B_j); L_2(0, X_j)|^2 \leq C_2 |h; L_2(0, X_j)|^2,\ C_2 = 1 + 2|b|^2 X_j.$$

Therefore $V(B_j)$ is the Bessel system [19] in the space $L_2(0, X_j)$.

Analogously we can prove that the biorthogonal system which consists of root functions of adjoint problem

$$-z^{(2)}(x_j) = \overline{\mu} z(x_j),\ z^{(1)}(0) + z^{(1)}(X_j) = 0,\ z(0) + z(X_j) + b(z(0) - z(X_j)) = 0$$

is Bessel system in the space $L_2(0, X_j)$. Therefore if we apply Bari's theorem [19] we obtain the following lemma.

Lemma 3. *For any fixed $b_j \in \mathbb{R}$ spectra of operators B_j, $A_{0,j}$ coincide and system of functions $V(B_j)$ forms a Riesz basis in the space $L_2(0, X_j)$, $j = 1, 2, \ldots, m$.*

5 Transformation operators of ordinary differential equation of second order

Let us consider any sequence of real numbers $\{\theta_{k_j}\}_{k_j=1}^{\infty}$ and consider in the space $L_2(0, X_j)$ operator $A_{1,j}$. An eigenvalues of this operator coinside with eigenvalues of the operator $A_{0,j}$ and root functions can be defined by equations

$$v_{1,k,j}(x_j, A_{1,j}) = \frac{2}{\sqrt{2X_j}} \cos \rho_{k,j} X_j^{-1} x_j, \tag{14}$$

$$v_{0,k,j}(x_j, A_{1,j}) = (1 + \theta_{k_j}(2x_j - 1))\frac{2}{\sqrt{2X_j}} \sin \rho_{k,j} X_j^{-1} x_j,\ k_j = 1, 2, \ldots. \tag{15}$$

Let $R\left(A_{1,j}\right) = E_j + S\left(A_{1,j}\right)$ be the operator which acts by rule $V(A_{0,j}) \to V(A_{1,j})$. From definition of the operator $R(A_{1,j})$ we obtain: $S^2(A_{1,j}) = 0$. Therefore $R^{-1}(A_{1,j}) = E_j - S(A_{1,j})$ exists.

Lemma 4. *For any sequence* $\{\theta_{k_j}\}_{k_j=1}^{\infty} \subset \mathbb{R}$ *the system of functions* $V\left(A_{1,j}\right)$ *are total and minimal in the space* $L_2(0, X_j)$.

Proof. Let us suppose that function $h = h_0 + h_1$, $h_r \in L_{2,r}(0, X_j)$ exists and is orthogonal to all elements of the system $V(A_{1,j})$. Since functions (14) are elements of orthnormal basis of the space $L_{2,1}(0, X_j)$, we obtain $h_1 = 0$ So $h = h_0 \in L_{2,0}(0, X_j)$.

Since function h is orthogonal to elements of the system $V(A_{1,j})$, we have:

$$(h, v_{0,k}(x_j, A_{1,j}); L_2(0, X_j)) = (h_0, \tau_{0,k,j}(x_j); L_2(0, X_j)) = 0,\ k = 1, 2, \dots.$$

The system $T_{j,0} = \{\tau_{0,k,j}(x_j) \in L_2(0, X_j),\ k = 1, 2, \dots\}$ is an orthonormal basis in the space $L_{2,0}(0, X_j)$. So we obtain that $h_1 = 0$.

Therefore $h \equiv 0$. □

Lemma 5. *The system of functions* $V(A_{1,j})$ *is a Riesz basis in the space* $L_2(0, X_j)$ *if and only if the sequence* $\{\theta_k\}_{k=1}^{\infty}$ *is bounded.*

Proof. Necessity. If the system of functions $V\left(A_{1,j}\right)$ is a Riesz basis in the space $L_2(0, X_j)$, then it is almost normalized.

If we take into consideration (14), (15), we have inequality

$$0 < 1 \le \|v_{0,q}(x_j, A_{1,j}); L_2(0, X_j)\|^2 = 1 + |\theta_q|^2 \le C_3 < \infty,\ C_3 := 1 + \max \theta_q^2.$$

Sufficiency. If we take into consideration formulas (12), (13) then for any functions $h \in L_2(0, X_j)$ we have inequality

$$\sum_{k_j=1}^{\infty} \sum_{s=0}^{1} \|(R(A_{1,j}^*)h, \tau_{s,k_j}(x_j); L_2(0, X_j\|^2$$
$$= \sum_{k_j=1}^{\infty} \sum_{s=0}^{1} |(h, v_{s,k_j}(x_j, A_{1,j}); L_2(0, X_j))|^2 \le C_3 \|h; L_2(0, X_j)\|^2.$$

Therefore the operator $R(A_{1,j}^*)$ is adjoint to $R(A_{1,j})$ and bounded in the space $L_2(0, X_j) \to L_2(0, X_j)$. So operators $R(A_{1,j})$, $R^{-1}(A_{1,j}) = 2E - R(A_{1,j})$ are also bounded.

If we take into consideration Lemma 4 and Bari's theorem [19] we get: the system of function $V(A_{1,j})$ is a Riesz basis in the space $L_2(0, X_j)$. □

6 NONSELFADJOINT PROBLEM FOR ORDINARY DIFFERENTIAL EQUATION OF EVEN ORDER

Let us consider for any $j \in \{1, 2, \dots, m\}$, $p \in \{1, 2, \dots, n\}$, $b \in \mathbb{R}$, the problem

$$L(D)y := \sum_{|\beta| \le n} a_\beta D^{2\beta} y = \lambda y,\ \lambda \in \mathbb{C}, \tag{16}$$

$$\ell_{1,s,q} y := D_q^{2s-2} y|_{x_q=0} + D_q^{2s-2} y|_{x_q=X_q} = 0,\ q \ne j, s = 1, \dots, n, q = 1, \dots, m, \tag{17}$$

$$\ell_{1,n+s,q} y := D_q^{2s-1} y|_{x_q=0} + D_q^{2s-1} y|_{x_q=X_q} = 0,\ s \ne p, q \ne j, q = 1, \dots, m, s = 1, \dots, n, \tag{18}$$

$$\ell_{1,n+s,j} y := D_j^{2s-2} y|_{x_j=0} + D_j^{2s-2} y|_{x_j=X_j} = 0,\ s \ne p, s = 1, 2, \dots, n, \tag{19}$$

$$\ell_{1,n+s,j} y := D_j^{2s-1} y|_{x_j=0} + D_j^{2s-1} y|_{x_j=X_j} = 0, \tag{20}$$

$$\ell_{1,n+p,j} y := D_j^{2p-1} y|_{x_j=0} + D_j^{2p-1} y|_{x_j=X_j} + b(D_j^{2p-1} y|_{x_j=0} - D_j^{2p-1} y|_{x_j=X_j}) = 0. \tag{21}$$

Let $L_{1,p,j}$ be an operator of the problem (16) – (21), $V\left(L_{1,p,j}\right)$ be the system of root functions of operator $L_{1,p,j}$. This operator acts in a following manner $L_{1,p,j}y := L(D)y,\ y \in D(L_{1,p,j})$, $D(L_{1,p,j}) := \{y \in W_2^{2n}(G) : \ell_{1,r,j}y = 0,\ r = 1,2,\ldots,2n,\ j = 1,2,\ldots,m\}$.

Let us consider by fixing $k(j) := (k_1, k_2, \ldots, k_{j-1}, k_{j+1}, \ldots, k_m) \in \mathbb{N}^{m-1}$ solutions of the problem (16) – (21) in a form of product

$$y(x) := z(x_j) \prod_{r=1, r\neq j}^{m} \tau_{q_r,k_r}(x_r),\ k_r = 1,2,\ldots, j \neq r, r = 1,2,\ldots,m. \tag{22}$$

For determination of an unknown function $z(x_j)$ we have the following problem

$$\sum_{|\beta|\leq n} a_\beta (-1)^{\beta_r} \prod_{r=1, r\neq j}^{m} (-1)^{\beta_r} \rho_{k,r}^{2\beta_r} z^{(2\beta_j)}(x_j) = \lambda z(x_j),\ \lambda \in \mathbb{C}, \tag{23}$$

$$\ell_{1,s,j}z := z^{(2s-2)}|_{x_j=0} + z^{(2s-2)}|_{x_j=X_j} = 0,\ s = 1,2,\ldots,n, \tag{24}$$

$$\ell_{1,n+s,j}z := z^{(2s-1)}|_{x_j=0} + z^{(2s-1)}|_{x_j=X_j} = 0,\ s \neq p, s = 1,2,\ldots,n, \tag{25}$$

$$\ell_{1,n+p,j}z := z^{(2p-1)}|_{x_j=0} + z^{(2p-1)}|_{x_j=X_j} + b(z^{(2p-1)}|_{x_j=0} - z^{(2p-1)}|_{x_j=X_j}) = 0. \tag{26}$$

Let $L_{1,k(j)}$ be the operator of the broblem (23) – (26). The operator $L_{0,(k_j)}$ is partial case of operator $L_{1,k(j)}$, if $b = 0$.

So

$$L_{1,k(j)}z(x_j) := \sum_{|\beta|\leq n} a_\beta (-1)^{\beta_s} \prod_{s=1, s\neq j}^{m} (\rho_{k,s})^{2\beta_s} z^{(2\beta_j)}(x_j),\ z \in D(L_{1,k(j)}),$$
$$D(L_{1,k(j)}) := \{y \in W_2^{2n}(0, X_j) : \ell_{1,s,j}z = 0,\ s = 1,2,\ldots,2n\}.$$

Lemma 6. *For any $a_\beta \in \mathbb{R}$, $|\beta| \leq n$, $k(j) \in \mathbb{Z}_0^{m-1}$, $p \in \{1,2,\ldots,n\}$, $b \in \mathbb{R}$, the operator $L_{1,k(j)}$ has eigenvalues* (5) *and a system of root functions $V(L_{1,k(j)})$, which is a Riesz basis in the space $L_2(0, X_j)$.*

Proof. The root $\omega_r(k(j), \lambda)$ of the equation

$$\sum_{|\beta|\leq n} a_\beta \prod_{s=1, s\neq j}^{m} (\rho_{k,s})^{2\beta_s} \omega^{2\beta_j} = \lambda, \tag{27}$$

which is characteristic for the equation (23), we will chose from the condition $\operatorname{Re}\omega_n(k(j), \lambda) \leq \operatorname{Re}\omega_{n-1}(k(j), \lambda) \leq \ldots \leq \operatorname{Re}\omega_1(k(j), \lambda) \leq 0$.

Let us consider functions

$$z_{0,1}(x_j, \lambda) = (2x_j - X_j)\frac{2}{\sqrt{2X_j}} \sin \omega_1(k(j), \lambda) X_j^{-1} x_j,$$
$$z_{0,q}(x_j, \lambda) := \frac{1}{2}(1 - e^{\omega_q(k(j),\lambda)X_j})^{-1}(e^{\omega_q(k(j),\lambda)x_j} + e^{\omega_q(k(j),\lambda)(X_j - x_j)}) \in L_{2,0}(0, X_j),\ q = \overline{2,n},$$
$$z_{0,n+1}(x_j, \lambda) = (2x_j - X_j))\frac{2}{\sqrt{2X_j}} \cos \omega_1(k(j), \lambda) X_j^{-1} x_j,$$
$$z_{0,n+q}(x_j, \lambda) := \frac{1}{2}(1 + e^{\omega_q(k(j),\lambda)X_j})^{-1}(e^{\omega_q(j,k(j),\lambda)x_j} - e^{\omega_q(k(j),\lambda)(X_j - x_j)}) \in L_{2,1}(0, X_j),\ q = \overline{2,n}.$$

If we substitute this expressions in boundary conditions (24) – (26), we will get eigenvalues (5) and eigenfunctions of the operator $L_{1,k(j)}$

$$v_{1,k_j}(x_j, L_{1,k(j)}) = \frac{2}{\sqrt{2X_j}} \cos \rho_{k,j} X_j^{-1} x_j,\ k_j = 1, 2, \dots. \tag{28}$$

Let $\omega_r(k_j, \lambda_k)$ are roots of the equation (27), $\lambda = \lambda_k$ obtained from the equation $\omega_1(k_j, \lambda_k) = \imath\pi(2k_j - 1)X_j^{-1}$ and $\operatorname{Re}\omega_n(k_j, \lambda_k) \le \operatorname{Re}\omega_{n-1}(k_j, \lambda_k) \le \ldots \le \operatorname{Re}\omega_1(k_j, \lambda_k) \le 0$.

Let us consider system of functions

$$z_{0,1}(x_j, k_j, \lambda_k) = (2x_j - X_j))\frac{2}{\sqrt{2X_j}} \sin \rho_{k,j} X_j^{-1} x_j,\ k_j = 1, 2, \dots, \tag{29}$$

$$z_{0,q}(x_j, k_j, \lambda_k) := \frac{1}{2}(1 + e^{\omega_{q,k_j}\lambda_k X_j})^{-1}((e^{\omega_{q,k_j}(\lambda_k)x_j} - e^{\omega_{q,k_j}(\lambda_k)(X_j - x_j)})),\ k_j = 1, 2, \dots, \tag{30}$$

and a square matrix of the order n, which elements we can define as follows: p-th row defined by functions (29), (30) and elements of other rows defined by numbers

$$\eta_{q,r,k_j} = (\rho_{k,j})^{1-2r}\ell_{1,n+r,j} z_{1,q}(x_j, k_j, \lambda_k) = (\omega_j(k_j, \lambda_k)X_j)^{2r-1},$$
$$\eta_{1,r,k_j} = (-1)^r\sqrt{2X_j},\ k_j = 1, 2, \dots,\ q = 2, \dots, n,\ r \ne p, r = 1, 2, \dots, n.$$

Determinant of obtained matrix we will denote by $y_{1,p,k_j}(x_j, \lambda_k)$, $k_j = 1, 2, \dots$.

Remark 1. *For any fixed $k(j) \in \mathbb{N}^{m-1}$, if $k_j \to \infty$, we get*

$$\delta_{1,k_j}(\lambda_k) = \omega_1(k_j, \lambda_k)(2\pi k_j X_j^{-1})^{-1} = \imath,$$
$$\delta_{q,k_j}(\lambda_k) = \omega_q(k_j, \lambda_k)(2\pi k_j X_j^{-1})^{-1} = \varepsilon_q X_j(1 + O(k_j^{-1})),$$

where ε_q are roots of the equation $(-1)^n(\varepsilon)^{2n} = 1$, $\operatorname{Im}\varepsilon_q < 0$, $q = 2, 3, \dots, n$.

If we substitute function $y_{1,p,k_j}(x_j, \lambda_k)$ in boundary conditions (23) – (26), we will get equalities

$$\ell_{1,s,j} y_{1,p,k_j} = 0,\ j \ne n + p, k_j = 1, 2, \dots, \tag{31}$$

$$c_{p,k_j} := \ell_{1,n+p,j} y_{1,p,k_j} = \sqrt{2X_j}\rho_{k_j}^{2p-1} W_{k_j}(\lambda_k) \prod_{q=1}^{n} \delta_{q,k_j}(\lambda_k),\ k_j = 1, 2, \dots, \tag{32}$$

where $W_{k_j}(\lambda_k)$ is a Wandermond determinant of the order n, which is constructed by numbers $-1, \delta_{q,k_j}(\lambda_k)^2$, $q = 2, 3, \dots, n$.

Remark 2. *For any fixed $k(j) \in \mathbb{N}^{m-1}$ number sequence $W_{k_j,n}(\lambda_k)$ converges to Wandermond determinant $W(\varepsilon_2^2, \dots, \varepsilon_n^2)$, if $k_j \to \infty$, which is constructed using numbers $\varepsilon_2^2, \dots, \varepsilon_n^2$.*

Under this conditions sequence $\delta_{q,k_j}(\lambda_k)$ converges to ε_q, $q = 1, 2, \dots, n$.

Then there exist positive numbers C_4, C_5 such that following inequality takes place:

$$0 < C_4 \le |c_{p,k_j}|^{-1}\rho_{k_j}^{1-2p} \le C_5 < \infty,\ k_j = 1, 2, \dots. \tag{33}$$

Let us choose function $y_{2,p,k_j}(x_j, \lambda_k)$ so that the equality

$$\ell_{1,n+p,j} y_{2,p}(x_j, k_j, \lambda_k) = \left(\sqrt{2X_j}\right)^{-1} \rho_{k_j}^{2p-1} \tag{34}$$

takes place. An adjoint function $v_{0,k_j}(x_j, L_{1,(k_j)})$ of the operator $L_{1,(k_j)}$ we define by sum

$$v_{0,k_j}(x_j, L_{1,(k_j)}) := \frac{2}{\sqrt{2X_j}} \sin(2k_j - 1)\pi X_j^{-1} x_j + \eta_{1,p,j,k} y_{2,p,k_j}(x_j, \lambda_k),\ k_j = 1, 2, \ldots. \tag{35}$$

To define unknown parameters $\eta_{1,p,j,k}$ we substitute expression (35) in boundary conditions (25), (26). If we consider formula (34), we obtain

$$\eta_{1,p,j,k} = (-1)^p b,\ k_j = 1, 2, \ldots. \tag{36}$$

Remark 3. *Functions* $y_{1,p,k_j}(x_j, \lambda_k)$ *and* $y_{2,p,k_j}(x_j, \lambda_k)$ *connected with notions*

$$y_{2,p,k_j}(x_j, \lambda_k) = \chi_{p,j,k} y_{1,p,k_j}(x_j, \lambda_k), \tag{37}$$

where $C_6 \le |\chi_{p,j,k}| \le C_7,\ p = 1, 2, \ldots, n, j = 1, 2, \ldots, m, k \in \mathbb{N}^m$.

Therefore operator $L_{1,(k_j)}$ has a system $V(L_{1,(k_j)})$ of root functions (29), (35), (36) in sense of equations

$$L_{1,(k_j)} v_{0,k_j}(x_j, L_{1,(k_j)}) = \lambda_k v_{0,k_j}(x_j, L_{1,(k_j)}) + \xi_{p,k_j} v_{1,k_j}(x_j, L_{1,(k_j)}),$$
$$\xi_{p,k_j} := \det(\eta_{q,r,k_j})_{r=\overline{1,n}, r\ne p}^{q=\overline{2,n}} \frac{\partial \lambda_k}{\partial \rho_{j,k}} \chi_{p,j,k} \eta_{1,p,j,k} c_{p,k_j},\ p = 1, 2, \ldots, n, j = 1, 2, \ldots, m, k \in \mathbb{N}^m.$$

For problem (21) – (26) there exists an adjoint problem which has a system of root functions that is biorthogonal to $V(L_{1,(k_j)})$. Therefore the system $V(L_{1,(k_j)})$ is total and minimal in the space $L_2(0, X_j)$.

Let $H_{p,(k_j)}$ be a root subspace of the operator $V(L_{1,(k_j)})$ which corresponds to two-fold eigenvalue λ_k. According to the Shkalikov theorem [29] the system of subspaces $\{H_{p,(k_j)}\}_{k_j=1}^{\infty}$ is a Riesz basis of subspaces.

Let $y_{3,p,k_j}(x_j, \lambda_k) := v_{0,k_j}(x_j, L_{1,(k_j)}) - (v_{0,k_j}, v_{1,k_j}; L_2(0, X_j)) v_{1,k_j}(x_j, L_{1,(k_j)})$. Let us notice that functions $y_{3,p,k_j}(x_j, \lambda_k)$ and $v_{1,k_j}(x_j, L_{1,(k_j)})$ are orthogonal in the space $L_2(0, X_j)$. Let

$$y_{4,p,k_j}(x_j, \lambda_k) := \varphi_{p,k_j} y_{3,p,k_j}(x_j, \lambda_k), \tag{38}$$

where φ_{p,k_j} satisfy condition $\|y_{4,p,k_j}(x_j, \lambda_k); L_2(0, X_j)\| = 1$. Therefore functions $y_{4,p,k_j}(x_j, \lambda_k)$ and $v_{1,k_j}(x_j, L_{1,(k_j)})$ form an orthonormal basis in the space $H_{p,(k_j)}$. Since (17) takes place we obtain following: system of functions $\{v_{1,k_j}(x_j, L_{1,(k_j)}) y_{4,p,k_j(x_j}, \lambda_k)\}_{k_j=1}^{\infty}$ is a Riesz basis in the space $L_2(0, X_j)$.

Therefore such positive numbers $C_8,\ C_9$ exist that for any function $\varphi \in L_2(0, X_j)$ inequality

$$C_8 \|\varphi; L_2(0, X_j)\|^2 \le \sum_{k_j=1}^{\infty} ((\varphi, v_{1,k_j}; L_2(0, X_j))^2 + (\varphi, y_{4,p,k_j}; L_2(0, X_j))^2 \le C_9 \|\varphi; L_2(0, X_j)\|^2$$

takes place. If we consider equation (38) and inequality $1 < \varphi_{p,k_j} < \infty$ for any function $\varphi \in L_2(0, X_j)$, we obtain the following estimation

$$C_{10} \|\varphi; L_2(0, X_j)\|^2 \le \sum_{k_j=1}^{\infty} ((\varphi, v_{1,k_j}; L_2(0, X_j))^2 + (\varphi, y_{3,p,k_j}; L_2(0, X_j))^2 \le C_{11} \|\varphi; L_2(0, X_j)\|^2. \tag{39}$$

Let us prove that the system of functions $V(L_{1,(k_j)})$ is Bessel in the space $L_2(0, X_j)$, so there exists a positive number C_{12} such that following inequality

$$\sum_{k_j=1}^{\infty}((\varphi, v_{0,k_j}; L_2(0, X_j))^2 + (\varphi, v_{1,k_j}; L_2(0, X_j))^2 \leq C_{12}\|\varphi; L_2(0, X_j)\|^2 \tag{40}$$

takes place. From the Cauchy inequality and the definition of y_{3,p,k_j} we have the following estimation

$$(\varphi, v_{0,k_j}; L_2(0, X_j))^2 \leq 2((\varphi, y_{3,p,k_j}; L_2(0, X_j))^2 + (\varphi, v_{1,k_j}; L_2(0, X_j))^2)(v_{0,k_j}, v_{1,k_j}; L_2(0, X_j))^2.$$

The system of functions $V(L_{1,(k_j)})$ is orthonormal in the space $L_2(0, X_j)$. Then inequality $|(v_{0,k_j}, v_{1,k_j}; L_2(0, X_j))|^2 \leq C_{13} < \infty$ takes place. If we consider the last inequality and (38) we will get (40) if $C_{12} = 3C_9 + 2C_{13}$.

Let $R(L_{1,k(j)}) := E_j + S(L_{1,k(j)}) : L_2(0, X_j) \to L_2(0, X_j)$ be an operator that acts $V(A_{0,j}) \to V(L_{1,k(j)})$. If we consider operator $S(L_{1,k(j)}) : L_{2,0}(0, X_j) \to L_{2,1}(0, X_j)$ for $L_{2,1}(0, X_j) \to 0$, we will obtain $S(L_{1,k(j)})^2 = 0$. Therefore $R^{-1}(L_{1,k(j)}) = E_j - S(L_{1,k(j)})$ exists.

Operator $R(L_{1,k(j)}) := E_j + S(L_{1,k(j)}) : L_2(0, X_j) \to L_2(0, X_j)$ is bounded since the system is Bessel. Therefore the operator $R^{-1}(L_{1,k(j)}) : L_2(0, X_j) \to L_2(0, X_j)$ is also bounded. Therefore the system of functions $V(L_{1,(k_j)})$ is a Riesz basis in the space $L_2(0, X_j)$. □

7 TRANSFORMATION OPERATORS FOR DIFFERENTIAL EQUATIONS OF EVEN ORDER

Let us consider a sequence $\left\{\theta_{k_j}\right\}_{k_j=1}^{\infty} \subset \mathbb{R}$ and consider an operator $A_{2,p,j}$ such that its eigenvalues coincide with eigenvalues of operator $A_{0,j}$ and root functions are defined by equations

$$v_{1,k_j}\left(x_j, A_{2,p,j}\right) = \frac{2}{\sqrt{2X_j}}\cos\rho_{k,j}x_j, \tag{41}$$

$$v_{0,k_j}\left(x_j, A_{2,p,j}\right) = \frac{2}{\sqrt{2X_j}}\sin\rho_{k,j}x_j + \theta_{k_j}y_{1,p,k_j}\left(x_j, \lambda_k\right),\ k_j = 1, 2, \ldots. \tag{42}$$

Let $R(A_{2,p,j}) = E + S(A_{2,p,j})$ be an operator in the space $L_2(0, X_j)$ defined by $V(A_{0,j}) \to V(A_{2,p,j})$. From the definition of the operator $R(A_{2,p,j})$ we obtain $S^2(A_{2,p,j}) = 0$. Therefore, the operator $R^{-1}(A_{2,p,j}) = E - S(A_{2,p,j})$ exists.

Lemma 7. *For any $a_\beta \in \mathbb{R}$, $|\beta| \leq n, j \in \{1, 2, \ldots, m\}$, $k(j) \in \mathbb{N}^{m-1}$ and for any sequence $\{\theta_{k_j}\}_{k_j=1}^{\infty} \subset \mathbb{R}$ a system of functions $V(A_{2,p,j})$ is total and minimal in the space $L_2(0, X_j)$. The system of functions $V(A_{2,p,j})$ is a Riesz basis in the space $L_2(0, X_j)$ if and only if the sequence $\{\theta_{k_j}\}_{k_j=1}^{\infty}$ is bounded.*

Proof. First part can be proved analogously to Lemma 4. □

We will denote by $\Phi_p(L_{0,(k_j)})$ a set of all operators $A_{2,p,j}$ defined by (41), (42). Also we denote by $\Gamma_p(L_{0,(k_j)})$ a set of all operators $R(A_{2,p,j})$, which are generated by operators $A_{2,p,j} \in \Phi(L_{0,(k_j)})$. From formula (35) we obtain $R(L_{1,k(j)}) := E_j + S(L_{1,k(j)}) \in \Gamma_p(L_{0,(k_j)})$. Let us consider two sequences $\{\theta^1_{k_j}\}_{k_j=1}^{\infty}$, $\{\theta^2_{k_j}\}_{k_j=1}^{\infty}$ and define two transformation operators $R_q =$

$E_j + S_q \in \Gamma_p(L_{0,(k_j)})$, $q = 1,2$. Let us consider equality $S_q^2 = 0$, $q = 1,2$, we can define $\Gamma_p(L_{0,(k_j)})$ an operation of multiplication on the space

$$R_1 R_2 = E_j + S_1 + S_2. \tag{43}$$

From $R_q^{-1} = E_j - S_q$ it follows that the set $\Gamma_p(A_{0,j})$ is a group. Since the equality $R_1 R_2 = E_j + S_1 + S_2 = R_2 R_1$ takes place we obtain that the set $\Gamma_p(L_{0,(k_j)})$ is a commutative group.

Lemma 8. *For any fixed $a_\beta \in \mathbb{R}$, $|\beta| \le n$, $j = 1,2,\dots,m$, $k(j) \in \mathbb{N}^{m-1}$ the system of functions $V(A_{2,p,j})$ is a Riesz basis in the space $L_2(0, X_j)$ if and only if the sequence $\{\theta_{k_j}\}_{k_j=1}^{\infty}$ is bounded.*

This lemma can be proved analogously to Lemma 7.

Let us choose an arbitrary n sequences of real numbers $\{\theta_{p,k_j}\}_{k_j=1}^{\infty}$, $p = 1,2,\dots,m$, and consider an operator $A_{3,j}$. This operator eigenvalues coincides with eigenvalues of $A_{0,j}$ and roots of the function are defined by equations

$$v_{1,k_j}(x_j, A_{3,j}) = \frac{2}{\sqrt{2X_j}} \cos \rho_{k,j} x_j, \tag{44}$$

$$v_{0,k_j}(x_j, A_{3,j}) = \frac{2}{\sqrt{2X_j}} \sin \rho_{k,j} x_j + \sum_{p=1}^{n} \theta_{p,k_j} y_{1,p,k_j}(x_j, \lambda_k),\ k_j = 1,2,\dots. \tag{45}$$

Let $R(A_{3,j}) = E + S(A_{3,j})$ be the operator defined in the space $L_2(0, X_j)$ by $V(A_{0,j}) \to V(A_{3,j})$. From the definition of $R(A_{3,j})$ we obtain $S^2(A_{3,j}) = 0$. Therefore operator $R^{-1}(A_{3,j}) = E - S(A_{3,j})$ exists.

Lemma 9. *For any $a_\beta \in \mathbb{R}$, $|\beta| \le n$ and sequences $\{\theta_{p,k_j}\}_{k_j=1}^{\infty} \subset \mathbb{R}$, $p = 1,2,\dots,m$ a system of functions $V(A_{3,j})$ is total and minimal in the space $L_2(0, X_j)$. The system of functions $V(A_{3,j})$ is a Riesz basis in the space $L_2(0, X_j)$ if and only if any sequence $\{\theta_{p,k_j}\}_{k_j=1}^{\infty}$, $p = 1,2,\dots,m$ is bounded.*

We can prove this lemma analogously to Lemma 7.

Let us define root functions of the operator $L_{1,p,j}$ by equalities

$$v_k(x, L_{1,p,j}) = v_{k_j}(x_j, L_{1,k(j)}) \prod_{r=1, r\ne j}^{m} \tau_{r,k_r}(x_r),\ k \in \mathbb{N}^m.$$

By the system $V(L_{1,p,j})$ of root functions of the operator $L_{1,p,j}$ we define operator $R(L_{1,p,j}) := E + S(L_{1,p,j})$, which acts in the space $L_2(G)$. This operator transfer system of functions $V(L_0)$ to the system $V(L_{1,p,j})$. So we obtain $R(L_{1,p,j} v_k(x, (L_0)) := v_k(x, L_{1,p,j})$, $k \in \mathbb{N}^m$. The operator $R(L_{1,p,j})$ is defined by equality

$$R(L_{1,p,j}) = E_1 \otimes \dots \otimes E_{j-1} \otimes R(L_{1,k(j)}) \otimes E_{j+1} \dots E_m, \tag{46}$$

where E_s is the identical transformation in the space $L_2(0, X_s)$, $s = 1,2,\dots,m$.

Let us denote by $\Gamma_{p,j}(L_0)$ a set of the operators which is defined by formula $E_1 \otimes \dots \otimes E_{j-1} \otimes R(A_{2,p,j}) \otimes E_{j+1} \dots \otimes E_m$, where $R(A_{2,p,j}) \in \Gamma(k(j))$. A set of the operators $R(L_0) = R_1 \otimes R_2 \dots \otimes R_m$ we will denote by $\Gamma_p(L_0)$.

Remark 4. *We define multiplication on the set* $\Gamma_p(L_0)$ *according to formula* (43) *such that this set will be abelian group.*

Theorem 4. *Let assumption* P_1 *takes place. Then for any fixed* $a_\beta \in \mathbb{R}$, $|\beta| \leq n$ *an operator* $L_{1,p,j}$ *has eigenvalues* (5) *and system of root functions* $V(L_{1,p,j})$*, which is total and minimal in the space* $L_2(G)$.

If assumptions P_2, P_3 *take place then the system of functions* $V(L_{1,p,j})$ *is a Riesz basis in the space* $L_2(G)$.

Proof. According to Lemma 6 for any $k(j) \in \mathbb{N}^{m-1}$ there exists a system of functions $W(L_{1,k(j)}) = \{w_{k_j}(x_j, L_{1,k(j)}),\ k_j = 1,2,\dots\}$, which is biorthogonal to the system $V(L_{1,k(j)})$.

Therefore we can define elements of the system $W(L_{1,p,j})$, which is biorthogonal in the space $L_2(G)$ to the system by $V(L_{1,p,j})$

$$w_k(x, L_{1,p,j}) = w_{k_j}(x_j, L_{1,k(j)}) \prod_{r=1, r\neq j}^{m} \tau_{r,k_r}(x_r),\ k \in \mathbb{N}^m.$$

So the system $V(L_{1,p,j})$ is total and minimal in the space $L_2(G)$. If assumptions P_2, P_3 take place then root functions (44) of the operator $L_{1,p,j}$ are normalized for any $k(j) \in \mathbb{N}^{m-1}$ and a system $V(L_{1,p,j})$ is a Riesz basis of the space $L_2(G)$. □

8 PERTURBATED BOUNDARY PROBLEM WITH ACCENTED VARIABLE

Let us consider for any fixed $j = 1,2,\dots,m$, $p = 1,2,\dots,n$ equation (16) and problem with boundary conditions

$$\ell_{2,s,r}y := D_r^{2s-2}\, y|_{x_r=0} + D_r^{2s-2}\, y|_{x_r=X_r} = 0,\ r \neq j, s = 1,2,\dots,n, r = 1,2,\dots,m, \tag{47}$$

$$\ell_{2,n+s,r}y := D_r^{2s-1}\, y|_{x_r=0} + D_r^{2s-1}\, y|_{x_r=X_r} = 0,\ j \neq r, s = 1,2,\dots,n, r = 1,2,\dots,m, \tag{48}$$

$$\ell_{2,n+p,j}y := D_j^{2p-1}\, y|_{x_j=0} + D_j^{2p-1}\, y|_{x_j=X_j} + \sum_{r=0}^{1}\sum_{q=0}^{m_{p,j}} b_{q,r,p,j}\, D_j^q y\Big|_{x_j=rX_j} = 0. \tag{49}$$

Let $L_{2,p,j}$ be the operator of the problem (16), (47) – (49), $L_{2,p,j}y := L(D)\,y, y \in D\left(L_{2,p,j}\right)$, $D(L_2) := \left\{y \in W_2^{2n}(G) : \ell_{2,s,j}y = 0,\ s = 1,2,\dots,2n, j = 1,2,\dots,m\right\}$. Let $V(L_2)$ be the system of root functions of L_2.

Consider for any fixed $k(j) \in \mathbb{N}^{m-1}$ solutions of the spectral problem for operator L_2 in a form of product (22). To define unknown function $z(x_j)$ we obtain the following problem

$$\sum_{|\beta|\leq n} (-1)^{\beta_j} a_\beta \prod_{s=1, s\neq j}^{m} \mu_{k,s}^{\beta_s} z^{(2\beta_j)}(x_j) = \lambda z(x_j),\ \lambda \in \mathbb{C}, \tag{50}$$

$$\ell_{2,s,j}z := z^{(2s-2)}\, |_{x_j=0} + z^{(2s-2)}\, |_{x_j=X_j} = 0,\ s = 1,2,\dots,n, \tag{51}$$

$$\ell_{2,n+s,r}z := z^{(2s-1)}\Big|_{x_j=0} + z^{(2s-1)}\Big|_{x_j=X_j} = 0,\ s \neq p, s = 1,2,\dots,n, \tag{52}$$

$$\ell_{2,n+p,j}z := z^{(2p-1)}\Big|_{x_j=0} + z^{(2p-1)}\Big|_{x_j=X_j} + \sum_{r=0}^{1}\sum_{q=0}^{m_{p,j}} b_{q,r,p,j}\, z^{(q)}\Big|_{x_j=rX_j} = 0. \tag{53}$$

Let $L_{2,k(j)}$ be an operator of the problem (50) – (53). Therefore,

$$L_{2,k(j)}z := \sum_{|\beta|\le n} (-1)^{\beta_j} a_\beta \prod_{s=1,s\neq j}^{m} \mu_{k,s}^{\beta_s} z^{(2\beta_j)}\left(x_j\right),\ z\in D\left(L_{2,k(j)}\right),$$
$$D\left(L_{2,k(j)}\right) := \left\{y\in W_2^{2n}\left(0,X_j\right) : \ell_{2,s,j}z=0,\ s=1,2,\dots,2n\right\}.$$

Lemma 10. *Let assumption P_1 holds true. Then for any $a_\beta\in\mathbb{R}$, $b_{q,0,p,j}\in\mathbb{R}$, $|\beta|\le n, q=0,1,\dots,m_{p,j}$, $k(j)\in\mathbb{N}^{m-1}$ operator $L_{2,k(j)}$ has eigenvalues* (5) *and a system of root functions $V\left(L_{2,k(j)}\right)$, which is total and minimal in the space $L_2(0,X_j)$.*

If assumtion P_2 holds true then the system of functions $V(L_{2,k(j)})$ is a Riesz basis in the space $L_2(0,X_j)$.

Proof. The isospectrality of operators $L_{0,(k_j)}$ and $L_{2,k(j)}$ can be proved by the same way as in Theorem 9.

If assumtion P_1 holds true, then eigenfunctions of the operator $L_{2,k(j)}$ are following

$$v_{1,k_j}\left(x_j,L_{2,k(j)}\right) = \tfrac{2}{\sqrt{2X_j}}\cos\rho_{k_j}X_j^{-1}x_j,\ k_j=1,2,\dots. \tag{54}$$

Root functions $v_{0,k_j}(x_j,L_{2,(k_j)})$ of the operator $L_{2,(k_j)}$ are defined by

$$v_{0,k_j}(x_j,L_{2,(k_j)}) = \frac{2}{\sqrt{2X_j}}\sin\rho_{k_j}X_j^{-1}x_j + \eta_{2,p,j,k}y_{1,p,k_j}\left(x_j,\lambda_k\right),\ k_j=1,2,\dots. \tag{55}$$

To define $\eta_{2,p,j,k}$ we can substitute expression (55) into boundary conditions (51)–(53). If we consider formulas (32), (33) we obtain

$$\eta_{2,p,j,k} = (c_{p,k_j})^{-1}\ell_{p,j}^2\tau_{2k_j-1,j}(x_j),\ k_j=1,2,\dots. \tag{56}$$

Therefore, operator $L_{2,k(j)}$ has a system of eigenfunction (54) – (56). If we consider formulas (31), (32) we obtain that the operator $L_{2,(k_j)}$ is a partial case of the operator $A_{2,p,j}$. Therefore, from Lemma 7 it follows that the system $V(L_{2,k(j)})$ is total and minimal in the space $L_2(0,X_j)$.

Let assumption P_1 holds. Since equations (33), (49) take place, we get $\left|\eta_{2,p,j,k}\right|\rho_{k,j}^{2p-m_{p,j}-1}\le C_{14}<\infty$. So from Theorem 9 it follows that the system $V\left(L_{2,k(j)}\right)$ is a Riesz basis in the space $L_2(0,X_j)$. $\square$

Let us define root functions of the operator $L_{2,p,j}$ by equalities

$$v_k\left(x,L_{2,p,j}\right) = v_{k_j}\left(x_j,L_{2,k(j)}\right)\prod_{r=1,r\neq j}^{m}\tau_{r,k_r}\left(x_r\right),\ k\in\mathbb{N}^m. \tag{57}$$

Using the system $V\left(L_{2,p,j}\right)$ of root functions (57) of the operator $L_{2,p,j}$ we can define an operator $R(L_{2,p,j}) := E+S(L_{2,p,j})$, which acts from system of functions $V(L_0)$ to the system $V(L_{2,p,j})$. The operator $R(L_{2,p,j})$ can be defined by equation (46). If assumption P_1 holds true, then for any $k(j)\in\mathbb{N}^{m-1}$ there exists a system of functions $W(L_{2,k(j)}) = \{w_{k_j}(x_j,L_{2,k(j)}),\ k_j=1,2,\dots\}$, which is biorthogonal to the system $W(L_{2,k(j)})$.

Therefore, we can define elements of the biorthogonal system $W\left(L_{2,p,j}\right)$ in the space $L_2(G)$ to the system $V(L_{2,p,j})$ by equality

$$w_k\left(x, L_{2,p,j}\right) = w_{k_j}\left(x_j, L_{2,k(j)}\right) \prod_{r=1, r\neq j}^{m} \tau_{r,k_r}\left(x_r\right),\ k \in \mathbb{N}^m.$$

So the system of functions $V(L_{2,p,j})$ is total and minimal in the space $L_2(G)$.

If assumptions P_1–P_3 take place then system of functions $V(L_{2,p,j})$ is Bessel in the space $L_2\left(G\right)$, since the transformation operator $R(L_{2,p,j})$ is continuous with action $L_2(G) \to L_2(G)$. A converse operator is also bounded. Therefore the system of functions $V(L_{2,p,j})$ is a Riesz basis of the space $L_2(G)$. So we prove following theorem.

Theorem 5. *Let assumptions P_1–P_2 take place. Then for any fixed $a_\beta \in \mathbb{R}$, $b_{q,0,p,j} \in \mathbb{R}$, $|\beta| \leq n$ operator $L_{2,p,j}$ has eigenvalues* (5) *a system of root functions $V(L_{2,p,j})$, which is total and minimal in the space $L_2(G)$.*

2. If assumptions P_1–P_3 take place, then system of functions $V(L_{2,p,j})$ is Riesz basis in the space $L_2(G)$.

Consider boundary problem

$$L\left(D\right)y := \sum_{|\beta|\leq n} a_\beta D^{2\beta} y = \lambda y, \tag{58}$$

$$\ell_{3,s,r}y := D_r^{2s-2}\, y|_{x_r=0} + D_r^{2s-2}\, y|_{x_r=X_r} = 0,\ s = 1,2,\ldots,n,\ r = 1,2,\ldots,m, \tag{59}$$

$$\ell_{3,n+s,r}y := D_r^{2s-1}\, y|_{x_r=0} + D_r^{2s-1}\, y|_{x_r=X_r} = 0,\ s = 1,2,\ldots,n, r \neq j,\ r = 1,2,\ldots,m, \tag{60}$$

$$\ell_{3,n+p,j}y := D_j^{2p-1}\, y|_{x_j=0} + D_j^{2p-1}\, y|_{x_j=X_j} + l_{p,j}^2 y = 0,\ p = 1,2,\ldots,n, \tag{61}$$

$$\ell_{p,j}^1 y := \sum_{r=0}^{1}\sum_{q=0}^{m_{p,j}} b_{q,r,p,j}\, D_j^q y\Big|_{x_j=rX_j}.$$

Let $L_{3,j}$ be the operator of the problem (58) – (61), $V\left(L_{3,j}\right)$ be a system of root functions of the operator $L_{3,j}$. Let $L_3 y := L\left(D\right)y,\ y \in D\left(L_3\right), D(L_{3,j}) := \{y \in W_2^{2n}(G) : \ell_{3,s,j}y = 0,$ $s = 1,2,\ldots,2n, j = 1,2,\ldots,m\}$. Let us consider for fixed $k\left(j\right) \in \mathbb{N}^{m-1}$ solutions of spectral problem for operator L_3 in a form of product (22).

To define an unknown function $z(x_j)$ we have the following problem

$$\sum_{|\beta|\leq n} a_\beta \prod_{s=1, s\neq j}^{m} \left(k_s \pi X_s^{-1}\right)^{2\beta_s} z^{(2\beta_j)}\left(x_j\right) \ = \lambda z\left(x_j\right),\ \lambda \in \mathbb{C}, \tag{62}$$

$$\ell_{3,s,j}y := z^{(2s-1)}\Big|_{x_j=0} - z^{(2s-1)}\Big|_{x_j=X_j} = 0,\ s = 1,2,\ldots,n, \tag{63}$$

$$\ell_{3,n+p,j}z := z^{(2p-1)}\Big|_{x_j=0} + z^{(2p-1)}\Big|_{x_j=X_j} + \ell_{p,j}^2 z = 0,\ p = 1,2,\ldots,n. \tag{64}$$

Let $L_{3,k(j)}$ be the operator of the problem (62) – (64) and

$$L_{3,k(j)}z := \sum_{|\beta|\leq n} a_\beta \prod_{s=1, s\neq j}^{m} \left(\rho_{k,s}\right)^{2\beta_s} z^{(2\beta_j)}\left(x_j\right), z \in D(L_{3,k(j)}),$$

$$D\left(L_{3,k(j)}\right) := \left\{y \in W_2^{2n}\left(0, X_j\right) : \ell_{3,s,j}z = 0; s = 1,2,\ldots,2n\right\}.$$

Lemma 11. *Let assumption P_1 holds true. Then for any fixed $a_\beta \in \mathbb{R}$, $b_{q,0,p,j} \in \mathbb{R}$, $|\beta| \leq n$, $k(j) \in \mathbb{N}^{m-1}$ operator $L_{3,k(j)}$ has eigenvalues* (5) *and the system of root functions $V(L_{3,k(j)})$, which is total and minimal in the space $L_2(0, X_j)$.*

2. If assumption P_2 holds true then system of functions $V(L_{3,k(j)})$ is a Riesz basis in the space $L_2(0, X_j)$.

Proof. The Isospectrality of operators $L_{0,(k_j)}$ and $L_{3,k(j)}$ can be proved in the same way as in Lemma 4.

If assumption P_1 holds true we obtain following eigenfunctions of $L_{3,k(j)}$

$$v_{1,k_j}\left(x_j, L_{3,k(j)}\right) = \frac{2}{\sqrt{2X_j}} \cos \rho_{k,j} x_j X_j^{-1},\ k_j = 1, 2, \ldots. \tag{65}$$

Root functions $v_{0,k_j}(x_j, L_{3,k(j)})$ of the operator $L_{3,k(j)}$ we defined by

$$v_{0,k_j}\left(x_j, L_{3,(k_j)}\right) = \frac{2}{\sqrt{2X_j}} \sin \rho_{k,j} X_j^{-1} x_j + \sum_{p=1}^{n} \eta_{2,p,j,k} y_{1,p,k_j}\left(x_j, \lambda_k\right),\ k_j = 1, 2, \ldots, \tag{66}$$

where numbers $\eta_{2,p,j,k}$ defined by equation (56). Therefore, operator $L_{3,k(j)}$ has a system of root functions (65), (66). If we consider formulas (31), (32), it is easy to see that operator $L_{3,k(j)}$ is a partial case of operator $A_{2,p,j}$. Therefore from Lemma 7 it follows that the system $V(L_{3,k(j)})$ is total and minimal in the space $L_2(0, X_j)$ and the biorthogonal system $W(L_{3,k(j)}) := \{w_{k_j}(x_j, L_{3,(k_j)}) \in L_2(0, X_j),\ k_j = 1, 2, \ldots\}$ exists.

Let assumption P_2 takes place. From (49) we can get $\sum\limits_{p=1}^{n} \left|c_{1,p,j,k}\right|^2 \leq C_{15} < \infty$. Then the system $V(L_{3,k(j)})$ is normalized and from Lemma 9 it follows that the system $V(L_{3,k(j)})$ is a Riesz basis in the space $L_2(0, X_j)$. □

Let us define root functions of the operator $L_{3,j}$ by equations

$$v_k\left(x, L_{3,j}\right) = v_{k_j}\left(x_j, L_{3,k(j)}\right) \prod_{r=1, r\neq j}^{m} \tau_{r,k_r}\left(x_r\right),\ k \in \mathbb{N}^m. \tag{67}$$

Using the system $V(L_{3,j})$ of root functions (67) we can define an operator $R(L_{3,j}) := \prod\limits_{p=1}^{n} R(L_{2,p,j}) \in \Gamma_j(L_0)$, $S(L_{3,j}) := \sum\limits_{p=1}^{n} S(L_{2,p,j})$, which acts from system of functions $V\left(L_0\right)$ to the system $V\left(L_{3,j}\right)$.

Theorem 6. *Let assumption P_1 holds. Then for any fixed $a_\beta \in \mathbb{R}$, $b_{q,0,p,j} \in \mathbb{R}$, $|\beta| \leq n$ the operator $L_{3,j}$ has eigenvalues* (6) *and the system of eigenfunctions $V(L_{3,j})$, which is total and minimal in the space $L_2(G)$.*

2. If assumptions P_1–P_3 hold, then the system of functions $V(L_{3,j})$ is a Riesz basis in the space $L_2(G)$.

Proof. Let assumption P_1 holds. Then according to Lemma 11 for any $k(j) \in \mathbb{N}^{m-1}$ there exists a system of functions $W(L_{3,k(j)}) = \{w_{k_j}(x_j, L_{3,k(j)}),\ k_j = 1, 2, \ldots,\}$ which is biorthogonal to the system $W(L_{3,k(j)})$. Therefore we can define elements of the system $W\left(L_3\right)$, which is biorthogonal in the space $L_2(G)$ to the system $V(L_3)$, by the following way

$$w_k(x, L_{3,j}) = w_{k_j}(x_j, L_{3,k(j)}) \prod_{r=1, r\neq j}^{m} \tau_{r,k_r}\left(x_r\right), \; k \in \mathbb{N}^m.$$

So the system of functions $V\left(L_{3,j}\right)$ is total and minimal in the space $L_2(G)$. Last part of the proof can be made analogously to Theorem 5. □

9 PROOFS OF THE MAIN THEOREMS

Let us consider spectral problem for $p = 1, 2, \dots, n, j = 1, 2, \dots, m$

$$L\left(D\right) y := \sum_{|\beta|\leq n} a_\beta D^{2\beta} y = \lambda y, \tag{68}$$

$$\ell_{p,j} y := D_j^{2p-2} \, y|_{x_j=0} + D_j^{2p-2} \, y|_{x_j=X_j} = 0, \; p = 1, 2, \dots, n, j = 1, 2, \dots, m, \tag{69}$$

$$\ell_{n+p,j} y := D_j^{2p-1} y|_{x_j=0} + D_j^{2p-1} y|_{x_j=X_j} + \sum_{r=0}^{1} \sum_{q=0}^{m_{p,j}} b_{q,r,p,j} D_j^q y|_{x_j=rX_j} = 0. \tag{70}$$

Proof. Proof of Theorem 1.

Let

$$R\left(L\right) := \prod_{j=1}^{m} R_j\left(L_3\right), \; R\left(L\right) := E + \sum_{j=1}^{m} S_j(L_3) \in \Gamma\left(L_0\right). \tag{71}$$

Root functions of the operator L of problem (68) – (70) we can define in the form

$$\begin{aligned} v_k\left(x, L\right) &= \prod_{j=1}^{m} v_{k_j}\left(x_j, L_{3,k(j)}\right), \; k \in \mathbb{N}^m, \\ v_k\left(x, L\right) &= v_k\left(x, L_0\right) + \sum_{j=1}^{m} S(L_{3,j}) v_k\left(x, L_0\right). \end{aligned} \tag{72}$$

So $V\left(L\right) := \{v_k\left(x, L\right) \in L_2\left(G\right) : v_k\left(x, L\right) = R\left(L\right) v_k\left(x, L_0\right), k \in \mathbb{N}^m\}$ is a system of root functions. Since biorthogonal system of functions $w_k\left(x, L\right) = \prod_{j=1}^{m} w_{k_j}\left(x_j, L_{3,k(j)}\right), k \in \mathbb{N}^m$, exists, then we have the proof of the theorem. □

If assumptions of Theorem 2 hold true, Theorem 6 takes place too. So $R(L_{3,j}) \in [L_2(G)]$, $j = 1, 2, \dots, m$. If we consider equation (71), we will obtain $R(L), R^{-1}(L) \in [L_2(G)]$. Therefore $V(L)$ is a Riesz basis of the space $[L_2(G)]$ by definition.

Remark 5. *There exist positive numbers $C_1(L), C_2(L)$ such that for any function*

$$f\left(x\right) = \sum_{|k|=0}^{\infty} f_k v_k\left(x, L\right) \in L_2(G), f_k = \left(f, w_k; L_2\left(G\right)\right), k \in \mathbb{N}^m,$$

holds the following inequality

$$C_{16} || f; L_2\left(G\right) ||^2 \sum_{|k|=0}^{\infty} |f_k|^2 \leq C_{17} || f; L_2\left(G\right) ||^2. \tag{73}$$

Let us consider boundary problem

$$L(D)\,y := \sum_{|\beta|\le n} a_\beta D^{2\beta} y = f, \tag{74}$$

$$\ell_{s,j} y := D_j^{2s-2} y\,|_{x_j=0} + D_j^{2s-2} y\,|_{x_j=X_j} = 0, j = 1,2,\ldots,m, \tag{75}$$

$$\ell_{s+n,j} y := D_j^{2s-1} y\,|_{x_j=0} + D_j^{2s-1} y\,|_{x_j=X_j} + \sum_{r=0}^{1}\sum_{q=0}^{m_s} b_{q,r,s,j} D_j^q y\,|_{x_j=x_{j,r}} = 0, \tag{76}$$

$$f(x) = \sum_{|k|=0}^{\infty}\sum_{r\in Q_m} f_{r,k} v_{r,k}(x,L),\ f_{r,k} = (u, w_{r,k}(x,L;L_2(G)),\ r \in Q_m, k \in \mathbb{N}^m. \tag{77}$$

We will search a solution of the problem in the form of series

$$u(x) = \sum_{|k|=0}^{\infty}\sum_{r\in Q_m} u_{r,k} v_{r,k}(x,L). \tag{78}$$

We will use the following notations: $f_{r+s,k} := \prod_{j=1}^m f_{r_j+s_j,k_j}$, $\xi_{r,k} := \prod_{j=1}^m \xi_{r_j,k_j}$, $\xi_{r_j,k_j} := D_{\rho_{k,j}} \sum_{|\beta|\le n} a_\beta \rho_k^{2\beta}$. If we substitute series (77), (78) into formula (74), we can get

$$\begin{aligned} u_{r,k} &= \sum_{j=1}^{m}\sum_{r_j+s_j\le 1} \lambda_k^{-1-|s|} f_{r+s,k}\xi_{r+s,k},\ k\in\mathbb{N}^m,\ r\in Q_m, \\ u(x) &= \sum_{|k|=0}^{\infty}\sum_{r\in Q_m}\sum_{j=1}^{m}\sum_{r_j+s_j\le 1} \lambda_k^{-1-|s|} f_{r+s,k}\xi_{r+s,k} v_k(x,L). \end{aligned} \tag{79}$$

If we apply Cauchy inequality to (78) we will obtain the inequality $|u_{r,k}|^2 \le C_{18}\sum_{q\in Q_m}|u_{q,k}|^2$. Therefore, using inequality (73) we can get

$$||u;L_2(G)||^2 \le C_{19}(L)\,||f;L_2,\ (G)||^2,\ C_{19} = C_{18}C_{17}C_{16}^{-1}. \tag{80}$$

Let us suppose that coefficient of derivative $D_{x_m}^{2n}$ equals 1 and give a proof for variable x_m. Let us show that $D_m^{2n} u(x) \in L_2(G)$. For any fixed $k(m) \in \mathbb{N}^{m-1}$ we consider boundary problem

$$\sum_{|\beta|\le n} (-1)^{\beta_m} a_\beta \prod_{s=1}^{m-1} \mu_{k,s}^{\beta_s} y^{(2\beta_m)}(x_m) \ = f(x_m), \tag{81}$$

$$\ell_{2,s,m} y := y^{(2s-2)}\,|_{x_m=0} + y^{(2s-2)}\,|_{x_m=X_m} = 0,\ s = 1,2,\ldots,n, \tag{82}$$

$$\ell_{2,n+s,m} y := y^{(2s-1)}\Big|_{x_m=0} + y^{(2s-1)}\Big|_{x_j=X_m} = 0,\ s\ne p, s = 1,2,\ldots,n, \tag{83}$$

$$\ell_{2,n+p,m} y := y^{(2p-1)}\Big|_{x_m=0} + y^{(2p-1)}\Big|_{x_m=X_m} + \ell_{p,m}^1 z = 0, \tag{84}$$

$$\ell_{p,m}^2 y := \sum_{r=0}^{1}\sum_{q=0}^{m_{p,m}} b_{q,r,p,j}\, y^{(q)}\Big|_{x_m=x_{m,r}}. \tag{85}$$

Let us consider functions $y(x_m), f(x_m)$ in a form of series which is constructed using system of root functions of operator $L_{2,k(m),p} := L_{2,k(m)}$:

$$y = \sum_{k_m=1}^{\infty}\sum_{s=0}^{1} y_{s,k_m,p} v_{s,k_m}(x, L_{2,k(m),p}),\ f = \sum_{k_m=1}^{\infty}\sum_{s=0}^{1} f_{s,k_m,p} v_{s,k_m}(x, L_{2,k(m),p}).$$

If we substitute this expressions into equation (81) we can obtain $y_{0,k_m,p} = \lambda_k^{-1} f_{0,k_m}$, $y_{1,k_m} = -\lambda_k^{-2}\xi_{0,k_m} f_{1,k_m,p} + \lambda_k^{-1} f_{1,k_m,p}$. Therefore,

$$y = \sum_{k_m=1}^{\infty} (\lambda_k^{-1} f_{0,k_m,p} v_{0,k_m}(x, L_{2,k(m),p}) + (-\lambda_k^{-2}\xi_{0,k_m} f_{1,k_m,p} + \lambda_k^{-1} y_{1,k_m,p}) v_{1,k_m}(x, L_{2,k(m)}, p)).$$

Let us consider sequence of numbers $\gamma_{m,k} := \rho_{m,k}\lambda_k^{-1}$, $k_m = 1, 2, \ldots$. The sequence $\gamma_{m,k} = \rho_{m,k}\lambda_k^{-1} = 1 - \lambda_k^{-1} \sum\limits_{|\beta|\le n, \beta_m < n} a_\beta \prod\limits_{s=1}^{m} (\rho_{m,s})^{2\beta_s} \to 1$, $k_m \to \infty$ is convergent.

Therefore $0 < C_{21} \le \gamma_{m,k} \le C_{20} < \infty$.

Consider the system of functions

$$V_{m,k,p} := \{v_{r,k_m,p}(x) \in L_2(0, X_m) : v_{r,k_m,p}(x) := \gamma_{m,k} v_{r,k_m}(x, L_{2,k(m),p}),\ k_m = 1, 2, \ldots\}.$$

If assumption of Lemma 10 holds, from the last inequality it follows that the system $V_{m,k,p}$ is a Riesz basis in the space $L_2(0, X_m)$.

Let $V_{1,m,k,p} := \{v_{1,r,k_m,p}(x) \in L_2(0, X_m) : v_{1,r,k_m,p}(x) := \lambda_k^{-1} D_m^{2n} v_{r,k_m}(x, L_{2,k(m),p}),\ r = 0, 1,\ k_m = 1, 2, \ldots\}$.

Since operator D_m^{2n} commutate with the involution I_m, then analogously to Lemma 4 we can prove that the system $V_{1,m,k,p}$ is total and minimal in the space $L_2(0, X_m)$.

Let $v_{2,r,k_m,p}(x) := v_{1,r,k_m,p}(x) - v_{0,r,k_m,p}(x)$, $r = 0, 1, k_m = 1, 2, \ldots$. From formulas (28) – (30) it follows $v_{2,0,k_m,p}(x) = \vartheta_{1,k_m,p}\rho_{m,k,p}^{-1} z_{0,q}(x_j, k_j, \lambda_k) + \sum\limits_{s=2}^{m} \vartheta_{s,k_m,p} z_{0,q}(x_j, k_j, \lambda_k)$, where $|\vartheta_{s,k_m,p}| \le C_{22} < \infty$. Therefore, $\sum\limits_{k_m=1}^{\infty}\sum\limits_{r=0}^{1} (v_{2,r,k_m,p}(x); L_2(0, X_m))^2 < \infty$.

So the system $V_{m,k,p}$ is a Riesz basis in the space $L_2(0, X_m)$ and the system $V_{1,m,k,p}$ is total and minimal in the space $L_2(0, X_m)$. Therefore, from Bari's theorem [6] we obtain: the system $V_{1,m,k,p}$ is a Riesz basis in $L_2(0, X_m)$ and therefore, the operator $R(V_{1,m,k,p}) : V(L_{0,k_m} \to V_{1,m,k}$ is bounded. The product of this operators is also continuous in the space $L_2(0, X_m)$.

So for any fixed $k(m) \in \mathbb{N}^{m-1}$ inequality

$$\sum_{k_m=1}^{\infty}\sum_{r=0}^{1} (D_m^{2n} u, v_{r,k}(x, L); L_2(G))^2 \le C_{23} \sum_{k_m=1}^{\infty}\sum_{r=0}^{1} (f, v_{r,k}(x, L); L_2(G))^2$$

takes place. If we summarize by $k(m) \in \mathbb{N}^{m-1}$, we will get

$$D_{x_m}^{2n} |u; L_2(G)|^2 \le C_{23} |f; L_2(G)|^2.$$

The assumption made in the inequality proof is insignificant since if assumption P_3 holds coefficients of the hiest degree derivatives is nonequal to zero and has the same sign.

Analogously we can prove that $D_j^{2n} u(x) \in L_2(G)$, $j = 1, 2, \ldots, m-1$ for any other variables. So using the definition of norm in the space $L_2(G)$, we obtain the proof of the Theorem 3.

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Баранецький Я.О., Івасюк І.Я., Каленюк П.І., Соломко А.В. *Нелокальна крайова задача зі збуреннями умов антиперіодичності для еліптичного рівняння з постійними коефіцієнтами* // Карпатські матем. публ. — 2018. — Т.10, №2. — С. 215–234.

У роботі в обмеженому m-вимірному паралелепіпеді методом Фур'є досліджується задача з нелокальними крайовими умовами, які є збуреннями умов антиперіодичності. Вивчено властивості оператора перетворення $R : L_2(G) \to L_2(G)$, який встановлює зв'язок між самоспряженим оператором L_0 задачі з умовами антиперіодичності та оператором L збуреної нелокальної задачі $RL_0 = LR$.

Також побудовано комутативну групу операторів перетворення $\Gamma(L_0)$. Встановлено, що кожному операторові перетворення $R \in \Gamma(L_0) : L_2(G) \to L_2(G)$ відповідає деяка абстрактна нелокальна задача і навпаки. Побудовано систему $V(L)$ кореневих функцій оператора L, яка містить нескінченне число приєднаних функцій. Визначено умови, при яких система $V(L)$ повна та мінімальна в просторі $L_2(G)$, та умови, при яких вона є базою Ріса у просторі $L_2(G)$.

У випадку, якщо система $V(L)$ є базою Ріса в просторі $L_2(G)$, встановлено достатні умови, при яких нелокальна задача має єдиний розв'язок у вигляді ряду Фур'є за системою $V(L)$.

Ключові слова і фрази: диференціально-операторне рівняння, власні функції, база Ріса.

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BEDRATYUK L., BEDRATYUK A.

THE INVERSE AND DERIVATIVE CONNECTING PROBLEMS FOR SOME HYPERGEOMETRIC POLYNOMIALS

Given two polynomial sets $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ such that $\deg(P_n(x)) = \deg(Q_n(x)) = n$. The so-called connection problem between them asks to find coefficients $\alpha_{n,k}$ in the expression $Q_n(x) = \sum_{k=0}^{n} \alpha_{n,k} P_k(x)$. The connection problem for different types of polynomials has a long history, and it is still of interest. The connection coefficients play an important role in many problems in pure and applied mathematics, especially in combinatorics, mathematical physics and quantum chemical applications. For the particular case $Q_n(x) = x^n$ the connection problem is called the inversion problem associated to $\{P_n(x)\}_{n\geq 0}$. The particular case $Q_n(x) = P'_{n+1}(x)$ is called the derivative connecting problem for polynomial family $\{P_n(x)\}_{n\geq 0}$. In this paper, we give a closed-form expression of the inversion and the derivative coefficients for hypergeometric polynomials of the form

$$ {}_2F_1\left[\begin{matrix}-n, a\\ b\end{matrix}\middle| z\right], \quad {}_2F_1\left[\begin{matrix}-n, n+a\\ b\end{matrix}\middle| z\right], \quad {}_2F_1\left[\begin{matrix}-n, a\\ \pm n+b\end{matrix}\middle| z\right], $$

where ${}_2F_1\left[\begin{matrix}a, b\\ c\end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$ is the Gauss hypergeometric function and $(x)_n$ denotes the Pochhammer symbol defined by $(x)_n = \begin{cases} 1, & n = 0, \\ x(x+1)(x+2)\cdots(x+n-1), & n > 0. \end{cases}$

All polynomials are considered over the field of real numbers.

Key words and phrases: connection problem, inversion problem, derivative connecting problem, connecting coefficients, hypergeometric functions, hypergeometric polynomials.

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INTRODUCTION

Given two polynomial sets $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ such that

$$\deg(P_n(x)) = \deg(Q_n(x)) = n.$$

The connection problem between them consists in finding the coefficients $\alpha_{n,k}$ in the expansion

$$Q_n(x) = \sum_{k=0}^{n} \alpha_{n,k} P_k(x).$$

For the particular case $Q_n(x) = x^n$ the connection problem is called the inversion problem associated to $\{P_n(x)\}_{n\geq 0}$. The particular case $Q_n(x) = P'_{n+1}(x)$ is called the derivative connecting problem for polynomial family $\{P_n(x)\}_{n\geq 0}$.

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The study of such a problem has attracted lot of interest in the last few years. The inverse problem for classical orthogonal polynomials are considered in [6], for more general case see [7]. The connection coefficients have been computed explicitly for classical orthogonal polynomials in [6] and [8].

The derivation connection problem (with respect to parameter derivatives) for hypergeometric polynomials ${}_2F_1\left[\begin{matrix} -n & a \\ & b \end{matrix}\middle| z\right]$ was solved in [9]. In [10, 11] the first author solved the derivation connection problem for the Fibonacci, Lucas and Kravchuk polynomials and use the solutions to produce new combinatorial identities for these polynomials.

Our aim in this paper is to compute the inversion and derivative connection coefficients for hypergeometric polynomials of the forms

$$
{}_2F_1\left[\begin{matrix} -n & a \\ & b \end{matrix}\middle| z\right], \quad {}_2F_1\left[\begin{matrix} -n & n+a \\ & b \end{matrix}\middle| z\right], \quad {}_2F_1\left[\begin{matrix} -n & a \\ & \pm n+b \end{matrix}\middle| z\right],
$$

where

$$
{}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}
$$

is the Gauss hypergeometric function.

The main results of this paper are gathered together in the following two theorems.

Theorem 1. *The following identities hold:*

$$
\begin{aligned}
&(i) \quad z^n = \frac{(b)_n}{(a)_n} \sum_{i=0}^{n} (-1)^i \binom{n}{i}\, {}_2F_1\left[\begin{matrix} -i & a \\ & b \end{matrix}\middle| z\right], \\
&(ii) \quad z^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (a+2i) \frac{(b)_n}{(a+i)_{n+1}}\, {}_2F_1\left[\begin{matrix} -i & i+a \\ & b \end{matrix}\middle| z\right], \\
&(iii) \quad z^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (b+2n-1) \frac{(b+i)_{n-1}}{(a)_n}\, {}_2F_1\left[\begin{matrix} -i & a \\ & i+b \end{matrix}\middle| z\right], \\
&(iv) \quad z^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (b-1) \frac{(b-i)_{n-1}}{(a)_n}\, {}_2F_1\left[\begin{matrix} -i & a \\ & -i+b \end{matrix}\middle| z\right].
\end{aligned}
$$

Theorem 2. *The following identities hold:*

$$
\begin{aligned}
&(i) \quad \frac{d}{dz}\, {}_2F_1\left[\begin{matrix} -n & a \\ & b \end{matrix}\middle| z\right] = \sum_{i=0}^{n-2} \frac{n!}{i!} \frac{a-b}{(a+i)_{n-i}}\, {}_2F_1\left[\begin{matrix} -i & a \\ & b \end{matrix}\middle| z\right] - n\frac{b+n-1}{a+n-1}\, {}_2F_1\left[\begin{matrix} -(n-1) & a \\ & b \end{matrix}\middle| z\right], \\
&(ii) \quad \frac{d}{dz}\, {}_2F_1\left[\begin{matrix} -n & n+a \\ & b \end{matrix}\middle| z\right] = \sum_{i=0}^{n-1} \Bigg((-n+i) \binom{n}{i} \frac{(a+2i)(n+a)_{i+1}}{(b+i)\,(a+i)_{i+1}} \\
&\qquad\qquad \times {}_3F_2\left[\begin{matrix} -n+i+1 & b+i & a+i+n+1 \\ & b+i+1 & a+2i+1 \end{matrix}\middle| 1\right]\Bigg)\, {}_2F_1\left[\begin{matrix} -i & i+a \\ & b \end{matrix}\middle| z\right], \\
&(iii) \quad \frac{d}{dz}\, {}_2F_1\left[\begin{matrix} -n & a \\ & -n+b \end{matrix}\middle| z\right] = \sum_{i=0}^{n-2} (-1)^{n+i} \frac{n!}{i!} \frac{(b-1)}{(b-n)_{n-i}}\, {}_2F_1\left[\begin{matrix} -i & a \\ & -i+b \end{matrix}\middle| z\right] \\
&\qquad\qquad + \frac{n(a+n-1)}{(n-b)}\, {}_2F_1\left[\begin{matrix} -(n-1) & a \\ & -(n-1)+b \end{matrix}\middle| z\right].
\end{aligned}
$$

1 BASIC DEFINITIONS AND IDENTITIES

The generalized hypergeometric series is defined by

$$ {}_pF_q\left[\begin{matrix} a_1 & a_2 & \dots & a_p \\ b_1 & b_2 & \dots & b_q \end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k}\frac{z^k}{k!}, $$

where a_i, b_i are complex parameters and $(x)_n$ denotes the Pochhammer symbol (or shifted factorial) defined by

$$ (x)_n = \begin{cases} 1, & n = 0, \\ x(x+1)(x+2)\cdots(x+n-1), & n > 0. \end{cases} $$

It is assumed that b_i are not negative integers or zero.

The partial case ${}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}\middle| z\right]$ is called the Gauss hypergeometric function. The series converges when $|z| < 1$ and also when $z = 1$ provided that $\mathrm{Re}(c - a - b) > 0$. In this case the Gauss summation identity holds:

$$ {}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}\middle| 1\right] = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}, \tag{1} $$

where $\Gamma(z)$ is the Gamma function defined by the equality $\Gamma(z+1) = z\Gamma(z)$.

When $a = -n$ or $b = -n$ is a negative integer the series terminates and reduces to a polynomial of degree n, called a hypergeometric polynomial:

$$ {}_2F_1\left[\begin{matrix} -n & a \\ & b \end{matrix}\middle| z\right] = \sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{(a)_i}{(b)_i}z^i. $$

For the hypergeometric polynomial the summation identity becomes

$$ {}_2F_1\left[\begin{matrix} -n & a \\ & b \end{matrix}\middle| 1\right] = \frac{(b-a)_n}{(b)_n}, \tag{2} $$

and this is equivalent to Vandermonde's theorem. If the hypergeometric function is differentiated of z, it gives

$$ \frac{d}{dz}\,{}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}\middle| z\right] = \frac{ab}{c}\,{}_2F_1\left[\begin{matrix} a+1 & b+1 \\ & c+1 \end{matrix}\middle| z\right]. \tag{3} $$

We also need the following properties of the Pochhammer symbol

$$ (x)_n = \frac{\Gamma(x+n)}{\Gamma(n)}, \tag{4} $$

$$ (x)_n = n!\binom{x+n-1}{n}(-x)_n = (-1)^n n!\binom{x}{n}, \tag{5} $$

$$ (-x)_n = (-1)^n(x-n+1)_n, \tag{6} $$

$$ (x)_{n+m} = (x)_m(x+m)_n, \tag{7} $$

(see [1,2] for more details). We will also often use the summation interchange formula

$$ \sum_{i=1}^{n}\sum_{j=1}^{i} a_i b_j = \sum_{j=1}^{n}\left(\sum_{i=j}^{n} a_i\right) b_j, \tag{8} $$

(see [3]).

2 INVERSE PROBLEM

A solution of the inverse problem for the family $P_n(z) = \sum\limits_{k=0}^{n} p_{n,k} z^k$, namely

$$z^n = \sum_{k=0}^{n} \alpha_i P_i(z) = \sum_{i=0}^{n} \alpha_i \left(\sum_{k=0}^{i} p_{i,k} z^k \right) = \sum_{k=0}^{n} \left(\sum_{i=k}^{n} \alpha_i p_{i,k} \right) z^k, \tag{9}$$

defines the orthogonal relation

$$\sum_{i=k}^{n} \alpha_i p_{i,k} = \delta_{n,k},$$

where $\delta_{n,k}$ is the Kronecker delta. Similar orthogonal relations are frequently encountered in combinatorial problems and have been extensively studied by Riordan [4]. Thus, to solve the inverse problems we will check whether the numbers α_i and the coefficients of the corresponding hypergeometric polynomials are orthogonal.

Let us prove Theorem 1. For the item (i) we just check an orthogonality. We have

$$\begin{aligned}
\frac{(b)_n}{(a)_n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} {}_2F_1\left[\begin{matrix} -i, a \\ b \end{matrix} \middle| z \right] &= \frac{(b)_n}{(a)_n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \sum_{k=0}^{i} \frac{(-i)_k (a)_k}{(b)_k} \frac{z^k}{k!} \\
&= \frac{(b)_n}{(a)_n} \sum_{k=0}^{n} \left(\sum_{i=k}^{n} (-1)^i \binom{n}{i} \frac{(-i)_k}{k!} \frac{(a)_k}{(b)_k} \right) z^k \\
&= \frac{(b)_n}{(a)_n} \sum_{k=0}^{n} \frac{(a)_k}{(b)_k} \left(\sum_{i=k}^{n} (-1)^i \binom{n}{i} \frac{(-i)_k}{k!} \right) z^k \\
&= \frac{(b)_n}{(a)_n} \sum_{k=0}^{n} \frac{(a)_k}{(b)_k} \left(\sum_{i=k}^{n} (-1)^{i+k} \binom{n}{i} \binom{i}{k} \right) z^k \\
&= \frac{(b)_n}{(a)_n} \sum_{k=0}^{n} \frac{(a)_k}{(b)_k} \delta_{n,k} z^k = z^n,
\end{aligned}$$

as required. Here we have used (8) and the well known (see [4]) orthogonal relation

$$\sum_{i=k}^{n} (-1)^{i+k} \binom{n}{i} \binom{i}{k} = \delta_{n,k}.$$

(ii) We have

$$\begin{aligned}
\sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{(a+2i)(b)_n}{(a+i)_{n+1}} {}_2F_1\left[\begin{matrix} -i, i+a \\ b \end{matrix} \middle| z \right] &= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{(a+2i)(b)_n}{(a+i)_{n+1}} \sum_{k=0}^{i} \frac{(-i)_k (i+a)_k}{(b)_k} \frac{z^k}{k!} \\
= \sum_{k=0}^{n} \left(\sum_{i=k}^{n} (-1)^i \binom{n}{i} (a+2i) \frac{(b)_n}{(a+i)_{n+1}} \frac{(-i)_k}{k!} \frac{(i+a)_k}{(b)_k} \right) z^k & \\
= \sum_{k=0}^{n} \frac{(b)_n}{(b)_k} \left(\sum_{i=k}^{n} (-1)^{i+k} \binom{n}{i} \binom{i}{k} \frac{a+2i}{(a+i+k)_{n-k+1}} \right) z^k & \\
= \sum_{k=0}^{n} \frac{(b)_n}{(b)_k (n-k+1)!} \left(\sum_{i=k}^{n} (-1)^{i+k} (a+2i) \binom{n}{i} \binom{i}{k} \binom{a+n+i}{n-k+1}^{-1} \right) z^k & \\
= \sum_{k=0}^{n} \frac{(b)_n}{(b)_k (n-k+1)!} \binom{n}{k} \left(\sum_{i=k}^{n} (-1)^{i+k} (a+2i) \binom{n-k}{n-i} \binom{a+n+i}{n-k+1}^{-1} \right) z^k. &
\end{aligned}$$

Let us prove the orthogonal relation

$$\sum_{i=k}^{n}(-1)^{i+k}(a+2i)\binom{n-k}{n-i}\binom{a+n+i}{n-k+1}^{-1}=\delta_{n,k}.$$

Rewrite the relation in an equivalent form by shifting the index of summation from i to $i+k$:

$$\sum_{i=0}^{n-k}(-1)^{i}(a+2(i+k))\binom{n-k}{i}\binom{a+n+i+k}{n-k+1}^{-1}=\delta_{n,k}.$$

Now we again perform the shifts $n-k\mapsto n$ and $a+n+2k\to a$ and will get the relation in such a simplified form

$$\sum_{i=0}^{n}(-1)^{i}(a+2i)\binom{n}{i}\binom{a+n+i}{n+1}^{-1}=\delta_{n,0}.$$

For $n=0$ the both sides are equal to 1. Let us prove that the sum equals 0 for $n>0$. Indeed, we have

$$\begin{aligned}\sum_{i=0}^{n}(-1)^{i}(a+2i)\binom{n}{i}\binom{a+n+i}{n+1}^{-1}&=\sum_{i=0}^{n}\frac{(-1)^{i}(a+2i)n!(n+1)!(a+i-1)!}{i!(n-i)!(a+n+i)!}\\&=\frac{(a-1)!n!}{(a+n)!}\sum_{i=0}^{n}\frac{(a+2i)(-n)_i(a)_i}{i!(a+n+1)_i}.\end{aligned}$$

Now to calculate the last sum we divide it into two sums and then express them by hypergeometric functions

$$\begin{aligned}\sum_{i=0}^{n}\frac{(a+2i)(-n)_i(a)_i}{i!(a+n+1)_i}&=a\sum_{i=0}^{n}\frac{(-n)_i(a)_i}{i!(a+n+1)_i}+2\sum_{i=0}^{n}\frac{i(-n)_i(a)_i}{i!(a+n+1)_i}\\&=a\,{}_2F_1\left[\begin{matrix}-n,a\\n+a+1\end{matrix}\middle|1\right]+2\sum_{i=1}^{n}\frac{(-n)_i(a)_i}{(i-1)!(a+n+1)_i}\\&=a\,{}_2F_1\left[\begin{matrix}-n,a\\n+a+1\end{matrix}\middle|1\right]+\frac{2a\,(-n)}{(a+n+1)}\,{}_2F_1\left[\begin{matrix}-n+1,a+1\\n+a+2\end{matrix}\middle|1\right]\\&=a\left(\frac{\Gamma(a+n+1)\Gamma(2n+1)}{\Gamma(2n+a+1)\Gamma(n+1)}-\frac{\Gamma(a+n+1)\Gamma(2n+1)}{\Gamma(2n+a+1)\Gamma(n+1)}\right)=0.\end{aligned}$$

(iii) Since

$${}_2F_1\left[\begin{matrix}-n\ \ a\\n+b\end{matrix}\middle|z\right]=\sum_{k=0}^{n}\frac{(-i)_k(a)_k}{(i+b)_k}\frac{z^k}{k!}=(-1)^k\binom{i}{k}\frac{(a)_k}{(i+b)_k},$$

we have to prove the following orthogonal relation:

$$\sum_{i=k}^{n}(-1)^{i+k}\binom{n}{i}\binom{i}{k}(b+2n-1)\frac{(b+i)_{n-1}}{(a)_n}\frac{(a)_k}{(b+i)_k}=\delta_{n,k}. \tag{10}$$

After simplification we obtain

$$(b+2n-1)\sum_{i=k}^{n}(-1)^{i+k}\binom{n-k}{n-i}(b+i+k)_{n-1-k}=\delta_{n,k}.$$

By index shifting like as in (ii) we get the identity

$$(b-1)\sum_{i=0}^{n}(-1)^i\binom{n}{i}(b+i)_{n-1}=\delta_{n,0}.$$

For $n=0$ taking into account

$$(b)_{-1}=\frac{\Gamma(b-1)}{\Gamma(b)}=\frac{(b-2)!}{(b-1)!}=\frac{1}{b-1},$$

we have that the identity is true.

For $n>0$ taking into account

$${}_2F_1\left[\begin{matrix}-n & b\\ & c\end{matrix}\middle|1\right]=\frac{(c-b)_n}{c_n},$$

we get

$$\begin{aligned}\sum_{i=0}^{n}(-1)^i\binom{n}{i}(b+i)_{n-1}&=\sum_{i=0}^{n}(-1)^i\frac{n!}{i!(n-i)!}(b+i)_{n-1}=\sum_{i=0}^{n}\frac{(-n)_i}{i!}(b+i)_{n-1}\\&=(b)_{n-1}\sum_{i=0}^{n}\frac{(-n)_i(b+n-1)_i}{(b)_i i!}\\&=(b)_{n-1}{}_2F_1\left[\begin{matrix}-n, b+n-1\\ b\end{matrix}\middle|1\right]=(b)_{n-1}\frac{(-n+1)_n}{(b)_n}=0.\end{aligned}$$

This complete the proof of the item (iii).

(iv) Since

$${}_2F_1\left[\begin{matrix}-n & a\\ -n+b\end{matrix}\middle|z\right]=\sum_{k=0}^{n}\frac{(-i)_k(a)_k}{(-i+b)_k}\frac{z^k}{k!}=(-1)^k\binom{i}{k}\frac{(a)_k}{(-i+b)_k},$$

we have to prove the following orthogonal relation:

$$(b-1)\sum_{i=k}^{n}(-1)^{k+i}\binom{n}{i}\binom{i}{k}(b-i+k)_{n-1-k}=\delta_{n,k}.$$

The proof techniques are similar to the one of the identity (10) and we omit it.

3 THE DERIVATIVE CONNECTING PROBLEM

Let us prove Theorem 2.

Proof. (i) We first prove the auxiliary combinatorial identity:

$$S_{n,k}=\sum_{i=k}^{n-2}\frac{1}{(i-k)!(a+i)_{n-i}}=\frac{1}{(a+k)\,(a+n-1)\,(n-(k+2))!}. \tag{11}$$

Simplify

$$\begin{aligned} S_{n,k} &= \sum_{i=k}^{n-2} \frac{1}{(i-k)!(a+i)_{n-i}} = \sum_{i=k}^{n-2} \frac{1}{(i-k)!(n-i)!\binom{a+n-1}{n-i}} \\ &= \frac{1}{(n-k)!} \sum_{i=k}^{n-2} \frac{(n-k)!}{(i-k)!(n-i)!} \binom{a+n-1}{n-i}^{-1} \\ &= \frac{1}{(n-k)!} \sum_{i=k}^{n-2} \binom{n-k}{i-k} \binom{a+n-1}{n-i}^{-1} = \frac{1}{(n-k)!} \sum_{i=0}^{n-k-2} \binom{n-k}{i} \binom{a+n-1}{n-i-k}^{-1}. \end{aligned}$$

Put

$$S'_{n,k} = (n-k)! S_n = \sum_{i=0}^{n-k-2} \frac{\binom{n-k}{i}}{\binom{a+n-1}{n-i-k}}.$$

We prove by double induction on n and then on k that

$$S'_{n,k} = \frac{(n-k)!}{(a+k)\,(a+n-1)\,(n-(k+2))!} = \frac{(n-k)(n-k-1)}{(a+k)\,(a+n-1)}. \tag{12}$$

Firstly we fix k and use the induction on n. The base case $n = k+2$ is obviously true. Assume that the identity

$$S'_{n-1,k} = \sum_{i=0}^{n-k-3} \frac{\binom{n-k-1}{n-i-k-1}}{\binom{a+n-1}{n-i-k}} = \frac{(n-k-1)(n-k-2)}{(a+k)\,(a+n-2)},$$

holds. Then by standard combinatorial technique we have

$$\begin{aligned} S'_{n,k} = \sum_{i=0}^{n-k-2} \frac{\binom{n-k}{i}}{\binom{a+n-1}{n-i-k}} &= \frac{\binom{n-k}{n-k-2}}{\binom{a+n-1}{2}} + \sum_{i=0}^{n-k-3} \frac{\binom{n-k}{n-i-k}}{\binom{a+n-1}{n-i-k}} \\ &= \frac{\binom{n-k}{2}}{\binom{a+n-1}{2}} + \sum_{i=0}^{n-k-3} \frac{\frac{n-k}{n-i-k}\binom{n-k-1}{n-i-k-1}}{\frac{a+n-2}{n-i-k}\binom{a+n-2}{n-i-k-1}} \\ &= \frac{\binom{n-k}{2}}{\binom{a+n-1}{2}} + \frac{n-k}{a+n-1} \sum_{i=0}^{n-k-3} \frac{\binom{n-k-1}{n-i-k-1}}{\binom{a+n-1}{n-i-k}} \\ &= \frac{(n-k)(n-k-1)}{(a+n-1)(a+n-2)} + \frac{n-k}{a+n-1} S'_{n-1,k} \\ &= \frac{(n-k)(n-k-1)}{(a+n-1)(a+n-2)} + \frac{n-k}{a+n-1} \frac{(n-k-1)(n-k-2)}{(a+k)\,(a+n-2)} \\ &= \frac{(n-k)(n-k-1)}{(a+n-1)(a+n-2)} \left(1 + \frac{n-k-2}{a+k}\right) = \frac{(n-k)(n-k-1)}{(a+k)\,(a+n-1)}. \end{aligned}$$

Thus, for a fixed k and all n the following relation

$$S_{n,k} = \frac{1}{(n-k)!}\frac{(n-k)(n-k-1)}{(a+k)\,(a+n-1)} = \frac{1}{(a+k)\,(a+n-1)\,(n-(k+2))!}$$

holds.

Now let us fix n. The induction on k is true due to obvious identity $S'_{n,k+1} = S'_{n-1,k}$. This completes the proof of (12).

Let us show that for the coefficients α_i

$$\alpha_i = \frac{n!}{i!}\frac{a-b}{(a+i)_{n-i}}, \quad \alpha_{n-1} = -n\frac{b+n-1}{a+n-1},$$

the following identity holds:

$$\sum_{i=k}^{n-1} \alpha_i (-i)_k = \frac{(-n)_{k+1}(b+k)}{a+k}. \tag{13}$$

Indeed, by (11) we obtain

$$\begin{aligned}\sum_{i=k}^{n-2} \alpha_i(-i)_k &= (a-b)n!\sum_{i=k}^{n-2}\frac{(-i)_k}{i!(a+i)_{n-i}} = (a-b)n!(-1)^k\sum_{i=k}^{n-2}\frac{1}{(i-k)!(a+i)_{n-i}}\\ &= \frac{(a-b)n!(-1)^k}{(a+k)\,(a+n-1)\,(n-(k+2))!} = \frac{(k-n+1)\,(a-b)(-n)_{k+1}}{(a+k)\,(a+n-1)}.\end{aligned}$$

Taking into account the identity

$$\frac{(k-n+1)\,(a-b)}{(a+k)\,(a+n-1)} + \frac{b+n-1}{a+n-1} = \frac{b+k}{a+k},$$

we get

$$\begin{aligned}\sum_{i=k}^{n-1}\alpha_i(-i)_k &= \sum_{i=k}^{n-2}\alpha_i(-i)_k + (-n)\frac{b+n-1}{a+n-1}(-(n-1))_k\\ &= \frac{(k-n+1)\,(a-b)(-n)_{k+1}}{(a+k)\,(a+n-1)} + \frac{b+n-1}{a+n-1}(-n)_{k+1} = \frac{(-n)_{k+1}(b+k)}{a+k}.\end{aligned}$$

This establishes the identity (13).

Now we can prove Theorem 2, item (i). Taking into account

$$\alpha_i = \frac{n!}{i!}\frac{a-b}{(a+i)_{n-i}}, \quad \alpha_{n-1} = -n\frac{b+n-1}{a+n-1},$$

let us expand the sum

$$\begin{aligned}\sum_{i=0}^{n-2}\frac{n!}{i!}\frac{a-b}{(a+i)_{n-i}}\,{}_2F_1\left[\begin{matrix}-i,a\\ b\end{matrix}\middle| z\right] - n\frac{b+n-1}{a+n-1}{}_2F_1\left[\begin{matrix}-(n-1),a\\ b\end{matrix}\middle| z\right] &= \sum_{i=0}^{n-1}\alpha_i\,{}_2F_1\left[\begin{matrix}-i,a\\ b\end{matrix}\middle| z\right]\\ &= \sum_{k=0}^{n-1}\left(\sum_{i=k}^{n-1}\alpha_i\frac{(-i)_k(b)_k}{(a)_k}\right)\frac{z^k}{k!} = \sum_{k=0}^{n-1}\left(\sum_{i=k}^{n-1}\alpha_i(-i)_k\right)\frac{(b)_k}{(a)_k}\frac{z^k}{k!}\\ &= \sum_{k=0}^{n-1}\frac{(-n)_{k+1}(b+k)}{a+k}\frac{(b)_k}{(a)_k}\frac{z^k}{k!} = \frac{-nb}{a}\sum_{k=0}^{n-1}\frac{(-n+1)_k(b+1)_k}{(a+1)_k}\frac{z^k}{k!}\\ &= \frac{-nb}{a}\,{}_2F_1\left[\begin{matrix}-n+1,b+1\\ a+1\end{matrix}\middle| z\right] = \frac{d}{dz}\,{}_2F_1\left[\begin{matrix}-n,a\\ b\end{matrix}\middle| z\right].\end{aligned}$$

(ii). Let us find the differential connecting coefficients for the family of polynomials ${}_2F_1\left[\begin{matrix}-n & n+a\\ & b\end{matrix}\middle| z\right]$ by using the solution of the corresponding inverse problem (Theoren 1,(ii)):

$$z^k = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(a+2i)\frac{(b)_k}{(a+i)_{k+1}}\,{}_2F_1\left[\begin{matrix}-i & i+a\\ & b\end{matrix}\middle| z\right].$$

Taking into account

$$\frac{(a+1)_k}{(a)_k} = \frac{a+k}{a},\quad \frac{(b)_k}{(b+1)_k} = \frac{b}{b+k},\quad \frac{(-n)_{k+1}}{k!} = (-1)^{k+1}\binom{n}{k}(n-k),$$

we have

$$\begin{aligned}
\frac{d}{dz}\,{}_2F_1\left[\begin{matrix}-n, n+a\\ b\end{matrix}\middle| z\right] &= \frac{-n(n+a)}{b}\,{}_2F_1\left[\begin{matrix}-(n-1), n+a+1\\ b+1\end{matrix}\middle| z\right]\\
&= \frac{-n(n+a)}{b}\sum_{k=0}^{n-1}\frac{(-n+1)_k(n+a+1)_k}{(b+1)_k}\frac{z^k}{k!}\\
&= \frac{-n(n+a)}{b}\sum_{k=0}^{n-1}\frac{(-n+1)_k(n+a+1)_k}{k!(b+1)_k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(a+2i)\frac{(b)_k}{(a+i)_{k+1}}\,{}_2F_1\left[\begin{matrix}-i, i+a\\ b\end{matrix}\middle| z\right]\\
&= \frac{-n(n+a)}{b}\sum_{i=0}^{n-1}\left(\sum_{k=i}^{n-1}(-1)^i\binom{k}{i}\frac{(-n+1)_k(n+a+1)_k}{k!(b+1)_k}\frac{(a+2i)(b)_k}{(a+i)_{k+1}}\right){}_2F_1\left[\begin{matrix}-i, i+a\\ b\end{matrix}\middle| z\right]\\
&= \sum_{i=0}^{n-1}\left(\sum_{k=i}^{n-1}(-1)^{i+k+1}\binom{k}{i}\binom{n}{k}(n-k)\frac{(a+2i)(n+a)_{k+1}}{(b+k)(a+i)_{k+1}}\right){}_2F_1\left[\begin{matrix}-i, i+a\\ b\end{matrix}\middle| z\right]\\
&= \sum_{i=0}^{n-1}\binom{k}{i}\left(\sum_{k=i}^{n-1}(-1)^{i+k+1}\binom{n-i}{n-k}(n-k)\frac{(a+2i)(n+a)_{k+1}}{(b+k)(a+i)_{k+1}}\right){}_2F_1\left[\begin{matrix}-i, i+a\\ b\end{matrix}\middle| z\right].
\end{aligned}$$

In the internal sum we perform the shift of the index of summation as $k \mapsto k+i$:

$$\begin{aligned}
&\sum_{k=i}^{n-1}(-1)^{i+k+1}\binom{n-i}{n-k}(n-k)\frac{(a+2i)(n+a)_{k+1}}{(b+k)(a+i)_{k+1}}\\
&\qquad = \sum_{k=0}^{n-1-i}(-1)^{k+1}\binom{n-i}{n-(k+i)}(n-(k+i))\frac{(a+2i)(n+a)_{k+1+i}}{(b+k+i)(a+i)_{k+1+i}}.
\end{aligned}$$

By using the relations

$$\frac{(a+2i)(n+a)_{k+1+i}}{(a+i)_{k+1+i}} = \frac{(a+2\,i)\,(n+a)_{i+1}}{(a+i)_{i+1}}\cdot\frac{(a+n+i+1)_k}{(a+2\,i+1)_k},$$

$$\frac{1}{(b+k+i)} = \frac{1}{(b+i)}\cdot\frac{(b+i)_k}{(b+i+1)_k},$$

$$(-1)^{k+1}\binom{n-i}{n-(k+i)}(n-(k+i)) = (-n+i)\,\frac{(-n+i+1)_k}{k!},$$

we rewrite the sum in the form

$$\begin{aligned}
&(-n+i)\,(a+2\,i)\,\frac{(n+a)_{i+1}}{(b+i)\,(a+i)_{i+1}}\sum_{k=0}^{n-1-i}\frac{(-n+i+1)_k(b+i)_k(a+n+i+1)_k}{k!(b+i+1)_k(a+2\,i+1)_k}\\
&= (-n+i)\,(a+2\,i)\,\frac{(n+a)_{i+1}}{(b+i)\,(a+i)_{i+1}}\,{}_3F_2\left[\begin{matrix}-n+i+1, b+i,, a+i+n+1\\ b+i+1, a+2\,i+1\end{matrix}\middle| 1\right].
\end{aligned}$$

Finally, we get

$$\frac{d}{dz}\,{}_2F_1\left[\begin{matrix}-n,n+a\\ b\end{matrix}\middle|\,z\right] = \sum_{i=0}^{n-1}\left((-n+i)\binom{n}{i}\frac{(a+2i)\,(n+a)_{i+1}}{(b+i)\,(a+i)_{i+1}}\right.$$
$$\left.\times\;{}_3F_2\left[\begin{matrix}-n+i+1,b+i,,a+i+n+1\\ b+i+1,a+2\,,i+1\end{matrix}\middle|\,1\right]\right){}_2F_1\left[\begin{matrix}-i,i+a\\ b\end{matrix}\middle|\,z\right],$$

as reguired.

(iii) We have to prove that

$$\frac{d}{dz}\,{}_2F_1\left[\begin{matrix}-n,a\\ -n+b\end{matrix}\middle|\,z\right] = \sum_{i=0}^{n-2}(-1)^{n+i}\frac{n!}{i!}\frac{(b-1)}{(b-n)_{n-i}}\,{}_2F_1\left[\begin{matrix}-i,a\\ -i+b\end{matrix}\middle|\,z\right]$$
$$-\frac{n(a+n-1)}{(b-n)}{}_2F_1\left[\begin{matrix}-(n-1),a\\ -(n-1)+b\end{matrix}\middle|\,z\right].$$

We find the differential connecting coefficients for the family of polynomials ${}_2F_1\left[\begin{matrix}-n\;\;a\\ -n+b\end{matrix}\middle|\,z\right]$ by using the solution of the corresponding inverse problem (Theoren 1, item (iii)):

$$z^k = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(b-1)\frac{(b-i)_{k-1}}{(a)_k}\,{}_2F_1\left[\begin{matrix}-i\;\;a\\ -i+b\end{matrix}\middle|\,z\right].$$

We have

$$\frac{d}{dz}\,{}_2F_1\left[\begin{matrix}-n,a\\ -n+b\end{matrix}\middle|\,z\right] = \frac{-na}{-n+b}\,{}_2F_1\left[\begin{matrix}-n+1,a+1\\ -n+1+b\end{matrix}\middle|\,z\right] = \frac{-na}{-n+b}\sum_{k=0}^{n-1}\frac{(-n+1)_k(a+1)_k}{(-n+b+1)_k}\frac{z^k}{k!}$$
$$= \frac{-na}{-n+b}\sum_{k=0}^{n-1}\frac{(-n+1)_k(a+1)_k}{k!(-n+b+1)_k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(b-1)\frac{(b-i)_{k-1}}{(a)_k}\,{}_2F_1\left[\begin{matrix}-i,a\\ -i+b\end{matrix}\middle|\,z\right]$$
$$= \frac{-na}{-n+b}\sum_{i=0}^{n-1}\left(\sum_{k=i}^{n-1}\frac{(-n+1)_k(a+1)_k}{k!(-n+b+1)_k}(-1)^i\binom{k}{i}(b-1)\frac{(b-i)_{k-1}}{(a)_k}\right){}_2F_1\left[\begin{matrix}-i,a\\ -i+b\end{matrix}\middle|\,z\right].$$

Put

$$\alpha_{n,i} = \sum_{k=i}^{n-1}\frac{-na}{-n+b}\frac{(-n+1)_k(a+1)_k}{k!(-n+b+1)_k}(-1)^i\binom{k}{i}(b-1)\frac{(b-i)_{k-1}}{(a)_k}$$
$$= \frac{-n}{-n+b}\sum_{k=i}^{n-1}(-1)^i\binom{k}{i}(b-1)\frac{(a+k)(-n+1)_k(b-i)_{k-1}}{k!(-n+b+1)_k}$$

For $i=n-1$ we have

$$\alpha_{n,n-1} = \frac{-n}{-n+b}(-1)^{n-1}(b-1)\frac{(a+n-1)(-n+1)_{n-1}(b-(n-1))_{n-2}}{(n-1)!(-n+b+1)_{k-1}} = -\frac{n(a+n-1)}{(b-n)}.$$

For $i<n-1$ we have

$$\alpha_{n,i} = (-1)^{n+i}\frac{n!}{i!}\frac{(b-1)}{(b-n)_{n-i}},$$

thus

$$\frac{-n}{-n+b}\sum_{k=i}^{n-1}(-1)^i\binom{k}{i}(b-1)\frac{(a+k)(-n+1)_k(b-i)_{k-1}}{k!(-n+b+1)_k}=(-1)^{n+i}\frac{n!}{i!}\frac{(b-1)}{(b-n)_{n-i}}$$

or

$$\sum_{k=i}^{n-1}\binom{k}{i}\frac{(a+k)(-n)_{k+1}(b-i)_{k-1}}{k!(-n+b+1)_k}=(-1)^n\frac{n!}{i!}\frac{(b-n)}{(b-n)_{n-i}}.$$

Divide the sum into the two sums

$$\begin{aligned}\sum_{k=i}^{n-1}\binom{k}{i}\frac{(a+k)(-n)_k(b-i)_{k-1}}{k!(-n+b+1)_k}&=a\sum_{k=i}^{n-1}\binom{k}{i}\frac{(-n)_{k+1}(b-i)_{k-1}}{k!(-n+b+1)_k}\\&+\sum_{k=i}^{n-1}\binom{k}{i}\frac{k(-n)_{k+1}(b-i)_{k-1}}{k!(-n+b+1)_k}\end{aligned}$$

and calculate them separately. Taking into account

$$\begin{gathered}(b-i)_{k-1}=(b-i)_i(b)_{k-1-i}=(b-i)_i(k-1-i)!\binom{b+k-i-2}{k-1-i},\\(-n+b+1)_k=k!\binom{-n+b+1}{k},\quad (b)_{k-1}=\frac{(b+k-2)!}{(b-1)!},\end{gathered}$$

for the first sum we have

$$\begin{aligned}\sum_{k=i}^{n-1}\binom{k}{i}\frac{(-n)_{k+1}(b-i)_{k-1}}{k!(-n+b+1)_k}&=\sum_{k=i}^{n-1}(-1)^{k+1}\binom{k}{i}\binom{n}{k}\frac{(n-k)(b-i)_{k-1}}{(-n+b+1)_k}\\&=\binom{n}{i}\sum_{k=i}^{n-1}(-1)^{k+1}\binom{n-i}{n-k}(n-k)\frac{(b-i)_{k-1}}{(-n+b+1)_k}\\&=\binom{n}{i}(b-i)_i\sum_{k=i}^{n-1}(-1)^{k+1}\binom{n-i}{n-k}(n-k)\frac{(b)_{k-1-i}}{(-n+b+1)_k}.\end{aligned}$$

Now we shift the summation indexes $k\mapsto k+i$ and $n\mapsto n+i$:

$$\begin{aligned}\sum_{k=i}^{n-1}(-1)^{k+1}\binom{n-i}{n-k}\frac{(n-k)(b)_{k-1-i}}{(-n+b+1)_k}&=\sum_{k=0}^{n-i-1}(-1)^{k+i+1}\binom{n-i}{k}\frac{(n-k-i)(b)_{k-1}}{(-n+b+1)_{k+i}}\\&=\sum_{k=0}^{n-1}(-1)^{k+i+1}\binom{n}{k}\frac{(n-k)(b)_{k-1}}{(-n-i+b+1)_{k+i}}\\&=(-1)^{i+1}(-n+b-i)!\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}\frac{(n-k)(b)_{k-1}}{(-n+b+k)!}\\&=(-1)^{i+1}\frac{(-n+b-i)!}{(b-1)!}\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k)\frac{(b+k-2)!}{(-n+b+k)!}.\end{aligned}\tag{14}$$

Let $[z^n]f(z)$ denote the operation of extracting the coefficient of z^n in a formal power series $f(z)$. It is clear that $[z^n]$ is a linear operation and the following well known properties holds:

$$[z^p](1+z)^q=\binom{q}{p},\quad [z^{p-q}]f(z)=[z^p]z^qf(z),$$

see [5]. By using these properties let us prove that the sum (14) is equal to 0. We have

$$\begin{aligned}
\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k)\frac{(b+k-2)!}{(b+k-n)!} &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k\frac{(b+n-k-2)!}{(b-k)!}\\
&= (-1)^n n(n-2)!\sum_{k=1}^{n}(-1)^k\binom{n-1}{k-1}\binom{b+n-k-2}{b-k}\\
&= (-1)^{n+1} n(n-2)!\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{b+n-k-3}{b-k-1}\\
&= (-1)^{n+1} n(n-2)!\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{-n+1}{b-k+1}\\
&= (-1)^{n+1} n(n-2)!\sum_{k=0}^{n-1}\binom{n-1}{k}[z^{b-k+1}](1+z)^{-n+1}\\
&= (-1)^{n+1} n(n-2)!\,[z^{b+1}](1+z)^{-n+1}\sum_{k=0}^{n-1}\binom{n-1}{k}z^k\\
&= (-1)^{n+1} n(n-2)!\,[z^{b+1}](1+z)^{-n+1}(1+z)^{n-1}\\
&= (-1)^{n+1} n(n-2)!\,[z^{b+1}]1 = 0,
\end{aligned}$$

and the claim follows.

The second identity

$$\sum_{k=i}^{n-1}\binom{k}{i}\frac{k(-n)_{k+1}(b-i)_{k-1}}{k!(-n+b+1)_k} = (-1)^n\frac{n!}{i!}\frac{(b-n)}{(b-n)_{n-i}}$$

can be proved using the same arguments used in (11), so we will omit it here. □

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Розглянемо послідовності многочленів $\{P_n(x)\}_{n\geq 0}$, $\{Q_n(x)\}_{n\geq 0}$ такі, що $\deg(P_n(x)) = n$, $\deg(Q_n(x)) = n$. Задача зв'язності для них полягає у знаходженні коефіцієнтів $\alpha_{n,k}$ у виразі $Q_n(x) = \sum\limits_{k=0}^{n} \alpha_{n,k} P_k(x)$. Задача зв'язності для різних типів многочленів має довгу історію і продовжує викликати інтерес в різних галузях математики, зокрема в комбінаториці, математичній фізиці, квантовій хімії. Для часткового випадку $Q_n(x) = x^n$ задача зв'язності називається оберненою задачею для $\{P_n(x)\}_{n\geq 0}$. Частковий випадок $Q_n(x) = P'_{n+1}(x)$ має назву диференціальної задачі зв'язності для послідовності многочленів $\{P_n(x)\}_{n\geq 0}$. В пропонованій статті ми знаходимо у замкненому вигляді коефіцієнти оберненої і диференціальної задач зв'язності для гіпергеометричних многочленів вигляду

$$ {}_2F_1\left[\begin{matrix} -n, a \\ b \end{matrix} \middle| z\right], \quad {}_2F_1\left[\begin{matrix} -n, n+a \\ b \end{matrix} \middle| z\right], \quad {}_2F_1\left[\begin{matrix} -n, a \\ \pm n + b \end{matrix} \middle| z\right], $$

де ${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} \middle| z\right] = \sum\limits_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$ – гіпергеометрична функція Гауса, а $(x)_n$ позначає символ Похгаммера, який визначається формулою $(x)_n = \begin{cases} 1, & n = 0, \\ x(x+1)(x+2)\cdots(x+n-1), & n > 0. \end{cases}$

Всі многочлени розглядаються над полем дійсних чисел.

Ключові слова і фрази: гіпергеометрична фунція, коефіцієнти зв'язності, обернена задача, задача диференціальної зв'язності, гіпергеометричний многочлен.

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(p,q)TH ORDER ORIENTED GROWTH MEASUREMENT OF COMPOSITE p-ADIC ENTIRE FUNCTIONS

Let $\mathbb{K}$ be a complete ultrametric algebraically closed field and let $\mathcal{A}(\mathbb{K})$ be the $\mathbb{K}$-algebra of entire functions on $\mathbb{K}$. For any p-adic entire function $f \in \mathcal{A}(\mathbb{K})$ and $r > 0$, we denote by $|f|(r)$ the number $\sup\{|f(x)| : |x| = r\}$, where $|\cdot|(r)$ is a multiplicative norm on $\mathcal{A}(\mathbb{K})$. For any two entire functions $f \in \mathcal{A}(\mathbb{K})$ and $g \in \mathcal{A}(\mathbb{K})$ the ratio $\frac{|f|(r)}{|g|(r)}$ as $r \to \infty$ is called the comparative growth of f with respect to g in terms of their multiplicative norms. Likewise to complex analysis, in this paper we define the concept of (p,q)th order (respectively (p,q)th lower order) of growth as $\rho^{(p,q)}(f) = \limsup\limits_{r\to+\infty} \frac{\log^{[p]}|f|(r)}{\log^{[q]} r}$ (respectively $\lambda^{(p,q)}(f) = \liminf\limits_{r\to+\infty} \frac{\log^{[p]}|f|(r)}{\log^{[q]} r}$), where p and q are any two positive integers. We study some growth properties of composite p-adic entire functions on the basis of their (p,q)th order and (p,q)th lower order.

Key words and phrases: p-adic entire function, growth, (p,q)th order, (p,q)th lower order, composition.

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INTRODUCTION AND DEFINITIONS

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0, complete with respect to a p-adic absolute value $|\cdot|$ (example $\mathbb{C}_p$). For any $\alpha \in \mathbb{K}$ and $R \in (0,+\infty)$, the closed disk $\{x \in \mathbb{K} : |x-\alpha| \le R\}$ and the open disk $\{x \in \mathbb{K} : |x-\alpha| < R\}$ are denoted by $d(\alpha,R)$ and $d(\alpha,R^-)$ respectively. Also $C(\alpha,r)$ denotes the circle $\{x \in \mathbb{K} : |x-\alpha| = r\}$. Moreover $\mathcal{A}(\mathbb{K})$ represent the $\mathbb{K}$-algebra of analytic functions on $\mathbb{K}$, i.e. the set of power series with an infinite radius of convergence. For the most comprehensive study of analytic functions inside a disk or in the whole field $\mathbb{K}$, we refer the reader to the books [9, 10, 15, 18]. During the last several years the ideas of p-adic analysis have been studied from different aspects and many important results were gained (see [1–6], [8, 11–14, 19]).

Let $f \in \mathcal{A}(\mathbb{K})$ and $r > 0$, then we denote by $|f|(r)$ the number $\sup\{|f(x)| : |x| = r\}$ where $|\cdot|(r)$ is a multiplicative norm on $\mathcal{A}(\mathbb{K})$. For any two entire functions $f \in \mathcal{A}(\mathbb{K})$ and $g \in \mathcal{A}(\mathbb{K})$ the ratio $\frac{|f|(r)}{|g|(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of their multiplicative norms.

For any $x \in [0,\infty)$ and $k \in \mathbb{N}$, we define recursively $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ and $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$, where $\mathbb{N}$ stands for the set of all positive integers. We also denote $\log^{[0]} x = x$ and $\exp^{[0]} x = x$. Throughout the paper, log denotes the Neperian logarithm.

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Taking this into account the order (resp. lower order) of an entire function $f \in \mathcal{A}(\mathbb{K})$ is given by (see [4])

$$\begin{matrix} \rho(f) \\ \lambda(f) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[2]}|f|(r)}{\log r}.$$

The above definition of order (resp. lower order) does not seem to be feasible if an entire function $f \in \mathcal{A}(\mathbb{K})$ is of order zero. To overcome this situation and in order to study the growth of an entire function $f \in \mathcal{A}(\mathbb{K})$ precisely, one may introduce the concept of logarithmic order (resp. logarithmic lower order) by increasing $\log^{+}$ once in the denominator following the classical definition of logarithmic order (see, for example, [7]). Therefore the logarithmic order $\rho_{\log}(f)$ and logarithmic lower order $\lambda_{\log}(f)$ of an entire function $f \in \mathcal{A}(\mathbb{K})$ are define as

$$\begin{matrix} \rho_{\log}(f) \\ \lambda_{\log}(f) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[2]}|f|(r)}{\log^{[2]} r}.$$

Further the concept of (p,q)th order (p and q are any two positive integers with $p \geq q$) is not new and was first introduced by Juneja et al. [16,17]. In the line of Juneja et al. [16,17], now we shall introduce the definitions of (p,q)th order and (p,q)th lower order respectively of an entire function $f \in \mathcal{A}(\mathbb{K})$ where $p,q \in \mathbb{N}$. In order to keep accordance with the definition of logarithmic order we will give a minor modification to the original definition of (p,q)-order introduced by Juneja et al. [16,17].

Definition 1. *Let $f \in \mathcal{A}(\mathbb{K})$ and $p,q \in \mathbb{N}$. Then the (p,q)th order and (p,q)th lower order of f are respectively defined as:*

$$\begin{matrix} \rho^{(p,q)}(f) \\ \lambda^{(p,q)}(f) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]}|f|(r)}{\log^{[q]} r}.$$

These definitions extend the generalized order $\rho^{[l]}(f)$ and generalized lower order $\lambda^{[l]}(f)$ of $f \in \mathcal{A}(\mathbb{K})$ for each integer $l \geq 2$ since these correspond to the particular case $\rho^{[l]}(f) = \rho^{(l,1)}(f)$ and $\lambda^{[l]}(f) = \lambda^{(l,1)}(f)$. Clearly $\rho^{(2,1)}(f) = \rho(f)$ and $\lambda^{(2,1)}(f) = \lambda(f)$. The above definition avoid the restriction $p > q$ and give the idea of generalized logarithmic order.

However in this connection we just introduce the following definition which is analogous to the definition of Juneja et al. [16,17].

Definition 2. *An entire function $f \in \mathcal{A}(\mathbb{K})$ is said to have index-pair (p,q), where p and $q \in \mathbb{N}$, if $b < \rho^{(p,q)}(f) < \infty$ and $\rho^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ otherwise. Moreover if $0 < \rho^{(p,q)}(f) < \infty$, then*

$$\begin{cases} \rho^{(p-n,q)}(f) = \infty & \text{for } n < p, \\ \rho^{(p,q-n)}(f) = 0 & \text{for } n < q, \\ \rho^{(p+n,q+n)}(f) = 1 & \text{for } n = 1,2,\ldots. \end{cases}$$

Similarly for $0 < \lambda^{(p,q)}(f) < \infty$, one can easily verify that

$$\begin{cases} \lambda^{(p-n,q)}(f) = \infty & \text{for } n < p, \\ \lambda^{(p,q-n)}(f) = 0 & \text{for } n < q, \\ \lambda^{(p+n,q+n)}(f) = 1 & \text{for } n = 1,2,\ldots. \end{cases}$$

The main aim of this paper is to establish some results related to the growth properties of composite p-adic entire functions on the basis of (p,q)th order and (p,q)th lower order, where $p,q \in \mathbb{N}$.

1 Lemma

In this section we present the following lemma which can be found in [4] or [5] and will be needed in the sequel.

Lemma 1. *Let $f, g \in \mathcal{A}(\mathbb{K})$. Then for all sufficiently large values of r the following equality holds*

$$|f \circ g|(r) = |f|(|g|(r)).$$

2 Main results

Theorem 1. *Let $f, g \in \mathcal{A}(\mathbb{K})$ be such that $\rho^{(m,n)}(g) < \lambda^{(p,q)}(f) \leq \rho^{(p,q)}(f) < \infty$, where $p, q, m, n \in \mathbb{N}$. Then*

$$(i) \quad \lim_{r\to+\infty} \frac{\log^{[p]} |f \circ g| \left(\exp^{[n-1]} r\right)}{\log^{[p-1]} |f|(\exp^{[q-1]} r)} = 0 \quad \text{if } q \geq m$$

and

$$(ii) \quad \lim_{r\to+\infty} \frac{\log^{[p+m-q-1]} |f \circ g| \left(\exp^{[n-1]} r\right)}{\log^{[p-1]} |f|(\exp^{[q-1]} r)} = 0 \quad \text{if } q < m.$$

Proof. We get from Lemma 1, for all sufficiently large positive numbers of r that

$$\log^{[p]} |f \circ g| \left(\exp^{[n-1]} r\right) = \log^{[p]} |f| \left(|g| \left(\exp^{[n-1]} r\right)\right)$$

i.e.,

$$\log^{[p]} |f \circ g| \left(\exp^{[n-1]} r\right) \leqslant \left(\rho^{(p,q)}(f) + \varepsilon\right) \log^{[q]} |g| \left(\exp^{[n-1]} r\right). \tag{1}$$

Now the following two cases may arise.

Case I. Let $q \geqslant m$. Then we have from (1) for all sufficiently large positive numbers of r that

$$\log^{[p]} |f \circ g| \left(\exp^{[n-1]} r\right) \leqslant \left(\rho^{(p,q)}(f) + \varepsilon\right) \log^{[m-1]} |g| \left(\exp^{[n-1]} r\right) \tag{2}$$

i.e.,

$$\log^{[p]} |f \circ g| \left(\exp^{[n-1]} r\right) \leqslant \left(\rho^{(p,q)}(f) + \varepsilon\right) r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)}. \tag{3}$$

Case II. Let $q < m$. Then for all sufficiently large positive numbers of r we get from (1) that

$$\log^{[p]} |f \circ g| \left(\exp^{[n-1]} r\right) \leqslant \left(\rho^{(p,q)}(f) + \varepsilon\right) \exp^{[m-q]} \log^{[m]} |g| \left(\exp^{[n-1]} r\right). \tag{4}$$

Further for all sufficiently large positive numbers of r, it follows that

$$\log^{[m]} |g| \left(\exp^{[n-1]} r\right) \leqslant \log \left(r^{\rho^{(m,n)}(g)+\varepsilon}\right)$$

i.e.,

$$\exp^{[m-q]} \log^{[m]} |g| \left(\exp^{[n-1]} r\right) \leqslant \exp^{[m-q-1]} \left(r^{\rho^{(m,n)}(g)+\varepsilon}\right). \tag{5}$$

Now from (4) and (5) we have for all sufficiently large positive numbers of r that

$$\log^{[p]} |f \circ g| \left(\exp^{[n-1]} r\right) \leq \left(\rho^{(p,q)}(f) + \varepsilon\right) \exp^{[m-q-1]} \left(r^{\rho^{(m,n)}(g)+\varepsilon}\right)$$

i.e.,

$$\log^{[p+1]}|f\circ g|\left(\exp^{[n-1]}r\right)\leqslant\exp^{[m-q-2]}\left(r^{\rho^{(m,n)}(g)+\varepsilon}\right)+O(1)$$

i.e.,

$$\log^{[p+1]}|f\circ g|\left(\exp^{[n-1]}r\right)\leqslant\exp^{[m-q-2]}\left(r^{\rho^{(m,n)}(g)+\varepsilon}\right)\left(1+\frac{O(1)}{\exp^{[m-q-2]}\left(r^{\rho^{(m,n)}(g)+\varepsilon}\right)}\right)$$

i.e.,

$$\log^{[p+m-q-1]}|f\circ g|\left(\exp^{[n-1]}r\right)\leqslant r^{\rho_g(m,n)+\varepsilon}\left(1+\frac{O(1)}{\exp^{[m-q-2]}\left(r^{\rho^{(m,n)}(g)+\varepsilon}\right)}\right). \tag{6}$$

Also from the definition of $\lambda^{(p,q)}(f)$, we get for all sufficiently large positive numbers of r that

$$\log^{[p-1]}|f|(\exp^{[q-1]}r)\geqslant r^{(\lambda^{(p,q)}(f)-\varepsilon)}. \tag{7}$$

Now combining (3) of Case I and (7) we get for all sufficiently large positive numbers of r that

$$\frac{\log^{[p]}|f\circ g|\left(\exp^{[n-1]}r\right)}{\log^{[p-1]}|f|(\exp^{[q-1]}r)}\leq\frac{\left(\rho_f\left(p,q\right)+\varepsilon\right)r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)}}{r^{(\lambda^{(p,q)}(f)-\varepsilon)}}. \tag{8}$$

Since $\rho^{(m,n)}(g)<\lambda^{(p,q)}(f)$ we can choose $\varepsilon\,(>0)$ in such a way that

$$\rho^{(m,n)}(g)+\varepsilon<\lambda^{(p,q)}(f)-\varepsilon. \tag{9}$$

Therefore in view of (9) it follows from (8) that

$$\lim_{r\to+\infty}\frac{\log^{[p]}|f\circ g|\left(\exp^{[n-1]}r\right)}{\log^{[p-1]}|f|(\exp^{[q-1]}r)}=0.$$

Hence the first part of the theorem follows.

Further combining (6) of Case II and (7) we obtain for all sufficiently large positive numbers of r that

$$\frac{\log^{[p+m-q-1]}|f\circ g|\left(\exp^{[n-1]}r\right)}{\log^{[p-1]}|f|(\exp^{[q-1]}r)}\leq\frac{r^{\rho^{(m,n)}(g)+\varepsilon}\left(1+\frac{O(1)}{\exp^{[m-q-2]}\left(r^{\rho^{(m,n)}(g)+\varepsilon}\right)}\right)}{r^{(\lambda^{(p,q)}(f)-\varepsilon)}}. \tag{10}$$

Therefore in view of (9) we get from above that

$$\lim_{r\to+\infty}\frac{\log^{[p+m-q-1]}|f\circ g|\left(\exp^{[n-1]}r\right)}{\log^{[p-1]}|f|(\exp^{[q-1]}r)}=0\,.$$

Thus the theorem follows. □

Theorem 2. *Let $f,g\in\mathcal{A}(\mathbb{K})$ be such that $\lambda^{(m,n)}(g)<\lambda^{(p,q)}(f)\leq\rho^{(p,q)}(f)<\infty$, where $p,q,m,n\in\mathbb{N}$. Then*

$$(i)\qquad\varliminf_{r\to+\infty}\frac{\log^{[p]}|f\circ g|\left(\exp^{[n-1]}r\right)}{\log^{[p-1]}|f|(\exp^{[q-1]}r)}=0\quad\text{if } q\geq m$$

and

$$(ii)\qquad\varliminf_{r\to+\infty}\frac{\log^{[p+m-q-1]}|f\circ g|\left(\exp^{[n-1]}r\right)}{\log^{[p-1]}|f|(\exp^{[q-1]}r)}=0\quad\text{if } q<m.$$

The proof of Theorem 2 is omitted as it can be carried out in the line of Theorem 1.

Theorem 3. *Let* $f, g \in \mathcal{A}(\mathbb{K})$ *be such that* $0 < \lambda^{(p,q)}(f) \leq \rho^{(p,q)}(f) < \infty$ *and* $\rho^{(m,n)}(g) < \infty$, *where* $p, q, m, n \in \mathbb{N}$. *Then*

$$(i) \quad \varlimsup_{r\to+\infty} \frac{\log^{[p+1]} |f\circ g| \left(\exp^{[n-1]} r\right)}{\log^{[p]} |f|(\exp^{[q-1]} r)} \leqslant \frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)} \quad \text{if } q \geq m$$

and

$$(ii) \quad \varlimsup_{r\to+\infty} \frac{\log^{[p+m-q]} |f\circ g| \left(\exp^{[n-1]} r\right)}{\log^{[p]} |f|(\exp^{[q-1]} r)} \leqslant \frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)} \quad \text{if } q < m.$$

Proof. In view of the definition $\lambda^{(p,q)}(f)$, we have for all sufficiently large positive numbers of r that

$$\log^{[p]} |f|(\exp^{[q-1]} r) \geq \left(\lambda^{(p,q)}(f) - \varepsilon\right) \log r. \tag{11}$$

Case I. If $q \geqslant m$, then from (3) and (11) we get for all sufficiently large positive numbers of r that

$$\frac{\log^{[p+1]} |f\circ g| \left(\exp^{[n-1]} r\right)}{\log^{[p]} |f|(\exp^{[q-1]} r)} \leqslant \frac{\left(\rho^{(m,n)}(g) + \varepsilon\right) \log r + \log\left(\rho^{(p,q)}(f) + \varepsilon\right)}{\left(\lambda^{(p,q)}(f) - \varepsilon\right) \log r}.$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\varlimsup_{r\to+\infty} \frac{\log^{[p+1]} |f\circ g| \left(\exp^{[n-1]} r\right)}{\log^{[p]} |f|(\exp^{[q-1]} r)} \leqslant \frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}.$$

This proves the first part of the theorem.

Case II. If $q < m$ then from (6) and (11) we obtain for all sufficiently large positive numbers of r that

$$\frac{\log^{[p+m-q]} |f\circ g| \left(\exp^{[n-1]} r\right)}{\log^{[p]} |f|(\exp^{[q-1]} r)} \leqslant \frac{\left(\rho^{(m,n)}(g) + \varepsilon\right) \log r + \log\left(1 + \frac{O(1)}{\exp^{[m-q-2]}\left(r^{\rho^{(m,n)}(g)+\varepsilon}\right)}\right)}{\left(\lambda^{(p,q)}(f) - \varepsilon\right) \log r}.$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\varlimsup_{r\to+\infty} \frac{\log^{[p+m-q]} |f\circ g| \left(\exp^{[n-1]} r\right)}{\log^{[p]} |f|(\exp^{[q-1]} r)} \leqslant \frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}.$$

Thus the second part of the theorem is established. $\square$

Theorem 4. *Let* $f, g \in \mathcal{A}(\mathbb{K})$ *be such that* $0 < \lambda^{(p,q)}(f) \leq \rho^{(p,q)}(f) < \infty$ *and* $\lambda^{(m,n)}(g) > 0$, *where* $p, q, m, n \in \mathbb{N}$. *Then for any positive integer* l, *we have*

$$(i) \quad \lim_{r\to\infty} \frac{\log^{[p]} |f\circ g|(\exp^{[n-1]} r)}{\log^{[p+1]} |f| \left(\exp^{[l]} r\right)} = \infty \quad \text{if } q < m \text{ and } q \geqslant l;$$

$$(ii) \quad \lim_{r\to\infty} \frac{\log^{[p]} |f\circ g|(\exp^{[n-1]} r)}{\log^{[p-q-l+1]} |f| \left(\exp^{[l]} r\right)} = \infty \quad \text{if } q < m \text{ and } q < l;$$

$$(iii) \quad \lim_{r\to\infty} \frac{\log^{[p+m-q-1]} |f\circ g|(\exp^{[n-1]} r)}{\log^{[p+1]} |f| \left(\exp^{[l]} r\right)} = \infty \quad \text{if } q > m \text{ and } q < l;$$

and

$$(iv) \quad \lim_{r\to\infty} \frac{\log^{[p+m-q-1]} |f\circ g|(\exp^{[n-1]} r)}{\log^{[p+1]} |f| \left(\exp^{[l]} r\right)} = \infty \quad \text{if } q > m \text{ and } q \geqslant l.$$

Proof. Let us choose $0 < \varepsilon < \min\left\{\lambda^{(p,q)}(f), \lambda^{(m,n)}(g)\right\}$. Now for all sufficiently large positive numbers of r we get from Lemma 1,

$$\log^{[p]} |f\circ g|(\exp^{[n-1]} r) \geqslant (\lambda^{(p,q)}(f) - \varepsilon) \log^{[q]} |g| \left(\exp^{[n-1]} r\right). \tag{12}$$

Further from the definition of (m,n)th lower order of g we have for all sufficiently large positive numbers of r that

$$\log^{[m]} |g| \left(\exp^{[n-1]} r\right) \geqslant \log r^{(\lambda^{(m,n)}(g)-\varepsilon)}. \tag{13}$$

Now the following two cases may arise.

Case I. Let $q < m$. Then from (12) and (13) we obtain for all sufficiently large positive numbers of r that

$$\log^{[p]} |f\circ g|(\exp^{[n-1]} r) \geqslant (\lambda^{(p,q)}(f) - \varepsilon) \exp^{[m-q]} \log^{[m]} |g| \left(\exp^{[n-1]} r\right) \tag{14}$$

i.e.,

$$\log^{[p]} |f\circ g|(\exp^{[n-1]} r) \geqslant (\lambda^{(p,q)}(f) - \varepsilon) \exp^{[m-q]} \log r^{(\lambda^{(m,n)}(g)-\varepsilon)}$$

$$\log^{[p]} |f\circ g|(\exp^{[n-1]} r) \geqslant (\lambda^{(p,q)}(f) - \varepsilon) \exp^{[m-q-1]} r^{(\lambda^{(m,n)}(g)-\varepsilon)}. \tag{15}$$

Case II. Let $q > m$. Then from (12) and (13) it follows for all sufficiently large positive numbers of r that

$$\log^{[p]} |f\circ g|(\exp^{[n-1]} r) \geqslant (\lambda^{(p,q)}(f) - \varepsilon) \log^{[q-m]} \log r^{(\lambda^{(m,n)}(g)-\varepsilon)}$$

i.e.,

$$\log^{[p+m-q-1]} |f\circ g|(\exp^{[n-1]} r) \geqslant r^{(\lambda^{(m,n)}(g)-\varepsilon)}. \tag{16}$$

Again from the definition of $\rho^{(p,q)}(f)$ we get for all sufficiently large positive numbers of r that

$$\log^{[p]} |f| \left(\exp^{[l]} r\right) \leq \left(\rho^{(p,q)}(f) + \varepsilon\right) \log^{[q]} \exp^{[l]} r. \tag{17}$$

Now the following two cases may arise.

Case III. Let $q \geqslant l$. Then we have from (17) for all sufficiently large positive numbers of r that

$$\log^{[p]} |f| \left(\exp^{[l]} r\right) \leq \left(\rho^{(p,q)}(f) + \varepsilon\right) \log^{[q-l]} r$$

i.e.,

$$\log^{[p+1]} |f| \left(\exp^{[l]} r\right) \leq \log^{[q-l+1]} r + \log\left(\rho^{(p,q)}(f) + \varepsilon\right). \tag{18}$$

Case IV. Let $q < l$. Then we have from (17) for all sufficiently large positive numbers of r that

$$\log^{[p]} |f| \left(\exp^{[l]} r\right) \leq \left(\rho^{(p,q)}(f) + \varepsilon\right) \exp^{[l-q]} r$$

i.e.,

$$\log^{[p+1]}|f|\left(\exp^{[l]} r\right) \leq \exp^{[l-q-1]} r + \log\left(\rho^{(p,q)}(f)+\varepsilon\right)$$

i.e.,

$$\log^{[p-q+l+1]}|f|\left(\exp^{[l]} r\right) \leq \log r + O(1). \tag{19}$$

Now combining (15) of Case I and (18) of Case III it follows for all sufficiently large positive numbers of r that

$$\frac{\log^{[p]}|f\circ g|(\exp^{[n-1]} r)}{\log^{[p+1]}|f|\left(\exp^{[l]} r\right)} \geq \frac{(\lambda^{(p,q)}(f)-\varepsilon)\exp^{[m-q-1]} r^{(\lambda^{(m,n)}(g)-\varepsilon)}}{\log^{[q-l+1]} r + \log\left(\rho^{(p,q)}(f)+\varepsilon\right)}.$$

Since $q < m$, we get from the above that

$$\lim_{r\to+\infty}\frac{\log^{[p]}|f\circ g|(\exp^{[n-1]} r)}{\log^{[p+1]}|f|\left(\exp^{[l]} r\right)} = \infty.$$

This proves the first part of the theorem.

Again in view of (15) of Case I and (19) of Case IV we have for all sufficiently large positive numbers of r that

$$\frac{\log^{[p]}|f\circ g|(\exp^{[n-1]} r)}{\log^{[p-q+l+1]}|f|\left(\exp^{[l]} r\right)} \geq \frac{(\lambda^{(p,q)}(f)-\varepsilon)\exp^{[m-q-1]} r^{(\lambda^{(m,n)}(g)-\varepsilon)}}{\log r + O(1)}. \tag{20}$$

When $q < m$ and $q < l$ then we get from (20) that

$$\lim_{r\to+\infty}\frac{\log^{[p]}|f\circ g|(\exp^{[n-1]} r)}{\log^{[p-q+l+1]}|f|\left(\exp^{[l]} r\right)} = \infty.$$

This establishes the second part of the theorem.

Now in view of (16) of Case II and (18) of Case III we get for all sufficiently large positive numbers of r that

$$\frac{\log^{[p+m-q-1]}|f\circ g|(\exp^{[n-1]} r)}{\log^{[p+1]}|f|\left(\exp^{[l]} r\right)} \geq \frac{r^{(\lambda^{(m,n)}(g)-\varepsilon)}}{\log^{[q-l+1]} r + \log\left(\rho^{(p,q)}(f)+\varepsilon\right)}$$

i.e.,

$$\lim_{r\to+\infty}\frac{\log^{[p+m-q-1]}|f\circ g|(\exp^{[n-1]} r)}{\log^{[p+1]}|f|\left(\exp^{[l]} r\right)} = \infty,$$

from which the third part of the theorem follows.

Again from (16) of Case II and (19) of Case IV we have for all sufficiently large positive numbers of r that

$$\frac{\log^{[p+m-q-1]}|f\circ g|(\exp^{[n-1]} r)}{\log^{[p-q+l+1]}|f|\left(\exp^{[l]} r\right)} \geq \frac{r^{(\lambda^{(m,n)}(g)-\varepsilon)}}{\log r + O(1)}$$

i.e.,

$$\lim_{r\to+\infty}\frac{\log^{[p+m-q-1]}|f\circ g|(\exp^{[n-1]} r)}{\log^{[p-q+l+1]}|f|\left(\exp^{[l]} r\right)} = \infty.$$

This proves the fourth part of the theorem. Thus the theorem follows. □

Theorem 5. *Let $f,g,h,k \in \mathcal{A}(\mathbb{K})$ be such that $0 < \rho^{(a,b)}(h) < \infty$, $\lambda^{(p,q)}(f) > 0$, $\lambda^{(m,n)}(g) > 0$ and $\rho^{(c,d)}(k) < \lambda^{(m,n)}(g)$, where $a,b,c,d,p,q,m,n \in \mathbb{N}$. Then*

$$(i) \quad \lim_{r\to+\infty} \frac{\log^{[p]} |f\circ g|(\exp^{[n-1]} r)}{\log^{[a]} |h\circ k|(r)} = \infty \quad \text{if } b \geqslant c \text{ and } q < m,$$

$$(ii) \quad \lim_{r\to+\infty} \frac{\log^{[p]} |f\circ g|(\exp^{[n-1]} r)}{\log^{[a+c-b-1]} |h\circ k|(r)} = \infty \quad \text{if } b < c \text{ and } q < m,$$

$$(iii) \quad \lim_{r\to+\infty} \frac{\log^{[p+m-q-1]} |f\circ g|(\exp^{[n-1]} r)}{\log^{[a]} |h\circ k|(r)} = \infty \quad \text{if } b \geqslant c \text{ and } q \geqslant m,$$

$$\text{and } (iv) \quad \lim_{r\to+\infty} \frac{\log^{[p+m-q-1]} |f\circ g|(\exp^{[n-1]} r)}{\log^{[a+c-b-1]} |h\circ k|(r)} = \infty \quad \text{if } b < c \text{ and } q \geqslant m.$$

Proof. In view of Lemma 1 we obtain for all sufficiently large positive numbers of r that

$$\log^{[a]} |h\circ k|(r) \leqslant \left(\rho^{(a,b)}(h)+\varepsilon\right) \log^{[b]} |k|(r). \tag{21}$$

Now from the definition of (c,d)th order of k we get for arbitrary positive ε and for all sufficiently large positive numbers of r that

$$\log^{[c]} |k|(r) \leqslant \left(\rho^{(c,d)}(k)+\varepsilon\right) \log^{[d]} r$$

i.e.,

$$\log^{[c]} |k|(r) \leqslant \left(\rho^{(c,d)}(k)+\varepsilon\right) \log r \tag{22}$$

i.e.,

$$\log^{[c-1]} |k|(r) \leqslant r^{\left(\rho^{(c,d)}(k)+\varepsilon\right)}. \tag{23}$$

Now the following cases may arise.

Case I. Let $b \geqslant c$. Then we have from (21) for all sufficiently large positive numbers of r that

$$\log^{[a]} |h\circ k|(r) \leqslant \left(\rho^{(a,b)}(h)+\varepsilon\right) \log^{[c-1]} |k|(r). \tag{24}$$

So from (23) and (24), it follows for all sufficiently large positive numbers of r that

$$\log^{[a]} |h\circ k|(r) \leqslant \left(\rho^{(a,b)}(h)+\varepsilon\right) r^{\left(\rho^{(c,d)}(k)+\varepsilon\right)}. \tag{25}$$

Case II. Let $b < c$. Then we get from (21) for all sufficiently large positive numbers of r that

$$\log^{[a]} |h\circ k|(r) \leqslant \left(\rho^{(a,b)}(h)+\varepsilon\right) \exp^{[c-b]} \log^{[c]} |k|(r). \tag{26}$$

Now from (22) and (26) we obtain for all sufficiently large positive numbers of r that

$$\log^{[a]} |h\circ k|(r) \leqslant \left(\rho^{(a,b)}(h)+\varepsilon\right) \exp^{[c-b]} \log r^{\left(\rho^{(c,d)}(k)+\varepsilon\right)}$$

i.e.,

$$\log^{[a+c-b-1]} |h\circ k|(r) \leqslant r^{\left(\rho^{(c,d)}(k)+\varepsilon\right)} + O(1). \tag{27}$$

Since $\rho^{(c,d)}(k) < \lambda^{(m,n)}(g)$ we can choose $\varepsilon\ (>0)$ in such a way that

$$\rho^{(c,d)}(k) + \varepsilon < \lambda^{(m,n)}(g) - \varepsilon. \tag{28}$$

Now combining (25) of Case I, (15) and in view of (28) it follows for all sufficiently large positive numbers of r that

$$\frac{\log^{[p]} |f \circ g|(\exp^{[n-1]} r)}{\log^{[a]} |h \circ k|\,(r)} \geq \frac{(\lambda^{(p,q)}(f) - \varepsilon) \exp^{[m-q-1]} r^{(\lambda^{(m,n)}(g)-\varepsilon)}}{\left(\rho^{(a,b)}(h) + \varepsilon\right) r^{\left(\rho^{(c,d)}(k)+\varepsilon\right)}}$$

i.e.,

$$\lim_{r\to+\infty} \frac{\log^{[p]} |f \circ g|(\exp^{[n-1]} r)}{\log^{[a]} |h \circ k|\,(r)} = \infty,$$

from which the first part of the theorem follows.

Again combining (27) of Case II, (15) and in view of (28) we obtain for all sufficiently large positive numbers of r that

$$\frac{\log^{[p]} |f \circ g|(\exp^{[n-1]} r)}{\log^{[a+c-b-1]} |h \circ k|\,(r)} \geq \frac{(\lambda^{(p,q)}(f) - \varepsilon) \exp^{[m-q-1]} r^{(\lambda^{(m,n)}(g)-\varepsilon)}}{r^{\left(\rho^{(c,d)}(k)+\varepsilon\right)} + O(1)}$$

i.e.,

$$\lim_{r\to+\infty} \frac{\log^{[p]} |f \circ g|(\exp^{[n-1]} r)}{\log^{[a+c-b-1]} |h \circ k|\,(r)} = \infty.$$

This establishes the second part of the theorem.

Further in view of (25) of Case I and (16) we get for all sufficiently large positive numbers of r that

$$\frac{\log^{[p+m-q-1]} |f \circ g|(\exp^{[n-1]} r)}{\log^{[a]} |h \circ k|\,(r)} \geq \frac{r^{(\lambda^{(m,n)}(g)-\varepsilon)}}{\left(\rho^{(a,b)}(h) + \varepsilon\right) r^{\left(\rho^{(c,d)}(k)+\varepsilon\right)}}. \tag{29}$$

So from (28) and (29) we obtain that

$$\lim_{r\to+\infty} \frac{\log^{[p+m-q-1]} |f \circ g|(\exp^{[n-1]} r)}{\log^{[a]} |h \circ k|\,(r)} = \infty,$$

from which the third part of the theorem follows.

Again combining (27) of Case II and (16) it follows for all sufficiently large positive numbers of r that

$$\frac{\log^{[p+m-q-1]} |f \circ g|(\exp^{[n-1]} r)}{\log^{[a+c-b-1]} |h \circ k|\,(r)} \geq \frac{r^{(\lambda^{(m,n)}(g)-\varepsilon)}}{r^{\left(\rho^{(c,d)}(k)+\varepsilon\right)} + O(1)}. \tag{30}$$

Now in view of (28) we obtain from (30) that

$$\lim_{r\to+\infty} \frac{\log^{[p+m-q-1]} |f \circ g|(\exp^{[n-1]} r)}{\log^{[a+c-b-1]} |h \circ k|\,(r)} = \infty.$$

This proves the fourth part of the theorem. Thus the theorem follows. $\square$

Theorem 6. *Let $f,\ g \in \mathcal{A}(\mathbb{K})$ be such that $\rho^{(a,b)}(f \circ g) < \infty$ and $\lambda^{(m,n)}(g) > 0$, where $a,b,m,n \in \mathbb{N}$. Then*

$$\lim_{r\to+\infty} \frac{\left[\log^{[a]} |f \circ g|(\exp^{[b-1]} r)\right]^2}{\log^{[m-1]} |g|(\exp^{[n]} r) \cdot \log^{[m]} |g|(\exp^{[n-1]} r)} = 0.$$

Proof. For any $\varepsilon > 0$ we have $\log^{[a]} |f \circ g|(\exp^{[b-1]} r) \leq \left(\rho^{(a,b)}(f \circ g) + \varepsilon\right) \log^{[b]} \exp^{[b-1]} r$, i.e.,

$$\log^{[a]} |f \circ g|(\exp^{[b-1]} r) \leq \left(\rho^{(a,b)}(f \circ g) + \varepsilon\right) \log r. \tag{31}$$

Again we obtain that $\log^{[m]} |g|(\exp^{[n-1]} r) \geq \left(\lambda^{(m,n)}(g) - \varepsilon\right) \log^{[n]} \exp^{[n-1]} r$, i.e.,

$$\log^{[m]} |g|(\exp^{[n-1]} r) \geq \left(\lambda^{(m,n)}(g) - \varepsilon\right) \log r. \tag{32}$$

Similarly we have $\log^{[m]} |g|(\exp^{[n]} r) \geq \left(\lambda^{(m,n)}(g) - \varepsilon\right) \log^{[n]} \exp^{[n]} r$, i.e.,

$$\log^{[m-1]} |g|(\exp^{[n]} r) \geq \exp\left[\left(\lambda^{(m,n)}(g) - \varepsilon\right) r\right]. \tag{33}$$

From (31) and (32) we have for all sufficiently large positive numbers of r that

$$\frac{\log^{[a]} |f \circ g|(\exp^{[b-1]} r)}{\log^{[m]} |g|(\exp^{[n-1]} r)} \leq \frac{\left(\rho^{(a,b)}(f \circ g) + \varepsilon\right) \log r}{\left(\lambda^{(m,n)}(g) - \varepsilon\right) \log r}.$$

As $\varepsilon\ (> 0)$ is arbitrary we obtain from the above that

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[a]} |f \circ g|(\exp^{[b-1]} r)}{\log^{[m]} |g|(\exp^{[n-1]} r)} \leq \frac{\rho^{(a,b)}(f \circ g)}{\lambda^{(m,n)}(g)}. \tag{34}$$

Again from (31) and (33) we get for all sufficiently large positive numbers of r that

$$\frac{\log^{[a]} |f \circ g|(\exp^{[b-1]} r)}{\log^{[m-1]} |g|(\exp^{[n]} r)} \leq \frac{\left(\rho^{(a,b)}(f \circ g) + \varepsilon\right) \log r}{\exp\left[\left(\lambda^{(m,n)}(g) - \varepsilon\right) r\right]}.$$

Since $\varepsilon\ (> 0)$ is arbitrary it follows from the above that

$$\lim_{r\to+\infty} \frac{\log^{[a]} |f \circ g|(\exp^{[b-1]} r)}{\log^{[m-1]} |g|(\exp^{[n]} r)} = 0. \tag{35}$$

Thus the theorem follows from (34) and (35). □

Theorem 7. *Let $f, g \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda^{(p,q)}(f) \leq \rho^{(p,q)}(f) < \infty$ and $0 < \lambda^{(m,n)}(g) \leq \rho^{(m,n)}(g) < \infty$, where $p,q,m,n \in \mathbb{N}$. Then*

$$(i) \quad \frac{\lambda^{(p,q)}(f) \cdot \lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)} \leq \underline{\lim_{r\to+\infty}} \frac{\log^{[p]} |f \circ g|(r)}{\log^{[p]} |f|(r)} \leqslant \min\left\{\rho^{(m,n)}(g), \frac{\rho^{(p,q)}(f) \cdot \lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\right\};$$

$$\max\left\{\lambda^{(m,n)}(g), \frac{\lambda^{(p,q)}(f) \cdot \rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\} \leq \overline{\lim_{r\to+\infty}} \frac{\log^{[p]} |f \circ g|(r)}{\log^{[p]} |f|(r)} \leq \frac{\rho^{(p,q)}(f) \cdot \rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)},$$

when $q = m = n$,

$$(ii)\quad \frac{\lambda^{(p,q)}(f)\cdot\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)} \le \varliminf_{r\to+\infty}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)} \leqslant \min\left\{\rho^{(m,n)}(g),\frac{\rho^{(p,q)}(f)\cdot\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\right\};$$

$$\max\left\{\lambda^{(m,n)}(g),\frac{\lambda^{(p,q)}(f)\cdot\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\} \le \varlimsup_{r\to+\infty}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)} \le \frac{\rho^{(p,q)}(f)\cdot\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)},$$

when $q=m>$ *or* $<n$,

$$(iii)\quad \frac{\lambda^{(p,q)}(f)}{\rho^{(p,q)}(f)} \le \varliminf_{r\to+\infty}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)} \leqslant \min\left\{1,\frac{\rho^{(p,q)}(f)}{\lambda^{(p,q)}(f)}\right\};$$

$$\max\left\{1,\frac{\lambda^{(p,q)}(f)}{\rho^{(p,q)}(f)}\right\} \le \varlimsup_{r\to+\infty}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)} \le \frac{\rho^{(p,q)}(f)}{\lambda^{(p,q)}(f)},$$

when $q>m$,

$$(iv)\quad \frac{\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)} \le \varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \min\left\{\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)},\frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\}$$
$$\le \max\left\{\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)},\frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\} \le \varlimsup_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \le \frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)},$$

when $m>q=n$,

$$(v)\quad \frac{\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)} \le \varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)} \leqslant \min\left\{\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)},\frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\}$$
$$\le \max\left\{\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)},\frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\} \le \varlimsup_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)} \le \frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)},$$

when $m>q>n$, *and*

$$(vi)\quad \frac{\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)} \le \varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)} \leqslant \min\left\{\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)},\frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\}$$
$$\le \max\left\{\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)},\frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\} \le \varlimsup_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}$$
$$\le \frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)},$$

when $m>q<n$.

Proof. From the definitions of (p,q)th order and (p,q)th lower order of f, we have for all sufficiently large positive numbers of r that

$$\log^{[p]}|f| \le \left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r\,, \tag{36}$$
$$\log^{[p]}|f| \ge \left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r \tag{37}$$

and also for a sequence of positive numbers of r tending to infinity we get that

$$\log^{[p]}|f| \ge \left(\rho^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r\,, \tag{38}$$
$$\log^{[p]}|f| \le \left(\lambda^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r\,. \tag{39}$$

Now in view of Lemma 1, we have for all sufficiently large positive numbers of r that

$$\log^{[p]} |f \circ g| (r) \leqslant \left(\rho^{(p,q)}(f) + \varepsilon\right) \log^{[q]} |g| (r) \tag{40}$$

and also we get for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]} |f \circ g| (r) \leqslant \left(\lambda^{(p,q)}(f) + \varepsilon\right) \log^{[q]} |g| (r) . \tag{41}$$

Similarly, in view of Lemma 1, it follows for all sufficiently large positive numbers of r that

$$\log^{[p]} |f \circ g| (r) \geq \left(\lambda^{(p,q)}(f) - \varepsilon\right) \log^{[q]} |g| (r) \tag{42}$$

and also we obtain for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]} |f \circ g| (r) \geq \left(\rho^{(p,q)}(f) - \varepsilon\right) \log^{[q]} |g| (r) . \tag{43}$$

Now the following two cases may arise.

Case I. Let $q = m = n$. Then we have from (40) for all sufficiently large positive numbers of r that

$$\log^{[p]} |f \circ g| (r) \leqslant \left(\rho^{(p,q)}(f) + \varepsilon\right) \left(\rho^{(m,n)}(g) + \varepsilon\right) \log^{[n]} r, \tag{44}$$

and for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]} |f \circ g| (r) \leqslant \left(\rho^{(p,q)}(f) + \varepsilon\right) \left(\lambda^{(m,n)}(g) + \varepsilon\right) \log^{[n]} r. \tag{45}$$

Also we obtain from (41) for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]} |f \circ g| (r) \leqslant \left(\lambda^{(p,q)}(f) + \varepsilon\right) \left(\rho^{(m,n)}(g) + \varepsilon\right) \log^{[n]} r. \tag{46}$$

Further it follows from (42) for all sufficiently large positive numbers of r that

$$\log^{[p]} |f \circ g| (r) \geq \left(\lambda^{(p,q)}(f) - \varepsilon\right) \left(\lambda^{(m,n)}(g) - \varepsilon\right) \log^{[n]} r, \tag{47}$$

and for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]} |f \circ g| (r) \geq \left(\lambda^{(p,q)}(f) - \varepsilon\right) \left(\rho^{(m,n)}(g) - \varepsilon\right) \log^{[n]} r. \tag{48}$$

Moreover, we obtain from (43) for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]} |f \circ g| (r) \geq \left(\rho^{(p,q)}(f) - \varepsilon\right) \left(\lambda^{(m,n)}(g) - \varepsilon\right) \log^{[n]} r. \tag{49}$$

Therefore from (37) and (44), we have for all sufficiently large positive numbers of r that

$$\begin{aligned}\frac{\log^{[p]} |f \circ g| (r)}{\log^{[p]} |f| (r)} &\leqslant \frac{\left(\rho^{(p,q)}(f) + \varepsilon\right) \left(\rho^{(m,n)}(g) + \varepsilon\right) \log^{[n]} r}{\left(\lambda^{(p,q)}(f) - \varepsilon\right) \log^{[q]} r} \\ &= \frac{\left(\rho^{(p,q)}(f) + \varepsilon\right) \left(\rho^{(m,n)}(g) + \varepsilon\right) \log^{[q]} r}{\left(\lambda^{(p,q)}(f) - \varepsilon\right) \log^{[q]} r}\end{aligned}$$

i.e.,

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f|(r)} \leqslant \frac{\rho^{(p,q)}(f)\cdot\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)} . \tag{50}$$

Similarly from (38) and (44), for a sequence of positive numbers of r tending to infinity it follows that

$$\begin{aligned}\frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f|(r)} &\leqslant \frac{\left(\rho^{(p,q)}(f)+\varepsilon\right)\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]} r}{\left(\rho^{(p,q)}(f)-\varepsilon\right)\log^{[q]} r}\\ &= \frac{\left(\rho^{(p,q)}(f)+\varepsilon\right)\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[q]} r}{\left(\rho^{(p,q)}(f)-\varepsilon\right)\log^{[q]} r},\end{aligned}$$

$$\underline{\lim_{r\to+\infty}} \frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f|(r)} \leqslant \rho^{(m,n)}(g). \tag{51}$$

Also from (37) and (45), we obtain for a sequence of positive numbers of r tending to infinity that

$$\begin{aligned}\frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f|(r)} &\leqslant \frac{\left(\rho^{(p,q)}(f)+\varepsilon\right)\left(\lambda^{(m,n)}(g)+\varepsilon\right)\log^{[n]} r}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]} r}\\ &= \frac{\left(\rho^{(p,q)}(f)+\varepsilon\right)\left(\lambda^{(m,n)}(g)+\varepsilon\right)\log^{[q]} r}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]} r},\end{aligned}$$

$$\underline{\lim_{r\to+\infty}} \frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f|(r)} \leqslant \frac{\rho^{(p,q)}(f)\cdot\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}, \tag{52}$$

Further from (37) and (46), for a sequence of positive numbers of r tending to infinity we have that

$$\begin{aligned}\frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f|(r)} &\leqslant \frac{\left(\lambda^{(p,q)}(f)+\varepsilon\right)\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]} r}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]} r}\\ &= \frac{\left(\lambda^{(p,q)}(f)+\varepsilon\right)\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[q]} r}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]} r},\end{aligned}$$

$$\underline{\lim_{r\to+\infty}} \frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f|(r)} \leqslant \rho^{(m,n)}(g). \tag{53}$$

Thus from (51), (52) and (53) it follows that

$$\underline{\lim_{r\to+\infty}} \frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f|(r)} \leqslant \min\left\{\rho^{(m,n)}(g), \frac{\rho^{(p,q)}(f)\cdot\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\right\}. \tag{54}$$

Further from (36) and (47), for all sufficiently large positive numbers of r we have that

$$\frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f|(r)} \geq \frac{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]} r}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]} r},$$

i.e.,

$$\underline{\lim}_{r\to+\infty} \frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\lambda^{(p,q)}(f)\cdot\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)}. \tag{55}$$

Similarly, from (39) and (47) we obtain that

$$\overline{\lim}_{r\to+\infty} \frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \lambda^{(m,n)}(g). \tag{56}$$

Also from (36) and (48), for a sequence of positive numbers of r tending to infinity we obtain that

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\left(\rho^{(m,n)}(g)-\varepsilon\right)\log^{[n]} r}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]} r}$$

i.e.,

$$\overline{\lim}_{r\to+\infty} \frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\lambda^{(p,q)}(f)\cdot\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}, \tag{57}$$

and from (36) and (49), for a sequence of positive numbers of r tending to infinity we have that

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\left(\rho^{(p,q)}(f)-\varepsilon\right)\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]} r}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]} r}$$

i.e.,

$$\overline{\lim}_{r\to+\infty} \frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \lambda^{(m,n)}(g). \tag{58}$$

Thus from (56), (57) and (58) it follows that

$$\overline{\lim}_{r\to+\infty} \frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \max\left\{\lambda^{(m,n)}(g), \frac{\lambda^{(p,q)}(f)\cdot\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\}. \tag{59}$$

Therefore the first part of the theorem follows from (50), (54), (55) and (59).

Case II. Let $q = m$ and $m > n$ or $n < m$. Now from (37) and (44), for all sufficiently large positive numbers of r we have that

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]} r\right)} \leqslant \frac{\left(\rho^{(p,q)}(f)+\varepsilon\right)\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]} r}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[n]} r}$$

i.e.,

$$\overline{\lim}_{r\to+\infty} \frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]} r\right)} \leqslant \frac{\rho^{(p,q)}(f)\cdot\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}. \tag{60}$$

Similarly, from (38) and (44) for a sequence of positive numbers of r tending to infinity it follows that

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]} r\right)} \leqslant \frac{\left(\rho^{(p,q)}(f)+\varepsilon\right)\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]} r}{\left(\rho^{(p,q)}(f)-\varepsilon\right)\log^{[n]} r}$$

i.e.,

$$\varliminf_{r\to+\infty} \frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f|\left(\exp^{[q-n]} r\right)} \leqslant \rho^{(m,n)}(g). \tag{61}$$

Also from (37) and (45), for a sequence of positive numbers of r tending to infinity we obtain that

$$\frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f|\left(\exp^{[q-n]} r\right)} \leqslant \frac{\left(\rho^{(p,q)}(f)+\varepsilon\right)\left(\lambda^{(m,n)}(g)+\varepsilon\right)\log^{[n]} r}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[n]} r}$$

i.e.,

$$\varliminf_{r\to+\infty} \frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f|\left(\exp^{[q-n]} r\right)} \leqslant \frac{\rho^{(p,q)}(f)\cdot\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}, \tag{62}$$

and from (37) and (46), for a sequence of positive numbers of r tending to infinity we have that

$$\frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f|\left(\exp^{[q-n]} r\right)} \leqslant \frac{\left(\lambda^{(p,q)}(f)+\varepsilon\right)\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]} r}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[n]} r}$$

i.e.,

$$\varliminf_{r\to+\infty} \frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f|\left(\exp^{[q-n]} r\right)} \leqslant \rho^{(m,n)}(g). \tag{63}$$

Thus from (61), (62) and (63) it follows that

$$\varliminf_{r\to+\infty} \frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f|\left(\exp^{[q-n]} r\right)} \leqslant \min\left\{\rho^{(m,n)}(g), \frac{\rho^{(p,q)}(f)\cdot\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\right\}. \tag{64}$$

Further from (36) and (47), for all sufficiently large positive numbers of r we have that

$$\frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f|\left(\exp^{[q-n]} r\right)} \geq \frac{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]} r}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[n]} r}$$

i.e.,

$$\varliminf_{r\to+\infty} \frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f|\left(\exp^{[q-n]} r\right)} \geq \frac{\lambda^{(p,q)}(f)\cdot\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)}. \tag{65}$$

Similarly, from (39) and (47) for a sequence of positive numbers of r tending to infinity it follows that

$$\frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f|\left(\exp^{[q-n]} r\right)} \geq \frac{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]} r}{\left(\lambda^{(p,q)}(f)+\varepsilon\right)\log^{[n]} r}$$

i.e.,

$$\varlimsup_{r\to+\infty} \frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f|\left(\exp^{[q-n]} r\right)} \geq \lambda^{(m,n)}(g). \tag{66}$$

Also from (36) and (48), for a sequence of positive numbers of r tending to infinity we obtain that

$$\frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f|\left(\exp^{[q-n]} r\right)} \geq \frac{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\left(\rho^{(m,n)}(g)-\varepsilon\right)\log^{[n]} r}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[n]} r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\geq\frac{\lambda^{(p,q)}(f)\cdot\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}. \tag{67}$$

Similarly from (36) and (49), we get that

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\geq\lambda^{(m,n)}(g). \tag{68}$$

Thus from (66), (67) and (68) it follows that

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\geq\max\left\{\lambda^{(m,n)}(g),\frac{\lambda^{(p,q)}(f)\cdot\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\}. \tag{69}$$

Thus the second part of the theorem follows from (60), (64), (65) and (69).

Case III. Let $q>m$. Then from (40) for all sufficiently large positive numbers of r we have

$$\log^{[p]}|f\circ g|(r)\leqslant\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q-m]}\left[\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r\right]$$

i.e.,

$$\log^{[p]}M\left(r,f\circ g\right)\leqslant\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q-m+n]}r+O(1) \tag{70}$$

and for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]}|f\circ g|(r)\leqslant\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q-m+n]}r+O(1). \tag{71}$$

Also for the same reasoning, from (41) for a sequence of positive numbers of r tending to infinity we obtain that

$$\log^{[p]}|f\circ g|(r)\leqslant\left(\lambda^{(p,q)}(f)+\varepsilon\right)\log^{[q-m+n]}r+O(1). \tag{72}$$

Further from (42), for all sufficiently large positive numbers of r it follows that

$$\log^{[p]}|f\circ g|(r)\geq\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q-m+n]}r+O(1), \tag{73}$$

and for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]}|f\circ g|(r)\geq\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q-m+n]}r+O(1). \tag{74}$$

Moreover from (43) for a sequence of positive numbers of r tending to infinity we obtain that

$$\log^{[p]}|f\circ g|(r)\geq\left(\rho^{(p,q)}(f)-\varepsilon\right)\log^{[q-m+n]}r+O(1). \tag{75}$$

Now from (37) and (70), for all sufficiently large positive numbers of r we have that

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\leqslant\frac{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q-m+n]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q-m+n]}r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\leqslant\frac{\rho^{(p,q)}(f)}{\lambda^{(p,q)}(f)}. \tag{76}$$

Similarly, from (38) and (70) for a sequence of positive numbers of r tending to infinity it follows that

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\leqslant\frac{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q-m+n]}r+O(1)}{\left(\rho^{(p,q)}(f)-\varepsilon\right)\log^{[q-m+n]}r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\leqslant 1. \tag{77}$$

Also from (37) and (71) for a sequence of positive numbers of r tending to infinity we obtain

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\leqslant\frac{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q-m+n]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q-m+n]}r}$$

i.e.,

$$\underline{\lim_{r\to+\infty}}\frac{\log^{[p]}M(r,f\circ g)}{\log^{[p]}M\left(\exp^{[m-n]}r,f\right)}\leqslant\frac{\rho^{(p,q)}(f)}{\lambda^{(p,q)}(f)}, \tag{78}$$

and from (37) and (72) for a sequence of positive numbers of r tending to infinity also we have

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\leqslant\frac{\left(\lambda^{(p,q)}(f)+\varepsilon\right)\log^{[q-m+n]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q-m+n]}r}$$

i.e.,

$$\underline{\lim_{r\to+\infty}}\frac{\log^{[p]}M(r,f\circ g)}{\log^{[p]}M\left(\exp^{[m-n]}r,f\right)}\leqslant 1. \tag{79}$$

Thus from (77), (78) and (79) it follows that

$$\underline{\lim_{r\to+\infty}}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\leqslant\min\left\{1,\frac{\rho^{(p,q)}(f)}{\lambda^{(p,q)}(f)}\right\}. \tag{80}$$

Further from (36) and (73), for all sufficiently large positive numbers of r we have that

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\geq\frac{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q-m+n]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q-m+n]}r}$$

i.e.,

$$\underline{\lim_{r\to+\infty}}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\geq\frac{\lambda^{(p,q)}(f)}{\rho^{(p,q)}(f)}. \tag{81}$$

Similarly, from (39) and (73) for a sequence of positive numbers of r tending to infinity it follows that

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\geq\frac{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q-m+n]}r+O(1)}{\left(\lambda^{(p,q)}(f)+\varepsilon\right)\log^{[q-m+n]}r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\geq 1. \tag{82}$$

Also from (36) and (74), for a sequence of positive numbers of r tending to infinity we obtain

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\geq\frac{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q-m+n]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q-m+n]}r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\geq\frac{\lambda^{(p,q)}(f)}{\rho^{(p,q)}(f)},\tag{83}$$

and from (36) and (75) for a sequence of positive numbers of r tending to infinity also we have

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\geq\frac{\left(\rho^{(p,q)}(f)-\varepsilon\right)\log^{[q-m+n]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q-m+n]}r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\geq 1.\tag{84}$$

Thus from (82), (83) and (84) it follows that

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\geq\max\left\{1,\frac{\lambda^{(p,q)}(f)}{\rho^{(p,q)}(f)}\right\}.\tag{85}$$

Hence the third part of the theorem follows from (76), (80), (65) and (85).

Case IV. Let $m>q=n$. Then from (40) for all sufficiently large positive numbers of r we have

$$\log^{[p+m-q]}|f\circ g|(r)\leqslant\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r+O(1),\tag{86}$$

and for a sequence of positive numbers of r tending to infinity that

$$\log^{[p+m-q]}|f\circ g|(r)\leqslant\left(\lambda^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r+O(1).\tag{87}$$

Also from (41) for a sequence of positive numbers of r tending to infinity we obtain that

$$\log^{[p+m-q]}|f\circ g|(r)\leqslant\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r+O(1).\tag{88}$$

Further, from (42) for all sufficiently large positive numbers of r it follows that

$$\log^{[p+m-q]}|f\circ g|(r)\geq\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r+O(1),\tag{89}$$

and for a sequence of positive numbers of r tending to infinity that

$$\log^{[p+m-q]}|f\circ g|(r)\geq\left(\rho^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r+O(1).\tag{90}$$

Moreover, from (43) for a sequence of positive numbers of r tending to infinity we obtain that

$$\log^{[p+m-q]}|f\circ g|(r)\geq\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r+O(1).\tag{91}$$

Therefore from (37) and (86), for all sufficiently large positive numbers of r we have that

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \frac{\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r} = \frac{\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[q]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\varlimsup_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\,. \tag{92}$$

Similarly, from (38) and (86) for a sequence of positive numbers of r tending to infinity it follows that

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \frac{\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r+O(1)}{\left(\rho^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r} = \frac{\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[q]}r+O(1)}{\left(\rho^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}. \tag{93}$$

Also from (37) and (87) for a sequence of positive numbers of r tending to infinity we obtain

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \frac{\left(\lambda^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r} = \frac{\left(\lambda^{(m,n)}(g)+\varepsilon\right)\log^{[q]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}, \tag{94}$$

and from (37) and (88) for a sequence of positive numbers of r tending to infinity also we have

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \frac{\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r} = \frac{\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[q]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\,. \tag{95}$$

Thus from (93), (94) and (95) it follows that

$$\varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \min\left\{\frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)},\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)},\frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\right\}. \tag{96}$$

Further from (36) and (89), for all sufficiently large positive numbers of r we have that

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r} = \frac{\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[q]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)}\,. \tag{97}$$

Similarly, from (39) and (89) for a sequence of positive numbers of r tending to infinity it follows that

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]} r + O(1)}{\left(\lambda^{(p,q)}(f)+\varepsilon\right)\log^{[q]} r} = \frac{\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[q]} r + O(1)}{\left(\lambda^{(p,q)}(f)+\varepsilon\right)\log^{[q]} r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\lambda_g(m,n)}{\lambda^{(p,q)}(f)}. \tag{98}$$

Also from(36) and (90) for a sequence of positive numbers of r tending to infinity we obtain

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\left(\rho^{(m,n)}(g)-\varepsilon\right)\log^{[n]} r + O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]} r} = \frac{\left(\rho^{(m,n)}(g)-\varepsilon\right)\log^{[q]} r + O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]} r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}, \tag{99}$$

and from (36) and (91) for a sequence of positive numbers of r tending to infinity also we have

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]} r + O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]} r} = \frac{\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[q]} r + O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]} r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)}. \tag{100}$$

Thus from (98), (99) and (100) it follows that

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \max\left\{\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}, \frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}, \frac{\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\}. \tag{101}$$

Therefore the fourth part of the theorem follows from (92), (96), (98) and (101).

Case V. Let $m > q > n$. Currently from (37) and (86), we have for all sufficiently large positive numbers of r that

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]} r\right)} \leqslant \frac{\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]} r + O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[n]} r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]} r\right)} \leqslant \frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}. \tag{102}$$

Similarly, from (38) and (86) for a sequence of positive numbers of r tending to infinity it follows that

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]} r\right)} \leqslant \frac{\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]} r + O(1)}{\left(\rho^{(p,q)}(f)-\varepsilon\right)\log^{[n]} r}$$

i.e.,

$$\varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\leqslant\frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}\,. \tag{103}$$

Also from (37) and (87), for a sequence of positive numbers of r tending to infinity we obtain that

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\leqslant\frac{\left(\lambda^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[n]}r}$$

i.e.,

$$\varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\leqslant\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}, \tag{104}$$

and from (37) and (88) for a sequence of positive numbers of r tending to infinity also we have

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\leqslant\frac{\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[n]}r}$$

i.e.,

$$\varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\leqslant\frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\,. \tag{105}$$

Thus from (103), (104) and (105) it follows that

$$\varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\leqslant\min\left\{\frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)},\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)},\frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\right\}\,. \tag{106}$$

Further from (36) and (89), for all sufficiently large positive numbers of r we have that

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\geq\frac{\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[n]}r}$$

i.e.,

$$\varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\geq\frac{\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)}\,. \tag{107}$$

Similarly, from (39) and (89) for a sequence of positive numbers of r tending to infinity it follows that

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\geq\frac{\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r+O(1)}{\left(\lambda^{(p,q)}(f)+\varepsilon\right)\log^{[n]}r}$$

i.e.,

$$\varlimsup_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\geq\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\,. \tag{108}$$

Also from (36) and (90), for a sequence of positive numbers of r tending to infinity we obtain

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\geq\frac{\left(\rho^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[n]}r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)} \geq \frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}, \tag{109}$$

and from (36) and (91) for a sequence of positive numbers of r tending to infinity also we have

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)} \geq \frac{\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[n]}r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)} \geq \frac{\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)}. \tag{110}$$

Thus from (98), (99), and (100) it follows that

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)} \geq \max\left\{\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}, \frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}, \frac{\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\}. \tag{111}$$

Thus the fifth part of the theorem follows from (102), (106), (107) and (111).

Case VI. Let $m > q < n$. At this instant case from (37) and (86) for all sufficiently large positive numbers of r we have that

$$\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)} \leqslant \frac{\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[q]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)} \leqslant \frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}. \tag{112}$$

Similarly, from (38) and (86) for a sequence of positive numbers of r tending to infinity it follows that

$$\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)} \leqslant \frac{\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[q]}r+O(1)}{\left(\rho^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\underline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)} \leqslant \frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}. \tag{113}$$

Also from (37) and (87) for a sequence of positive numbers of r tending to infinity we obtain

$$\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)} \leqslant \frac{\left(\lambda^{(m,n)}(g)+\varepsilon\right)\log^{[q]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\underline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)} \leqslant \frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}, \tag{114}$$

and from (37) and (88) for a sequence of positive numbers of r tending to infinity also we have

$$\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)}\leqslant\frac{\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[q]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\underline{\lim}_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)}\leqslant\frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\,. \tag{115}$$

Thus from (113), (114) and (115) it follows that

$$\underline{\lim}_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)}\leqslant\min\left\{\frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)},\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)},\frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\right\}\,. \tag{116}$$

Further from (36) and (89), for all sufficiently large positive numbers of r we have that

$$\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)}\geq\frac{\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[q]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\underline{\lim}_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)}\geq\frac{\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)}. \tag{117}$$

Similarly, from (39) and (89) for a sequence of positive numbers of r tending to infinity it follows that

$$\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)}\geq\frac{\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[q]}r+O(1)}{\left(\lambda^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\overline{\lim}_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)}\geq\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}. \tag{118}$$

Also from (36) and (90), for a sequence of positive numbers of r tending to infinity we obtain

$$\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)}\geq\frac{\left(\rho^{(m,n)}(g)-\varepsilon\right)\log^{[q]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\overline{\lim}_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)}\geq\frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}, \tag{119}$$

and from (36) and (91) for a sequence of positive numbers of r tending to infinity also we have

$$\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)}\geq\frac{\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[q]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]} |f\circ g| \left(\exp^{[n-q]} r\right)}{\log^{[p]} |f| (r)} \geq \frac{\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)}. \tag{120}$$

Thus from (98), (99) and (100) it follows that

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]} |f\circ g| \left(\exp^{[n-q]} r\right)}{\log^{[p]} |f| (r)} \geq \max\left\{ \frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}, \frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}, \frac{\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)} \right\}. \tag{121}$$

Hence the sixth part of the theorem follows from (112), (116), (118) and (121). □

Theorem 8. *Let $f, g \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda^{(p,q)}(f) \leq \rho^{(p,q)}(f) < \infty$ and $0 < \lambda^{(m,n)}(g) \leq \rho^{(m,n)}(g) < \infty$, where $p, q, m, n \in \mathbb{N}$. Then*

$$(i) \quad \frac{\lambda^{(p,q)}(f)\cdot\lambda^{(m,n)}(g)}{\rho^{(m,n)}(g)} \leq \underline{\lim_{r\to+\infty}} \frac{\log^{[p]} |f\circ g| (r)}{\log^{[m]} |g| (r)} \leq \min\left\{ \rho^{(p,q)}(f), \frac{\lambda^{(p,q)}(f)\cdot\rho^{(m,n)}(g)}{\lambda^{(m,n)}(g)} \right\};$$

$$\max\left\{ \lambda^{(p,q)}(f), \frac{\rho^{(p,q)}(f)\cdot\lambda^{(m,n)}(g)}{\rho^{(m,n)}(g)} \right\} \leq \overline{\lim_{r\to+\infty}} \frac{\log^{[p]} |f\circ g| (r)}{\log^{[m]} |g| (r)} \leq \frac{\rho^{(p,q)}(f)\cdot\rho^{(m,n)}(g)}{\lambda^{(m,n)}(g)},$$

when $q = m$,

$$(ii) \quad \frac{\lambda^{(p,q)}(f)}{\rho^{(m,n)}(g)} \leq \underline{\lim_{r\to+\infty}} \frac{\log^{[p]} |f\circ g| \left(\exp^{[q-m]} r\right)}{\log^{[m]} |g| (r)} \leqslant \min\left\{ \frac{\rho^{(p,q)}(f)}{\rho^{(m,n)}(g)}, \frac{\rho^{(p,q)}(f)}{\lambda^{(m,n)}(g)}, \frac{\lambda^{(p,q)}(f)}{\lambda^{(m,n)}(g)} \right\};$$

$$\max\left\{ \frac{\rho^{(p,q)}(f)}{\rho^{(m,n)}(g)}, \frac{\lambda^{(p,q)}(f)}{\rho^{(m,n)}(g)}, \frac{\lambda^{(p,q)}(f)}{\lambda^{(m,n)}(g)} \right\} \leq \overline{\lim_{r\to+\infty}} \frac{\log^{[p]} |f\circ g| \left(\exp^{[q-m]} r\right)}{\log^{[m]} |g| (r)} \leq \frac{\rho^{(p,q)}(f)}{\lambda^{(m,n)}(g)},$$

when $q > m$, and

$$(iii) \quad \frac{\lambda^{(m,n)}(g)}{\rho^{(m,n)}(g)} \leq \underline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]} |f\circ g| (r)}{\log^{[m]} |g| (r)} \leq 1 \leq \overline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]} |f\circ g| (r)}{\log^{[m]} |g| (r)} \leq \frac{\rho^{(m,n)}(g)}{\lambda^{(m,n)}(g)},$$

when $m > q$.

We omit the proof of Theorem 8 as it can easily be deduced in the line of Theorem 7.

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Бісвас Т. *Оцінка орієнтованого росту складених p-адичних цілих функцій, що залежить від (p,q)-го порядку* // Карпатські матем. публ. — 2018. — Т.10, №2. — С. 248–272.

Нехай $\mathbb{K}$ — повне ультраметричне алгебраїчно замкнуте поле, $\mathcal{A}(\mathbb{K})$ — $\mathbb{K}$-алгебра цілих функцій на $\mathbb{K}$. Для довільної p-адичної цілої функції $f \in \mathcal{A}(\mathbb{K})$ і $r > 0$ позначимо $|f|(r)$ число $\sup\{|f(x)| : |x| = r\}$, де $|\cdot|(r)$ є мультиплікативною нормою на $\mathcal{A}(\mathbb{K})$. Для довільних двох цілих функцій $f \in \mathcal{A}(\mathbb{K})$ та $g \in \mathcal{A}(\mathbb{K})$ співвідношення $\frac{|f|(r)}{|g|(r)}$ при $r \to \infty$ називають порівняльним ростом f відносно g в сенсі їхніх мультиплікативних норм. Аналогічно до того, як це роблять в комплексному аналізі, в цій статті ми визначаємо поняття (p,q)-го порядку (відповідно (p,q)-го нижнього порядку) росту наступним чином $\rho^{(p,q)}(f) = \limsup\limits_{r\to+\infty} \frac{\log^{[p]}|f|(r)}{\log^{[q]} r}$ (відпоідно $\lambda^{(p,q)}(f) = \liminf\limits_{r\to+\infty} \frac{\log^{[p]}|f|(r)}{\log^{[q]} r}$), де p і q два довільні натуральні числа. Ми досліджуємо деякі властивості росту складених p-адичних цілих функцій на основі їхнього (p,q)-го порядку і (p,q)-го нижнього порядку.

Ключові слова і фрази: p-адична ціла функція, ріст, (p,q)-й порядок, (p,q)-й нижній порядок, композиція.

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BURTNYAK I.V., MALYTSKA H.P.

APPLICATION OF THE SPECTRAL THEORY AND PERTURBATION THEORY TO THE STUDY OF ORNSTEIN-UHLENBECK PROCESSES

The theoretical bases of this paper are the theory of spectral analysis and the theory of singular and regular perturbations. We obtain an approximate price of Ornstein-Uhlenbeck double barrier options with multidimensional stochastic diffusion as expansion in eigenfunctions using infinitesimal generators of a $(l+r+1)$-dimensional diffusion in Hilbert spaces. The theorem of accuracy estimation of options prices approximation is established. We also obtain explicit formulas for derivatives price based on the expansion in eigenfunctions and eigenvalues of self-adjoint operators using boundary value problems for singular and regular perturbations.

Key words and phrases: spectral theory, singular perturbation theory, regular perturbation theory, Sturm-Liouville theory, infinitesimal generator, multidimensional diffusion.

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INTRODUCTION

In 1956 McKean H.P. constructed a spectral presentation for general one-dimensional diffusion [1]. Since then, spectral theory has become an important instrument for the analysis of financial diffusion models, as investigation of expansion in eigenfunctions of linear operators. Many problems concerning derivatives estimation are solved using methods of spectral theory; as a result it is widely used in financial mathematics.

Spectral theory has been extensively applied by many scientists, namely, to forecast call option price [2], to find interest rates on securities [3] and model volatility of financial assets. Both spectral theory and stochastic volatility models have become an indispensable tool in mathematics of finance [4], due to the fact that prices of double barrier options are subjected to the Brownian motion and are correlated with volatility [5]. Therefore, it is employed in an investigation of stochastic volatility, in particular the asset volatility, which is the basis of controlled and nonlocal diffusion [6]. Applying methods of spectral theory, theories of singular and regular perturbations, we can obtain approximate price of Ornstein-Uhlenbeck double barrier options with multidimensional volatility, as expansion in eigenfunctions using infinitesimal generators of a $(l+m+1)$-dimensional diffusion, $l \geq 1$, $r \geq 1$, $l \in \mathbb{N}$, $r \in \mathbb{N}$, i.e. the diffusion depends on one local variable, l-dimensional fast variable and n-dimensional slow variable. This paper develops the following researches [7–9], in [9] it is considered the case $l = 1$ and $m = 1$.

The purpose of the article is to elaborate algorithms for evaluating approximate price of double barrier options and to find explicit formulas for derivatives estimation as expansion in

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eigenfunctions and eigenvalues of self-adjoint operators using boundary problems for singular and regular perturbations. The theorem of accuracy estimation of options prices approximation is established.

1 RESULTS

Let (Q, F, P) be the probability space that supports a correlated Brownian motion $(W^x, W^{y_1}, \dots, W^{y_l}, W^{z_1}, \dots, W^{z_r})$ and an exponential random variable $\varepsilon \sim \exp(1)$, which is independent of $(W^x, W^{y_1}, \dots, W^{y_l}, W^{z_1}, \dots, W^{z_r})$. We will assume that the economy with $(l + r + 1)$ factors is described by the homogeneous time and continuous Markov process $\boldsymbol{X} = (X, Y_1, \dots, Y_l, Z_1, \dots, Z_n)$, which is defined in some state space $E = I \times R^l \times R^r$, where $(Y_1, \dots, Y_l) \in \mathrm{R}^l$, $(Z_1, \dots, Z_r) \in \mathrm{R}^r$, I is the interval at R with points e_1 and e_2, such that $-\infty < e_1 < e_2 < \infty$. We assume that X has the beginning at E and instantly disappears once X goes beyond I. In particular, the dynamics of X with physical measure $\mathbb{P}$ is as follows:

$$\boldsymbol{X}_t = \begin{cases} (X_t, Y_{1t}, \dots, Y_{lt}, Z_{1t}, \dots, Z_{rt}), & \tau_I > t, \\ \Delta, & \tau_I \leq t, \end{cases}$$

$\tau_I = \inf\{t > 0 : \ X_t \notin I\}$, where $(X, Y_1, \dots, Y_l, Z_1, \dots, Z_r)$ are set

$$\begin{cases} dX_t = v\left(X_t\right) dt + a\left(X_t\right) f\left(Y_{1t}, \dots, Y_{lt}, Z_{1t}, \dots, Z_{rt}\right) dW_t^x, & \\ dY_{jt} = \frac{1}{\varepsilon_j} \alpha_j\left(Y_{jt}\right) dt + \frac{1}{\sqrt{\varepsilon_j}} \beta_j\left(Y_{jt}\right) dW_t^{y_j}, & j = \overline{1, l}, \\ dZ_{it} = \delta_i c_i\left(Z_{it}\right) dt + \sqrt{\delta_i} g_i\left(Z_{it}\right) dW_t^{z_i}, & i = \overline{1, r}, \\ d(W^x, W^{y_j})_t = \rho_{xy_j} dt, & j = \overline{1, l}, \\ d(W^x, W^{z_i})_t = \rho_{xz_i} dt, & i = \overline{1, r}, \\ d(W^{y_j}, W^{z_i})_t = \rho_{y_j z_i} dt, & j = \overline{1, l}, i = \overline{1, r}, \\ d(W^{y_j}, W^{y_r})_t = \rho_{y_j y_s} dt, & j = \overline{1, l}, s = \overline{1, l}, \\ d(W^{z_i}, W^{z_k})_t = \rho_{z_i z_k} dt, & i = \overline{1, n}, k = \overline{1, r}, \\ (X, Y_1, \dots, Y_l, Z_1, \dots, Z_r) = (x, y_1, \dots, y_l, z_1, \dots, z_r) \in E, & \end{cases}$$

where $\rho_{y_j y_s} = 0$, $j \neq r$, $\rho_{z_i z_k} = 0$, $i \neq k$, ρ_{xy_j}, ρ_{xz_i}, $\rho_{y_j z_i}$ meet the conditions $\|\rho_{xy_j}| \leq 1$, $|\rho_{xz_i}| \leq 1$, $|\rho_{y_j z_i}| \leq 1$, and correlation matrices of the form

$$\begin{pmatrix} 1 & \rho_{xy_j} & \rho_{xz_i} \\ \rho_{y_j x} & 1 & \rho_{y_j z_i} \\ \rho_{z_i x} & \rho_{z_i y_j} & 1 \end{pmatrix}$$

semipositively defined, that is $1 + 2\rho_{xy_j}\rho_{xz_i}\rho_{y_j z_i} - {\rho_{xy_j}}^2 - {\rho_{xz_i}}^2 - {\rho_{y_j z_i}}^2 \geq 0$, $j = \overline{1, l}$, $i = \overline{1, r}$. Process X may represent many economic phenomena and processes.

For example, the reserve size, the index price and reliable short-term interest rates, etc. Even more broadly, X is an external factor that characterizes the value of any of the above-mentioned processes. Physical measure $\mathbb{P}$ of process X is understood as the process X, which has an instant drift $v\left(X_t\right)$ and stochastic volatility $a\left(X_t\right) f\left(Y_{1t}, \dots, Y_{lt}, Z_{1t}, \dots, Z_{nt}\right) > 0$, which contains both components: local $a(X_t)$ and nonlocal $f(Y_{1t},\dots,Y_{lt},Z_{1t},\dots,Z_{nt})$. Note that infinitesimal generators for Y_j and Z_i have the form

$$\mathfrak{L}_{Y_j}^{\varepsilon_j}=\frac{1}{\varepsilon_j}\left(\frac{1}{2}{\beta_j}^2\left(y_j\right)\partial^2_{y_jy_j}+\alpha_j\left(y_j\right)\partial_{y_j}\right),\quad \mathfrak{L}_{Z_i}^{\delta_i}=\delta_i\left(\frac{1}{2}{g_i}^2\left(z_i\right)\partial^2_{z_iz_i}+c_i\left(z_i\right)\partial_{z_i}\right),\quad \forall\, i,j,$$

and are characterized by the measures $\frac{1}{\varepsilon_j}$ and δ_i, respectively. Thus, $Y_1,\dots,Y_l$ and $Z_1,\dots,Z_n$ have an internal time scale $\varepsilon_j>0$ and $\frac{1}{\delta_i}>0$. We consider $\varepsilon_j<<1$ and $\delta_i<<1$, so that the internal time scale Y_j is small, and the internal time scale Z_i is large. Consequently, Y_j, $j=\overline{1,l}$, are fast variables, and Z_i, $i=\overline{1,n}$, are slow variables. Note that $\mathfrak{L}_{Y_j}^{\varepsilon_j}$ and $\mathfrak{L}_{Z_i}^{\delta_i}$ have the form

$$\mathfrak{L}=\tfrac{1}{2}a^2\left(x\right)\partial^2_{xx}+b\left(x\right)\partial_x-k(x),\ x\in(e_1,e_2),\ \text{ç}\,k\left(x\right)=0,$$

for all $x\in I$, are always self-adjoint in the Hilbert space $H=L^2(I,m)$, where $I\in R$ is the interval with the points e_1 and e_2 and m is the diffusion density rate. Note,

$$\mathrm{Dom}\left(\mathfrak{L}\right)=\left\{f\in L^2\left(I,m\right):f,\partial_xf\in AC_{\mathrm{loc}}\left(I\right),\ \mathfrak{L}f\in L^2\left(I,m\right),\ BCs \text{ on } e_1 \text{ and } e_2\right\},$$

where $AC_{\mathrm{loc}}\left(I\right)$ is the space of functions which are absolutely continuous on each compact subinterval I (see [9]). The boundary conditions for e_1 and e_2 are applied on the output, input, and regular bounds.

We will evaluate the derivatives with payoff at time $t>0$, which may depend on the trajectory of X. In particular, we will consider the forms of payoff: Payoff $=H(X_t)\mathbb{I}_{(\tau>t)}$, where τ is a random moment of time during which there is a failure to make a payment of premium. Since we are interested in the derivatives estimation, we must determine the dynamics $(X,Y_1,\dots,Y_l,Z_1,\dots,Z_r)$ under the evaluation of the degree of neutral risk, which we denote as $\widetilde{\mathbb{P}}$. We have the following dynamics

$$\begin{cases} dX_t=\left(b\left(X_t\right)-a\left(X_t\right)f\left(Y_{1t},\dots,Y_{lt},Z_{1t},\dots,Z_{rt}\right)\Omega\left(Y_{1t},\dots,Y_{lt},Z_{1t},\dots,Z_{rt}\right)\right)dt \\ \qquad\qquad +a\left(X_t\right)f\left(Y_{1t},\dots,Y_{lt},Z_{1t},\dots,Z_{rt}\right)d\tilde{W}_t^x, \\ dY_{jt}=\left(\frac{1}{\varepsilon_j}\alpha_j\left(Y_{jt}\right)-\frac{1}{\sqrt{\varepsilon_j}}\beta_j(Y_{jt})\Lambda_j\left(Y_{1t},\dots,Y_{lt},Z_{1t},\dots,Z_{rt}\right)\right)dt+\frac{1}{\sqrt{\varepsilon_j}}\beta_j\left(Y_{jt}\right)d\tilde{W}_t^{y_j}, \\ dZ_{it}=\left(\delta_ic_i\left(Z_{it}\right)-\sqrt{\delta_i}g_i\left(Z_{it}\right)\Gamma_i\left(Y_{1t},\dots,Y_{lt},Z_{1t},\dots,Z_{rt}\right)\right)dt+\sqrt{\delta_i}g_i\left(Z_{it}\right)d\tilde{W}_t^{z_i}, \\ d\langle\tilde{W}^x,\tilde{W}^{y_j}\rangle_t=\rho_{xy_j}dt, \qquad j=\overline{1,l}, \\ d\langle\tilde{W}^x,\tilde{W}^{z_i}\rangle_t=\rho_{xz_i}dt, \qquad i=\overline{1,r}, \\ d\left\langle\tilde{W}^{y_j},\tilde{W}^{z_i}\right\rangle_t=\rho_{y_jz_i}dt, \qquad j=\overline{1,l},\quad i=\overline{1,r}, \\ d\left\langle\tilde{W}^{y_j},\tilde{W}^{y_s}\right\rangle_t=\rho_{y_jy_s}dt, \qquad j=\overline{1,l},\quad s=\overline{1,l}, \\ d\left\langle\tilde{W}^{z_i},\tilde{W}^{z_k}\right\rangle_t=\rho_{z_iz_k}dt, \qquad i=\overline{1,n},\quad k=\overline{1,n}, \\ (X_0,\ Y_1,\dots,Y_l,Z_1,\dots,Z_{r0})=(x,y_1,\dots y_l,z_1,\dots,z_r)\in E, \end{cases} \tag{1}$$

where $\rho_{y_jy_s}=0$, $j\neq s$, $\rho_{z_iz_k}=0$, $i\neq k$, and

$$d\tilde{W}_t^x:=dW_t^x+\left(\frac{v\left(X_t\right)-b(X_t)}{a\left(X_t\right)f\left(Y_{1t},\dots,Y_{lt},Z_{1t},\dots,Z_{nt}\right)}+\Omega\left(Y_{1t},\dots,Y_{lt},Z_{1t},\dots,Z_{rt}\right)\right)dt,$$

$$d\tilde{W}_t^{y_j}:=dW_t^{y_j}+\Lambda_j\left(Y_{1t},\dots,Y_{lt},Z_{1t},\dots,Z_{rt}\right)dt,$$

$$d\tilde{W}_t^{z_i}:=dW_t^{z_i}+\Gamma_i\left(Y_{1t},\dots,Y_{lt},Z_{1t},\dots,Z_{rt}\right)dt.$$

We establish such conditions so that the system (1) has the only strong solution.

Random time τ is the time of the derivative asset. In our case, default can occur in one of two ways:

1) when X fall outside the interval I,

2) at random time τ_h, which is managed by the risk level $h(X_t) \geq 0$.

This can be expressed as follows

$$\begin{cases} \tau = \tau_I \wedge \tau_h, \\ \tau_I = inf\ t \geq 0 : \ X_t \notin I\ , \\ \tau_h = inf\ t \geq 0 : \ \int_0^t h\left(X_s\right) ds \geq \varepsilon\left(X, Y_1, \ldots, Y_l, Z_1, \ldots, Z_n\right),\ \varepsilon \sim \exp(1). \end{cases}$$

Note that the random variable ε is independent of $(X, Y_1, \ldots, Y_l, Z_1, \ldots, Z_n)$.

To track τ_h, we use the process indicator $D_t = \mathbb{I}_{t \geq \tau_h}$, where $\mathbb{D} = \mathfrak{D}_t,\ t \geq 0$, is a filter generated by D and $\mathbb{F} = \mathfrak{F}_t,\ t \geq 0$ is filter generator $(W^x, W^{y_1}, \ldots W^{y_l}, W^{z_1}, \ldots, W^{z_n})$. We use the filtering $\mathbb{G} = \mathfrak{G}_t, t \geq 0$, where $\mathfrak{G}_t = \mathfrak{F}_t \bigvee \mathfrak{D}_t$. Note that $(X, Y_1, \ldots, Y_l, Z_1, \ldots, Z_n)$ are applied to $\mathbb{G}$ and τ is a stopping time ($\tau \leq t \in \mathfrak{G}_t$ for all $t \geq 0$).

We will evaluate the derivative asset of some payoff (payment) using the neutral pricing risk and Markovian chain X, the price $u^{\overline{\varepsilon}, \overline{\delta'}}(t, x, y_1, \ldots y_l, z_1, \ldots, z_r)$ of some derivative assets at the initial moment of time has the form

$$u^{\overline{\varepsilon}, \overline{\delta'}}(t, x, y_1, \ldots y_l, z_1, \ldots, z_r) = \widetilde{\mathbb{E}}_{x, y_1, \ldots y_l, z_1, \ldots, z_r} \left[exp\left(- \int_0^t r\left(X_s\right) ds \right) H(X_t \mathbb{I}_{t > \tau}) \right],$$

where $\overline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_l)$, $\overline{\delta'} = (\delta_1, \ldots, \delta_r)$, and $(x, y_1, \ldots y_l, z_1, \ldots, z_r) \in E$ is a starting point of the process $(X, Y_1, \ldots, Y_l, Z_1, \ldots, Z_r)$. Using the Feynmann-Kac formulas, we can show that $u^{\overline{\varepsilon}, \overline{\delta'}}(t, x, y_1, \ldots y_l, z_1, \ldots, z_r)$ satisfies the following Cauchy problem (see [9])

$$(-\partial_t + \mathfrak{L}^{\overline{\varepsilon}, \overline{\delta'}}) u^{\overline{\varepsilon}, \overline{\delta'}} = 0, \quad (y_1, \ldots y_l, z_1, \ldots, z_r) \in E,\ t \in \mathbb{R}^+, \tag{2}$$

$$u^{\overline{\varepsilon}, \overline{\delta'}}(0, x, y_1, \ldots y_l, z_1, \ldots, z_r) \ = H(x), \tag{3}$$

where the operator $\mathfrak{L}^{\overline{\varepsilon}, \overline{\delta'}}$ has the form

$$\mathfrak{L}^{\overline{\varepsilon}, \overline{\delta'}} = \sum_{j=1}^{l} \frac{1}{\varepsilon_j} \mathfrak{L}_{0j} + \sum_{j=1}^{l} \frac{1}{\sqrt{\varepsilon_j}} \mathfrak{L}_{1j} + \mathfrak{L}_{2j} + \sum_{i,j} \sqrt{\frac{\delta_i}{\varepsilon_j}} \mathfrak{M}_{3ij} + \sum_i \sqrt{\delta_i} \mathfrak{M}_{1i} + \sum_i \delta_i \mathfrak{M}_{2i},$$

$$\mathfrak{L}_{0j} = \frac{1}{2} \beta_j^2\left(y_j\right) \partial^2_{y_j y_j} + \alpha_j\left(y_j\right) \partial_{y_j}, \quad j = \overline{1, l},$$

$$\mathfrak{L}_{1j} = \beta_j(y_j) (\rho_{x y_j} a\left(x\right) f\left(y_1, \ldots y_l, z_1, \ldots, z_r\right) \partial_x - \Lambda_j(y_1, \ldots y_l, z_1, \ldots, z_r)) \partial_{y_j},$$

$$\mathfrak{L}_{2j} = \frac{1}{2} a^2\left(x\right) f^2\left(y_1, \ldots y_l, z_1, \ldots, z_r\right) \partial^2_{xx}$$

$$+ \left(b\left(x\right) - a\left(x\right) \Omega\left(y_1, \ldots y_l, z_1, \ldots, z_r\right) f(y_1, \ldots y_l, z_1, \ldots, z_r)\right) \partial_x - k\left(x\right),$$

$$\mathfrak{M}_{3ij} = \rho_{x z_i} \beta_j\left(y_j\right) g_i\left(z_i\right) \partial^2_{y_j z_i},$$

$$\mathfrak{M}_{1i} = g_i\left(z_i\right) \left(\rho_{x z_i} a\left(x\right) f\left(y_1, \ldots y_l, z_1, \ldots, z_r\right) \partial_x - \Gamma_i\left(y_1, \ldots y_l, z_1, \ldots, z_r\right)\right) \partial_{z_i},$$

$$\mathfrak{M}_{2i} = \frac{1}{2} g_i^2\left(z_i\right) \partial^2_{z_i z_i} + c_i\left(z_i\right) \partial_{z_i}, \quad k\left(x\right) = r\left(x\right) + h\left(x\right), \quad \mathfrak{L}_{0j} = \mathfrak{L}^1_{Y_j}.$$

We assume that the diffusion with the infinitesimal generator $\mathfrak{L}^1_{Y_j}$ has an invariant distribution Π with density

$$\pi_j\left(y_j\right) = \frac{2}{\beta_j^2(y_j)}\exp\int_{y_{j0}}^{y_j}\frac{2\alpha_j(\theta)}{\beta_j^2(\theta)}d\theta\ , \quad \forall j = \overline{1,l}.$$

Besides the initial condition (3), the function $u^{\overline{\varepsilon},\overline{\delta'}}(t,x,y_1,\dots y_l,z_1,\dots,z_r)$ must meet boundary conditions at the points e_1 and e_2 of the interval I. The boundary conditions at points e_1 and e_2 belong to the domain $\mathfrak{L}^{\overline{\varepsilon},\overline{\delta'}}$ and will depend on the nature of process X on the points of I and are classified as natural, output, input or regular [10]. The Cauchy problem (2)–(3) for $(f,\alpha_1,\dots,\alpha_l,\beta_1,\dots,\beta_r,\Lambda_1,\dots,\Lambda_l,c_1,\dots,c_r,g_1,\dots,g_r,\Gamma_1,\dots,\Gamma_r)$ has no analytical solution. However, for fixed $\overline{\delta'}$, the conditions containing $\overline{\varepsilon}$ and are arbitrarily deviated in the $\overline{\varepsilon}$-axis, which causes singular perturbations. For a fixed ε_j condition containing δ_i are small for some small $\overline{\delta'}$-axis, which causes regular perturbations. Thus, the $\overline{\varepsilon}$-axis and $\overline{\delta'}$-axis yields the combined singular-regular perturbation of $\mathcal{O}(1)$ of the operator $\mathfrak{L}_2$.To find the asymptotic solution of the Cauchy problem (2)–(3), we develop $u^{\overline{\varepsilon},\overline{\delta'}}$ in orders $\sqrt{\varepsilon_j}$ and $\sqrt{\delta_i}$ [11]:

$$u^{\overline{\varepsilon},\overline{\delta'}} = \sum_{i_1\geq 0}\cdots\sum_{i_l\geq 0}\sum_{j_1\geq 0}\cdots\sum_{j_r\geq 0}\sqrt{\varepsilon_1}^{j_1}\dots\sqrt{\varepsilon_l}^{j_l}\sqrt{\delta_1}^{i_1}\dots\sqrt{\delta_r}^{i_r}u_{j_1,\dots,j_n,i_1,\dots,i_l}$$

where

$$\sum_{i_1\geq 0}\cdots\sum_{i_l\geq 0}\sum_{j_1\geq 0}\cdots\sum_{j_r\geq 0}\sqrt{\varepsilon_1}^{j_1}\dots\sqrt{\varepsilon_l}^{j_l}\sqrt{\delta_1}^{i_1}\dots\sqrt{\delta_r}^{i_r}u_{j_1,\dots,j_r,i_1,\dots,i_l}$$
$$= \lim\sum_{i_1\geq 0}^{m_1}\cdots\sum_{i_l\geq 0}^{m_l\geq 0}\sum_{j_1\geq 0}^{m_{l+1}\geq 0}\cdots\sum_{j_n\geq 0}^{m_{l+r}}\sqrt{\varepsilon_1}^{j_1}\dots\sqrt{\varepsilon_l}^{j_l}\sqrt{\delta_1}^{i_1}\dots\sqrt{\delta_r}^{i_r}u_{j_1,\dots,j_r,i_1,\dots,i_l},\quad \forall\, m_n\to\infty.$$

The approximate price is calculated

$$u^{\overline{\varepsilon},\overline{\delta'}} \approx u_{\overline{0},\overline{0'}} + \sum_{j=1}^{l}\sqrt{\varepsilon_j}u_{\overline{1_j},\overline{0'}} + \sum_{i=1}^{r}\sqrt{\delta_i}u_{\overline{0},\overline{1'_i}}.$$

The choice of development in half-integer orders ε_j and δ_i are natural for $\mathfrak{L}^{\overline{\varepsilon},\overline{\delta'}}$.

By conducting an analysis of singular perturbations at the corresponding levels, we obtain that $u_{\overline{0},\overline{0'}}$, $u_{\overline{1_j},\overline{0'}}$, $u_{\overline{0},\overline{1'_i}}$ do not depend on $y_1,\dots,y_l$. The basic findings of the asymptotic analysis are given using the following formulas

$$\mathcal{O}\left(1\right):\ \sum_{j=1}\mathfrak{L}_{0j}u_{\overline{2_j},\overline{0'}} + \left(-\partial_t + \langle\mathfrak{L}_2\rangle\right)u_{\overline{0},\overline{0'}} = 0,\ \ u_{\overline{0},\overline{0'}}\left(0,x,z_1,\dots,z_r\right) = H\left(x\right), \tag{4}$$

$$\mathcal{O}\left(\sqrt{\varepsilon_j}\right):\mathfrak{L}_{0j}u_{\overline{3_j},\overline{0'}} + \mathfrak{L}_{1j}u_{\overline{2_j},\overline{0'}} + \left(-\partial_t + \langle\mathfrak{L}_2\rangle\right)u_{\overline{1_j},\overline{0'}} + \sum_{k\neq j}\mathfrak{L}_{1k}u_{\overline{1_{kj}},\overline{0'}} + \sum_{i\neq j}\mathfrak{L}_{1i} = \mathcal{A}_j u_{\overline{0},\overline{0'}},$$

$$u_{\overline{1_j},\overline{0'}}\left(0,x,z_1,\dots,z_n\right) = 0, \overline{1}_{kj} = \left(\underbrace{0,\dots 1}_{k}\ \underbrace{0,1,0,\dots.0}_{j}\right). \tag{5}$$

According to the analysis of regular perturbations we have

$$\mathcal{O}\left(\sqrt{\delta_i}\right):\ \left(-\partial_t + \langle\mathfrak{L}_2\rangle\right)u_{\overline{0},\overline{1'_i}} = \mathcal{B}_i\partial_{z_i}u_{\overline{0},\overline{0'}},\ \ u_{\overline{0},\overline{1'_i}}\left(0,x,z_1,\dots,z_r\right) = 0,\quad i = \overline{1,r}. \tag{6}$$

Operators $\langle\mathfrak{L}_2\rangle$, $\mathcal{A}_j$, $\mathcal{B}_i$ and ∂_{z_i} are defined by the formulas

$$\langle\mathfrak{L}_2\rangle=\frac{1}{2}\overline{\sigma}^2a^2\left(x\right)\partial^2_{xx}+\Big(b\left(x\right)-\overline{f\Omega}a(x)\Big)\partial_x-k\left(x\right),\quad x\in\left(e_1,e_2\right),$$

$$\mathcal{A}_j=-\nu_{3j}a\left(x\right)\partial_x a^2\left(x\right)\partial^2_{xx}-\nu_{2j}a^2\left(x\right)\partial^2_{xx}-\mathcal{U}_{2j}a\left(x\right)\partial_x a\left(x\right)\partial_x-\mathcal{U}_{1j}a\left(x\right)\partial_x,$$

$$\mathcal{B}_i=-\nu_{1i}a\left(x\right)\partial_x-\nu_{0i},\quad \partial_{z_i}=\partial_{z_i}\overline{\sigma}\partial_{\overline{\sigma}}+\overline{f\Omega}'\partial_{\overline{f\Omega}},\quad \nu_{1i}:=g_i\rho_{xz_i}\left\langle f\right\rangle,\quad \nu_0=g_i\left\langle \Gamma_i\right\rangle,\quad \forall\, i=\overline{1,n},$$

and norm function is defined by

$$\left\langle\mathcal{X}\right\rangle_j:=\int\mathcal{X}\left(y_1,\dots y_l\right)\pi_j\left(y_j\right)dy_j,\quad \forall\, j=\overline{1,l},$$

$$\left\langle\mathcal{X}\right\rangle_{1,2}=\int_{R^2}\mathcal{X}\left(y_1,\dots y_l\right)\pi_1\left(y_1\right)\pi_2\left(y_2\right)dy_1dy_2,\dots,$$

$$\left\langle\mathcal{X}\right\rangle_{l-1,l}=\int_{R^l}\mathcal{X}\left(y_1,\dots y_l\right)\pi_1\left(y_1\right)\dots\pi_l\left(y_l\right)dy_1\dots dy_l,$$

$$\left\langle\mathcal{X}\right\rangle_{l-1,l}=\left\langle\mathcal{X}\right\rangle,\left\langle f\Omega\right\rangle:=\overline{f\Omega},\left\langle f^2\right\rangle=\overline{\sigma}^2.$$

We find solutions of the equations (4)–(6) on the basis of eigenfunctions, eigenvalues of the operator $\langle\mathfrak{L}_2\rangle$, each of which meets the corresponding Poisson equation

$$\mathfrak{L}_{01}\varphi_1=f^2-\left\langle f^2\right\rangle_1,\ \mathfrak{L}_{02}\varphi_2=\left\langle f^2\right\rangle_1-\left\langle f^2\right\rangle_{1,2},\dots,\mathfrak{L}_{0l}\varphi_l=\left\langle f^2\right\rangle_{l-2,l-1}-\left\langle f^2\right\rangle_{l-1,l},$$

$$\mathfrak{L}_{01}\eta_1=f\Omega-\left\langle f\Omega\right\rangle_1,\dots,\mathfrak{L}_{0j}\eta_j=\left\langle f\Omega\right\rangle_{j-2,j-1}-\left\langle f\Omega\right\rangle_{j-1,j},\dots,\mathfrak{L}_{0l}\eta_l=\left\langle f\Omega\right\rangle_{l-2,l-1}-\left\langle f\Omega\right\rangle_{l-1,l}.$$

Theorem 1. *Assume that we can solve the following equation to find an eigenvalue*

$$-\left\langle\mathfrak{L}_2\right\rangle\psi_n=\lambda_n\psi_n,\qquad \psi_n\in dom\left(\left\langle\mathfrak{L}_2\right\rangle\right),\tag{7}$$

and also that $H\in\mathcal{H}$. Then the solution $u_{\overline{0},\overline{0}'}$ has the form

$$u_{\overline{0},\overline{0}'}=\sum_{n=1}^{\infty}c_n\psi_nT_n,\qquad c_n=\left(\psi_n,H\right),\qquad T_n=e^{-t\lambda_n}.$$

Proof. Since $u_{\overline{0},\overline{0}'}$ satisfies the differential equation (4), suppose that occurs (7), the boundary conditions are fulfilled $u_{\overline{0},\overline{0}'}\left(0,x,z_1,\dots,z_n\right)=H\left(z_1,\dots,z_n\right),T_n\left(0,z_1,\dots,z_n\right)=1$, $\varphi\left(\lambda\right)=\mathrm{Id}$, this means

$$\mathrm{Id}f=\sum_{n=1}^{\infty}\left(\psi_n,f\right)\psi_n,\quad \forall\, f\in\mathcal{H},$$

this is equivalent to having its eigenfunctions ψ_n dense self-adjoint operators in $\mathcal{H}$ form the Schauder basis. In fact, the basis can be chosen orthonormal $\left(\psi_n,\psi_m\right)=\delta_{n,m}$. Also note that $\varphi\left(\lambda\right)=R_\lambda$ gives the actual representation of the resolvent of the operator

$$R_\lambda f=\sum_{n=1}^{\infty}\frac{\left(\psi_n,f\right)}{\lambda_n-\lambda}\psi_n,\quad \forall f\in\mathcal{H},\ \lambda\in\rho\left(\mathfrak{L}\right),$$

to payoff function H :

$$\mathrm{Id}\ H=\sum_{n=1}^{\infty}\left(\psi_n,f\right)\psi_n=\sum_{n=1}^{\infty}c_n\psi_n.$$

□

Theorem 2. *Let c_n, ψ_n, T_n be described using Theorem 1. We define*

$$\mathcal{A}_{jk,n} := \left(\psi_k, \mathcal{A}_j \psi_n\right), \quad U_{k,n} := \frac{T_k - T_n}{\lambda_k - \lambda_n}.$$

Then the solution $u_{\overline{1}_j,\overline{0'}}$ of equation (5) *has the form*

$$u_{\overline{1}_j,\overline{0'}} = \sum_n \sum_{k \neq n} c_n A_{jk,n} \psi_k U_{k,n} - \sum_n c_n \mathcal{A}_{jn,n} \psi_n t T_n.$$

Note that $u_{\overline{1}_j,\overline{0'}}$ is linear in the parameter group $(\vartheta_{3j}, \vartheta_{2j}, u_{2j}, u_{1j})$.

Proof. Let us show that $u_{\overline{1}_j,\overline{0'}}$ satisfies the differential equation and boundary conditions (5). It is clear that the boundary conditions for $u_{\overline{1}_j,\overline{0'}}(0, x, z_1, \ldots, z_n) = 0$ are executed. To show that $u_{\overline{1}_j,\overline{0'}}$ satisfies the differential equation (6), we note that

$$\mathcal{A}_j u_{\overline{1}_j,\overline{0'}} = \sum_n c_n \left(\mathcal{A}_j \psi_n\right) T_n = \sum_n \sum_k c_n \mathcal{A}_{jk,n} \psi_k T_n,$$

according to the proof of Theorem 1. Now, using (7) and the following equality

$$\left(-\partial_t - \lambda_k\right) U_{k,n} = T_n, \quad \left(-\partial_t - \lambda_n\right) t T_n = -T_n,$$

it is easy to see that

$$\left(-\partial_t + \langle \mathfrak{L}_2 \rangle\right) u_{\overline{1}_j,\overline{0'}} = \mathcal{A}_j u_{\overline{1}_j,\overline{0'}} = \sum_n c_n \left(\mathcal{A}_j \psi_n\right) T_n = \sum_n \sum_k c_n \mathcal{A}_{jk,n} \psi_k T_n.$$

□

Theorem 3. *Let c_n, ψ_n and T_n be defined with Theorem 1, and $U_{k,n}$ with Theorem 2. We have*

$$\widetilde{\mathcal{B}}_{ik,n} := \left(\psi_k,\ \mathcal{B}_i \partial_{Z_i} \psi_n\right), \mathcal{B}_{ik,n} := \left(\psi_k,\ \mathcal{B}_i \psi_n\right), V_{ik,n} := \frac{T_k - T_n}{\left(\lambda_k - \lambda_n\right)^2} + \frac{t T_n}{\lambda_k - \lambda_n}.$$

Then the solution $u_{\overline{0},\overline{1'_i}}$ has the form

$$\begin{aligned} u_{\overline{0},\overline{1'_i}} = & \sum_n \sum_{k \neq n} c_n \widetilde{\mathcal{B}}_{ik,n} \psi_k U_{ik,n} - \sum_n c_n \widetilde{\mathcal{B}}_{in,n} \psi_n t T_n \\ & + \sum_n \sum_{k \neq n} (\partial_{Z_i} c_n) \mathcal{B}_{ik,n} \psi_k U_{ik,n} - \sum_n (\partial_{Z_i} c_n) \mathcal{B}_{in,n} \psi_n t T_n \\ & + \sum_n \sum_{k \neq n} c_n \mathcal{B}_{ik,n} \psi_k (\partial_{Z_i} \lambda_n) V_{ik,n} - \sum_n c_n \mathcal{B}_{in,n} \psi_n (\partial_{Z_i} \lambda_n) \frac{1}{2} t^2 T_n. \end{aligned}$$

Proof. We need to show that $u_{\overline{0},\overline{1'_i}}$ satisfies the differential equation and boundary conditions (5). We see that the boundary condition $u_{\overline{0},\overline{1'_i}}(0, x, z_1, \ldots, z_n) = 0$ is executed. To show that $u_{\overline{0},\overline{1'_i}}$ satisfies the differential equation, we note that

$$\begin{aligned} \mathcal{B}_i \partial_{z_i} u_{\overline{0},\overline{0'}} = & \sum_n c_n \left(\mathcal{B}_i \partial_{z_i} \psi_n\right) T_n + \sum_n (\partial_{z_i} c_n)(\mathcal{B}_i \psi_n) T_n \\ & + \sum_n c_n (\mathcal{B}_i \psi_n)(\partial_{z_i} T_n) = \sum_n \sum_k c_n \widetilde{\mathcal{B}}_{ik,n} \psi_k T_n \\ & + \sum_n \sum_k (\partial_{z_i} c_n)(\mathcal{B}_{ik,n} \psi_k) T_n - \sum_n \sum_k c_n \mathcal{B}_{ik,n} \psi_k \left(\partial_{z_i} \lambda_n\right) t T_n, \quad i = \overline{1, n}, \end{aligned}$$

where we used

$$\text{Id } H = \sum_{n=1}^{\infty} (\psi_n, f)\, \psi_n = \sum_{n=1}^{\infty} c_n \psi_n,$$

in the second equality using $-\langle \mathfrak{L}_2 \rangle \psi_n = \lambda_n \psi_n$ and equality

$$(-\partial_t - \lambda_k)\, U_{k,n} = T_n, \quad (-\partial_t - \lambda_n)\, tT_n = -T_n,$$

$$(-\partial_t - \lambda_k)\, V_{k,n} = -tT_n, \quad (-\partial_t - \lambda_k)\, \frac{1}{2} t^2 T_n = -tT_n,$$

one can see that

$$(-\partial_t + \langle \mathfrak{L}_2 \rangle)\, u_{\overline{0},\overline{1'_i}} = \mathcal{B}_i \partial_{z_i} u_{\overline{0},\overline{0'}} = \sum_n c_n \left(\mathcal{B}_n \partial_{z_n} \psi_n \right) T_n$$

$$\sum_n (\partial_{z_n} c_n)(\mathcal{B}_n \psi_n) T_n + \sum_n c_n (\mathcal{B}_n \psi_n)(\partial_{z_n} T_n) = \sum_n \sum_k c_n \widetilde{\mathcal{B}}_{ik,n} \psi_k T_n$$

$$+ \sum_n \sum_k \partial_{z_i} c_n) \mathcal{B}_{ik,n} \psi_k T_n - \sum_n \sum_k c_n \mathcal{B}_{ik,n} \psi_k \left(\partial_{z_i} \lambda_n \right) tT_n, \quad i = \overline{1,n}.$$

Note that $u_{\overline{0},\overline{1'_i}}$ is linear in $(\nu_{1i}\overline{\sigma}',\ \nu_{1i}\overline{f\Omega'},\ \nu_{0i}\overline{\sigma}',\ \nu_{0i}\overline{f\Omega'})$.

We have obtained the approximate solution $u^{\overline{\varepsilon},\overline{\delta'}} \approx u_{\overline{0},\overline{0'}} + \sum_{j=1}^{l} \sqrt{\varepsilon_j} u_{\overline{1_j},\overline{0'}} + \sum_{i=1}^{n} \sqrt{\delta_i} u_{\overline{0},\overline{1'_i}}$ for the derivative asset pricing. □

For a more exact result we assume that the Payoff function $H(x)$ and its derivative are smooth and limited functions. Thus, we restrict our derivative analysis to a smooth and limited payoff; in this case, the closeness estimates is based on the following theorem.

Theorem 4. *For the fixed $(t, x, y_1, \ldots, y_l, z_1, \ldots, z_r)$ there exists an invariable C such that for any $\varepsilon_j \leq 1$, $\delta_i \leq 1$ we have*

$$\left| u^{\overline{\varepsilon},\overline{\delta'}} - \left(u_{\overline{0},\overline{0'}} + \sum_{j=1}^{l} \sqrt{\varepsilon_j} u_{\overline{1_j},\overline{0'}} + \sum_{i=1}^{n} \sqrt{\delta_i} u_{\overline{0},\overline{1'_i}} \right) \right| \leq C \left(\sum_{j=1}^{l} \varepsilon_j + \sum_{i=1}^{r} \delta_i \right).$$

Proof. Before setting the main result of accuracy we formulate such a lemma.

Lemma 1. *Let $J(y_1, \ldots, y_l, z_1, \ldots, z_n)$ grows polynomially. Then for every $(y_1, \ldots, y_l, z_1, \ldots, z_n)$, $s < t$, there is a positive start $C < \infty$ such that for any $\varepsilon_j \leq 1$, $\delta_i \leq 1$, the following inequality holds*

$$\widetilde{\mathbb{E}}_{y_1, \ldots, y_l, z_1, \ldots, z_n} \left[|J(Y_{1s}, \ldots, Y_{ls}, Z_{1s}, \ldots, Z_{ns})| \right] \leq C.$$

Proof. It is enough to consider $J(y_1, \ldots, y_l, z_1, \ldots, z_n) = y_j^k$ and $J(y_1, \ldots, y_l, z_1, \ldots, z_n) = z_i^k$, $k \in \mathbb{N}$. For the second one we have the following. Physically $\mathbb{P}$ we understand as

$$\mathbb{E}\left[|Z_{is}|^k \right] = \mathbb{E}\left[\left| Z^{(1)}_{i\delta_i s} \right|^k \right] \leq \sup_{\delta_i \leq 1} \mathbb{E}\left[\left| Z^{(1)}_{i\delta_i s} \right|^k \right] \leq C_i(s,k) \leq C_i(t,k), \quad i = \overline{1,n}.$$

Now we define exponential martingales $M_t^{(\Gamma_i)}$, which connect the dynamics Z_i at neutral risk of measurement $\widetilde{\mathbb{P}}$ in its dynamics according to physical measure $\mathbb{P}$. We have

$$M_t^{\Gamma_i} := \exp\left(-\int_0^t \Gamma_i(Y_{1s}, \ldots, Y_{ls}, Z_{1s}, \ldots, Z_{ns}) dW_s^{z_i} - \frac{1}{2} \int_0^t \Gamma_i^2(Y_{1s}, \ldots, Y_{ls}, Z_{1s}, \ldots, Z_{ns}) ds \right) = \left. \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} \right|_{\mathcal{F}_t}.$$

$|Z_s|^k$ can be found as follows

$$\widetilde{\mathbb{E}}\left[|Z_{is}|^k\right] = \mathbb{E}\left[|Z_{is}|^k M_s^{(\Gamma_{\rm i})}\right]$$
$$= \mathbb{E}\left[|Z_{is}|^k \exp\left(\frac{1}{2}\int_0^s \Gamma_{\rm i}^2\left(Y_{1u},\ldots,Y_{lu},Z_{1u},\ldots,Z_{nu}\right)du\right)\left(M_s^{(2\Gamma_{\rm i})}\right)^{1/2}\right]$$
$$\leq \left(\mathbb{E}\left[|Z_{is}|^{2k}\exp\left(\int_0^s \Gamma_{\rm i}^2\left(Y_{1u},\ldots,Y_{lu},Z_{1u},\ldots,Z_{nu}\right)du\right)\right]\right)^{1/2}$$

$\left(\mathbb{E}\left[M_s^{(2\Gamma_{\rm i})}\right]\right)^{\frac{1}{2}}$ (by Cauchy-Schwartz)

$$= \left(\mathbb{E}\left[|Z_{is}|^{2k}\exp\left(\int_0^s \Gamma_{\rm i}^2\left(Y_{1u},\ldots,Y_{lu},Z_{1u},\ldots,Z_{nu}\right)du\right)\right]\right)^{1/2}$$

$(M^{(2\Gamma_{\rm i})} - is\ \ \mathbb{P}-$martingale)

$$\leq \left(\mathbb{E}\left[\left|Z^{(1)}_{i\delta_i s}\right|^{2k}\right]\exp(s\|\Gamma_{\rm i}\|^2_\infty))\right)^{\frac{1}{2}} \leq C.$$

Consider now the case $J\left(y_1,\ \ldots,y_l,z_1,\ldots,z_n\right) = y_j^k$. We have

$$\mathbb{E}\left[\left|Y_{js}\right|^k\right] = \mathbb{E}\left[\left|Y^{(1)}_{js/\varepsilon_j}\right|^k\right] \leq \sup_{\varepsilon_j\leq 1}\mathbb{E}\left[\left|Y^{(1)}_{js/\varepsilon_j,}\right|^k\right] \leq C_j\left(k\right).$$

Using the above considerations is easy to show that

$$\widetilde{\mathbb{E}}\left[\left|Y_{js}\right|^k\right] = \mathbb{E}\left[\left|Y_{js}\right|^k M_s^{(\Lambda_{\rm j})}\right] \leq \left(\mathbb{E}\left[\left|Y^{(1)}_{js/\varepsilon_j,}\right|^{2k}\right]\exp(s\left\|\Lambda_{\rm j}\right\|^2_\infty))\right)^{\frac{1}{2}} \leq C_j.$$

The Lemma 1 is proved. ☐

Let us return to the proof of the Theorem 4. We start with the definition of the remainder term $R^{\overline{\varepsilon},\overline{\delta'}}$

$$u_{\overline{0},\overline{0'}} + \sum_{j=1}^l \sqrt{\varepsilon_j}u_{\overline{1_j},\overline{0'}} + \sum_{i=1}^n \sqrt{\delta_i}u_{\overline{0},\overline{1'_i}} + \sum_{j=1}^l \varepsilon_j\left(u_{\overline{2_j},\overline{0'}} + \sum_{j=1}^l \sqrt{\varepsilon_j}u_{\overline{3_j},\overline{0'}}\right)$$
$$u^{\overline{\varepsilon},\overline{\delta'}} + \sum_{k\neq j}^l \sqrt{\varepsilon_k}\sqrt{\varepsilon_j}u_{\overline{1_{kj}},\overline{0'}} + \sum_{j=1}^l\sum_{i=1}^n \sqrt{\delta_i}\left(\sqrt{\varepsilon_j}u_{\overline{1_j}\overline{1'_i}} + \sum_{j=1}^l \varepsilon_j u_{\overline{2_j}\overline{1'_i}}\right)$$
$$+ \sum_{k\neq j}^l\sum_{j=1}^l\sum_{i=1}^n \sqrt{\delta_i}\sqrt{\varepsilon_k}\sqrt{\varepsilon_j}u_{\overline{1_{kj}},\overline{1'_i}} + R^{\overline{\varepsilon},\overline{\delta'}}.$$

Functions $u_{\overline{0},\overline{0'}}$, $u_{\overline{1_j},\overline{0'}}$, $u_{\overline{0},\overline{1'_i}}$ are the only solutions of equations (4)–(6), respectively. Function $u_{\overline{2_j},\overline{0'}}$, $w_{\overline{2_j},\overline{0'}}$ is a solution of the Poisson equation $0 = \mathfrak{L}_{0j}u_{\overline{0},\overline{0'}}$, $j = \overline{1,l}$. To characterize $u_{\overline{1_j}\overline{1'_i}}$, $u_{\overline{2_j}\overline{1'_i}}$ continue the singular analysis of perturbations.

$$0 = \mathfrak{L}_{0j}u_{\overline{3_j}\overline{1'_i}} + \mathfrak{L}_{1j}u_{\overline{2_j}\overline{1'_i}} + \sum_{k\neq j}^l \mathfrak{L}_{1k}u_{\overline{1_{kj}},\overline{1'_i}} + \left(-\partial_t + \mathfrak{L}_2\right)u_{\overline{1_j}\overline{1'_i}}\mathcal{O} + \mathfrak{M}_{3ij}u_{\overline{2_j},\overline{0'}} + \mathfrak{M}_{1i}u_{\overline{1_j},\overline{0'}}. \quad (8)$$

Equation (8) is a Poisson equation. In order to determine the solution for (8) $u_{\overline{3_j 1_i'}}$ in the space $L^2(\mathbb{R},\pi)$, the centering condition must meet $0=\mathfrak{L}_{0j}u+\mathcal{X},\ j=\overline{1,l}$. In (8) the condition of centering is

$$0=\left\langle \mathfrak{L}_{1j}u_{\overline{2_j 1_i'}}\right\rangle+\left(-\partial_t+\langle\mathfrak{L}_2\rangle\right)u_{\overline{1_j 1_i'}}+\left\langle \mathfrak{M}_{3ij}u_{\overline{2_j,0'}}\right\rangle+\langle\mathfrak{M}_{1i}\rangle\,u_{\overline{1_j,0'}}, \tag{9}$$

let us express $u_{\overline{2_j 1_i'}}$,

$$0=\left(-\partial_t+\langle\mathfrak{L}_2\rangle\right)u_{\overline{0,1_i'}}+\langle\mathfrak{M}_{1i}\rangle\,u_{\overline{0,0'}}-\sum_{j=1}^{l}\mathfrak{L}_{0j}u_{\overline{2_j 1_i'}}+\left(-\partial_t+\mathfrak{L}_2\right)u_{\overline{0,1_i'}}$$

$$+\mathfrak{M}_{1i}u_{\overline{0,0'}}+\sum_{j=1}^{l}\mathfrak{M}_{3ij}u_{\overline{1_j,0'}}+\mathfrak{L}_{1j}u_{\overline{1_j 1_i'}},$$

Let us put down similar terms

$$\begin{aligned}
0&=\sum_{j=1}^{l}\mathfrak{L}_{0j}u_{\overline{2_j 1_i'}}+\left(\mathfrak{L}_2-\langle\mathfrak{L}_2\rangle\right)u_{\overline{0,1_i'}}+\mathfrak{M}_{1i}u_{\overline{0,0'}}-\langle\mathfrak{M}_{1i}\rangle\,u_{\overline{0,0'}}=\sum_{j=1}^{l}\mathfrak{L}_{0j}u_{\overline{2_j 1_i'}}\\
&\quad+\left(\left(-\frac{1}{2}a^2\left(\overline{\sigma}^2-f^2\right)\right)\partial_{xx}^2+a(f\Omega-\overline{f\Omega})\partial_x\right)u_{\overline{0,1_i'}}\\
&\quad+\left[g_i\left(\rho_{xz_i}af\partial_x-\Gamma_i\right)\partial_{z_i}-g_i\left(\rho_{xz_i}a\langle f\rangle\partial_x-\langle\Gamma_i\rangle\right)\partial_{z_i}\right]u_{\overline{0,0'}}\\
&=\sum_{j=1}^{l}\mathfrak{L}_{0j}u_{\overline{2_j 1_i'}}+\left(\left(-\frac{1}{2}a^2\left(\overline{\sigma}^2-f^2\mp\overline{\sigma}_1^2\mp\overline{\sigma}_{12}^2\mp\ldots\mp\overline{\sigma}_{l-2,l-1}^2\right)\right)\partial_{xx}^2\right.\\
&\quad\left.+a(f\Omega-\overline{f\Omega}\mp\overline{f\Omega}_1\mp\overline{f\Omega}_{12}\mp\cdots\mp\overline{f\Omega}_{l-2,l-1})\partial_x\right)u_{\overline{0,1_i'}}\\
&\quad+\left[g_i\left(\rho_{xz_i}a(f-\langle f\rangle\mp\langle f\rangle_1\mp\langle f\rangle_{12}\mp\cdots\mp\langle f\rangle_{l-2,l-1})\partial_x\right.\right.\\
&\quad\left.\left.-\left(\Gamma-\langle\Gamma\rangle\mp\langle\Gamma\rangle_1\mp\langle\Gamma\rangle_{12}\mp\cdots\mp\langle\Gamma\rangle_{l-2,l-1}\right)\partial_{z_i}\right)u_{\overline{0,0'}}\right].
\end{aligned}$$

Consider such systems of Poisson equations

$$\mathfrak{L}_{01}\varphi_1=f^2-\overline{\sigma}_1^2,\quad \mathfrak{L}_{02}\varphi_2=\overline{\sigma}_1^2-\overline{\sigma}_{12}^2,\ \ldots,\ \mathfrak{L}_{0j}\varphi_j=\overline{\sigma}_{j-2,j-1}^2-\overline{\sigma}_{j-1,j}^2,\ldots,$$

$$\mathfrak{L}_{0l}\varphi_l=\overline{\sigma}_{l-2,l-1}^2-\overline{\sigma}_{l-1,l}^2\ \ \mathfrak{L}_{01}\eta_1=f\Omega-\overline{f\Omega}_1,\ldots,\mathfrak{L}_{0l}\eta_l=\overline{f\Omega}_{l-2,l-1}-\overline{f\Omega},$$

$$\mathfrak{L}_{01}\xi_1=f-\langle f\rangle_1,\quad \mathfrak{L}_{02}\xi_2=\langle f\rangle_1-\langle f\rangle_{12},\ \ldots,\mathfrak{L}_{0l}\xi_l=\langle f\rangle_{l-2,l-1}-\langle f\rangle$$

$$\mathfrak{L}_{01}\zeta_1=\Gamma-\langle\Gamma\rangle_1,\ldots,\mathfrak{L}_{0l}\zeta_l=\langle\Gamma\rangle_{l-2,l-1}-\langle\Gamma\rangle.$$

Functions $\xi_j(y_1,\ \ldots,y_l,z_1,\ldots,z_n)$, $\eta_j(y_1,\ \ldots,y_l,z_1,\ldots,z_n)$ are solutions of the corresponding Poisson equations, the formula holds

$$\sum_{j=1}^{l}\mathfrak{L}_{0j}u_{\overline{2_j 1_i'}}+\sum_{j=1}^{l}\mathfrak{L}_{0j}\left(-\frac{1}{2}a^2\varphi_j\partial_{xx}^2+a\eta_j\partial_x\right)u_{\overline{0,1_i'}}+\left[g_i\left(\rho_{xz_i}a\xi_j\partial_x-\zeta_j\right)\partial_{z_i}\right]u_{\overline{0,0'}}=0.$$

Therefore, the formula holds

$$u_{\overline{2_j 1_i'}}=-\left(\frac{1}{2}a^2\varphi_j\partial_{xx}^2-a\eta_j\partial_x\right)u_{\overline{0,1_i'}}-g_j\left(\rho_{xz_i}a\xi_j\partial_x-\zeta_j\right)\partial_{z_i}u_{\overline{0,0'}}+D_j,\ \forall j=\overline{1,l},\ \ i=\overline{1,n}. \tag{10}$$

$D_j(x, z_1, \ldots, z_n)$ is a constant that does not depend on y. Substituting (10) in (9) we find $u_{\overline{1_j 1_i'}}$ knowing $u_{\overline{0,0'}}, u_{\overline{1_j,0'}}, u_{\overline{0,1_i'}}$. $u_{\overline{1_j 1_i'}}$ is a solution (9) with boundary conditions $u\left(0, x, z_1, \ldots, z_n\right) = 0$.

Take $u_{\overline{1k_j 0'}} \equiv 0,\ \forall\, k \neq j,\ k = \overline{1, l}$.

Let us calculate

$$0 = (-\partial_t + \mathfrak{L}^{\bar{\varepsilon},\bar{\delta}'})u^{\bar{\varepsilon},\bar{\delta}'}\left(-\partial_t + \mathfrak{L}^{\bar{\varepsilon},\bar{\delta}'}\right)R^{\bar{\varepsilon},\bar{\delta}'} + \sum_{j=1}^{l}\frac{1}{\varepsilon_j}F_{0j} + \sum_{j=1}^{l}\frac{1}{\sqrt{\varepsilon_j}}F_{1j} + \sum_{j=1}^{l}F_{2j}$$
$$+ \sum_{i=1}^{n}\sum_{j=1}^{l}\sqrt{\delta_i}\left(\frac{1}{\varepsilon_j}F_{3ji} + \frac{1}{\sqrt{\varepsilon_j}}F_{4ji} + F_{5ji}\right) + \sum_{j=1}^{l}\varepsilon_j R_{1j}^{\varepsilon_j} + \sum_{i=1}^{n}\sum_{j=1}^{l}\sqrt{\varepsilon_j\delta_i}R_{2j}^{\varepsilon_j} + \sum_{i=1}^{n}\delta_i R_{3j}^{\varepsilon_j},$$

where

$$F_{0j} = \mathfrak{L}_{0j}u_{\overline{0,0'}}, \quad F_{1j} = \mathfrak{L}_{0j}u_{\overline{1_j,0'}} + \mathfrak{L}_{1j}u_{\overline{0,0'}},$$
$$\sum_{j=1}^{l}F_{2j} = \sum_{j=1}^{l}\mathfrak{L}_{0j}u_{\overline{2_j,0'}} + \sum_{j=1}^{l}\mathfrak{L}_{1j}u_{\overline{1_j,0'}} + \left(-\partial_t + \mathfrak{L}_2\right)u_{\overline{0,0'}},$$
$$F_{3ji} = \mathfrak{L}_{0j}u_{\overline{0,1_i'}},$$
$$F_{4ji} = \mathfrak{L}_{0j}u_{\overline{1_j 1_i'}} + \mathfrak{L}_{1j}u_{\overline{0,1_i'}} + \mathfrak{M}_{3ij}u_{\overline{0,0'}},$$
$$\sum_{j=1}^{l}F_{5ji} = \sum_{j=1}^{l}\mathfrak{L}_{0j}u_{\overline{2_j 1_i'}} + \sum_{k\neq j}\mathfrak{L}_{1j}u_{\overline{1_j 1_i'}} + \mathfrak{M}_{3ij}u_{\overline{1_j,0'}} + \mathfrak{M}_{1i}u_{\overline{0,0'}} + \left(-\partial_t + \mathfrak{L}_2\right)u_{\overline{0,1_i'}},$$
$$R_{1j}^{\varepsilon_j} = \left(-\partial_t + \mathfrak{L}_2\right)u_{\overline{2_j,0'}} + \mathfrak{L}_{1j}u_{\overline{3_j,0'}} + \sqrt{\varepsilon_j}\left(-\partial_t + \mathfrak{L}_2\right)u_{\overline{3_j,0'}},$$
$$R_{2ij}^{\varepsilon_j} = \left(-\partial_t + \mathfrak{L}_2\right)u_{\overline{1_j 1_i'}} + \mathfrak{L}_{1j}u_{\overline{2_j 1_i'}} + \mathfrak{M}_{1i}u_{\overline{1_j,0'}} + \mathfrak{M}_{3ij}u_{\overline{2_j,0'}}$$
$$+\sqrt{\varepsilon_j}\left(\left(-\partial_t + \mathfrak{L}_2\right)u_{\overline{2_j,1'}} + \mathfrak{M}_{1i}u_{\overline{2_j,0'}} + \mathfrak{M}_{3j}u_{\overline{3_{ij},0'}}\right) + \varepsilon_j\mathfrak{M}_{1j}u_{\overline{3_j,0'}},$$
$$R_{3ij}^{\varepsilon_j} = \mathfrak{M}_{1j}u_{\overline{0,1_i'}} + \mathfrak{M}_{2j}u_{\overline{0,0'}} + \mathfrak{M}_{3j}u_{\overline{1_j 1_i'}}$$
$$+\sqrt{\varepsilon_j}\left(\mathfrak{M}_{1j}u_{\overline{1_j 1_i'}} + \mathfrak{M}_{2j}u_{\overline{1_j,0'}} + \mathfrak{M}_{3j}u_{\overline{1_j 1_i'}}\right) + \varepsilon_j\left(\mathfrak{M}_{1j}u_{\overline{2_j 1_i'}} + \mathfrak{M}_{2j}u_{\overline{2_j,0'}}\right).$$

It is easy to see, $F_{0j} = F_{1j} = F_{3ij} = F_{4ij} = F_{5ij} = 0$.

So we have

$$0 = \left(-\partial_t + \mathfrak{L}^{\bar{\varepsilon},\bar{\delta}'}\right)R^{\bar{\varepsilon},\bar{\delta}'} + \sum_{j=1}^{l}\varepsilon_j R_{1j}^{\varepsilon_j} + \sum_{j=1}^{l}\sum_{i=1}^{n}\left(\sqrt{\varepsilon_j\delta_i}R_{2ij}^{\varepsilon_j} + \delta_i R_{3ij}^{\varepsilon_j}\right),$$

$$R^{\bar{\varepsilon},\bar{\delta}'}\left(0, x, y_1, \ \ldots, y_l, z_1, \ldots, z_n\right) = \sum_{j=1}^{l}\varepsilon_j G_{1j}^{\varepsilon_j}\left(x, y_1, \ \ldots, y_l, z_1, \ldots, z_n\right)$$
$$+ \sum_{j=1}^{l}\sum_{i=1}^{n}\sqrt{\varepsilon_j\delta_i}G_{2ij}^{\varepsilon_j}\left(x, y_1, \ \ldots, y_l, z_1, \ldots, z_n\right), \tag{11}$$

where

$$G_{1j}^{\varepsilon_j}\left(x, y_1, \ldots, y_l, z_1, \ldots, z_n\right) := -\, u_{\overline{2_j,0'}}\left(0, x, y_1, \ldots, y_l, z_1, \ldots, z_n\right)$$
$$- \sqrt{\varepsilon_j}u_{\overline{3_j,0'}}\left(0, x, y_1, \ldots, y_l, z_1, \ldots, z_n\right),$$

$$G^{\varepsilon_j}_{2ij}\left(x,y_1,\dots,y_l,z_1,\dots,z_n\right) := -\,u_{\overline{1_j 1'_i}}\left(0,x,y_1,\dots,y_l,z_1,\dots,z_n\right) - \sqrt{\varepsilon_j}u_{\overline{2_j 1'_i}}\left(0,x,y_1,\dots,y_l,z_1,\dots,z_n\right).$$

Using the formulas of Feynman-Kats, let us express $R^{\overline{\varepsilon},\overline{\delta'}}(t,x,y_1,\ \dots,y_l,z_1,\dots,z_n)$ solving equation (11) with boundary conditions as a mathematical expectation

$$R^{\overline{\varepsilon},\overline{\delta'}}(t,x,y_1,\ \dots,y_l,z_1,\dots,z_n)$$

$$=\sum_{j=1}^{l}\varepsilon_j\widetilde{\mathbb{E}}_{x,y_1,\ \dots,y_l,z_1,\dots,z_n}\left[e^{-\int_0^t k(X_s)ds}G^{\varepsilon_j}_{1j}\left(X_t,Y_{1t},\dots,Y_{lt},Z_{1t},\dots,Z_{nt}\right)\right]$$

$$+\int_0^t e^{-\int_0^s k(X_u)du}R^{\varepsilon_j}_{1j}\left(s,X_s,Y_{1s},\dots,Y_{ls},Z_{1s},\dots,Z_{ns}\right)ds+\sqrt{\varepsilon\delta}\widetilde{\mathbb{E}}_{x,y_1,\ \dots,y_l,z_1,\dots,z_n}$$

$$\left[e^{-\int_0^t k(X_s)ds}G^{\varepsilon}_2\left(X_t,Y_{1t},\dots,Y_{lt},Z_{1t},\dots,Z_{nt}\right)\right.$$

$$\left.+\int_0^t e^{-\int_0^s k(X_u)du}R^{\varepsilon_j}_{2ij}\left(s,X_s,Y_{1s},\dots,Y_{ls},Z_{1s},\dots,Z_{ns}\right)ds\right]$$

$$+\delta\widetilde{\mathbb{E}}_{x,y_1,\ \dots,y_l,z_1,\dots,z_n}\left[e^{-\int_0^s k(X_u)du}R^{\varepsilon_j}_{3ij}\left(s,X_s,Y_{1s},\dots,Y_{ls},Z_{1s},\dots,Z_{ns}\right)ds\right].$$

We can conclude that the functions $\left(R^{\varepsilon_j}_{1j},R^{\varepsilon_j}_{2ij},R^{\varepsilon_j}_{3ij},G^{\varepsilon_j}_{1j},G^{\varepsilon_j}_{2ij}\right)$ limited by x and polynomially increase by $(y_1,\ \dots,y_l,z_1,\dots,z_n)$ [4]. Thus, according to Lemma 1 we have

$$\left|R^{\overline{\varepsilon},\overline{\delta'}}\right|\le\sum_{j=1}^{l}\varepsilon_jC_{1j}+\sum_{j=1}^{l}\sum_{i=1}^{n}\sqrt{\varepsilon_j\delta_i}C_{2ij}+\sum_{i=1}^{n}\delta_iC_{3i}\le\left(\sum_{j=1}^{l}\varepsilon_j+\sum_{i=1}^{n}\delta_i\right)C_4.$$

and hence

$$\left|u^{\overline{\varepsilon},\overline{\delta'}}-\left(u_{\overline{0},\overline{0'}}+\sum_{j=1}^{l}\varepsilon_ju_{\overline{1_j},\overline{0'}}+\sum_{i=1}^{n}\delta_iu_{\overline{0},\overline{1'_i}}\right)\right|\le\left|R^{\overline{\varepsilon},\overline{\delta'}}\right|$$

$$+\left|\sum_{j=1}^{l}\varepsilon_ju_{\overline{2_j},\overline{0'}}+\sum_{j=1}^{l}\varepsilon_j^{3/2}u_{\overline{3_j},\overline{0'}}+\sum_{j=1}^{l}\sum_{i=1}^{n}\sqrt{\varepsilon_j\delta_i}u_{\overline{1_j1'_i}}+\sum_{j=1}^{l}\sum_{i=1}^{n}\sqrt{\delta_i}\varepsilon_j\,u_{\overline{2_j1'_i}}\right|$$

$$\le\left(\sum_{j=1}^{l}\varepsilon_j+\sum_{i=1}^{n}\delta_i\right)C_4+\sum_{j=1}^{l}\varepsilon_j\left|u_{\overline{2_j},\overline{0'}}+\sqrt{\varepsilon_j}u_{\overline{3_j},\overline{0'}}\right|$$

$$+\sum_{j=1}^{l}\sum_{i=1}^{n}\sqrt{\varepsilon_j\delta_i}\left|u_{\overline{1_j1'_i}}+\sum_{j=1}^{l}\varepsilon_ju_{\overline{2_j1'_i}}\right|\le\left(\sum_{j=1}^{l}\varepsilon_j+\sum_{i=1}^{n}\delta_i\right)C.$$

The accuracy of the result is proved. □

Theorem 4 gives us information on how the approximate price behaves when $\varepsilon_j\to 0$ and $\delta_i\to 0$.

Let X be securities without payment on assets dividends (for example, share, index and so on). X is very often modelled as a geometric Brownian motion with constant volatility (for example, Black-Scholes option pricing model). Let us consider X as a geometric Brownian motion model with multidimensional stochastic volatility. Thus, $\widetilde{\mathbb{P}}$-dynamics in X are set

$$dX_t=rX_tdt+f\left(Y_1,\dots,Y_l,Z_1,\dots,Z_n\right)X_t\widetilde{dW}^x_t,h\left(X_t\right)=0.$$

We calculate the approximate price of the double barrier option defined on X.
We write operator $\langle \mathfrak{L}_2 \rangle$ and the density associated with t and rate $m(x)$

$$\langle \mathfrak{L}_2 \rangle = \frac{1}{2}\overline{\sigma}^2 x^2 \partial^2_{xx} + rx\partial_x - r, \quad m(x) = \frac{2}{\overline{\sigma}^2 x^2} exp\left(\frac{2r}{\overline{\sigma}^2}\ln x\right). \tag{12}$$

For a double barrier option with barrier value L and R, the payoff has the form

$$H(X_t)\,\mathbb{I}_{\tau>t} = (X_t - K)^+ \mathbb{I}_{\tau>t}, I = (L, R), 0 < L < K < R.$$

To calculate the value of this parameter, we must first find the eigenvalues of operator $\langle \mathfrak{L}_2 \rangle$ presented in (12) with boundary conditions

$$\lim_{\text{x}\to\text{L}} \psi_n(x) = 0, \quad \lim_{\text{x}\to\text{R}} \psi_n(x) = 0.$$

Note that we have introduced regular keeling (interrupt process) boundary conditions at the ends of L and R. The equation $-\langle \mathfrak{L}_2 \rangle \psi_n = \lambda_n \psi_n$, $\psi_n \in dom(\langle \mathfrak{L}_2 \rangle)$, with boundary conditions mentioned above can be found in [11]

$$\psi_n(x) = \frac{\overline{\sigma}\sqrt{x}}{\sqrt{\ln\left(\frac{R}{L}\right)}} \exp\left(\frac{-r}{\overline{\sigma}^2}\ln x\right) sin\left(\frac{n\pi \ln\left(\frac{x}{L}\right)}{\ln\left(\frac{R}{L}\right)}\right), \qquad n = 1, 2, 3, \ldots,$$

$$\lambda_n = \frac{1}{2}\left(\frac{n\pi\overline{\sigma}}{\ln(\frac{R}{L})}\right)^2 + \left(\frac{\nu^2}{2} + r\right), \qquad \nu = \frac{r}{\overline{\sigma}} - \frac{\overline{\sigma}}{2}.$$

We write the expressions for operators $\mathcal{A}_j$ and $\mathcal{B}_i$

$$\mathcal{A}_j = -\vartheta_{3j} x\partial_x x^2 \partial^2_{xx} - \vartheta_{2j} x^2 \partial^2_{xx}, \quad \mathcal{B}_i = -\vartheta_{1i} x\partial_x - \vartheta_{0i}. \tag{13}$$

On the basis of (5) we calculate $\mathcal{A}_{jk,n}$, $\mathcal{B}_{ik,n}$ and $\widetilde{\mathcal{B}}_{ik,n}$. For $k \neq n$ we obtain

$$\mathcal{A}_{jk,n} = -\vartheta_{3j}\left(\frac{(-1+(-1)^{k+n})kn(4n^2\pi^2\overline{\sigma}^4 + \left(-12r^2 + 4r\overline{\sigma}^2 + \overline{\sigma}^4\right)\ln^2(\frac{R}{L}))}{2(k^2-n^2)\overline{\sigma}^4 \ln^3(\frac{R}{L})}\right)$$

$$-\vartheta_{2j}\left(\frac{\left(-1+(-1)^{k+n}\right)knr}{(k^2-n^2)\,\overline{\sigma}^2 \ln\left(\frac{R}{L}\right)}\right),$$

$$\mathcal{B}_{ik,n} = \vartheta_{i1}\frac{2(-1+(-1)^{k+n})kn}{(k-n)(k+n)\ln\left(\frac{R}{L}\right)},$$

$$\widetilde{\mathcal{B}}_{ik,n} = -\vartheta_{1i}\overline{\sigma}'\left(\Upsilon_{k,n}\right) - \vartheta_{i0}\overline{\sigma}'\left(\frac{8\left(-1+(-1)^{k+n}\right)knr\ln\left(\frac{R}{L}\right)}{(k^2-n^2)^2\pi^2\overline{\sigma}^3}\right),$$

$$v_{k,n} := \frac{4nkr\left(\ln(L) - (-1)^{k+n}\ln(R)\right)}{(k^2-n^2)\,\overline{\sigma}^3 \ln\left(\frac{R}{L}\right)}$$

$$-\frac{2\left(-1+(-1)^{k+n}\right)kn\left((k-n)(k+n)\pi^2\overline{\sigma}^4 - 2r\left(-2r+\overline{\sigma}^2\right)\ln^2\left(\frac{R}{L}\right)\right)}{(k^2-n^2)^2\pi^2\overline{\sigma}^5\ln\left(\frac{R}{L}\right)},$$

and for $k = n$ we obtain

$$\mathcal{A}_{jn,n} = -\vartheta_{3j}\left(\frac{1}{\overline{\sigma}^3}\left(\frac{3n^2\pi^2\nu}{\ln^2\left(\frac{R}{L}\right)} - \nu^3\right) - \frac{1}{\overline{\sigma}^2}\left(\nu^2 - \frac{n^2\pi^2}{\ln^2\left(\frac{R}{L}\right)}\right) - \vartheta_{j2}\left(\frac{1}{\overline{\sigma}^2}\left(\nu^2 - \frac{n^2\pi^2}{\ln^2\left(\frac{R}{L}\right)}\right) + \frac{\nu}{\overline{\sigma}}\right)\right),$$

$$\mathcal{B}_{in,n} = \vartheta_{i1}\left(\frac{2r - \overline{\sigma}^2}{2\overline{\sigma}^2}\right) - \vartheta_{i0},$$

$$\widetilde{\mathcal{B}}_{in,n} = -\vartheta_{i1}\overline{\sigma}'\left(\frac{1}{\overline{\sigma}} - \frac{r\nu\left(\ln^2(R) - \ln^2(L)\right)}{\overline{\sigma}^4\ln\left(\frac{R}{L}\right)}\right) - \vartheta_{i0}\overline{\sigma}'\left(\frac{1}{\overline{\sigma}} - \frac{r\left(\ln^2(R) - \ln^2(L)\right)}{\overline{\sigma}^3\ln\left(\frac{R}{L}\right)}\right).$$

The calculation of c_n can be found in [12–14]

$$c_n = \left(\psi_n(x), \, (\mathrm{C} - K)^+\right) = \frac{L^{\frac{\nu}{\overline{\sigma}}}}{\log\left(\frac{R}{L}\right)}\left(L\Phi_n(\nu + \overline{\sigma}) - K\Phi_n(\nu)\right),$$

$$\Phi_n(\gamma) := \frac{2}{\omega_n^2 + z^2}\left(\exp(\mathfrak{K}\gamma)\;(\omega_n\cos(\omega_n\mathfrak{K}) - \gamma sin(\omega_n\mathfrak{K}) - \exp(\mathfrak{U}\gamma)\,(-1)^n\omega_n\right),$$

$$\omega_n := \frac{n\pi}{\mathfrak{U}}, \quad \mathfrak{K} := \frac{1}{\overline{\sigma}}\ln\left(\frac{K}{L}\right), \quad \mathfrak{U} := \frac{1}{\overline{\sigma}}\ln\left(\frac{R}{L}\right).$$

The approximate option price can be calculating applying Theorems 1–3.

Note that figures are constructed component-wise in each corresponding time scale, similarly to both components in works [9] and [13].

2 Conclusions

This paper expands methodology of approximate pricing for a wide range of derivative assets. Derivatives payoffs can be way dependent, and the process underlying it may have a jump. Jump intensity depends on multidimensionality of volatility. We have developed a general theory of pricing derivative options which are generated by diffusion processes, where diffusion depends on two groups of variables. An algorithm for approximate price calculation is given. The price accuracy is determined. A developed theory is applied to Ornstein-Uhlenbeck diffusion operator, which is expanded in eigenfunctions and eigenvalues.

The main advantage of our pricing methodology is that by combining methods of spectral theory, regular perturbation theory, and singular perturbation theory, we reduce everything to the solution of the equations to find eigenfunctions and eigenvalues.

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Буртняк І.В., Малицька Г. П. *Застосування спектральної теорії та теорії збурень до дослідження процесів Орнштейна-Уленбека* // Карпатські матем. публ. — 2018. — Т.10, №2. — С. 273–287.

В статті використано методи спектральної теорії та теорії сингулярних і регулярних збурень, знайдено наближену ціну двобар'єрних опціонів Орнштейна-Уленбека з багатофакторною волатильністю, як розвинення за власними функціями використовуючи інфінітезимальні генератори $(l+n+1)$ вимірної дифузії. Встановлено теорему оцінки точності наближення цін опціонів. Знайдено явні формули для знаходження вартості деривативів на основі розвинення за власними функціями та власними значеннями самоспряжених операторів з використанням крайових задач для сингулярних і регулярних збурень.

Ключові слова і фрази: спектральна теорія, сингулярна хвильова теорія, регулярна хвильова теорія, теорія Штурма-Ліувілля, інфінітезимальний генератор, багатофакторна дифузія.

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DEHGHANI M.A., DAVVAZ B.

ON CENTRAL AUTOMORPHISMS OF CROSSED MODULES

A crossed module (T, G, ∂) consist of a group homomorphism $\partial : T \to G$ together with an action $(g, t) \to {}^{g}t$ of G on T satisfying $\partial({}^{g}t) = g\partial(t)g^{-1}$ and ${}^{\partial(s)}t = sts^{-1}$, for all $g \in G$ and $s, t \in T$. The term crossed module was introduced by J. H. C. Whitehead in his work on combinatorial homotopy theory. Crossed modules and its applications play very important roles in category theory, homotopy theory, homology and cohomology of groups, algebra, K-theory etc. In this paper, we define Adeny-Yen crossed module map and central automorphisms of crossed modules. If C^* is the set of all central automorphisms of crossed module (T, G, ∂) fixing $Z(T, G, \partial)$ element-wise, then we give a necessary and sufficient condition such that $C^* = I_{nn}(T, G, \partial)$. In this case, we prove $Aut_C(T, G, \partial) \cong Hom((T, G, \partial), Z(T, G, \partial))$. Moreover, when $Aut_C(T, G, \partial) \cong Z(I_{nn}(T, G, \partial)))$, we obtain some results in this respect.

Key words and phrases: crossed module, central automorphism.

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1 INTRODUCTION

The term crossed module was introduced by J. H. C. Whitehead in his work on combinatorial homotopy theory [9]. So many mathematicians and many areas of mathematics have used crossed modules such as homotopy theory, homology and cohomology of groups, algebra, K-theory etc. Actor crossed module of algebroid was defined by Alp in [3]. Actions and automorphisms of crossed modules were studied by K. Norrie [2, 8]. Tensor product modulo q of two crossed modules defined by Conduché and Rodriguez-Fernandez [4]. Concepts of q-commutator and of q-center of a crossed module, q being a nonnegative integer, were defined by J.L. Doncel Juurez and A.R. Crondjeanl.-Valcarcel [6]. Adney and Yen in [1] obtained several sufficient conditions for a non-abelian p-group and introduced a special map. By using Adney-Yen map, in this paper, we introduce the concept of Adney-Yen crossed module map and central automorphisms of a crossed module and obtain some results in this respect.

2 CENTRAL AUTOMORPHISMS OF A CROSSED MODULE

We recall some basic definitions and properties of the category of crossed modules. A crossed module (T, G, ∂) consists of a group homomorphism $\partial : T \to G$ called the boundary map, together with an action $(g, t) \to {}^{g}t$ of G on T satisfying (1) $\partial({}^{g}t) = g\partial(t)g^{-1}$ and (2) ${}^{\partial(s)}t = sts^{-1}$, for all $g \in G$ and $s, t \in T$.

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The group automorphism $AutN$ of a group N comes equipped with the canonical homomorphism $\tau : N \to Aut(N)$ which has image $InnN$, the group of inner automorphism of N. The inner automorphism τ is one of the standard examples of a crossed module. Other standard examples of crossed modules are: the inclusion of a normal subgroup $N \to G$; a G-module M with the zero homomorphism $M \to G$; any epimorphism $E \to G$ with central kernel. We note at once certain consequences of the definition of a crossed module: (1) the kernel $\ker\partial$ lies in $Z(T)$, the center of T; (2) the image $\partial(T)$ is a normal subgroup of G; (3) the action of G on T induces a natural $(G/\partial(T))$-module structure on $Z(T)$; and $\ker\partial$ is a submodule of $Z(T)$.

We say that (S, H, ∂') is a sub-crossed module of the crossed module (T, G, ∂) if

- S is a subgroup of T, and H is a subgroup of G;
- ∂' is the restriction of ∂ to S;
- the action of H on S is included by the action of G on T.

A sub-crossed module (S, H, ∂) of (T, G, ∂) is normal if

- H is a normal subgroup of G;
- ${}^{g}s \in S$ for all $g \in G, s \in S$;
- ${}^{h}tt^{-1} \in S$ for all $h \in H, t \in T$.

In this case we consider the triple $(T/S, G/H, \bar{\partial})$, where $\bar{\partial} : T/S \to G/H$ is induced by ∂, and the new action is given by ${}^{gH}(tS) = ({}^{g}t)S$. This is the quotient crossed module of (T, G, ∂) by (S, H, ∂). A crossed module morphism $\langle \alpha, \varphi \rangle : (T, G, \partial) \to (T', G', \partial')$ is a commutative diagram of homomorphisms of groups

$$\begin{array}{ccc} T & \xrightarrow{\alpha} & T' \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial'} \\ G & \xrightarrow[\varphi]{} & G' \end{array}$$

such that for all $x \in G$ and $t \in T$; we have $\alpha({}^{x}t) = {}^{\varphi(x)}\, \alpha(t)$. We say that $\langle \alpha, \varphi \rangle$ is an isomorphism if α and φ are both isomorphisms. We denote the group of automorphisms of (T, G, ∂) by $Aut(T, G, \partial)$. The kernel of the crossed module morphism $\langle \alpha, \varphi \rangle$ is the normal sub-crossed module $(\ker\alpha, \ker\varphi, \partial)$ of (T, G, ∂), denoted by $\ker\langle \alpha, \varphi \rangle$. The image im⟨ff,′⟩ of $\langle \alpha, \varphi \rangle$ is the sub-crossed module (imff, im′, ∂') of (T', G', ∂'). For a crossed module (T, G, ∂), denote by $\mathrm{Der}(G, T)$ the set of all derivations from G to T, i.e., all maps $\chi : G \to T$ such that for all $x, y \in G$, $\chi(xy) = \chi(x)\,{}^{x}\chi(y)$. Each such derivation χ defines endomorphisms $\sigma = (\sigma_x)$ and $\theta (= \theta_x)$ of G, T respectively, given by $\sigma(x)) = \partial\chi(x)x$ and $\theta(t) = \chi\partial(t)t$, where $\sigma\partial(t) = \partial\theta(t)$, $\theta\chi(x) = \chi\partial(x)$ and $\theta({}^{x}t) = {}^{\sigma(x)}\theta(t)$. We define a multiplication in $\mathrm{Der}(G, T)$ by the formula $\chi_1 \circ \chi_2 = \chi$, where $\chi(x) = \chi_1\sigma_2(x)\chi_2(x)$ $(= \theta_1\chi_2(x)\chi_1(x))$. This turns $\mathrm{Der}(G, T)$ into a semigroup with identity element the derivation which maps each element of G into identity element of T. Moreover, if $\chi = \chi_1 \circ \chi_2$, then $\sigma = \sigma_1\sigma_2$. The whitehead group $D(G, T)$ is defined to be the group of units of $\mathrm{Der}(G, T)$, and the elements of $D(G, T)$ are called regular derivations.

Proposition 1. *The following statements are equivalent: (1) $\chi \in D(G,T)$; (2) $\sigma \in \mathrm{Aut}(G)$; (3) $\theta \in \mathrm{Aut}(T)$.*

The map $\Delta : D(G,T) \to \mathrm{Aut}(T,G,\partial)$ defined by $\Delta(X) = \langle \sigma, \theta \rangle$ is a homomorphisms of groups and there is an action of $\mathrm{Aut}(T,G,\partial)$ on $D(G,T)$ given by ${}^{\langle \alpha,\varphi \rangle}\chi = \alpha\chi\varphi^{-1}$ which makes $(D(G,T), \mathrm{Aut}(T,G,\partial), \Delta)$ a crossed module. This crossed module is called the actor crossed module $\mathcal{A}(T,G,\partial)$ of the crossed module (T,G,∂). There is a morphism of crossed modules $\langle \eta, \gamma \rangle : (T,G,\partial) \to \mathcal{A}(T,G,\partial)$ defined as follows. If $t \in T$, then $\eta_t : G \to T$ defined by $\eta_t(x) = t\,{}^{x}t^{-1}$ is a derivation and the map $t \to \eta_t$ defines a homomorphism $\eta : T \to D(G,T)$ of groups. Let $\gamma : G \to \mathcal{A}(T,G,\partial)$ be the homomorphism $y \to \langle \alpha_y, \varphi_y \rangle$, where $\alpha_y(t) = {}^{y}t$ and $\varphi_y(x) = yxy^{-1}$ for $t \in T$ and $y, x \in G$.

Definition 1. *Let (T,G,∂) be a crossed module. The center of (T,G,∂) is the crossed module kernel $Z(T,G,\partial)$ of $\langle \eta, \gamma \rangle$. Thus, $Z(T,G,\partial)$ is the crossed module $(T^G, St_G(T) \cap Z(G), \partial)$, where T^G denotes the fixed point subgroup of T, that is, $T^G = \{\, t \in T \mid {}^{x}t = t$ for all $x \in G\}$. $St_G(T)$ is the stabilizer in G of T, that is, $St_G(T) = \{\, x \in G \mid {}^{x}t = t$ for all $t \in T\}$ and $Z(T)$ is the center of G. Note that T^G is central in T.*

Definition 2. *Let (T,G,∂) be a crossed module. n-center of (T,G,∂), $Z^n(T,G)$ for n a nonnegative integer is the crossed module $\Big((T^G)^n, Z^n(G) \cap St_G(T), \partial\Big)$, where*

$$\begin{aligned} (T^G)^n &= \{t \in T | t^n = 1 \text{ and } {}^{g}t = t \text{ for all } g \in G\}, \\ Z^n(G) &= \{\, g \in Z(G) \mid g^n = 1\}, \\ St_G(T) &= \{g \in G \mid {}^{g}t = t \text{ for all } t \in T\}. \end{aligned}$$

The n-center of (T,G,∂) is a normal crossed submodule and is called the n-central crossed submodule of (T,G,∂).

Let (T,G,∂) be a crossed module, (T',G',∂) be a normal sub-crossed module of it, and $\langle \alpha, \varphi \rangle \in \mathrm{Aut}(T,G,\partial)$. Then $\langle \alpha, \varphi \rangle$ induces a $\langle \bar{\alpha}, \bar{\varphi} \rangle$ in $\mathrm{Aut}\left(T/T', G/G', \bar{\partial}\right)$ such that $\bar{\partial} : T/T' \to G/G'$, $\bar{\partial}(tT') = \partial(t)T'$.

Definition 3. *Let (T,G,∂) be a crossed module, $Z(T,G)$ be the center of it and $\langle \alpha, \varphi \rangle \in Aut(T,G,\partial)$. If $\langle \bar{\alpha}, \bar{\varphi} \rangle$, induced by $\langle \alpha, \varphi \rangle$ in $Aut\left(T/T^G, G/St_G(T) \cap Z(G), \bar{\partial}\right)$, is identity, then $\langle \alpha, \varphi \rangle$ is called central automorphism of crossed module (T,G,∂).*

Theorem 1. *If (T,G,∂) has trivial n-center, then its actor $\mathcal{A}(T,G,\partial)$ also has trivial n-center.*

Proof. Let us assume that $Z^n(T,G,\partial) = 1$ so that $(T^G)^n = 1$ and $Z^n(G) \cap St_G(T) = 1$. Now the n-center of $\mathcal{A}(T,G,\partial)$ is the crossed module

$$\begin{aligned} \mathcal{A}(T,G,\partial) &= (D(G,T), \mathrm{Aut}(T,G,\partial), \Delta)\,, Z^n(\mathcal{A}(T,G,\partial)) \\ &= \left(\left(D(G,T)^{\mathrm{Aut}(T,G,\partial)}\right)^n, Z^n(\mathrm{Aut}(T,G,\partial)) \cap St_{\mathrm{Aut}(T,G,\partial)}(D(G,T), \Delta\right). \end{aligned}$$

So, assume that $\chi \in \left(D(G,T)^{\mathrm{Aut}(T,G,\partial)}\right)^n$. Then, for all $\langle \alpha, \varphi \rangle \in \mathrm{Aut}(T,G,\partial)$, ${}^{\langle \alpha,\varphi \rangle}\chi = \chi$ and $\chi^n = 1$. In particular, this is true for all $\langle \alpha_y, \varphi_y \rangle$, where $y \in G$. But $\langle \alpha_y, \varphi_y \rangle_\chi = \eta_{\chi(y)^{-1}} \circ \chi$. So ${}^{\langle \alpha_y,\varphi_y \rangle}\chi = \chi$ implies that $\eta_{\chi(y)^{-1}} = 1$ for all $y \in G$. Then, we have $\chi(y)^{-1}\,{}^{x}\chi(y) = 1$, for all

$x, y \in G$. Now, since $(T^G)^n = 1$, χ is the trivial derivation, it follows that $\left(D(G,T)^{\langle \alpha_y, \varphi_y \rangle}\right)^n = 1$. Now, suppose that $\langle \alpha, \partial \rangle \in Z^n(\mathrm{Aut}(T,G,\partial)) \cap St_{\mathrm{Aut}(T,G,\partial)}(D(G,T))$. Then ${}^{\langle \alpha, \varphi \rangle}\chi = \chi$ for all $\chi D(G,T)$. In particular, ${}^{\langle \alpha, \varphi \rangle}\eta_t = \eta_t$, for all $t \in T$, that is $\eta_{\alpha(t)} = \eta_t$, which implies that $t^{-1}\alpha(t) \in (T^G)^n = 1$, for all $t \in T$. Thus $\alpha = 1_T$, the identity automorphism of T. Now we have $\langle \alpha, \varphi \rangle \in Z^n(\mathrm{Aut}(T,G,\partial))$. Hence, for all $y \in G$, $\langle \alpha, \varphi \rangle \langle \alpha_y, \varphi_y \rangle = \langle \alpha_y, \varphi_y \rangle \langle \alpha, \varphi \rangle$ and $\langle \alpha, \varphi \rangle^n = 1$ implying that $\varphi \varphi_y = \varphi_y \varphi$ for all $y \in G$. So we obtain $\varphi(xyx^{-1}) = y\varphi(x)y^{-1}$ for all $x, y \in G$. Since φ is an automorphism of G, it follows that $y^{-1}\varphi(y) \in Z(G)$, for all $y \in G$. Now, since $\langle \alpha, \varphi \rangle$ is a crossed module morphism, it follows that $\alpha({}^y t) = {}^{\varphi(y)}\alpha(t)$. But $\alpha^n = 1_T$ so that ${}^y t = \varphi(y)t$ for all $y \in G$ and $t \in T$. Thus, $y^{-1}\varphi(y) \in Z^n(G) \cap St_G(T) = 1$ so that $\varphi^n = 1_G$. Therefore $\langle \alpha^n, \varphi^n \rangle = \langle 1_T, 1_G \rangle$, and this completes the proof. □

A non-abelian group that has no non-trivial abelian direct factor is said to be purely non-abelian [1].

By using Adney-Yen map [1], we introduce the following definition.

Definition 4. *An Adney-Yen crossed module map is an onto map*

$$\langle \varphi_1, \varphi_2 \rangle \text{ from } Aut_C(T,G,\partial) \text{ to } Hom((T,G,\partial), Z(T,G,\partial))$$

such that $\langle \varphi_1, \varphi_2 \rangle \langle \alpha, \theta \rangle = \langle \varphi_1, \varphi_2 \rangle_{\langle \alpha, \theta \rangle}$ *and* $\langle \varphi_1, \varphi_2 \rangle_{\langle \alpha, \theta \rangle}$ *is the crossed module homomorphism of* (T,G,∂) *into* $Z(T,G,\partial) = (T^G, St_G(T) \cap Z(G), \partial)$. *Furthermore, such that* $\langle \varphi_1, \varphi_2 \rangle_{\langle \alpha, \theta \rangle} = \langle \varphi_{1\,\langle \alpha, \theta \rangle}, \varphi_{2\,\langle \alpha, \theta \rangle} \rangle$, *where* $\varphi_{1\,\langle \alpha, \theta \rangle} : T \to T^G$, $\varphi_{1\,\langle \alpha, \theta \rangle}(t) = t^{-1}\alpha(t)$ *and* $\varphi_{2\,\langle \alpha, \theta \rangle} : G \to St_G(T) \cap Z(G)$, $\varphi_{2\,\langle \alpha, \theta \rangle}(g) = g^{-1}\theta(g)$.

Theorem 2. *For purely non-abelian groups* T *and* G *an Adeny-Yen crossed module map is one-to-one correspondence of* $Aut_C(T,G,\partial)$ *onto* $Hom((T,G,\partial), Z(T,G,\partial))$.

Proof. The crossed module map $\langle \alpha, \theta \rangle \to \langle \varphi_1, \varphi_1 \rangle_{\langle \alpha, \theta \rangle}$ is a one-to-one crossed module map of $Aut_C(T,G,\partial)$ into the $Hom((T,G,\partial), Z(T,G,\partial))$.

Conversely, if $\langle f_1, f_2 \rangle \in Hom((T,G,\partial), Z(T,G,\partial))$, then $\langle \varphi_1, \varphi_2 \rangle_{\langle f_1, f_2 \rangle}(t) = tf_1(t)$, for all $t \in T$ and $\langle \varphi_1, \varphi_2 \rangle_{\langle f_1, f_2 \rangle}(g) = gf_2(g)$, for all $g \in G$, defines an endomorphism of (T,G,∂). The endomorphism $\langle \varphi_1, \varphi_2 \rangle_{\langle f_1, f_2 \rangle}$ is an automorphism if and only if $f_1(t) \neq t^{-1}$ and $f_2(g) \neq g^{-1}$ for every $g \in G$, $g \neq 1$ and $t \in T$, $t \neq 1$. But if T and G are direct product with an abelian factor, then there exists $\langle f_1, f_2 \rangle \in Hom((T,G,\partial), Z(T,G,\partial))$ such that $\langle f_1, f_2 \rangle \langle t, g \rangle = \langle t, g \rangle^{-1}$, for some $g \in G$, $g \neq 1$ and $t \in T$, $t \neq 1$. So, suppose that there exists $\langle f_1, f_2 \rangle \in Hom((T,G,\partial), Z(T,G,\partial))$ such that $f_1(t) = t^{-1}$ for some $t \in T$, $t \neq 1$ and $f_2(g) = g^{-1}$ for some $g \in G$, $g \neq 1$. Clearly, $t \in Z(T)$, $g \in Z(G)$. We assume that the order of t,$o(t) = p_1$ and $o(g) = p_2$ such that p_1, p_2 are primes. If

$$T/T' = T_{p_1}/T' \times T'_{p_1}/T' \text{ and } G/G' = G_{p_2}/G' \times G'_{p_2}/G',$$

where T_{p_1}/T' is the p_1-primary component of T/T' and G_{p_2}/G' is the P_2-primary component of G/G', then $tT' \in T_{p_1}/T'$, $tT' \neq T'$ and $gG' \in G_{p_2}/G'$, $gG' \neq G'$ for is contained in the kernel of f_1 and T' is contained in the kernel of f_2. But if the height of tT' in T_{p_1}/T' be $p_1^{k_1}$ and $t = x_1{}^{p_1^{k_1}}u$, where $x_1 \in T_{p_1}$ and $u \in T'$, gG' in G_{p_2}/G' be $p_2{}^{k_2}$ and $g = x_2{}^{p_2^{k_2}}v$, where $x_2 \in G_{p_2}$ and $v \in G'$. Then, $t^{-1} = f_1(t) = f_1(x_1{}^{p_1{}^{k_1}}u) = f_1(x_1)^{p_1{}^{k_1}}$ and $g^{-1} = f_2(g) = f_2(x_2{}^{p_2{}^{k_2}}v) = f_2(x_2)^{p_2{}^{k_2}}$. Set $y_1 = f_1(x_1)^{-1}$, $y_2 = f_2(x_2)^{-1}$. So $t = y_1{}^{p_1^{k_1}}$, $y_1 \in Z(T) \cap T_{p_1}$, $\{y_1\} \cap T' = 1$ and

$g = y_2{}^{p_2^{k_2}}$, $y_2 \in Z(G) \cap G_{p_2}$,$\{y_2\} \cap G' = 1$. By [7], y_1T' generates a direct factor of T_{p_1}/T' and y_2G' of G_{p_2}/G' too, say

$$T_{p_1}/T' = \{y_1T'\} \times H_{p_1}/T' \text{ and } G_{p_2}/G' = \{y_2G'\} \times H_{p_2}/G'.$$

Since $\{y_1\} \cap T' = 1$, $T = \{y_1\} \times (H_{p_1}T_{p_1'})$ is a direct decomposition of T, it follows that T has an abelian direct factor and G too, if the mapping $\langle f_1, f_2\rangle$ is not onto. □

Let C^* be the set of all central automorphisms of (T, G, ∂) fixing $Z(T, G, \partial)$ element wise.

Theorem 3. *If T is a finite p_1-group and G is a finite p_2-group, then $C^* = I_{nn}(T, G, \partial)$ if and only if T, G are abelian or T and G are nilpotent of class 2 and $Z(T)$, $Z(G)$ are cyclic.*

Proof. We have

$$C_{Aut_C(T,G,\partial)}(Z(T, G, \partial)) \cong Hom\left((T, G, \partial)/Z(T, G, \partial), Z(T, G, \partial)\right).$$

Since every element of $\langle f_1, f_2\rangle \in C^*$ fixes each element of $Z(T, G, \partial)$, for $\langle f_1, f_2\rangle \in C^*$, the map $\langle \sigma_1 f_1, \sigma_2 f_2\rangle$ of $(T, G, \partial)/Z(T, G, \partial)$ to $Z(T, G, \partial)$ defined by $\sigma_{1f_1} : T/T^G \to T^G$ such that $\sigma_{1f_1}(tT^G) = t^{-1}f_1(t)$ and $\sigma_{2f_2}(g(St_G(T) \cap Z(G))) = g^{-1}f_2(g)$ are well defined. It is obvious to see that $\sigma_1 : f_1 \to \sigma_{1f_1}$ and $\sigma_2 : f_2 \to \sigma_{2f_2}$ are injective homo morphisms. Now for each

$$\langle h_1, h_2\rangle \in Hom\left((T, G, \partial)/Z(T, G, \partial), Z(T, G, \partial)\right),$$

the crossed module map $\langle f_1, f_2\rangle$ defined by $f_1(t) = th_1(tT^G)$ for all $t \in T$ and $f_2(g) = gh_2(g(St_G(T) \cap Z(G)))$ for all $g \in G$ is a central automorphism fixing $Z(T, G, \partial)$ element-wise and $\langle \sigma_1, \sigma_2\rangle_{\langle f_1, f_2\rangle} = \langle h_1, h_2\rangle$. It follows $\langle \sigma_1, \sigma_2\rangle$ is a crossed module isomorphism and

$$C^* \cong Hom\left((T, G, \partial)/Z(T, G, \partial), Z(T, G, \partial)\right).$$

Now, suppose first that $C^* \cong I_{nn}((T, G, \partial))$ and T,G are non-abelian. If $t \in T$ and $g \in G$, then the inner automorphism $\langle \theta_1, \theta_2\rangle_{\langle t,g\rangle}$ induced by t,g is a central automorphism and so $[x, t] = x^{-1}\theta_{1t}(x) \in T^G$ for all $x \in T$ and $[y, g] = y^{-1}\theta_{2g}(y) \in St_G(T) \cap Z(G)$ for all $y \in G$. This shows that T and G are nilpotent of class 2. Since T is nilpotent of class 2, $\exp(T/T^G) = \exp(T') = p_1^{c_1}$ for some natural number c_1, and $\exp(G/St_G(T) \cap Z(G)) = \exp(G') = p_2^{c_2}$. Let T/T^G and T^G have ranks r_1 and s_1, respectively, and $G/St_G(T) \cap Z(G)$ and $St_G(T) \cap Z(G)$ have ranks r_2 and s_2, but T and G are nilpotent of class 2, it follows from [5] that T^G and $St_G(T) \cap Z(G)$ are cyclic. Conversely, if T and G are abelian, then it is clear that $C^* \cong I_{nn}((T, G, \partial)) = \langle 1, 1\rangle$. Assume that T and G are nilpotent of class 2 and $Z(T)$ and $Z(G)$ are cyclic. Since T/T^G and $G/Z(G)$ are abelian p_1 and p_2-groups of exponent $\mid T' \mid$ and $\mid G' \mid$, T' and G' are cyclic, $C^* \cong I_{nn}((T, G, \partial))$, since T and G are nilpotent of class 2, $I_{nn}((T, G, \partial)) \leq C^*$. Hence, $C^* \cong I_{nn}((T, G, \partial))$. □

Theorem 4. *For any non-abelian groups T and G the restriction of the Adeny-Yen crossed module map $\langle \varphi_1, \varphi_2\rangle : C^* \to Hom\left((T, G, \partial), (Z(T, G, \partial)\right)$ is a crossed module homomorphism.*

Proof. Suppose that $\langle \alpha_1, \theta_1\rangle$ and $\langle \alpha_2, \theta_2\rangle \in C^*$. On the other hand, we have the following diagrams

$$\begin{tikzcd} T \arrow[r, "\alpha_1"] \arrow[d, "\partial"'] & T^G \arrow[d, "\partial"] \\ G \arrow[r, "\theta_1"'] & St_G(T) \cap Z(G) \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\alpha_2"] \arrow[d, "\partial"'] & T^G \arrow[d, "\partial"] \\ G \arrow[r, "\theta_2"'] & St_G(T) \cap Z(G) \end{tikzcd}$$

Then, for any $t \in T,\ g \in G$ we have $\varphi_{1\langle\alpha_1,\theta_1\rangle\langle\alpha_2,\theta_2\rangle}(t) = t^{-1}(\alpha_1 \circ \alpha_2)(t) = t^{-1}(\alpha_1(\alpha_2(t)))$ and

$$\begin{aligned}\varphi_{1\langle\alpha_1,\theta_1\rangle} \circ \varphi_{1\langle\alpha_2,\theta_2\rangle}(t) &= \varphi_{1\langle\alpha_1,\theta_1\rangle}(t^{-1}\alpha_2(t)) = (t^{-1}\alpha_2(t))^{-1}\alpha_1(t^{-1}\alpha_2(t)) \\ &= \alpha_2(t^{-1})t\alpha_1(t^{-1})\alpha_1(\alpha_2(t)) = \alpha_2(t^{-1})t^{-1}\alpha_1(\alpha_2(t)) = t^{-1}\alpha_1(\alpha_2(t))\end{aligned}$$

Moreover, $\varphi_{1\langle\alpha_1,\theta_1\rangle\langle\alpha_2,\theta_2\rangle}(g) = g^{-1}(\alpha_1 \circ \alpha_2)(g) = g^{-1}(\alpha_1(\alpha_2(g)))$ and

$$\begin{aligned}\varphi_{1\langle\alpha_1,\theta_1\rangle} \circ \varphi_{1\langle\alpha_2,\theta_2\rangle}(g) &= \varphi_{1\langle\alpha_1,\theta_1\rangle}(g^{-1}\alpha_2(g)) = (g^{-1}\alpha_2(g))^{-1}\alpha_1(g^{-1}\alpha_2(g)) \\ &= \alpha_2(g^{-1})g\alpha_1(g^{-1})\alpha_1(\alpha_2(g)) = g\alpha_2(g^{-1})\alpha_1(g^{-1})\alpha_1(\alpha_2(g)) \\ &= g^{-1}\alpha_1(g^{-1})\alpha_1(\alpha_2(g)) = g^{-1}\alpha_1(\alpha_2(g)).\end{aligned}$$

So we have $\langle\varphi_1,\varphi_2\rangle_{\langle\alpha_1,\theta_1\rangle\langle\alpha_2,\theta_2\rangle} = \langle\varphi_1,\varphi_2\rangle_{\langle\alpha_1,\theta_1\rangle} \circ \langle\varphi_1,\varphi_2\rangle_{\langle\alpha_2,\theta_2\rangle}$. □

Theorem 5. *If T and G are purely non-abelian group and $Aut_C(T,G,\partial) = C^*$, then*

$$Aut_C(T,G,\partial) \cong Hom\left((T,G,\partial), Z(T,G,\partial)\right).$$

Proof. Since T and G are purely non-abelian, so by Theorem 2 the Adeny-Yen crossed module map from $Aut_C(T,G,\partial)$ to $Hom\left((T,G,\partial), Z(T,G,\partial)\right)$ is a bijection. But also $Aut_C(T,G,\partial) = C^*$, and thus by Theorem 4, $\langle\varphi_1,\varphi_2\rangle$ is a crossed module homomorphism. Therefore, $Aut_C(T,G,\partial) \cong Hom\left((T,G,\partial), Z(T,G,\partial)\right)$. □

Theorem 6. *Let T and G be purely non-abelian groups such that*

$$Aut_C(T,G,\partial) = Z(I_{nn}(T,G,\partial)).$$

Then

$$Aut_C(T,G,\partial) \cong Hom\left((T,G,\partial), Z(T,G,\partial)\right).$$

Proof. $Z(I_{nn}(T,G,\partial))$ is a sub-crossed module of $I_{nn}(T,G,\partial)$, which fixes $Z(T,G,\partial)$ pointwise, so $Aut_C(T,G,\partial) = C^*$. Thus, by Theorem 5 we have the desired conclusion. □

Theorem 7. *Let T and G be non-abelian group such that $Aut_C(T,G,\partial) = Z(I_{nn}(T,G,\partial))$. Then either T and G are purely non-abelian or T and G have purely non-abelian subgroups T_1 and G_1, with $|Z(T_1)|$ and $|Z(G_1)|$ odd such that $T = C_2 \times T_1$, $G = C_2 \times G_1$.*

Proof. Suppose that on the contrary $T = A \times T_1$ and $G = B \times G_1$, where T_1 and G_1 are purely non-abelian, $A,\ B$ are non-trivial abelian and either $A \neq C_2$ or $A = C_2$ or $B \neq C_2$ or $B = C_2$ and $|Z(T_1)|$ and $|Z(G_1)|$ are even. In these case, we claim (T,G,∂) has a central automorphism that is not inner, on the other hand $Aut_C(T,G,\partial) \neq Z(I_{nn}(T,G,\partial))$. If $A \neq C_2$ and $B \neq C_2$ and $(\varphi_1,\varphi_2) \in Aut(A,B,\partial) = Aut_C(A,B,\partial)$ is non-trivial, then for any $(a,t_1) \in A \times T_1$ and $(b,g_1) \in B \times G_1$ maps $(a,t_1) \to (\varphi_1(a),t_1)$ and $(b,g_1) \to (\varphi_2(b),g_1)$ give an automorphism of (T,G,∂) that is central but not inner. If $A = C_2$ and $B = C_2$ and $|Z(A)|$ and $|Z(B)|$ are even, take $z_1 \in Z(A)$ and $z_2 \in Z(B)$ with z_1 and z_2 or order 2, so maps $(1,t_1) \to (1,t_1)$, $(1,g_1) \to (1,g_1)$, $(a,t_1) \to (a,z_1t_1)$ and $(b,g_1) \to (b,z_2g_2)$ define an automorphism of (T,G,∂) that is central but not inner, since $(a,1) \to (a,z_1)$ and $(b,1) \to (b,z_2)$. □

Corollary 1. *Let p_1, p_2 be primes and T and G be non-abelian p_1-group and p_2-group respectively such that $Aut_C(T,G,\partial) = Z(I_{nn}(T,G,\partial))$. Then T and G are purely non-abelian.*

Theorem 8. *Let p_1, p_2 be primes, and T and G are non-abelian p_1-group and p_2-group respectively such that $Aut_C(T,G,\partial) = Z(I_{nn}(T,G,\partial))$. Then*

$$Aut_C(T,G,\partial) \cong Hom((T,G,\partial), Z(T,G,\partial)).$$

Proof. By Theorem 1, T and G are purely non-abelian and since

$$Aut_C(T,G,\partial) \cong Z(I_{nn}(T,G,\partial)))$$

is subcrossed module of $I_{nn}(T,G,\partial)$, which fixes $Z(T,G,\partial)$ point-wise, $Aut_C(T,G,\partial) = C^*$. Now, by Theorem 5 we have $Aut_C(T,G,\partial) \cong Hom\left((T,G,\partial), Z(T,G,\partial)\right)$. □

Corollary 2. *Let p_1, p_2 be primes, and T and G be finite p_1-group and p_2-group respectively such that $Aut_C(T,G,\partial) = C^*$. Then, T and G are purely non-abelian.*

Theorem 9. *Let p_1, p_2 be primes, and T and G be finite p_1-group and p_2-group respectively such that $Aut_C(T,G,\partial) = C^*$. Then*

$$Aut_C(T,G,\partial) \cong Hom\left((T,G,\partial), Z(T,G,\partial)\right).$$

Proof. By Corollary 2, T and G are purely non-abelian. Since $Aut_C(T,G,\partial) = C^*$, then by Theorem 5 we have $Aut_C(T,G,\partial) \cong Hom\left((T,G,\partial), Z(T,G,\partial)\right)$. □

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Перехресний модуль (T, G, ∂) складається з групового гомоморфізму $\partial : T \to G$ з дією $(g, t) \to {}^g t$ з G на T, яка задовольняє $\partial({}^g t) = g\partial(t)g^{-1}$ і ${}^{\partial(s)}t = sts^{-1}$ для всіх $g \in G$ і $s, t \in T$. Термін перехресного модуля введено Дж. Х. К. Уайтхедом у його роботі з комбінаторики теорії гомотопій. Перехресні модулі і їх застосування відіграють дуже важливу роль в теорії категорій, теорії гомотопій, гомології і когомологиї груп, алгебрі, К-теорії тощо. У даній роботі визначено відображення Адені-Єна перехресних модулів і центральні автоморфізми перехресних модулів. Якщо C^* — множина всіх центральних автоморфізмів перехресних модулів (T, G, ∂), які поточково фіксують $Z(T, G, \partial)$, то отримано необхідну і достатню умови щоб $C^* = I_{nn}(T, G, \partial)$. У цьому випадку доведено $Aut_C(T, G, \partial) \cong Hom((T, G, \partial), Z(T, G, \partial))$. Крім того, якщо $Aut_C(T, G, \partial) \cong Z(I_{nn}(T, G, \partial)))$, то отримані також певні результати в цьому напрямку.

Ключові слова і фрази: перехресний модуль, центральний автоморфізм.

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UNCONVENTIONAL ANALOGS OF SINGLE-PARAMETRIC METHOD OF ITERATIONAL AGGREGATION

When we solve practical problems that arise, for example, in mathematical economics, in the theory of Markov processes, it is often necessary to use the decomposition of operator equations using methods of iterative aggregation. In the studies of these methods for the linear equation $x = Ax + b$ the most frequent are the conditions of positiveness of the operator A, constant b and the aggregation functions, and also the implementation of the inequality $\rho(A) < 1$ for the spectral radius $\rho(A)$ of the operator A.

In this article for an approximate solution of a system composed of the equation $x = Ax + b$ represented in the form $x = A_1x + A_2x + b$, where $b \in E$, E is a Banach space, A_1, A_2 are linear continuous operators that act from E to E and the auxiliary equation $y = \lambda y - (\varphi, A_2x) - (\varphi, b)$ with a real variable y, where (φ, x) is the value of the linear functional $\varphi \in E^*$ on the elements $x \in E$, E^* is conjugation with space E, an iterative process is constructed and investigated

$$x^{(n+1)} = Ax^{(n)} + b + \frac{\sum\limits_{i=1}^{m} A_1^i x^{(n)}}{(\varphi, x^{(n)}) \sum\limits_{i=0}^{m} \lambda^i}(y^{(n)} - y^{(n+1)}) \quad (m < \infty),$$
$$y^{(n+1)} = \lambda y^{(n+1)} - (\varphi, A_2x^{(n)}) - (\varphi, b).$$

The conditions are established under which the sequences $x^{(n)}, y^{(n)}$, constructed with the help of these formulas, converge to x^*, y^* as a component of solving the system constructed from equations $x = A_1x + A_2x + b$ and the equation $y = \lambda y - (\varphi, A_2x) - (\varphi, b)$ not slower than the rate of convergence of the geometric progression with the denominator less than 1. In this case, it is required that the operator A be a compressive and constant by sign, and that the space E is semi-ordered. The application of the proposed algorithm to systems of linear algebraic equations is also shown.

Key words and phrases: aggregating functional, decomposition, iterative aggregation.

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INTRODUCTION

Actuality of the investigation of iterative aggregation methods connected with necessity of solving big dimensional problems with the aid of multiprocessor computable technical devices using decompositional algorithms for corresponding mathematical models. Multiparametric iterative aggregation has appeared to be an effective in mathematical economy, in investigation of Markov processes etc. (see [1–3, 6, 7, 13]) due to ability to make an acceptable results even

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in circumstances when convergence conditions of algorithms is unknown (see [5, p. 158]). The simplest single parametric method of iterative aggregation for equation

$$x = Ax + b \tag{1}$$

in [5, p. 155–158] was described by formula

$$x^{(n+1)} = \frac{(\varphi, b)}{(\varphi, x^{(n)} - Ax^{(n)})} Ax^{(n)} + b, \tag{2}$$

where (φ, x) are values of linear functional φ on elements x of Banach space E, $A : E \to E$. Instead of (2) we can consider

$$x^{(n+1)} = \frac{(\varphi, x^{(n+1)})}{(\varphi, x^{(n)})} Ax^{(n)} + b. \tag{3}$$

In [4, 8–12] it is launched method of algorithm (3) convergency investigation and its multi-parametric generalization under conditions of not semi ordered space E and inequality $\rho(A) < 1$ of spectral radius $\rho(A)$ of operator A does not demand.

1 Main suggestions

Let us suppose that equation (1) can be considered in the form

$$x = A_1 x + A_2 x + b, \tag{4}$$

where $b \in E$, E is a Banach space, A_1, A_2 are linear continuous operators that act from E to E. Let us denote by (φ, x) values of linear functional $\varphi \in E^*$ on elements $x \in E$, E^* is the adjoint space to E, A_1^* is the adjoint operator to A_1, E' is a set of real numbers. Let us consider the system formed by equation (4) and additional equation

$$y = \lambda y - (\varphi, A_2 x) - (\varphi, b) \tag{5}$$

with the real unknown y. Let us define a norm of $\{x, y\}$ $(x \in E, y \in E')$ by formula

$$||x, y|| = \sqrt{||x||^2 + |y|^2},$$

where $||x||$ is a norm of element $x \in E$, $|y|$ is an absolute value of number $y \in E'$. We denote by ε a set of pairs $\{x, y\}$ $(x \in E, y \in E')$ that satisfy the equation

$$(\varphi, x) + y = 0. \tag{6}$$

Theorem 1. *Let the following conditions hold*
1) pair (x^*, y^*) is the solution of system (4), (5) *in $\widetilde{E} = E \times E'$;*
2) the following equality takes place

$$A_1^* \varphi = \lambda \varphi, \quad \lambda \in E', \lambda \neq 1. \tag{7}$$

Then $(x^*, y^*) \in \varepsilon$.

Proof. From the condition 2) and the equalities (4), (5) for $x = x^*, y = y^*$ it follows that

$$\begin{aligned}(\varphi, x^*) + y^* &= (\varphi, A_1x^*) + (\varphi, A_2x^*) + (\varphi, b) + \lambda y^* - (\varphi, A_2x^*) - (\varphi, b)\\ &= (A_1^*\varphi, x^*) + \lambda y^* = \lambda[(\varphi, x^*) + y^*].\end{aligned}$$

Since $\lambda \neq 1$, then we obtain that (x^*, y^*) satisfies (6). □

Theorem 2. *Let us consider operator $a(x)w$ which is continuous by $x \in E$ and linear and continuous by $w \in E'$. Let us suppose that equality*

$$(\varphi, a(x)) = \lambda, \quad \lambda \in E', \lambda \neq 1 \tag{8}$$

takes place and condition 2) of Theorem 1 holds. If $\{x, y\} \in \varepsilon$, $x \in E$, $y \in E'$, then for pair $\{u, v\}$, which is the solution of system

$$u = A_1x + A_2x + b + a(x)(y - v), \tag{9}$$

$$v = \lambda v - (\varphi, A_2x) - (\varphi, b), \tag{10}$$

we can state that $\{u, v\} \in \varepsilon$.

Proof. Let us prove that (u, v) satisfies (6). Really,

$$\begin{aligned}(\varphi, u) + v = (\varphi, A_1x) &+ (\varphi, A_2x) + (\varphi, b) + (\varphi, a(x))y - (\varphi, a(x))v\\ &+ \lambda v - (\varphi, A_2x) - (\varphi, b) = \lambda[(\varphi, x) + y].\end{aligned}$$

Therefore $(u, v) \in \varepsilon$. □

Theorem 3. *If the condition 2) of Theorem 1 takes place, then the operator*

$$a(x) = \frac{\sum\limits_{i=1}^{m} A_1^i x}{(\varphi, x) \sum\limits_{i=0}^{m-1} \lambda^i} \quad (m < \infty) \tag{11}$$

satisfies equality (8).

Proof. Using (7) we obtain from (11) following:

$$\begin{aligned}(\varphi, a(x)) &= \frac{\sum\limits_{i=1}^{m} (\varphi, A_1^i x)}{(\varphi, x) \sum\limits_{i=0}^{m-1} \lambda^i} = \frac{\sum\limits_{i=1}^{m} (A_1^*\varphi, A_1^{i-1} x)}{(\varphi, x) \sum\limits_{i=0}^{m-1} \lambda^i}\\ &= \frac{\lambda \sum\limits_{i=1}^{m} (\varphi, A_1^{i-1} x)}{(\varphi, x) \sum\limits_{i=0}^{m-1} \lambda^i} = \frac{\lambda \sum\limits_{i=0}^{m-1} \lambda^i (\varphi, x)}{(\varphi, x) \sum\limits_{i=0}^{m-1} \lambda^i} = \lambda.\end{aligned}$$

The theorem is proved. □

2 ITERATIVE FORMULAS AND SUFFICIENT CONDITIONS OF CONVERGENCY

Let us construct sequence $\{x^{(n)}\}$, $\{y^{(n)}\}$ with starting approximation $(x^{(0)}, y^{(0)}) \in \varepsilon$ by formulas

$$x^{(n+1)} = Ax^{(n)} + b + \frac{\sum\limits_{i=1}^{m} A_1^i x^{(n)}}{(\varphi, x^{(n)}) \sum\limits_{i=0}^{m} \lambda^i}(y^{(n)} - y^{(n+1)}) \quad (m < \infty), \tag{12}$$

$$y^{(n+1)} = \lambda y^{(n+1)} - (\varphi, A_2 x^{(n)}) - (\varphi, b), \tag{13}$$

where $x \in E, y \in E', \lambda \in E', \lambda \neq 1$. From (5) and (13) we get

$$y^{(n+1)} - y^* = -\frac{1}{1-\lambda}(\varphi, A_2(x^{(n)} - x^*)).$$

From the Theorems 1 and 2 we obtain equality

$$y^{(n)} - y^* = -(\varphi, x^{(n)} - x^*).$$

From (12), (13) and (11) we get

$$\begin{aligned} x^{(n+1)} - x^* &= A(x^{(n)} - x^*) - a(x^{(n)})(\varphi, x^{(n)} - x^*) + a(x^{(n)})\frac{(\varphi, A_2(x^{(n)} - x^*))}{1-\lambda} \\ &= A(x^{(n)} - x^*) - \frac{a(x^{(n)})}{1-\lambda}(\varphi(I - A)(x^{(n)} - x^*)), \end{aligned}$$

or

$$x^{(n+1)} - x^* = A(x^{(n)} - x^*) - \frac{\sum\limits_{i=1}^{m} A_1^i x^{(n)}}{(1-\lambda)(\varphi, x) \sum\limits_{i=0}^{m-1} \lambda}(\varphi, (I - A)(x^{(n)} - x^*)), \tag{14}$$

where I is the identity operator.

Theorem 4. *Let the conditions of Theorems 1–3 take place. If for $(x, y) \in \varepsilon$, $w = x - x^*$ and operator $H_1(x)w$ defined by the formula*

$$H_1(x)w = Aw = \frac{\sum\limits_{i=1}^{m} A_1^i x}{(1-\lambda)(\varphi, x) \sum\limits_{i=0}^{m-1} \lambda_i}(\varphi, (I - A)w),$$

the inequality

$$||H_1(x)|| \leqslant q_1 \tag{15}$$

holds for $q_1 < 1$, then every sequence of $\{x^{(n)}\}$, $\{y^{(n)}\}$, constructed by formulas (12), (13)*, converges respectively to x^*, y^*, as a components of solution of system* (4), (5)*, not slowly then geometry progression with multiplier q_1.*

Proof. It is sufficient to use formulas (14), inequality (15) and condition $q_1 < 1$. □

3 APPLICATION TO A SYSTEM OF LINEAR ALGEBRAIC EQUATIONS

Let us consider case when A_1, A_2 are the squared matrices of order N, $N < \infty$. For $(x, y) \in \varepsilon, w \in E'$ let us define operator $H_2(x)w$ by the formula

$$H_2(x)w = [A - \frac{\sum\limits_{i=1}^{m} A^i x}{(1-\lambda)\varphi^T x \sum\limits_{i=0}^{m-1} \lambda_i} \varphi^T (I - A)]w,$$

where notation $\varphi^T x$ used instead of (φ, x), φ^T is a line vector, x is a row vector, T is the transposition symbol, $\lambda \in E', \lambda \neq 1$.

Theorem 5. *If for matrices A_1, A_2 conditions of theorems 1 — 3 take place and inequalities $||H_2(x)|| \leqslant q_2 < 1$ hold, then sequences $\{x^{(n)}\}$, $\{y^{(n)}\}$, constructed by formulas* (12)*,* (13) *converge to x^* and y^* respectively as a components of solution of system* (4)*,* (5) *not slowly then geometry progression with multiplier q_2.*

Proof. The theorem is a partial case of Theorem 4. □

4 EXPANSION ON CASE $m = \infty$

Let us change formula (12) as follows

$$x^{(n+1)} = Ax^{(n)} + b + \frac{A_1(I - A_1)^{-1}x^{(n)}}{(\varphi, x^{(n)})}(1 - \lambda), \tag{16}$$

where $\lambda \in E', \lambda \neq 1, x \in E$, and consider iterative process, which describes pair of formulas (16) and (13) with starting approximation $\{x^{(0)}, y^{(0)}\} \in \varepsilon$. Let us restrict ourselves to the situation, when $\lambda < 1$.

For $\{x, y\} \in \varepsilon, w = x - x^*$ let us define operator $H_3(x)w$ by the formula

$$H_3(x)w = Aw - (1 - \lambda)\frac{A_1(I - A_1)^{-1}x}{(\varphi, x)}(\varphi, (I - A)w). \tag{17}$$

Theorem 6. *Let the conditions of Theorems 1–3 take place and for operator $H_3(x)w$, defined by the formula* (17)*, following inequality holds*

$$||H_3(x)|| \leqslant q_3 < 1. \tag{18}$$

Then sequences $\{x^{(n)}\}$, $\{y^{(n)}\}$, constructed with the help of formulas (13)*,* (16)*, converge to x^* and y^* respectively as a components of solution of system* (4)*,* (5) *not slowly then geometry progression with multiplier q_3.*

Proof. The proof of the theorem can be obtained by notions (17), (18). □

Theorem 6 is an analogue of Theorem 4. Using similar way we can obtain analogue of Theorem 5 for systems of linear algebraic equations.

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Демків І.І., Копач М.І., Обшта А.Ф., Шувар Б.А. *Нетрадиційні аналоги однопараметричного методу ітеративного агрегування* // Карпатські матем. публ. — 2018. — Т.10, №2. — С. 296–302.

При розв'язанні практичних завдань, що виникають, наприклад, в математичній економіці, в терії марківських процесів, часто доводиться використовувати декомпозицію операторних рівнянь за допомогою методів ітеративного агрегування. В дослідженнях цих методів для лінійного рівняння $x = Ax + b$ найчастішими є вимоги додатності оператора A, вільного члена b та агрегуючих функціоналів, а також виконання нерівності $\rho(A) < 1$ для спектрального радіуса $\rho(A)$ оператора A.

В статті для наближеного розв'язання системи, складеної з рівняння $x = Ax + b$, представленого у вигляді $x = A_1x + A_2x + b$, де $b \in E$, E — банахів простір, A_1, A_2 — лінійні неперервні оператори, що діють з E в E, і допоміжного рівняння $y = \lambda y - (\varphi, A_2x) - (\varphi, b)$ з дійсним невідомим y, де (φ, x) — значення лінійного функціоналу $\varphi \in E^*$ на елементах $x \in E$, E^* — спряжений з E простір, побудовано і досліджено ітеративний процес

$$
\begin{aligned}
x^{(n+1)} &= Ax^{(n)} + b + \frac{\sum\limits_{i=1}^{m} A_1^i x^{(n)}}{(\varphi, x^{(n)}) \sum\limits_{i=0}^{m} \lambda^i} (y^{(n)} - y^{(n+1)}) \quad (m < \infty), \\
y^{(n+1)} &= \lambda y^{(n+1)} - (\varphi, A_2x^{(n)}) - (\varphi, b).
\end{aligned}
$$

Встановлено умови, при виконанні яких послідовності $x^{(n)}, y^{(n)}$, побудовані з допомогою цих формул, збігаються відповідно до x^*, y^* як компонент розв'язку системи, складеної з рівняння $x = A_1x + A_2x + b$ та рівняння $y = \lambda y - (\varphi, A_2x) - (\varphi, b)$, не повільніше від швидкості збіжності геометричної прогресії зі знаменником, меншим від одиниці. При цьому вимагається, щоб оператор A був стискуючим та знакосталим, а простір E напівупорядкованим. Показано також застосування запропонованого алгоритму до систем лінійних алгебраїчних рівнянь.

Ключові слова і фрази: декомпозиція, ітеративне агрегування, агрегуючі функціонали.

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HILBERT POLYNOMIALS OF THE ALGEBRAS OF SL_2-INVARIANTS

We consider one of the fundamental problems of classical invariant theory, the research of Hilbert polynomials for an algebra of invariants of Lie group SL_2. Form of the Hilbert polynomials gives us important information about the structure of the algebra. Besides, the coefficients and the degree of the Hilbert polynomial play an important role in algebraic geometry. It is well known that the Hilbert function of the algebra SL_n-invariants is quasi-polynomial. The Cayley-Sylvester formula for calculation of values of the Hilbert function for algebra of covariants of binary d-form $\mathcal{C}_d = \mathbb{C}[V_d \oplus \mathbb{C}^2]_{SL_2}$ (here V_d is the $d+1$-dimensional space of binary forms of degree d) was obtained by Sylvester. Then it was generalized to the algebra of joint invariants for n binary forms. But the Cayley-Sylvester formula is not expressed in terms of polynomials.

In our article we consider the problem of computing the Hilbert polynomials for the algebras of joint invariants and joint covariants of n linear forms and n quadratic forms. We express the Hilbert polynomials $\mathcal{H}(\mathcal{I}_1^{(n)}, i) = \dim(\mathcal{C}_1^{(n)})_i$, $\mathcal{H}(\mathcal{C}_1^{(n)}, i) = \dim(\mathcal{C}_1^{(n)})_i$, $\mathcal{H}(\mathcal{I}_2^{(n)}, i) = \dim(\mathcal{I}_2^{(n)})_i$, $\mathcal{H}(\mathcal{C}_2^{(n)}, i) = \dim(\mathcal{C}_2^{(n)})_i$ of those algebras in terms of quasi-polynomials. We also present them in the form of Narayana numbers and generalized hypergeometric series.

Key words and phrases: classical invariant theory, invariants, Hilbert function, Hilbert polynomials, Poincaré series, combinatorics.

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INTRODUCTION

Let $\mathbb{K}$ be a field of characteristic zero. Let V_d be the $d+1$-dimensional module of binary forms of degree d. Let $V_{\mathbf{d}} = V_{d_1} \oplus V_{d_2} \oplus \ldots \oplus V_{d_n}$, $\mathbf{d} := (d_1, d_2, \ldots d_n)$. Denote by $\mathbb{K}[V_{\mathbf{d}}]^{SL_2}$ the algebra of polynomial SL_2-invariant functions on $V_{\mathbf{d}}$. It is well known that $\mathcal{I}_{\mathbf{d}} := \mathbb{K}[V_{\mathbf{d}}]^{SL_2}$ is finitely generated and graded:

$$\mathcal{I}_{\mathbf{d}} := (\mathcal{I}_{\mathbf{d}})_0 \oplus (\mathcal{I}_{\mathbf{d}})_1 \oplus \ldots \oplus (\mathcal{I}_{\mathbf{d}})_i \oplus \ldots,$$

here $(\mathcal{I}_{\mathbf{d}})_i$ is a vector $\mathbb{K}$-space of invariants of degree i. The dimension of the vector space $(\mathcal{I}_{\mathbf{d}})_i$ is called *the Hilbert function* of the algebra $\mathcal{I}_{\mathbf{d}}$. It is defined as a function of the variable i :

$$\mathcal{H}(\mathcal{I}_{\mathbf{d}}, i) = \dim(\mathcal{I}_{\mathbf{d}})_i.$$

It is well known that the Hilbert function of an arbitrary finitely generated graded $\mathbb{K}$-algebra is a quasi-polynomial (starting from some i), see [7, 13, 15]. Since the algebra of invariants $\mathcal{I}_{\mathbf{d}}$ is finitely generated, we have

$$\mathcal{H}(\mathcal{I}_{\mathbf{d}}, i) = h_0(i) i^r + h_1(i) i^{r-1} + \ldots,$$

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where $h_k(i)$ is some periodic function with values in $\mathbb{Q}$. The quasi-polynomial $\mathcal{H}(\mathcal{I}_{\mathbf{d}}, i)$ is called *the Hilbert polynomial* of algebra of invariants $\mathcal{I}_{\mathbf{d}}$.

For the case of one binary form ($n = 1$) there exists classical Cayley-Sylvester formula for calculation of values of Hilbert function of $\mathcal{I}_d$:

$$\mathcal{H}(\mathcal{I}_d, i) = \omega_d(i, 0) - \omega_d(i, 2),$$

where $\omega_d(i, k)$ is the number of non-negative integer solutions of the system:

$$\begin{cases} \alpha_1 + 2\alpha_2 + \ldots + d\alpha_d = \dfrac{di - k}{2}, \\ \alpha_1 + \alpha_2 + \ldots + \alpha_d = i. \end{cases}$$

Also (see [8,14]) we have

$$\mathcal{H}(\mathcal{I}_d, i) = \left[q^{\frac{id}{2}}\right] \left(\frac{(1 - q^{d+1})\,(1 - q^{d+2})\ldots(1 - q^{d+i})}{(1 - q^2)\,(1 - q^3)\ldots(1 - q^i)} \right),$$

where $\left[q^{\frac{id}{2}}\right]$ denotes the coefficient of $q^{\frac{id}{2}}$. Generalizations of these formulas to the algebra $\mathcal{I}_{\mathbf{d}}$ was obtained in [1–4].

However, all these results are combinatorial formulas. They are not expressed in terms of Hilbert polynomials in i. Note that, it is hard to calculate for those formulas even for small values of d_k and i.

Although, Maple-procedure for computing of the Hilbert polynomials of the algebras of SL_2-invariants for small values of d was being offered in [5].

A partial characterization of Hilbert polynomials for non-standard graded algebras was obtained in [6].

Consider a direct sum of n linear forms $nV_1 = \underbrace{V_1 \oplus V_1 \oplus \ldots \oplus V_1}_{n \text{ times}}$. In the language of classical invariant theory the algebras $\mathcal{I}_1^{(n)} := \mathbb{C}[nV_1]^{SL_2}$ and $\mathcal{C}_1^{(n)} := \mathbb{C}[nV_1 \oplus \mathbb{C}^2]^{SL_2}$ are called *the algebra of joint invariants* and *the algebra of joint covariants for the n linear forms* respectively. Let V_2 be the complex vector space of quadratic binary forms endowed with the natural action of the special linear group SL_2. Consider the corresponding action of the group SL_2 on the algebras of polynomial functions $\mathbb{C}[nV_2]$ and $\mathbb{C}[nV_2 \oplus \mathbb{C}^2]$, where $nV_2 := \underbrace{V_2 \oplus V_2 \oplus \cdots \oplus V_2}_{n \text{ times}}$. Denote by $\mathcal{I}_2^{(n)} = \mathbb{C}[nV_2]^{SL_2}$ and by $\mathcal{C}_2^{(n)} = \mathbb{C}[nV_2 \oplus \mathbb{C}^2]^{SL_2}$ the corresponding algebras of invariant polynomial functions. In the language of classical invariant theory the algebras $\mathcal{I}_2^{(n)}$ and $\mathcal{C}_2^{(n)}$ are called *the algebra of joint invariants and the algebra of joint covariants for the n quadratic forms* respectively.

The algebras $\mathcal{C}_1^{(n)}, \mathcal{I}_1^{(n)}, \mathcal{C}_2^{(n)}$ and $\mathcal{I}_2^{(n)}$ are graded:

$$\mathcal{C}_1^{(n)} = (\mathcal{C}_1^{(n)})_0 + (\mathcal{C}_1^{(n)})_1 + \cdots + (\mathcal{C}_1^{(n)})_i + \cdots, \quad \mathcal{I}_1^{(n)} = (\mathcal{I}_1^{(n)})_0 + (\mathcal{I}_1^{(n)})_1 + \cdots + (\mathcal{I}_1^{(n)})_i + \cdots,$$
$$\mathcal{C}_2^{(n)} = (\mathcal{C}_2^{(n)})_0 + (\mathcal{C}_2^{(n)})_1 + \cdots + (\mathcal{C}_2^{(n)})_i + \cdots, \quad \mathcal{I}_2^{(n)} = (\mathcal{I}_2^{(n)})_0 + (\mathcal{I}_2^{(n)})_1 + \cdots + (\mathcal{I}_2^{(n)})_i + \cdots,$$

where each of the subspaces $(\mathcal{C}_1^{(n)})_i, (\mathcal{I}_1^{(n)})_i, (\mathcal{C}_2^{(n)})_i$ and $(\mathcal{I}_2^{(n)})_i$ is finite dimensional. The functions

$$\mathcal{H}(\mathcal{C}_1^{(n)}, i) = \dim(\mathcal{C}_1^{(n)})_i, \quad \mathcal{H}(\mathcal{I}_1^{(n)}, i) = \dim(\mathcal{I}_1^{(n)})_i,$$
$$\mathcal{H}(\mathcal{C}_2^{(n)}, i) = \dim(\mathcal{C}_2^{(n)})_i, \quad \mathcal{H}(\mathcal{I}_2^{(n)}, i) = \dim(\mathcal{I}_2^{(n)})_i$$

are called the Hilbert polynomials of the algebra of joint covariants for the n linear forms, the Hilbert polynomial of the algebra of joint invariants for the n linear forms, the Hilbert polynomial of the algebra of joint covariants for the n quadratic forms and the Hilbert polynomial of the algebra of joint invariants for the n quadratic forms, respectively. The formal power series

$$\mathcal{P}(\mathcal{C}_1^{(n)}, z) = \sum_{i=0}^{\infty} \mathcal{H}(\mathcal{C}_1^{(n)}, i)\, z^i, \quad \mathcal{P}(\mathcal{I}_1^{(n)}, z) = \sum_{i=0}^{\infty} \mathcal{H}(\mathcal{I}_1^{(n)}, i)\, z^i,$$
$$\mathcal{P}(\mathcal{C}_2^{(n)}, z) = \sum_{i=0}^{\infty} \mathcal{H}(\mathcal{C}_2^{(n)}, i)\, z^i, \; \mathcal{P}(\mathcal{I}_2^{(n)}, z) = \sum_{i=0}^{\infty} \mathcal{H}(\mathcal{I}_2^{(n)}, i)\, z^i$$

are called the Poincaré series of the algebras $\mathcal{C}_1^{(n)}, \mathcal{I}_1^{(n)}, \mathcal{C}_2^{(n)}$ and $\mathcal{I}_2^{(n)}$ respectively.

In the present paper we obtain explicit formulas for computation of the Hilbert polynomial of those algebras. We present some results in terms of *generalized hypergeometric functions.* A generalized hypergeometric function is given by a hypergeometric series, i.e., a series for which the ratio of successive terms can be written as follows

$${}_pF_q\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\middle|\, z\right] = \sum_{k=0}^{\infty} \frac{\prod_{h=1}^{p} (a_h)_k z^k}{\prod_{j=1}^{q} (b_j)_k k!},$$

where $(a)_k = a(a+1)\ldots(a+k-1)$ is the Pochhammer symbol or rising factorial.

If any a_j is a non-positive integer $(0, -1, -2, \ldots)$, then the series has only a finite number of terms and in fact is a polynomial of degree a_j. If any b_k is a non-positive integer (excepting the previous case with $b_k < a_j$), then the denominators become 0 and the series is undefined.

In the present paper we compute the Hilbert polynomials of the algebras of joint covariants and invariants for the n linear and quadratic forms:

$$\mathcal{H}(\mathcal{I}_1^{(n)}, i) = \begin{cases} N_{n+k-1,\, k+1}, & \text{if } i = 2k, \\ 0, & \text{if } i = 2k+1, \end{cases}$$
$$\mathcal{H}(\mathcal{C}_1^{(n)}, i) = \begin{cases} \binom{n+k-1}{k}^2, & \text{if } i = 2k, \\ \\ nN_{n+k,\, k+1}, & \text{if } i = 2k+1, \end{cases}$$

$$\mathcal{H}(\mathcal{I}_2^{(n)}, i) = \begin{cases} \sum_{k=0}^{[\frac{i}{2}]} \binom{n+k-1}{k}^2 \binom{n+i-2k-2}{n-2} \frac{3k-i+1}{k+1}, & \text{if } i > 1, \\ 1, & \text{if } i = 1, \end{cases}$$
$$\mathcal{H}(\mathcal{C}_2^{(n)}, i) = \sum_{k=0}^{[\frac{i}{2}]} \binom{n+k-1}{k}^2 \binom{n+i-2k-1}{n-1},$$

where $N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}, (1 \le k \le n)$ is the Narayana number.

We also express the Hilbert polynomials $\mathcal{H}(\mathcal{I}_2^{(n)}, i), \mathcal{H}(\mathcal{C}_2^{(n)}, i)$ in terms of generalized hypergeometric function:

$$\mathcal{H}(\mathcal{I}_2^{(n)}, i) = (1-n)\binom{n+i-2}{i} {}_5F_4\left[\begin{array}{c} n, n, -\frac{i}{2}, -\frac{i-1}{2}, -\frac{i}{3}+\frac{4}{3} \\ 1, -\frac{n+i-2}{2}, -\frac{n+i-3}{2}, -\frac{i}{3}+\frac{1}{3} \end{array}\middle| 1\right], \text{if } n > 3, i > 1,$$

$$\mathcal{H}(\mathcal{C}_2^{(n)}, i) = \binom{n+i-1}{i} {}_4F_3\left[\begin{array}{c} n, n, -\frac{i}{2}, -\frac{i-1}{2}, -\frac{n}{3} \\ 1, -\frac{n+i-1}{2}, -\frac{n+i-2}{2} \end{array}\middle| 1\right], \text{if } n > 2.$$

1 HILBERT POLYNOMIALS OF THE ALGEBRAS OF JOINT INVARIANTS AND COVARIANTS OF n LINEAR FORMS

Poincaré series for the algebras of joint invariants and covariants of n linear forms was derived by L. Bedratyuk in [2]. Using them, author found the following explicit formula for Poincaré series those algebras in [11]:

$$\mathcal{P}(\mathcal{I}_1^{(n)}, z) = \frac{N_{n-2}(z^2)}{(1-z^2)^{2n-3}} \quad \text{and} \quad \mathcal{P}(\mathcal{C}_1^{(n)}, z) = \frac{W_{n-1}(z^2) + nzN_{n-1}(z^2)}{(1-z^2)^{2n-1}},$$

where

$$N_n(z) = \sum_{k=1}^{n} \frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1} z^{k-1} \quad \text{and} \quad W_n(z) = \sum_{k=0}^{n} \binom{n}{k}^2 z^k$$

are the Narayana polynomials. Let us use these formulas to obtain the Hilbert polynomials of the algebras $\mathcal{I}_1^{(n)}$ and $\mathcal{C}_1^{(n)}$.

To prove Theorem 1, we need the following lemma.

Lemma 1 ([9, 11, 16]). *Let m, k, s be non-negative integers. Then the generalized Le Jen Shoo identity holds:*

$$\sum_{i=0}^{\min\{k,m\}} \binom{m}{i}\binom{m+2s}{i+s}\binom{k-i+2m+2s}{2m+2s} = \binom{m+k+s}{m+s}\binom{m+k+2s}{m+s}.$$

Theorem 1. *The following formulas hold*

$$(i) \quad \mathcal{H}(\mathcal{I}_1^{(n)}, i) = \begin{cases} N_{n+k-1,\,k+1}, & \text{if } i = 2k, \\ 0, & \text{if } i = 2k+1, \end{cases}$$

$$(ii) \quad \mathcal{H}(\mathcal{C}_1^{(n)}, i) = \begin{cases} \binom{n+k-1}{k}^2, & \text{if } i=2k, \\ nN_{n+k,\,k+1}, & \text{if } i=2k+1, \end{cases}$$

where $N_{n,\,k} = \dfrac{1}{n}\dbinom{n}{k}\dbinom{n}{k-1}$ are the Narayana numbers.

Proof. (i) Let us expand function

$$\mathcal{P}(\mathcal{I}_1^{(n)}, z) = \frac{N_{n-2}(z^2)}{(1-z^2)^{2n-3}}$$

into the Taylor series about $z = 0$:

$$\mathcal{P}(\mathcal{I}_1^{(n)}, z) = \sum_{k=0}^{n-3} \binom{n-3}{k}\binom{n-2}{k}\frac{z^{2k}}{k+1}\sum_{i=0}^{\infty}\binom{(2n-3)+i-1}{i}z^{2i}$$
$$= \sum_{k=0}^{\infty}\sum_{i=0}^{\min\{k,n-3\}}\binom{n-3}{i}\binom{n-2}{i}\binom{2n+k-i-4}{k-i}\frac{1}{i+1}z^{2k}$$
$$= \sum_{k=0}^{\infty}\frac{z^{2k}}{n-1}\sum_{i=0}^{\min\{k,n-3\}}\binom{n-3}{i}\binom{n-1}{i+1}\binom{2n+k-i-4}{k-i}.$$

Using Lemma 1 ($m = n-3$ and $s = 1$), we have:

$$\mathcal{P}(\mathcal{I}_1^{(n)}, z) = \sum_{k=0}^{\infty}\frac{1}{n-1}\binom{n+k-2}{n-2}\binom{n+k-1}{n-2}z^{2k}.$$

Statement (i) follows immediately from the definitions of Poincaré series, Hilbert polynomials and Narayana numbers.

Note that the identity $\mathcal{P}(\mathcal{I}_1^{(n)}, z) = \frac{N_{n-2}(z^2)}{(1-z^2)^{2n-3}}$ holds for $n \geq 3$. Then statement (i) holds for $n \geq 3$. Consider the case $n = 2$.We obtain that (x_1, y_1) are coordinates for the first V_1 and (x_2, y_2) are coordinates for the second one, both with respect to the canonical representation of SL_2. There is a single quadratic invariant $y_1x_2 - x_1y_2$. Hence

$$\mathcal{P}(\mathcal{I}_1^{(2)}, z) = \frac{1}{1-z^2} = 1 + z^2 + z^4 + z^6 + \ldots.$$

We have

$$\mathcal{H}(\mathcal{I}_1^{(2)}, i) = \cos^2\frac{\pi i}{2} = N_{2+[\frac{i}{2}]-1,\,2-1}\cos^2\frac{i\pi}{2}.$$

This proves that statement (i) holds for $n \geq 2$.

(ii) As above we use Poincaré series of the algebra $\mathcal{C}_1^{(n)}$ $(n > 1)$:

$$\mathcal{P}(\mathcal{C}_1^{(n)}, z) = \frac{\sum\limits_{k=0}^{n-1}\binom{n-1}{k}^2 z^{2k}}{(1-z^2)^{2n-1}} + \frac{\sum\limits_{k=0}^{n-2}\binom{n-2}{k}\binom{n}{k+1}z^{2k+1}}{(1-z^2)^{2n-1}}$$
$$= \sum_{k=0}^{n-1}\binom{n-1}{k}^2 z^{2k}\sum_{k=0}^{\infty}\binom{(2n-1)+i-1}{i}z^{2i}$$
$$+ \sum_{k=0}^{n-2}\binom{n-2}{k}\binom{n}{k+1}z^{2k+1}\sum_{k=0}^{\infty}\binom{(2n-1)+i-1}{i}z^{2i}$$
$$= \sum_{k=0}^{\infty}\sum_{i=0}^{\min\{k,n-1\}}\binom{n-1}{i}^2\binom{2n+k-i-2}{k-i}z^{2k}$$
$$+ \sum_{k=0}^{\infty}\sum_{i=0}^{\min\{k,n-2\}}\binom{n-2}{i}\binom{n}{i+1}\binom{k-i+2n-2}{2n-2}z^{2k+1}.$$

Using Lemma 1, we get:

$$\mathcal{P}(\mathcal{C}_1^{(n)}, z) = \sum_{k=0}^{\infty}\binom{n+k-1}{k}^2 z^{2k} + \sum_{k=0}^{\infty}\binom{n+k-1}{n-1}\binom{n+k}{n-1}z^{2k+1}.$$

This proves (ii) for $n>1$. Using a Maple-procedure for computing the Hilbert polynomials of the algebra $\mathcal{C}_1^{(1)}$, see [5], we get $\mathcal{H}(\mathcal{C}_1^{(1)},i)=1$. By formulas (ii) we have $\mathcal{H}(\mathcal{C}_1^{(1)},i)=1$, too. Hence, (ii) holds for $n\geq 1$. □

Corollary 1.

$$(i)\quad \mathcal{H}(\mathcal{I}_1^{(n)},i)=\frac{1}{(n-1)!(n-2)!}\sum_{m=1}^{n-1}\begin{bmatrix}n-1\\m\end{bmatrix}\left[\frac{i}{2}\right]^{m-1}\sum_{j=1}^{n-1}\begin{bmatrix}n-1\\j\end{bmatrix}\left(\left[\frac{i}{2}\right]+1\right)^{j-1},\ n>1,$$

$$(ii)\quad \mathcal{H}(\mathcal{C}_1^{(n)},i)=\frac{1}{(n-1)!^2}\sum_{m=1}^{n}\sum_{j=1}^{n}\begin{bmatrix}n\\m\end{bmatrix}\begin{bmatrix}n\\j\end{bmatrix}\frac{i^{m+j-2}\cos^2\frac{i\pi}{2}+(i+1)^{m-1}(i-1)^{j-1}\sin^2\frac{i\pi}{2}}{2^{m+j-2}},$$

where $\left[{n \atop m}\right]$ *are the unsigned Stirling numbers of the first kind.*

Proof. (i) Let us express the Narayana numbers in terms of the unsigned Stirling numbers of the first kind:

$$\begin{aligned}N_{n+[\frac{i}{2}]-1,\,n-1}&=\frac{1}{n-1}\binom{n+[\frac{i}{2}]-2}{n-2}\binom{n+[\frac{i}{2}]-1}{n-2}=\frac{\left([\frac{i}{2}]\right)_{n-1}\left([\frac{i}{2}]+1\right)_{n-1}}{(n-1)!(n-2)![\frac{i}{2}]([\frac{i}{2}]+1)}\\&=\frac{1}{(n-1)!(n-2)!}\sum_{m=1}^{n-1}\begin{bmatrix}n-1\\m\end{bmatrix}\left[\frac{i}{2}\right]^{m-1}\sum_{j=1}^{n-1}\begin{bmatrix}n-1\\j\end{bmatrix}\left(\left[\frac{i}{2}\right]+1\right)^{j-1}.\end{aligned}$$

In Theorem 1(i), we proved that $\mathcal{H}(\mathcal{I}_1^{(n)},i)=N_{n+[\frac{i}{2}]-1,\,n-1}\cos^2\frac{i\pi}{2}$ as $n>1$. Since $\cos^2\frac{i\pi}{2}=0$ as i is odd, it follows that

$$\begin{aligned}\mathcal{H}(\mathcal{I}_1^{(n)},i)&=\frac{1}{(n-1)!(n-2)!}\sum_{m=1}^{n-1}\sum_{j=1}^{n-1}\begin{bmatrix}n-1\\m\end{bmatrix}\begin{bmatrix}n-1\\j\end{bmatrix}\left(\left[\frac{i}{2}\right]+1\right)^{j-1}\left[\frac{i}{2}\right]^{m-1}\cos^2\frac{i\pi}{2}\\&=\frac{1}{(n-1)!(n-2)!}\sum_{m=1}^{n-1}\sum_{j=1}^{n-1}\begin{bmatrix}n-1\\m\end{bmatrix}\begin{bmatrix}n-1\\j\end{bmatrix}\frac{i^{m-1}(i+2)^{j-1}}{2^{m+j-2}}\cos^2\frac{i\pi}{2},\ \text{if } n>1.\end{aligned}$$

(ii) The proof of (ii) is completely analogous to that of (i). □

2 Hilbert polynomials of the algebras of joint invariants and covariants of n quadratic forms

The Poincaré series of the algebras of joint invariants and covariants of n quadratic forms are needed for the sequel. They were derived by L.Bedratyuk in [2]. Using them, the author obtained the following formulas in [12]:

$$\mathcal{P}(\mathcal{C}_2^{(n)},z)=\frac{W_{n-1}(z^2)}{(1-z)^n(1-z^2)^{2n-1}}\quad\text{and}\quad \mathcal{P}(\mathcal{I}_2^{(n)},z)=\frac{W_{n-1}(z^2)-nzN_{n-1}(z^2)}{(1-z)^n(1-z^2)^{2n-1}}.$$

Theorem 2. *Hilbert polynomials of the algebras of joint invariants and covariants of* n *quadratic forms are calculated by the following formula:*

$$(i)\quad \mathcal{H}(\mathcal{C}_2^{(n)},i)=\sum_{k=0}^{[\frac{i}{2}]}\binom{n+k-1}{k}^2\binom{n+i-2k-1}{n-1},$$

$$(ii)\quad \mathcal{H}(\mathcal{I}_2^{(n)},i)=\begin{cases}\displaystyle\sum_{k=0}^{[\frac{i}{2}]}\binom{n+k-1}{k}^2\binom{n+i-2k-2}{n-2}\frac{3k-i+1}{k+1}, & \text{if } i>1,\\ 1, & \text{if } i=1,\end{cases}\qquad \text{where } n>1.$$

Proof. This theorem can be proved basically in the same way as Theorem 1.

(i) Let us expand the Poincaré series of the algebra $\mathcal{C}_2^{(n)}$ into the Taylor series about $z = 0$. We have:

$$\mathcal{P}(\mathcal{C}_2^{(n)}, z) = \sum_{k=0}^{\infty} \binom{n+k-1}{k}^2 z^{2k} \sum_{i=0}^{\infty} \binom{n+i-1}{i} z^i$$
$$= \sum_{i=0}^{\infty} \sum_{k=0}^{\left[\frac{i}{2}\right]} \binom{n+k-1}{k}^2 \binom{n+i-2k-1}{i-2k} z^i.$$

(ii) Using Theorem 1 (ii), we get

$$\mathcal{P}(\mathcal{I}_2^{(n)}, z) = \left(\sum_{k=0}^{\infty} \binom{n+k-1}{k}^2 z^{2k} - \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} \binom{n+k}{n-1} z^{2k+1} \right) \sum_{i=0}^{\infty} \binom{n+i-1}{i} z^i$$
$$= \sum_{i=0}^{\infty} \sum_{k=0}^{[i/2]} \binom{n+k-1}{k}^2 \binom{n+i-2k-1}{n-1} z^i - \sum_{i=0}^{\infty} \sum_{k=0}^{[i/2]} \binom{n+k-1}{k} \binom{n+k}{n-1} \binom{n+i-2k-1}{n-1} z^{i+1}$$
$$= \sum_{i=0}^{\infty} \left(\sum_{k=0}^{[i/2]} \binom{n+k-1}{k}^2 \binom{n+i-2k-1}{n-1} - \sum_{k=0}^{[(i-1)/2]} \binom{n+k-1}{k} \binom{n+k}{n-1} \binom{n+i-2k-2}{n-1} \right) z^i.$$

By the definitions of Poincaré series and Hilbert polynomials,

$$\mathcal{H}(\mathcal{I}_2^{(n)}, i) = \sum_{k=0}^{[i/2]} \binom{n+k-1}{k}^2 \binom{n+i-2k-1}{n-1} - \sum_{k=0}^{[(i-1)/2]} \binom{n+k-1}{k} \binom{n+k}{n-1} \binom{n+i-2k-2}{n-1}.$$

Note that $\binom{n+i-2k-2}{n-1} = 0$, as $k > \left[\frac{i-1}{2}\right]$. We have

$$\mathcal{H}(\mathcal{I}_2^{(n)}, i) = \sum_{k=0}^{[i/2]} \binom{n+k-1}{k}^2 \binom{n+i-2k-1}{n-1} - \sum_{k=0}^{[i/2]} \binom{n+k-1}{k} \binom{n+k}{n-1} \binom{n+i-2k-2}{n-1}$$
$$= \sum_{k=0}^{[i/2]} \binom{n+k-1}{k}^2 \binom{n+i-2k-2}{n-2} \frac{3k-i+1}{k+1}.$$

We used the Poincaré series $\mathcal{I}_2^{(n)}$ and $\mathcal{C}_2^{(n)}$ for $n > 1$. Using Maple-procedure for computing the Hilbert polynomials of the algebras $\mathcal{C}_2^{(1)}$ and $\mathcal{I}_2^{(1)}$ (see [5]), we get

$$\mathcal{H}(\mathcal{I}_2^{(1)}, i) = \frac{1}{2}\cos(\pi i) + \frac{1}{2} = \cos\left(\frac{\pi i}{2}\right),$$
$$\mathcal{H}(\mathcal{C}_2^{(1)}, i) = \frac{i}{2} + \frac{1}{4}\cos(\pi i) + \frac{3}{4} = \left[\frac{i}{2}\right] + 1.$$

This completes the proof of Theorem 2. □

Let us express $\mathcal{H}(\mathcal{I}_2^{(n)}, i)$ in terms of a polynomial.

Corollary 2.

$$(i) \quad \mathcal{H}(\mathcal{C}_2^{(n)}, i) = \frac{1}{(n-1)!} \sum_{k=0}^{\left[\frac{i}{2}\right]} \sum_{m=1}^{n-1} \sum_{j=1}^{m-1} \binom{n+k-1}{k}^2 \binom{m-1}{j} \left[{n-1 \atop m} \right] i^j (-2k)^{m-j-1},$$

$$(ii) \quad \mathcal{H}(\mathcal{I}_2^{(n)}, i) = \frac{1}{(n-1)!} \sum_{k=0}^{\left[\frac{i}{2}\right]} \sum_{m=1}^{n-2} \binom{n+k-1}{k} \binom{n+k-1}{k+1} \left[{n-2 \atop m} \right] (i-2k)^{m-1} (3k-i+1).$$

Proof. (i) By Theorem $2(i)$ we have:

$$\mathcal{H}(\mathcal{C}_2^{(n)}, i) = \sum_{k=0}^{\left[\frac{i}{2}\right]} \binom{n+k-1}{k}^2 \binom{n+i-2k-1}{i-2k}.$$

Let us express $\binom{n+i-2k-1}{i-2k}$ in terms of the unsigned Stirling numbers of the first kind:

$$\begin{aligned}\binom{n+i-2k-1}{i-2k} &= \frac{\sum_{m=1}^{n-1} \left[{n-1 \atop m}\right](i-2k)^{m-1}}{(n-1)!} \\ &= \frac{1}{(n-1)!} \sum_{m=1}^{n-1} \sum_{j=0}^{m-1} \left[{n-1 \atop m}\right] \binom{m-1}{j} (-2k)^{m-j-1} i^j.\end{aligned}$$

(ii) The proof of (ii) is completely analogous to that of (i). $\square$

Let us express the Hilbert polynomial of the algebras of joint covariants and invariants for n quadratic forms in terms of generalized hypergeometric function:

Corollary 3.

(i) $\mathcal{H}(\mathcal{C}_2^{(n)}, i) = \binom{n+i-1}{i} {}_4F_3 \left[\begin{matrix} n, n, -\frac{i}{2}, -\frac{i-1}{2}, -\frac{n}{3} \\ 1, -\frac{n+i-1}{2}, -\frac{n+i-2}{2} \end{matrix} \middle| 1 \right]$, *if* $n > 2$,

(ii) $\mathcal{H}(\mathcal{I}_2^{(n)}, i) = (1-n)\binom{n+i-2}{i} {}_5F_4 \left[\begin{matrix} n, n, -\frac{i}{2}, -\frac{i-1}{2}, -\frac{i}{3}+\frac{4}{3} \\ 1, -\frac{n+i-2}{2}, -\frac{n+i-3}{2}, -\frac{i}{3}+\frac{1}{3} \end{matrix} \middle| 1 \right]$, *if* $n > 3$ *and* $i > 1$.

Proof. (i) By the above

$$\mathcal{H}(\mathcal{C}_2^{(n)}, i) = \sum_{k=0}^{\left[\frac{i}{2}\right]} \binom{n+k-1}{k}^2 \binom{n+i-2k-1}{i-2k}.$$

Let us remark that $\binom{n+i-2k-1}{i-2k} = 0$ as $2k > i$. It means that:

$$\mathcal{H}(\mathcal{C}_2^{(n)}, i) = \sum_{k=0}^{\infty} \binom{n+k-1}{k}^2 \binom{n+i-2k-1}{i-2k}.$$

Let us express $\sum_{k=0}^{\infty} \binom{n+k-1}{k}^2 \binom{n+i-2k-1}{i-2k}$ in terms of a generalized hypergeometric function in a way analogous to that used in [10]. Let us denote

$$a_k = \binom{n+k-1}{k}^2 \binom{n+i-2k-1}{i-2k}.$$

We have

$$\begin{aligned} a_0 &= \binom{n+i-1}{i}, \\ \frac{a_{k+1}}{a_k} &= \frac{(k+n)^2 (k-\frac{i}{2})(k-\frac{i-1}{2})}{(k+1)^2 (k-\frac{n+i-1}{2})(k-\frac{n+i-2}{2})}.\end{aligned}$$

It now follows that

$$\mathcal{H}(\mathcal{C}_2^{(n)}, i) = \binom{n+i-1}{i} {}_4F_3\left[\begin{matrix} n, n, -\frac{i}{2}, -\frac{i-1}{2} \\ 1, -\frac{n+i-1}{2}, -\frac{n+i-2}{2} \end{matrix}\middle| 1\right].$$

(ii) The proof of (ii) is completely analogous to that of (i). $\square$

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Ми розглядаємо одну з фундаментальних проблем класичної теорії інваріантів — дослідження многочленів Гільберта алгебри інваріантів групи Лі SL_2. Форма многочленів Гільберта несе важливу інформацію про структуру цієї алгебри. Крім того коефіцієнти і степінь многочленів Гільберта відіграють важливу роль в алгебраїчній геометрії. Відомо, що починаючи з деякого i функція Гільберта алгебри SL_n-інваріантів є квазімногочленом. Формула Келлі-Сільвестра для обчислення значень функції Гільберта алгебри коваріантів бінарної d-форми $\mathcal{C}_d = \mathbb{C}[V_d \oplus \mathbb{C}^2]^{SL_2}$ (тут V_d — комплексний $d+1$-вимірний векторний простір бінарних форм степеня d) була запропонована ще Сільвестром і пізніше узагальнена на алгебри спільних інваріантів скінченої кількості бінарних форм. Проте ці формули не виражають функції Гільберта як многочлен від i.

В нашій статті ми розглядаємо задачу обчислення в явній формі многочленів Гільберта алгебр спільних інваріантів та спільних коваріантів n лінійних форм і n квадратичних форм. Ми виразили многочлени Гільберта цих алгебр $\mathcal{H}(\mathcal{I}_1^{(n)}, i) = \dim(\mathcal{C}_1^{(n)})_i$, $\mathcal{H}(\mathcal{C}_1^{(n)}, i) = \dim(\mathcal{C}_1^{(n)})_i$, $\mathcal{H}(\mathcal{I}_2^{(n)}, i) = \dim(\mathcal{I}_2^{(n)})_i$, $\mathcal{H}(\mathcal{C}_2^{(n)}, i) = \dim(\mathcal{C}_2^{(n)})_i$ у вигяді квазімногочленів від i, а також подали їх у термінах відомих комбінаторних структур, таких як число Нараяна та узагальнений гіпергеометричний ряд.

Ключові слова і фрази: класична теорія інваріантів, інваріанти, функція Гільберта, многочлени Гільберта, квазімногочлени, ряди Пуанкаре, комбінаторика.

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COUPLED FIXED POINT RESULTS ON METRIC SPACES DEFINED BY BINARY OPERATIONS

In parallel with the various generalizations of the Banach fixed point theorem in metric spaces, this theory is also transported to some different types of spaces including ultra metric spaces, fuzzy metric spaces, uniform spaces, partial metric spaces, b-metric spaces etc. In this context, first we define a binary normed operation on nonnegative real numbers and give some examples. Then we recall the concept of T-metric space and some important and fundamental properties of it. A T-metric space is a 3-tuple $(X, T, \diamond)$, where X is a nonempty set, $\diamond$ is a binary normed operation and T is a T-metric on X. Since the triangular inequality of T-metric depends on a binary operation, which includes the sum as a special case, a T-metric space is a real generalization of ordinary metric space. As main results, we present three coupled fixed point theorems for bivariate mappings satisfying some certain contractive inequalities on a complete T-metric space. It is easily seen that not only existence but also uniqueness of coupled fixed point guaranteed in these theorems. Also, we provide some suitable examples that illustrate our results.

Key words and phrases: binary normed operation, T-metric space, coupled fixed point.

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1 INTRODUCTION

It is well known that the Banach contraction principle is a fundamental result in metrical fixed point theory. After this classical result, many authors have extended, generalized and improved this theorem using different contractive conditions (see [1, 3, 4, 6]). On the other hand, fixed and common fixed point results in different types of spaces including ultra metric spaces, fuzzy metric space, uniform space, partial metric space, b-metric space etc, have been developed (see [2, 5, 8, 9, 12]). An interesting generalization of metric space named as T-metric space has been recently introduced by [11] (see also [10]). Briefly, the concept of T-metric space is based on the fact that the triangle inequality in the metric definition depends on a binary operation.

This study was organized as follows: first, we recall the definition of T-metric and some properties of it. Finally, we prove some coupled fixed point theorems for single valued mappings in complete T-metric spaces satisfying different contractive type condition.

Here we will emphasize the concept of ultra metric because of it will be mentioned in the next. Let (X, d) be a metric space. If the metric d satisfies strong triangle inequality:

$$d(x, y) \leq \max\{d(x, z), d(z, y)\} \ \forall x, y, z \in X,$$

then d is called an ultra metric on X and the pair (X, d) is called an ultra metric space. An ultra metric space (X, d) is said to be spherically complete if every shrinking collection of balls (that is, every nested decreasing sequence of balls) in X has a nonempty intersection.

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2 A BINARY NORMED OPERATION AND T-METRIC SPACES

In this section, we define a binary normed operation and give some examples.

A *binary normed* operation is a mapping $\diamond : [0,\infty) \times [0,\infty) \to [0,\infty)$ which satisfies the following conditions:

(i) $\diamond$ is associative and commutative,

(ii) $\diamond$ is continuous,

(iii) $a \diamond 0 = a$ for all $a \in [0,\infty)$,

(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for each $a,b,c,d \in [0,\infty)$.

Some typical examples of $\diamond$ are as follows: let $a,b \in [0,\infty)$

(a) $a \diamond_1 b = \max\{a,b\}$,

(b) $a \diamond_2 b = \sqrt{a^2+b^2}$,

(c) $a \diamond_3 b = a+b$,

(d) $a \diamond_4 b = ab+a+b$,

(e) $a \diamond_5 b = (\sqrt{a}+\sqrt{b})^2$.

Straightforward calculations lead to the following relations among normed binary operations given above

$$a \diamond_1 b \leq a \diamond_2 b \leq a \diamond_3 b \leq a \diamond_4 b$$

and

$$a \diamond_3 b \leq a \diamond_5 b.$$

The following lemma defines a normed binary operation exploiting some properties of a self map on $[0,\infty)$.

Lemma 1. *Let $f : [0,\infty) \to [0,\infty)$ be any continuous, increasing and onto mapping. Let $\diamond : [0,\infty) \times [0,\infty) \to [0,\infty)$ be defined by*

$$a \diamond b = f^{-1}(f(a)+f(b))$$

for $a,b \in [0,\infty)$. Then $\diamond$ is a normed binary operation.

Proof. It follows immediately. □

Example 1. *Let $f : [0,\infty) \to [0,\infty)$ defined by $f(x) = e^x - 1$. Obviously f is a continuous and increasing map. Therefore by Lemma 1, $a \diamond b = \ln(e^a+e^b-1)$ defines a normed binary operation.*

We have the following simple observations about a normed binary operation.

Lemma 2. *The following statements hold for any normed binary operation.*
i) If $r,r' \geq 0$, then $r \leq r \diamond r'$.
ii) For $\delta \in (0,r)$, there exists $\delta' \in (0,r)$ such that $\delta' \diamond \delta < r$.
iii) For all $\varepsilon > 0$, there exists $\delta > 0$ such that $\delta \diamond \delta < \varepsilon$.

Proof. i) Since $r' \geq 0$, using properties (iii) and (iv) of a normed binary operation $\diamond$, we have $r \diamond r' \geq r \diamond 0 = r$.

ii) If we assume that every $\delta' > 0$ gives $\delta' \diamond \delta \geq r$. In particular, if we set $\delta' = \frac{1}{n}$, we get $\frac{1}{n} \diamond \delta \geq r$ which on taking the limit as $n \to \infty$ implies that $0 \diamond \delta \geq r$ which is a contradiction. Hence, by part (i) of this lemma we obtain $\delta' \leq \delta' \diamond \delta < r$.

iii) Assume the contrary, i.e., for all $\delta > 0$, $\delta \diamond \delta \geq \varepsilon$. For $\delta = \frac{1}{n}$ we have $\frac{1}{n} \diamond \frac{1}{n} \geq \varepsilon$ which on taking the limit as $n \to \infty$ gives $0 \geq \varepsilon$, which is a contradiction. Hence iii) follows. □

Now, we recall the concept of T-metric.

Definition 1 ([10]). *Let X be a nonempty set. A T-metric on X is a function $T : X^2 \to \mathbb{R}$ that satisfies the following conditions, for each $x, y, z \in X$,*

1. *$T(x, y) \geq 0$ and $T(x, y) = 0$ if and only if $x = y$,*
2. *$T(x, y) = T(y, x)$,*
3. *$T(x, y) \leq T(x, z) \diamond T(y, z)$.*

The 3-tuple $(X, T, \diamond)$ is called a T-metric space.

Example 2 ([11]). *i) Every ordinary metric d is a T-metric with $a \diamond b = a + b$.*

ii) Every ultra metric d is a T-metric with $a \diamond b = \max\{a, b\}$.

iii) Let $X = \mathbb{R}$ and $T(x, y) = \sqrt{|x - y|}$ for all $x, y \in \mathbb{R}$. If we take $a \diamond b = \sqrt{a^2 + b^2}$, then we have

$$T(x, y) = \sqrt{|x - y|} \leq \sqrt{|x - z| + |z - y|} = \sqrt{\sqrt{|x - z|^2} + \sqrt{|z - y|^2}} = T(x, z) \diamond T(z, y).$$

Therefore, the function T is a T-metric on X.

iv) Let $X = \mathbb{R}$ and $T(x, y) = (x - y)^2$ for every $x, y \in \mathbb{R}$. If we take $a \diamond b = (\sqrt{a} + \sqrt{b})^2$, then we get

$$T(x, y) = (x - y)^2 = |x - y|^2 \leq (|x - z| + |z - y|)^2 = \left(\sqrt{|x - z|^2} + \sqrt{|z - y|^2}\right)^2 = T(x, z) \diamond T(z, y).$$

Hence, the function T is a T-metric on X.

Remark 1 ([11]). *For a fixed $0 \leq \alpha \leq \frac{\pi}{4}$, if there exist β, γ such that $0 \leq \alpha \leq \beta + \gamma < \frac{\pi}{2}$, then*

$$\tan \alpha \leq \tan \beta + \tan \gamma + \tan \beta \tan \gamma.$$

Example 3 ([11]). *Let $X = [0, 1]$ and $T(x, y) = \tan(\frac{\pi}{4}|x - y|)$ for every $x, y \in X$. If we take $a \diamond b = a + b + ab$, then by Remark 1 we obtain*

$$\begin{aligned} T(x, y) &= \tan(\frac{\pi}{4}|x - y|) \\ &\leq \tan(\frac{\pi}{4}|x - z|) + \tan(\frac{\pi}{4}|z - y|) + \tan(\frac{\pi}{4}|x - z|) \tan(\frac{\pi}{4}|z - y|) \\ &= T(x, z) \diamond T(z, y). \end{aligned}$$

So, the function T is a T-metric on X.

Let $(X, T, \diamond)$ be a T-metric space. For $r > 0$ define

$$B_T(x,r) = \{y \in X : T(x,y) < r\}.$$

Definition 2 ([11]). *Let $(X, T, \diamond)$ be a T-metric space $r > 0$ and $A \subset X$.*

1. *The set $B_T(x,r)$ is called the open ball of a center x and a radius r.*
2. *If for all $x \in A$ there exists $r > 0$ such that $B_T(x,r) \subset A$, then the subset A is called an open subset of X.*
3. *The subset A of X is said to be T-bounded if there exists $r > 0$ such that $T(x,y) < r$ for all $x, y \in A$.*
4. *A sequence $\{x_n\}$ in X converges to x if $T(x_n, x) \to 0$ as $n \to \infty$ and we write $\lim_{n\to\infty} x_n = x$. That is for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $T(x_n, x) < \varepsilon$ for all $n \geq n_0$.*
5. *A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $T(x_n, x_m) < \varepsilon$ for all $n, m \geq n_0$.*
6. *The T-metric space $(X, T, \diamond)$ is said to be complete if every Cauchy sequence is convergent.*

Let τ be the set of all open subsets of X, then τ is a topology on X (induced by the T-metric T). Note that if A and B are open subsets of X and $x \in A \cap B$, then there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $B_T(x, \varepsilon_1) \subset A$ and $B_T(x, \varepsilon_2) \subset B$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\} > 0$, then by Lemma 2 (iii), there exists $\delta > 0$ such that $\delta \diamond \delta < \varepsilon$. In this case, we have $B_T(x, \delta \diamond \delta) \subset B_T(x, \varepsilon_1) \cap B_T(x, \varepsilon_2) \subset A \cap B$, hence $A \cap B$ is open.

Lemma 3 ([11]). *Let $(X, T, \diamond)$ be a T-metric space. If $r > 0$, then the open ball $B_T(x,r)$ with a center $x \in X$ and a radius r is an open set.*

Lemma 4 ([11]). *Let $(X, T, \diamond)$ be a T-metric space. If a sequence $\{x_n\}$ in X converges to x, then x is unique.*

Lemma 5 ([11]). *Let $(X, T, \diamond)$ be a T-metric space. Then every convergent sequence $\{x_n\}$ in X is a Cauchy sequence.*

Definition 3 ([11]). *Let $(X, T, \diamond)$ be a T-metric space. T is said to be continuous if*

$$\lim_{n\to\infty} T(x_n, y_n) = T(x,y),$$

whenever

$$\lim_{n\to\infty} T(x_n, x) = \lim_{n\to\infty} T(y_n, y) = 0.$$

Lemma 6. *Let $(X, T, \diamond)$ be a T-metric space. Then T is a continuous function.*

Proof. Assume that $\lim_{n\to\infty} T(x_n, x) = \lim_{n\to\infty} T(y_n, y) = 0$. By the triangular inequality we have

$$T(x_n, y_n) \leq T(x_n, x) \diamond T(x,y) \diamond T(y, y_n).$$

Hence we get

$$\limsup_{n\to\infty} T(x_n, y_n) \leq T(x,y).$$

Similarly, we obtain

$$T(x,y) \leq T(x,x_n) \diamond T(x_n,y_n) \diamond T(y_n,y)$$

and so

$$T(x,y) \leq \lim_{n\to\infty} \inf T(x_n,y_n).$$

Therefore

$$\lim_{n\to\infty} T(x_n,y_n) = T(x,y).$$

□

Henceforth, we assume that $\diamond$ is a binary operation on $[0,\infty) \times [0,\infty)$ such that
i) $\alpha(a \diamond b) = \alpha a \diamond \alpha b$ for every $\alpha \in \mathbb{R}^+$,
ii) there exists $h \geq 0$ such that $\underbrace{1 \diamond 1 \diamond \cdots \diamond 1}_{n} \leq n^h$.

Example 4. *Let* $a \diamond b = \max\{a,b\}$, $a \diamond b = \sqrt{a^2+b^2}$, $a \diamond b = a+b$ *and* $a \diamond b = (\sqrt{a}+\sqrt{b})^2$. *We take* $h \geq 0, h \geq \frac{1}{2}, h \geq 1$ *and* $h \geq 2$ *respectively in (ii). But if* $a \diamond b = a+b+ab$, *then is not necessary that* $\diamond$ *satisfies the above conditions.*

3 COUPLED FIXED POINT THEOREMS IN T-METRIC SPACES

Now, we remember the concept of a coupled fixed point on a T-metric space.

Definition 4 ([7]). *Let* X *be a nonempty set and* $F : X \times X \to X$ *be a function. An element* $(x,y) \in X \times X$ *is said to be a coupled fixed point of the map* F *if* $F(x,y) = x$ *and* $F(y,x) = y$.

Example 5. *Let* $X = \mathbb{R}$. *Define a map* F *on* $X \times X$ *by* $F(x,y) = xy^2$. *It is easy to see that* $(1,-1) \in X \times X$ *is a coupled fixed point of the mapping* F.

Theorem 1. *Let* $(X,T,\diamond)$ *be a complete* T*-metric space. Suppose that the map* $F : X \times X \to X$ *satisfies the following contractive condition for all* $x,y,u,v \in X$

$$T(F(x,y),F(u,v)) \leq kT(x,u) \diamond lT(y,v), \tag{1}$$

where k,l *are nonnegative constants with* $k \diamond l < 1$. *Then* F *has a unique coupled fixed point.*

Proof. Choose $x_0, y_0 \in X$ and set $x_1 = F(x_0,y_0)$ and $y_1 = F(y_0,x_0)$. We can define sequences $\{x_n\}$ and $\{y_n\}$ by $x_{n+1} = F(x_n,y_n)$ and $y_{n+1} = F(y_n,x_n)$. By (1) we have

$$\begin{aligned} T(x_n,x_{n+1}) &= T(F(x_{n-1},y_{n-1}),F(x_n,y_n)) \\ &\leq kT(x_{n-1},x_n) \diamond lT(y_{n-1},y_n). \end{aligned} \tag{2}$$

Similarly

$$\begin{aligned} T(y_n,y_{n+1}) &= T(F(y_{n-1},x_{n-1}),F(y_n,x_n)) \\ &\leq kT(y_{n-1},y_n) \diamond lT(x_{n-1},x_n). \end{aligned} \tag{3}$$

Letting

$$d_n = T(x_n,x_{n+1}) \diamond T(y_n,y_{n+1}), \tag{4}$$

we get

$$\begin{aligned} d_n &= T(x_n, x_{n+1}) \diamond T(y_n, y_{n+1}) \\ &\leq kT(x_{n-1}, x_n) \diamond lT(y_{n-1}, y_n) \diamond kT(y_{n-1}, y_n) \diamond lT(x_{n-1}, x_n) \\ &= (k \diamond l)[T(x_{n-1}, x_n) \diamond T(y_{n-1}, y_n)] \\ &= (k \diamond l) d_{n-1}. \end{aligned} \tag{5}$$

Consequently, if we set $\delta = k \diamond l$, then for each $n \in \mathbb{N}$ we obtain

$$d_n \leq \delta d_{n-1} \leq \delta^2 d_{n-2} \leq \cdots \leq \delta^n d_0. \tag{6}$$

If $d_0 = 0$ then $T(x_0, x_1) \diamond T(y_0, y_1) = 0$. Hence, we get $x_0 = x_1 = F(x_0, y_0)$ and $y_0 = y_1 = F(y_0, x_0)$, i.e., (x_0, y_0) is a coupled fixed point of F. Now suppose that $d_0 > 0$. For each $n > m$ we have

$$T(x_n, x_m) \leq T(x_n, x_{n-1}) \diamond T(x_{n-1}, x_{n-2}) \diamond \cdots \diamond T(x_{m+1}, x_m).$$

In the same manner, we get

$$T(y_n, y_m) \leq T(y_n, y_{n-1}) \diamond T(y_{n-1}, y_{n-2}) \diamond \cdots \diamond T(y_{m+1}, y_m).$$

Thus

$$\begin{aligned} T(x_n, x_m) &\leq T(x_n, x_m) \diamond T(y_n, y_m) \\ &\leq d_{n-1} \diamond d_{n-2} \diamond \cdots \diamond d_m \\ &\leq (\delta^{n-1} \diamond \delta^{n-2} \diamond \cdots \diamond \delta^m) d_0 \\ &\leq \delta^m d_0 \underbrace{(1 \diamond \cdots \diamond 1)}_{n-m} \\ &\leq \delta^m d_0 \underbrace{(1 \diamond \cdots \diamond 1)}_{n} \\ &\leq \delta^m d_0 n^h \longrightarrow 0. \end{aligned}$$

Hence for $\varepsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that for all $n > m \geq n_0$ we get $T(x_n, x_m) < \varepsilon$. Similarly, we can get $T(y_n, y_m) < \varepsilon$. It follows that $\{x_n\}$ and $\{y_n\}$ are Cauchy and by the completeness of X, $\{x_n\}$ and $\{y_n\}$ converge to u^* and v^* in X respectively. Thus

$$\lim_{n\to\infty} T(x_n, u^*) = \lim_{n\to\infty} T(y_n, v^*) = 0. \tag{7}$$

Using the triangular inequality and (1) we get

$$\begin{aligned} T(F(u^*, v^*), u^*) &\leq T(F(u^*, v^*), x_{n+1}) \diamond T(x_{n+1}, u^*) \\ &= T(F(u^*, v^*), F(x_n, y_n)) \diamond T(x_{n+1}, u^*) \\ &\leq kT(x_n, u^*) \diamond lT(y_n, v^*) \diamond T(x_{n+1}, u^*). \end{aligned}$$

Letting $n \to \infty$, then from (7), we obtain $T(F(u^*, v^*), u^*)) = 0$ and so $F(u^*, v^*) = u^*$. In the same maner, we have $F(v^*, u^*) = v^*$; i.e., (u^*, v^*) is a coupled fixed point of F. Now, if (u', v') is another coupled fixed point of F we get

$$T(u', u^*) = T(F(u', v'), F(u^*, v^*)) \leq kT(u', u^*) \diamond lT(v', v^*)$$

and

$$T(v', v^*) = T(F(v', u'), F(v^*, u^*)) \leq kT(v', v^*) \diamond lT(u', u^*).$$

Then

$$T(u', u^*) \diamond T(v', v^*) \leq (k \diamond l)[T(u', u^*) \diamond T(v', v^*)].$$

As $k \diamond l < 1$, we have $T(u', u^*) \diamond T(v', v^*) = 0$ and so $u' = u^*$ and $v' = v^*$. The proof of Theorem 1 is completed. $\square$

Example 6. *Let $X = \mathbb{R}$ and $T(x,y) = \sqrt{|x-y|}$ for all $x,y \in \mathbb{R}$. If we take $a \diamond b = \sqrt{a^2+b^2}$, then the function T is a T-metric on X. Let $F(x,y) = \dfrac{x+2y}{5} - 1$ for all $x,y \in X$. For all $x,y,u,v \in X$, we obtain*

$$\begin{aligned} T(F(x,y),F(u,v)) &= \sqrt{\frac{|(x-u)+2(y-v)|}{5}} \\ &\leq \frac{1}{\sqrt{5}}(\sqrt{|x-u|+2|y-v|}) \\ &= \frac{1}{\sqrt{5}}T(x,u) \diamond \frac{\sqrt{2}}{\sqrt{5}}T(y,v). \end{aligned}$$

Hence for $k = \dfrac{1}{\sqrt{5}}$ and $l = \dfrac{\sqrt{2}}{\sqrt{5}}$, we get $k \diamond l < 1$. It follows that all conditions of Theorem 1 hold, and $(-\frac{5}{2}, -\frac{5}{2}) \in X \times X$ is the unique coupled fixed point of the mapping F.

Example 7. *Let $X = \mathbb{R}$ and $T(x,y) = (x-y)^2$ for all $x,y \in \mathbb{R}$. If we take $a \diamond b = (\sqrt{a}+\sqrt{b})^2$, then the function T is a T-metric on X. Let $F(x,y) = \dfrac{x+2y}{5} - 1$ for all $x,y \in X$. For all $x,y,u,v \in X$, we obtain*

$$\begin{aligned} T(F(x,y),F(u,v)) &= \Big(\frac{x-u}{5} + 2\frac{y-v}{5}\Big)^2 \\ &\leq 2\Big(\frac{x-u}{5}\Big)^2 + 4\Big(\frac{y-v}{5}\Big)^2 \\ &= \frac{2}{25}(|x-u|)^2 + \frac{8}{25}(|y-v|)^2 \\ &\leq \Big(\frac{\sqrt{2}}{5}|x-u| + \frac{2\sqrt{2}}{5}|y-v|\Big)^2 \\ &= \frac{2}{25}T(x,u) \diamond \frac{8}{25}T(y,v). \end{aligned}$$

Hence for $k = \dfrac{2}{25}$ and $l = \dfrac{8}{25}$, we get $k \diamond l = \frac{18}{25} < 1$. It follows that the all conditions of Theorem 1 hold, and $\Big(-\frac{5}{2}, -\frac{5}{2}\Big) \in X \times X$ is the unique coupled fixed point of the mapping F.

Theorem 2. *Let $(X, T, \diamond)$ be a complete T-metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x,y,u,v \in X$*

$$T(F(x,y),F(u,v)) \leq kT(F(x,y),x) \diamond lT(F(u,v),u), \tag{8}$$

where k, l are nonnegative constants with $k \diamond l < 1$. Then F has a unique coupled fixed point.

Proof. Choose $x_0, y_0 \in X$ and set $x_1 = F(x_0,y_0)$ and $y_1 = F(y_0,x_0)$. We can define sequences $\{x_n\}$ and $\{y_n\}$ by $x_{n+1} = F(x_n,y_n)$ and $y_{n+1} = F(y_n,x_n)$. By (8), we have

$$\begin{aligned} T(x_n,x_{n+1}) &= T(F(x_{n-1},y_{n-1}),F(x_n,y_n)) \\ &\leq kT(F(x_{n-1},y_{n-1}),x_{n-1}) \diamond lT(F(x_n,y_n),x_n) \\ &= kT(x_n,x_{n-1}) \diamond lT(x_{n+1},x_n). \end{aligned}$$

If $T(x_{n+1}, x_n) \geq T(x_n, x_{n-1})$ then

$$\begin{aligned} T(x_n, x_{n+1}) &\leq kT(x_n, x_{n-1}) \diamond lT(x_{n+1}, x_n) \\ &\leq (k \diamond l)T(x_{n+1}, x_n) \\ &< T(x_{n+1}, x_n), \end{aligned}$$

which is a contradiction. Hence

$$T(x_n, x_{n+1}) \leq (k \diamond l)T(x_{n-1}, x_n) = \delta T(x_{n-1}, x_n).$$

Similarly

$$T(y_n, y_{n+1}) \leq (k \diamond l)T(y_{n-1}, y_n) = \delta T(y_{n-1}, y_n).$$

So, if $m > n$

$$\begin{aligned} T(x_n, x_m) &\leq T(x_n, x_{n+1}) \diamond T(x_{n+1}, x_{n+2}) \diamond \cdots \diamond T(x_{m-1}, x_m) \\ &\leq \delta^n T(x_0, x_1) \diamond \delta^{n+1} T(x_0, x_1) \diamond \cdots \diamond \delta^{m-1} T(x_0, x_1) \\ &= \delta^n T(x_0, x_1)(1 \diamond \delta \diamond \delta^2 \diamond \cdots \diamond \delta^{m-n-1}) \\ &\leq \delta^n T(x_0, x_1)(\underbrace{1 \diamond 1 \diamond 1 \diamond \cdots \diamond 1}_{m-n}) \\ &\leq \delta^n T(x_0, x_1)(\underbrace{1 \diamond 1 \diamond 1 \diamond \cdots \diamond 1}_{m}) \\ &\leq \delta^n T(x_0, x_1) m^h. \end{aligned}$$

It is easy to see that for all $m > n$ there exists $s > 0$ such that $m \leq n^s$. Thus

$$T(x_n, x_m) \leq \delta^n T(x_0, x_1) n^{hs} \to 0.$$

Hence for $\varepsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that for all $m > n \geq n_0$ we get $T(x_n, x_m) < \varepsilon$. Similarly, we can get $T(y_n, y_m) < \varepsilon$. It follows that $\{x_n\}$ and $\{y_n\}$ are Cauchy and by the completeness of X, $\{x_n\}$ and $\{y_n\}$ converge to u^* and v^* in X respectively. Thus

$$\lim_{n\to\infty} T(x_n, u^*) = \lim_{n\to\infty} T(y_n, v^*) = 0. \tag{9}$$

Applying the triangular inequality and (8) we get

$$\begin{aligned} T(F(u^*, v^*), u^*) &\leq T(F(u^*, v^*), x_{n+1}) \diamond T(x_{n+1}, u^*) \\ &= T(F(u^*, v^*), F(x_n, y_n) \diamond T(x_{n+1}, u^*) \\ &\leq kT(F(u^*, v^*), u^*) \diamond lT(F(x_n, y_n), x_n) \diamond T(x_{n+1}, u^*). \end{aligned}$$

Letting $n \to \infty$ and from (9) we obtain $T(F(u^*, v^*), u^*)) \leq kT(F(u^*, v^*), u^*))$ which implies that $T(F(u^*, v^*), u^*)) = 0$ and so $F(u^*, v^*) = u^*$. In the similar manner, we have $F(v^*, u^*) = v^*$, i.e; (u^*, v^*) is a coupled fixed point of F. Now, if (u', v') is another coupled fixed point of F, then

$$\begin{aligned} T(u', u^*) &= T(F(u', v'), F(u^*, v^*)) \\ &\leq kT(F(u', v'), u') \diamond lT(F(u^*, v^*), u^*) \\ &= kT(u', u') \diamond lT(u^*, u^*) = 0. \end{aligned}$$

This implies that $T(u', u^*) = 0$ and so $u' = u^*$. Siimilarly $v' = v^*$. The proof of Theorem 2 is completed. ☐

Theorem 3. *Let $(X, T, \diamond)$ be a complete T-metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$*

$$T(F(x,y), F(u,v)) \leq kT(F(x,y), u) \diamond lT(F(u,v), x), \tag{10}$$

where k, l are nonnegative constants with $k \diamond l \diamond l < 1$. Then F has a unique coupled fixed point.

Proof. Choose $x_0, y_0 \in X$ and set $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. We can define sequences $\{x_n\}$ and $\{y_n\}$ by $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$. By (10), we have

$$\begin{aligned} T(x_n, x_{n+1}) &= T(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\leq kT(F(x_{n-1}, y_{n-1}), x_n) \diamond lT(F(x_n, y_n), x_{n-1}) \\ &= kT(x_n, x_n) \diamond lT(x_{n+1}, x_{n-1}) \\ &= lT(x_{n+1}, x_{n-1}) \\ &\leq lT(x_{n+1}, x_n) \diamond T(x_n, x_{n-1}). \end{aligned}$$

If $T(x_{n+1}, x_n) \geq T(x_n, x_{n-1})$ then

$$\begin{aligned} T(x_n, x_{n+1}) &\leq lT(x_{n+1}, x_n) \diamond lT(x_{n+1}, x_n) \\ &\leq (l \diamond l)T(x_{n+1}, x_n) \\ &\leq (k \diamond l \diamond l)T(x_{n+1}, x_n) \\ &< T(x_{n+1}, x_n). \end{aligned}$$

which is contradiction. Hence

$$T(x_n, x_{n+1}) \leq (l \diamond l)T(x_{n-1}, x_n) = \delta T(x_{n-1}, x_n),$$

Similarly

$$T(y_n, y_{n+1}) \leq (l \diamond l)T(y_{n-1}, y_n) = \delta T(y_{n-1}, y_n),$$

where $\delta = l \diamond l \leq k \diamond l \diamond l < 1$. So, if $m > n$,

$$\begin{aligned} T(x_n, x_m) &\leq T(x_n, x_{n+1}) \diamond T(x_{n+1}, x_{n+2}) \diamond \cdots \diamond T(x_{m-1}, x_m) \\ &\leq \delta^n T(x_0, x_1) \diamond \delta^{n+1} T(x_0, x_1) \diamond \cdots \diamond \delta^{m-1} T(x_0, x_1) \\ &= \delta^n T(x_0, x_1)(1 \diamond \delta \diamond \delta^2 \diamond \cdots \diamond \delta^{m-n-1}) \\ &\leq \delta^n T(x_0, x_1)(\underbrace{1 \diamond 1 \diamond 1 \diamond \cdots \diamond 1}_{m-n}) \\ &\leq \delta^n T(x_0, x_1)(\underbrace{1 \diamond 1 \diamond 1 \diamond \cdots \diamond 1}_{m}) \\ &\leq \delta^n T(x_0, x_1) m^h. \end{aligned}$$

It is easy to see that for all $m > n$ there exists $s > 0$ such that $m \leq n^s$. Thus

$$T(x_n, x_m) \leq \delta^n T(x_0, x_1) n^{hs} \to 0.$$

Hence for $\varepsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that for all $m > n \geq n_0$ we get $T(x_n, x_m) < \varepsilon$. Similarly, we can get $T(y_n, y_m) < \varepsilon$. It follows that $\{x_n\}$ and $\{y_n\}$ are Cauchy and by the completeness of X, $\{x_n\}$ and $\{y_n\}$ converge to u^* and v^* in X respectively. Thus

$$\lim_{n \to \infty} T(x_n, u^*) = \lim_{n \to \infty} T(y_n, v^*) = 0. \tag{11}$$

Using the triangular inequality and (10) we get

$$\begin{aligned} T(F(u^*,v^*),u^*) &\leq T(F(u^*,v^*),x_{n+1}) \diamond T(x_{n+1},u^*) \\ &= T(F(u^*,v^*),F(x_n,y_n) \diamond T(x_{n+1},u^*) \\ &\leq kT(F(u^*,v^*),x_n) \diamond lT(F(x_n,y_n),u^*) \diamond T(x_{n+1},u^*). \end{aligned}$$

Letting $n \to \infty$, then from (11) we obtain $T(F(u^*,v^*),u^*)) \leq kT(F(u^*,v^*),u^*))$. This implies that $T(F(u^*,v^*),u^*)) = 0$ and so $F(u^*,v^*) = u^*$. In the similar manner, we have $F(v^*,u^*) = v^*$; i.e., (u^*,v^*) is a coupled fixed point of F. Now, if (u',v') is another coupled fixed point of F, then

$$\begin{aligned} T(u',u^*) &= T(F(u',v'),F(u^*,v^*)) \\ &\leq kT(F(u',v'),u^*) \diamond lT(F(u^*,v^*),u') \\ &= kT(u',u^*) \diamond lT(u^*,u') \\ &= (k \diamond l)T(u',u^*) \\ &\leq (k \diamond l \diamond l)T(u',u^*) \\ &< T(u',u^*). \end{aligned}$$

This implies that $T(u',u^*) = 0$ and so $u' = u^*$. Similarly $v' = v^*$. The proof of Theorem 3 is completed. □

If we set $a \diamond b = a + b$ and $T(x,y) = d(x,y)$ in Theorem 1 we have

Corollary 1. *Let (X,d) be a complete metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x,y,u,v \in X$*

$$d(F(x,y),F(u,v)) \leq kd(x,u) + ld(y,v),$$

where k, l are nonnegative constants with $k + l < 1$. Then F has a unique coupled fixed point.

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Карамі А., Шакері Р., Седхі С., Алтун І. *Результати про пари нерухомих точок на метричних просторах, визначених бінарними операціями* // Карпатські матем. публ. — 2018. — Т.10, №2. — С. 313–323.

Паралельно до різних узагальнень теореми Банаха про нерухому точку в метричних просторах, ця теорія є застосовною до різних типів просторів, зокрема, таких як ультраметричні простори, нечіткі метричні простори, рівномірні простори, частково метричні простори, b-метричні простори та ін. У цьому контексті спочатку ми визначаємо бінарну нормовану операцію на невід'ємних дійсних числах і даємо кілька прикладів. Тоді ми згадуємо поняття T-метричного простору та його важливі і фундаментальні властивості. T-метричний простір — це набір $(X, T, \diamond)$, де X є непорожньою множиною, $\diamond$ — бінарною нормованою операцією і T є деякою T-метрикою на X. Оскільки нерівність трикутника для T-метрики залежить від бінарної операції, для якої частковим випадком є сума, T-метричний простір є справжнім узагальненням звичайного метричного простору. Головними результатами, які ми представляємо, є три теореми для пар нерухомих точок для двохвимірних відображень, що задовольняють деякі нерівності стиску в повних T-метричних просторах. Легко бачити, що не тільки існування, але і єдиність пари нерухомих точок гарантується цими теоремами. Також ми представляємо деякі придатні приклади, що ілюструють наші результати.

Ключові слова і фрази: бінарна нормована операція, T-метричний простір, пара нерухомих точок.

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APPLICATION OF DUALITY THEORY TO SOLVE TWO-CRITERIA PROBLEM OF LINEAR PROGRAMMING FOR ECOLOGICAL-ECONOMIC SYSTEM

In the paper, we investigate two-criterion optimization problem: maximization of one target function and minimization of another target function. To solve the offered two-criterion problem, the method of the main criterion is applied. We consider the problem of production activity of the ecological-economic system with the maximization of the value of the final product as the first target function and the minimization of emissions of polluters into the environment as the second target function. We constructed of two production functions (economic and ecological). To construct the economic production function, we select maximal producing of the final products in a costing form as the most essential (main) criterion. Also, there is introduced the appropriate data of the criterion level total volume of emissions of polluters into the environment. After this two-criteria problem is reduced to one - criteria problem. For the construction of ecological production function, the main criterion in the problem of the minimal general volume of emissions of polluters into the environment is defined. We use a parameter of the criterion level of the second criterion and obtained one-criterion problem. Therefore, investigation of the appropriate dual problems explicitly provides economic and ecological production functions to the deduced one-criterion problems. These functions in input two-criterion problem give way to optimal manage of ecological-economic system.

Key words and phrases: Optimal management, two-criterion problem, dual problem, target function, main criterion method, ecological-economic equilibrium, production function.

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INTRODUCTION

Public production envisages not only a creation of material welfares but also activity connected with a decrease of environmental pollution and restoring of natural resources. High level of competitiveness in West European industries as well as a new policy of sustainable development causes a rethinking of management strategy of individual countries.

One of basic tasks, which a modern specialist-economist must be able to solve, is providing of the state, when economic and ecological requirements are balanced. That means providing equilibrium state of ecological economy. The ecological economy is a market economy that studies interaction between ecosystems, social associations and economic systems and also conditions that provide a steady, inexhausting state and progressive development of all three systems. The main task of the ecological economy is forming of fundamentally new directions of transformation of economy based on permanent recreation of innovative processes of reformation of production and consumption of products. The final goal of ecological economy is the sequential passing to ecologically more perfect production technologies, types of

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products, processes of consumption, economic relations, lifestyle [11]. At consideration of ecological economy activity of production can be presented as an ecological-economic production function in which, next to economic values ecological factors also have their market estimation. There is a necessity to build such a production function that has two constituents: economic and ecological.

ANALYSIS OF THE LAST RESEARCHES AND PUBLICATIONS

A modeling of the ecological-economic systems is the topic of many researches. The ecological-economic models by a structure can be divided into balance, optimization and simulation models. The balance models include the inter-branch Leontief-Ford model of the economy structure impact on the environment [6], and also its generalizations such as the Ayres-Kneese model [1] and the Willen model [23]. At the fact, the Willen model is a matrix balance in the Ayres-Kneese model. Among the simulation models we should mention the Meadows model [10] and the Forrester model [3]. J.W. Forrester attempted to analyze the interactions of demographic, industrial and agrarian systems. In the optimization ecological-economic models, the formation of the optimality criterion is fundamental. Particularly, I. Schimazu [21] proposed optimality criterion, which corresponds to three stages of society development. H. E. Dali [2] and T. H. Tietenberg [22] also considered ecological component in their models. Other scientists [12, 18–20] investigated regulation of the ecological consequences of economic growth and improvements for environmental management.

Among the Ukrainian scientists we have to mark works of I.Lyashenko. Exactly his school works actively on the problems of ecological-economic modeling. In spite of the large volume of publications, plenty of questions and problems for today are not studied enough, and the problem of the balanced economic, social and ecological development remains unsolved.

Now the problems of ecological-economic equilibrium are actively studied.

In [24, p. 446–455] the problem of manufactures ecologization is analyzed and the expansion of the classical model of interbranch balance and its transformation into the optimization model by inclusion of restrictions on emissions is carried out. This model is completed by marginal variables and by corresponding to them coefficients.

In [14, p. 931–938] the research of optimal trajectories of development of the ecological-economic system is carried out in case of equable division of labour resources between branches of material and nature protection productions.

In [15, p. 217–221], [16, p. 31] the technological structure of production in the ecological-economic system with taking into account introduction of technological innovations was researched. In particular, on the basis of distribution of production capacities according to technologies a corresponding equalization of dynamics, based on the set initial conditions and limits on economic and ecological resources, was built. The modelling of ecological-economic interaction in the process of realization of Kyoto protocol decisions which was made is very important [13].

In [17, p. 331–333] the parametrization of mathematical models of the ecological - economic systems is carried out in space of indexes of economic structure of society, prices and environmental pollution.

In [5, p. 170] the conditions for optimal interaction between basic and auxiliary productions in the ecological-economic systems have been investigated. The are the basis for prediction the of the investment level into auxiliary production, which provides its possible growth.

In [4, p. 184–185] the dynamic models of single sectorial economy taking into account the utilization processes of the created pollution and socio-economic structurization were developed.

In [8, p. 149–150] a comparative analysis of the most famous models of ecological-economic systems is presented. The approach to modelling of economic systems considering environmental factors, based on a modified model of Leontief-Ford and the principle of input-output is proposed.

The research of ecological-economic equilibrium is necessary nowadays, when the environmental pollution gained the global character. Using of ecological-economic function is very important in the researches as such function will reflect modelling of ecological-economic system.

The aim of our researches is a construction of optimal functioning model for ecological-economic system to find a solution of suitable two-criterion problem. The problem has solution as two ecological-production functions. We simultaneously consider the maximization of value of the final product and the minimization of emissions of polluters. Since it is difficult to create an effective production activity that takes into account not only economic benefits but also an environmental impact, we want to propose an ecological-economic model combining simultaneously these two factors. It is a necessary way of the problem solving.

THE MAIN RESULTS OF RESEARCHES

Let us consider at the ecological-economic system that includes a basic (material) production and auxiliary production (sewage treatment plants). The volumes of productions are limited to the present resources. The central core of the ecological-economic system is an inter-branch model of V. Leontief and D. Ford [7, p. 21]. It represents cooperation of industries of producing of products and industries that destroy harmful wastes. The efficiency of production activity is described by two criteria (economic and ecological). The first criterion is maximal producing of the final products in a cost form. The second criterion is minimal general volume of emissions of polluters into the environment.

We suggest to describe production activity of the ecological-economic system as following two-criteria problem of linear programming:

$$\begin{aligned} & f_1(x) = c^1 y^1 \to \max,\ f_2(x) = c^2 y^2 \to \min, \\ & x^1 = A_{11}x^1 + A_{12}x^2 + y^1,\ x^2 = A_{21}x^1 + A_{22}x^2 - y^2, \\ & B_1 x^1 + B_2 x^2 \le R,\ x^1 \ge 0,\ x^2 \ge 0,\ y^1 \ge 0,\ y^2 \ge 0, \end{aligned} \tag{1}$$

where c^1 is a vector of products' prices, c^2 is a vector of equivalence's coefficients of polluters (in relation to harm or in relation to the cost of destruction), x^1 is a vector of the gross producing of products, y^1 is a vector of producing of final products, x^2 is a vector of volumes of the destroyed polluters, y^2 is a vector of volumes of emissions of polluters in the environment, A_{11} and A_{12} are technological matrices of direct issue of products, A_{21} and A_{22} are technological matrices of the direct producing of pollutants (in a basic production and sewage treatment plants accordingly), B_1 and B_2 are matrices of costs of economic resources for the basic and the auxiliary productions, R is a vector of present economic resources.

The necessary and sufficient condition of non-negativity of solutions of the Leontief-Ford model at the productivity of block matrix $A = \begin{pmatrix} A_{11} \; A_{12} \\ A_{21} \; A_{22} \end{pmatrix} \geq 0$ and at the $y^1 > 0$, $y^2 \geq 0$ is the condition $x^2 = (E_2 - A_2)^{-1} \left(A_{21} (E_2 - A_2)^{-1} y^1 - y^2 \right) \geq 0$, where $A_2 = A_{22} + A_{21} (E_1 - A_{11})^{-1} A_{12}$, and E_1,E_2 are diagonal identity matrices [7, p. 27]. Besides from Leontief-Ford inter-branch balance we can get the next $x^1 = (I_1 - A_{11})^{-1} \cdot (A_{12} x^2 + y^1)$. It implies that if $x^2 \geq 0$ and $y^1 \geq 0$ then $x^1 \geq 0$. So, the productivity of ecological-economic system $\left(x^1 \geq 0, x^2 \geq 0\right)$ will be provided when $\left(y^1 \geq 0, y^2 \geq 0\right)$. Then the two-criterion problem (1) after the exclusion of variables x^1 and x^2 can be the following form:

$$\begin{aligned} & f_1(x) = c^1 y^1 \to \max, \; f_2(x) = c^2 y^2 \to \min, \\ & x^1 = A_{11} x^1 + A_{12} x^2 + y^1, \\ & x^2 = (E_2 - A_2)^{-1} \left(A_{21} (E_2 - A_2)^{-1} y^1 - y^2 \right) \geq 0, \\ & D_1 y^1 - D_2 y^2 \leq R, \; y^1 \geq 0, \; y^2 \geq 0, \end{aligned} \tag{2}$$

where

$$\begin{aligned} D_1 &= B_1 (E_1 - A_1)^{-1} + B_2 (E_2 - A_2)^{-1} A_{21} (E_1 - A_{11})^{-1} \geq 0, \\ D_2 &= B_1 (E_1 - A_1)^{-1} A_{12} (E_2 - A_{22})^{-1} + B_2 (E_2 - A_2)^{-1} \geq 0. \end{aligned}$$

For the investigation of two-criterion problem (2) we apply the method of main criterion [9, p. 47]. Thus, we build two production functions (economic and ecological).

For the construction of economic production function we will distinguish $f_1(x)$ as the main criterion and we will set admissible value $Z \geq 0$ for the criterion level $f_2(x)$, that is total volume of emissions of polluters into the environment.

We obtain the following one-criterion problem:

$$\begin{aligned} & c^1 y^1 \to \max, \\ & c^2 y^2 \leq Z, \\ & (E_2 - A_2)^{-1} A_{21} (E_2 - A_2)^{-1} y^1 - (E_2 - A_2)^{-1} y^2 \geq 0, \\ & D_1 y^1 - D_2 y^2 \leq R, \; y^1 \geq 0, \; y^2 \geq 0. \end{aligned} \tag{3}$$

Let us write the problem (3) in the next way:

$$\begin{aligned} & c^1 y^1 \to \max, \\ & c^2 y^2 \leq Z, \\ & - (E_2 - A_2)^{-1} A_{21} (E_2 - A_2)^{-1} y^1 + (E_2 - A_2)^{-1} y^2 \leq 0, \\ & D_1 y^1 - D_2 y^2 \leq R, \; y^1 \geq 0, \; y^2 \geq 0. \end{aligned} \tag{4}$$

The dual problem to problem (3) has the following form:

$$\begin{aligned} & pZ + rR \to \min, \\ & - q (E_2 - A_{22})^{-1} A_{21} (E_1 - A_{11})^{-1} y^1 + rD_1 \geq c^1, \\ & pc^2 + q (E_2 - A_{22})^{-1} - rD_2 \geq 0, \\ & p \geq 0, \; q \geq 0, \; r \geq 0, \end{aligned} \tag{5}$$

where $p \geq 0, q \geq 0, r \geq 0$ are dual variables (the cost of emissions of polluters into the environment, the cost of the destroyed polluters, the cost of economic resources).

For problem (5) we will find the basic feasible solution from the auxiliary system of linear equations:

$$\begin{aligned} &-q\left(E_2-A_{22}\right)^{-1} A_{21}\left(E_1-A_{11}\right)^{-1} y^1+rD_1-\lambda=c^1, \\ &pc^2+q\left(E_2-A_{22}\right)^{-1}-rD_2-\omega=0, \end{aligned} \tag{6}$$

where $\lambda \geq 0, \omega \geq 0$ are slack variables.

We find all basic feasible solutions of system (6), namely $\left(p_1^*, q_1^*, r_1^*\right), \left(p_2^*, q_2^*, r_2^*\right), \ldots,$ $\left(p_s^*, q_s^*, r_s^*\right)$. Then we explicitly write down an economic production function

$$F(Z,R)=c^1 y^{1*}=\min(p^* Z+r^* R)=\begin{cases} p_1^* Z+r_1^* R=f_1^1\left(x(Z,R)\right), x \in M_1, \\ \ldots \quad \ldots \quad \ldots \\ p_s^* Z+r_s^* R=f_1^s\left(x(Z,R)\right), x \in M_s, \end{cases}$$

where $\bigcup_{i=1}^s M_i=M$ is the domain of definition of the problem (3).

A production function depends on the parameter Z and is presented in dependence of the chosen set. At the different values of parameter Z we will obtain the different optimal solutions for the two-criterion problem (1).

For the construction of ecological production function we will define $f_2(x)$ as the main criterion in problem (2) and we will set admissible value $Q \geq 0$ of final producing products in a cost form (the first criterion).

We will get the following one-criterion problem

$$\begin{aligned} &c^2 y^2 \rightarrow \min, \\ &c^1 y^1 \geq Q, \\ &\left(E_2-A_2\right)^{-1} A_{21}\left(E_2-A_2\right)^{-1} y^1-\left(E_2-A_2\right)^{-1} y^2 \geq 0, \\ &D_1 y^1-D_2 y^2 \leq R,\ y^1 \geq 0,\ y^2 \geq 0. \end{aligned} \tag{7}$$

Let us write problem (7) in the next way:

$$\begin{aligned} &-c^2 y^2 \rightarrow \max, \\ &-c^1 y^1 \leq -Q, \\ &-\left(E_2-A_{22}\right)^{-1} A_{21}\left(E_1-A_{11}\right)^{-1} y^1+\left(E_2-A_{22}\right)^{-1} y^2 \leq 0, \\ &D_1 y^1-D_2 y^2 \leq R,\ y^1 \geq 0,\ y^2 \geq 0. \end{aligned} \tag{8}$$

The dual problem to problem (8) looks as following:

$$\begin{aligned} &-uQ+wR \rightarrow \min, \\ &-uc^1-v(E_2-A_{22})^{-1}A_{21}(E_1-A_{11})^{-1}+wD_1 \geq 0, \\ &v(E_2-A_{22})^{-1}-wD_2 \geq -c^2, \\ &u \geq 0, v \geq 0, w \geq 0, \end{aligned} \tag{9}$$

where $u \geq 0, v \geq 0, w \geq 0$ are dual variables (the cost of producing of final products, the cost of the destroyed polluters, the cost of economic resources).

For problem (9) we will find the basic feasible solutions from the auxiliary system of linear equations:

$$\begin{aligned} &-uc^1 - v(E_2 - A_{22})^{-1}A_{21}(E_1 - A_{11})^{-1} + wD_1 - \mu = 0, \\ &v(E_2 - A_{22})^{-1} - wD_2 - \nu = -c^2, \end{aligned} \tag{10}$$

where $\mu \geq 0, \nu \geq 0$ are slack variables.

We find all basic feasible solutions of system (10), namely (u_1^*, v_1^*, w_1^*), $(u_2^*, v_2^*,, w_2^*)$,..., (u_k^*, v_k^*, w_k^*). Then we explicitly write down an economic production function

$$F(Q,R) = c^2y^{2*} = \min(-u^*Q + w^*R) = \max(u^*Q - w^*R) =$$

$$= \begin{cases} u_1^*Q - w_1^*R = f_2^1\left(x(Q,R)\right), x \in \tilde{M}_1, \\ \dots \quad \dots \quad \dots \\ u_k^*Q - w_k^*R = f_2^k\left(x(Q,R)\right), x \in \tilde{M}_k, \end{cases}$$

where $\bigcup_{i=1}^k \tilde{M}_i = \tilde{M}$ is the domain of definition of the problem (7).

A production function depends on the parameter Q and in dependence of the chosen set. At the different values of the parameter Q we will get the different optimal solutions for the two-criterion problem (1).

Thus, investigation of the proposed two-criterion production activity model of ecological-economic system (1) is reduced to investigation of a pair of one-criterion problems (3) and (7). Applying theory of duality we obtained a pair of production functions for the ecological-economic system in an explicit form:

$$F(Z,R) = c^1y^{1*} = \min(p^*(Z,R) \cdot Z + r^*(Z,R) \cdot R),$$
$$F(Q,R) = c^2y^{2*} = \max(u^*(Q,R) \cdot Q - w^*(Q,R) \cdot R).$$

Example 1. *To demonstrate the construction of a pair of production functions for the ecological-economic system, we consider the following example. Let* (1) *be a given problem, where* $A_{11} = (0,4)$, $A_{12} = (0,2)$, $A_{21} = (0,2)$, $A_{22} = (0,1)$, $B_1 = (2)$, $B_2 = (1)$, $c^1 = (1)$, $c^2 = (2)$, $R = (R)$. *Then problem* (2) *looks as following*

$$\begin{aligned} &y^1 \to \max,\ y^2 \to \min, \\ &x^1 = 0,4x^1 + 0,2x^2 + y^1, \\ &x^2 = \frac{36}{125}y^1 - \frac{6}{5}y^2 \geq 0, \\ &4y^1 - 2y^2 \leq R,\ y^1 \geq 0,\ y^2 \geq 0. \end{aligned}$$

Setting admissible value $Z = (Z) \geq 0$ *(that is a total volume of emissions of polluters into the environment) we obtain one-criterion problem* (4) *in the following form:*

$$\begin{aligned} &y^1 \to max,\ y^2 \leq Z, \\ &-\frac{36}{125}y^1 + \frac{6}{5}y^2 \leq 0, \\ &4y^1 - 2y^2 \leq R,\ y^1 \geq 0,\ y^2 \geq 0. \end{aligned}$$

Let us write a dual problem to the problem:

$$\begin{aligned} &pZ + rR \to \min, \\ &-\frac{36}{125}q + 4r \geq 1, \\ &p + \frac{6}{5}q - 2r \geq 0, \\ &p \geq 0,\ q \geq 0,\ r \geq 0. \end{aligned}$$

Next, we construct an auxiliary system of linear equations and find the basic feasible solutions of the problem

$$-\frac{36}{125}q + 4r - \lambda = 1,$$
$$p + \frac{6}{5}q - 2t - \omega = 0.$$

To solve the system we examine it in these cases 1) $p \geq 0, q \geq 0$; *2)* $p \geq 0, r \geq 0$; *3)* $p \geq 0, \lambda \geq 0$; *4)* $p \geq 0, \omega \geq 0$; *5)* $q \geq 0, r \geq 0$; *6)* $q \geq 0, \lambda \geq 0$; *7)* $q \geq 0, \omega \geq 0$; *8)* $r \geq 0, \lambda \geq 0$; *9)* $r \geq 0, \omega \geq 0$; *10)* $\lambda \geq 0, \omega \geq 0$. *A feasible solution can be obtained in the cases 2 and 5. As consequence, there is the following economic production function* $F(Z,R) = \min\{\frac{1}{2}Z + \frac{1}{4}R; \frac{375}{164}R\}$.

We set $Q = Q(R) \geq 0$ *as an admissible value of final producing products in a cost form. Then problem* (7) *has form*

$$y^2 \to \min,\ y^1 \geq Q,$$
$$\frac{36}{125}y^1 - \frac{6}{5}y^2 \geq 0,$$
$$4y^1 - 2y^2 \leq R,\ y^1 \geq 0,\ y^2 \geq 0.$$

It can be rewritten as following

$$-y^2 \to \max,\ -y^1 \leq -Q,$$
$$-\frac{36}{125}y^1 + \frac{6}{5}y^2 \leq 0,$$
$$4y^1 - 2y^2 \leq R,\ y^1 \geq 0,\ y^2 \geq 0.$$

Find a dual problem to it

$$-uQ + wR \to \min,$$
$$-u - \frac{36}{125}v + 4w \geq 0,$$
$$\frac{6}{5}v - 2w \geq -1,$$
$$u \geq 0,\ v \geq 0,\ w \geq 0.$$

To find non-trivial feasible solutions of the system

$$-u - \frac{36}{125}v + 4w - \mu = 0,$$
$$\frac{6}{5}v - 2w - \nu = -1$$

we investigate the cases 1) $u \geq 0, v \geq 0$; *2)* $u \geq 0, w \geq 0$; *3)* $u \geq 0, \mu \geq 0$; *4)* $u \geq 0, \nu \geq 0$; *5)* $v \geq 0, w > 0$; *6)* $v \geq 0$; $\mu \geq 0$; *7)* $v \geq 0, \nu \geq 0$; *8)* $w \geq 0, \mu \geq 0$; *9)* $w \geq 0, \nu \geq 0$; *10)* $\mu \geq 0, \nu \geq 0$. *It is possible to solve the problem only in cases 2 та 8. As a result, we obtain an ecological production function* $F = \min\{-2Q + \frac{1}{2}R; \frac{1}{2}R\}$. *Thus, there is a constructed pair of productions functions for the ecological-economic system* $F(Z,R) = \min\{\frac{1}{2}Z + \frac{1}{4}R; \frac{375}{164}R\}$, $F(Q,R) = \min\{-2Q + \frac{1}{2}R; \frac{1}{2}R\}$.

Conclusions. The proposed model of the ecological-economic system gives an opportunity to treat the production activity as the problem of maximization of the final product value and as the problem of minimization of polluters emissions into the environment. In both cases, the ecological-economic production functions are constructed. The production functions describe the optimal performance of the ecological-economic system. Our optimization model allows to realize the largest volume of final product output, the best distribution of economic resources and the least amount of polluters emissions into the environment. The proposed approach to construct a pair of production functions best suits needs and is convenient for usage. Analysis of this pair of production functions provides acceptance of better and more effective decisions at the production management.

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Хрущ Л.З. *Застосування теорії двоїстості до розв'язування двокритерійної задачі лінійного програмування для еколого-економічної системи* // Карпатські матем. публ. — 2018. — Т.10, №2. — С. 324–332.

У статті досліджено оптимізаційну задачу з двома критеріями: максимізація однієї цільової функції та мінімізація іншої цільової функції. Для розв'язання запропонованої двокритерійної задачі застосовано метод головного критерію, причому розглянуто задачу виробничої діяльності еколого-економічної системи, в якій реалізується максимізація вартості кінцевого продукту, як перша цільова функція, і мінімізація викидів забруднювачів у навколишнє середовище, як друга цільова функція. Внаслідок цього здійснюється побудова двох виробничих функцій (економічної та екологічної). Для побудови економічної виробничої функції за найбільш суттєвий (головний) критерій виділено максимум випуску кінцевої продукції у вартісній формі й введено параметр із значенням експертно встановленого "порогового" (задовільного) рівня іншого критерію. Після цього двокритеріальну задачу приведено до однокритеріальної задачі. Аналогічно, для побудови екологічної виробничої функції за найбільш суттєвий (головний) критерій виділено мінімум загального обсягу викидів забруднювачів у навколишнє середовище, введено параметр "порогового" рівня другого критерію і отримано однокритеріальну задачу. Шляхом дослідження відповідних двоїстих задач до отриманих однокритерійних задач у явному вигляді записано економічну та екологічну виробничі функції. Такі функції відносно первинної двокритерійної задачі дають змогу здійснювати оптимальне управління еколого-економічною системою.

Ключові слова і фрази: Оптимальне управління, двокритерійна задача, двоїста задача, цільова функція, метод головного критерію, еколого-економічна рівновага, виробнича функція.

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ON FELLER SEMIGROUP GENERATED BY SOLUTION OF NONLOCAL PARABOLIC CONJUGATION PROBLEM

The paper deals with the problem of construction of Feller semigroup for one-dimensional inhomogeneous diffusion processes with membrane placed at a point whose position on the real line is determined by a given function that depends on the time variable. It is assumed that in the inner points of the half-lines separated by a membrane the desired process must coincide with the ordinary diffusion processes given there, and its behavior on the common boundary of these regions is determined by the nonlocal conjugation condition of Feller-Wentzell's type. This problem is often called a problem of pasting together two diffusion processes on a line.

In order to study the described problem we use analytical methods. Such an approach allows us to determine the desired operator family using the solution of the corresponding problem of conjugation for a linear parabolic equation of the second order (the Kolmogorov backward equation) with discontinuous coefficients. This solution is constructed by the boundary integral equations method under the assumption that the coefficients of the equation satisfy the Hölder condition with a nonzero exponent, the initial function is bounded and continuous on the whole real line, and the parameters characterizing the Feller-Wentzell conjugation condition and the curve defining the common boundary of the domains, where the equation is given, satisfies the Hölder condition with exponent greater than $\frac{1}{2}$.

Key words and phrases: Feller semigroup, diffusion process, parabolic problem of conjugation.

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Introduction

Consider on a plane (s, x) the set

$$S_t = \{(s, x): \quad 0 \le s < t \le T, \ -\infty < x < \infty\},$$

and denote by $\overline{S}_t$ the closure of S_t. Suppose that $\overline{S}_t$ contains a continuous curve $x = h(s)$, $0 \le s \le T$, which separates S_t into two domains:

$$S_t^{(1)} = \{(s, x): \quad 0 \le s < t \le T, \ -\infty < x < h(s)\}$$

and

$$S_t^{(2)} = \{(s, x): \quad 0 \le s < t \le T, \ h(s) < x < \infty\}.$$

Put $D_{1s} = (-\infty, h(s))$ and $D_{2s} = (h(s), \infty)$.

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Consider in S_T two uniformly parabolic operators with bounded coefficients

$$\frac{\partial}{\partial s} + L_s^{(i)} \equiv \frac{\partial}{\partial s} + \frac{1}{2} b_i(s,x) \frac{\partial^2}{\partial x^2} + a_i(s,x) \frac{\partial}{\partial x}, \; i = 1,2. \tag{1}$$

The problem is to find a solution $u(s,x,t)$ of the equation

$$\frac{\partial u}{\partial s} + L_s^{(i)} u = 0, \quad (s,x) \in S_t^{(i)}, \; i = 1,2, \tag{2}$$

which satisfies the 'initial' condition

$$\lim_{s \uparrow t} u(s,x,t) = \varphi(x), \quad x \in \mathbb{R}, \tag{3}$$

two conjugation conditions

$$u(s, h(s) - 0), t) = u(s, h(s) + 0, t), \quad 0 \le s \le t \le T, \tag{4}$$

$$\gamma(s) u(s, h(s), t) + \int\limits_{D_{1s} \cup D_{2s}} [u(s, h(s), t) - u(s, y, t)] \mu(s, dy) = 0, \quad 0 \le s \le t \le T, \tag{5}$$

and two fitting conditions

$$\varphi(h(t) - 0) = \varphi(h(t) + 0), \tag{6}$$

$$\gamma(t) \varphi(h(t)) + \int\limits_{D_{1t} \cup D_{2t}} [\varphi(h(t)) - \varphi(y)] \mu(t, dy) = 0. \tag{7}$$

The initial function $\varphi(x)$ in (3) is assumed to be bounded and continuous on $\mathbb{R}$ (in this case condition (6) holds automatically), the function $\gamma(s)$ and the Borel measure $\mu(s, \cdot)$ in (5) are nonnegative and such that $\gamma(s) + \mu(s, D_{1s} \cup D_{2s}) > 0$ for all $s \in [0, T]$.

The problem (2)–(7) arises, in particular, in the theory of diffusion processes in the construction of a one-dimensional model of the diffusion phenomenon with a membrane, or, what is the same, in solving using the analytical methods the so-called problem of pasting together two diffusion processes on a line [3,4,8,9]. In the considered case, the membrane is supposed to be moving, and it is placed at the point $x = h(s)$, which is at the same time the point of pasting together two given diffusion processes. If we assume that the solution $u(s,x,t) \equiv T_{st}\varphi(x)$ of (2)–(7) is a two-parameter Feller semigroup associated with some inhomogeneous Markov process on a line, then the validity for it of equation (2) implies that this process coincides in D_{is} with the diffusion processes given there by the differential operators $L_s^{(i)}$, $i = 1,2$, and initial condition (3) is in agreement with the equality $T_{ss} = I$, where I is the identity operator. Next, conjugation condition (4) is the reflection of the Feller property of the process and equality (5) is the Feller-Wentzell conjugation condition which has two terms. The local term is responsible for disappearance of the diffusing particle and the nonlocal one for the jump-like nature of the exit of process from the boundary of the region. Recall that in the general case the Feller-Wentzell conjugation condition contains also the derivatives of the unknown function in both variables, which correspond to the properties of the partial reflection at the common boundary of the regions and the phenomenon of 'viscosity' [1,6,11].

The classical solvability of problem (2)–(7) is proved under the assumption that the coefficients of equation (2) satisfy the Hölder condition with a nonzero exponent, the initial function

φ in (3) is bounded and continuous on the whole real line, and the parameters γ, μ characterizing the Feller-Wentzell conjugation condition (5) and the curve $x = h(s)$ defining the common boundary of the domains $S_t^{(1)}$ and $S_t^{(2)}$ satisfy the Hölder condition with exponent greater than $\frac{1}{2}$. In the investigations we use the fundamental solutions of the parabolic equations and the heat potentials generated by them [2, 5, 8]. As a result of their application, problem (2)–(7) is reduced to a system of two singular Volterra integral equations of the second kind which solution is obtained by the method of successive approximations.

Note that a similar problem was considered earlier in [9] for the case where the membrane is placed at a fixed point of the line. We also mention works [7, 10], which present the results concerning the construction of diffusion processes with jumps at the points of the boundary of the region by the methods of stochastic [7] and functional analysis [10].

Assume that the following conditions I–V are satisfied.

I. Equation (2) is a parabolic equation in the domain $\overline{S}_T$, i.e., there exist positive constants b and B such that

$$0 < b \leq b_i(s,x) \leq B < \infty, \; i = 1,2, \; (s,x) \in \overline{S}_T.$$

II. The coefficients $b_i(s,x)$ and $a_i(s,x)$, $i = 1,2$, are continuous in (s,x) and belong to the Hölder class $H^{\frac{\alpha}{2},\alpha}(\overline{S}_T)$, $0 < \alpha < 1$ (to recall the definitions of Hölder classes see [5]).

III. The initial function $\varphi(x)$ belongs to the space of bounded continuous functions, which we will denote by $C_b(\mathbb{R})$. The norm in this space is defined by the equality $\|\varphi\| = \sup\limits_{x\in\mathbb{R}} |\varphi(x)|$.

IV. In condition (5) the measure $\mu(s,\cdot)$ is nonnegative, $\mu(s, D_{1s} \cup D_{2s}) = 1$, $s \in [0,T]$ and for all $f \in C_b(\mathbb{R})$ the integrals

$$G_f^{(i)}(s) = \int\limits_{D_{is}} f(y)\mu(s,dy), \quad i = 1,2,$$

belong to the Hölder class $H^{\frac{1+\alpha}{2}}([0,T])$.

V. The functions $\gamma(s)$ and $h(s)$ are continuous and belong to $H^{\frac{1+\alpha}{2}}([0,T])$.

In view of IV condition (5) can be rewritten as follows

$$(\gamma(s)+1)u(s,h(s),t) = \int\limits_{D_{1s}\cup D_{2s}} u(s,y,t)\mu(s,dy). \tag{8}$$

Conditions I, II provide the existence of a fundamental solution for each of the equations in (2) (see [5,8]), i.e., the existence of a function $G_i(s,x,t,y)$, $i = 1,2$ $(0 \leq s < t \leq T;\; x,y \in \mathbb{R})$, which satisfies equation (2) for fixed $t \in (0,T]$, $y \in \mathbb{R}$ as a function of $(s,x) \in [0,t) \times \mathbb{R}$ and has the form

$$G_i(s,x,t,y) = Z_{i0}(s,x,t,y) + Z_{i1}(s,x,t,y), \quad i = 1,2, \tag{9}$$

where

$$Z_{i0}(s,x,t,y) = [2\pi b_i(t,y)(t-s)]^{-\frac{1}{2}} \exp\left\{-\frac{(y-x)^2}{2b_i(t,y)(t-s)}\right\}, \tag{10}$$

$$Z_{i1}(s,x,t,y) = \int_s^t d\tau \int_{\mathbb{R}} Z_{i0}(s,x,\tau,z)Q_i(\tau,z,t,y)dz, \tag{11}$$

and the function $Q_i(s,x,t,y)$ is a solution of some singular Volterra integral equation of the second kind.

Note that

$$\left|D_s^r D_x^p Z_{i0}(s,x,t,y)\right| \le C(t-s)^{-\frac{1+2r+p}{2}} \exp\left\{-c\frac{(y-x)^2}{t-s}\right\}, \tag{12}$$

$$\left|D_s^r D_x^p Z_{i1}(s,x,t,y)\right| \le C(t-s)^{-\frac{1+2r+p-\alpha}{2}} \exp\left\{-c\frac{(y-x)^2}{t-s}\right\}, \tag{13}$$

where $i=1,2,\ 0\le s<t\le T,\ x,y\in\mathbb{R},\ C$ i c are positive constants; in the sequel, various positive constants will be denoted by symbols C or c; r and p are nonnegative integers satisfying $2r+p\le 2,\ D_s^r$ is the partial derivative with respect to s of order $r,\ D_x^p$ is the partial derivative with respect to x of order p.

Given a fundamental solution $G_i(s,x,t,y),\ i=1,2$, and a function $h(s)$, we define the integrals

$$u_{i0}(s,x,t) = \int_{\mathbb{R}} G_i(s,x,t,y)\varphi(y)dy, \quad i=1,2, \tag{14}$$

$$u_{i1}(s,x,t) = \int_s^t G_i(s,x,\tau,h(\tau))V_i(\tau,t)d\tau, \quad i=1,2. \tag{15}$$

Here φ and $V_i,\ i=1,2$ are given functions, $0\le s<t\le T,\ x\in\mathbb{R}$. In the theory of parabolic equations the function $u_{i0}(s,x,t)$ is called the Poisson potential, and the function $u_{i1}(s,x,t)$ the parabolic simple-layer potential.

We recall some properties of functions $u_{i0}(s,x,t)$ and $u_{i1}(s,x,t),\ i=1,2$. Let $\varphi\in C_b(\mathbb{R})$. Then from the properties of the fundamental solution $G_i(s,x,t,y),\ i=1,2$, it follows that the potential u_{i0} exists and satisfies equation (2) and the 'initial' condition

$$\lim_{s\uparrow t} u_{i0}(s,x,t) = \varphi(x), \quad x\in\mathbb{R},\ i=1,2, \tag{16}$$

in the domain $(s,x)\in[0,t)\times\mathbb{R}$ for a fixed $t\in(0,T]$ as a function of arguments (s,x).

In addition, for the function $u_{i0}(s,x,t),\ i=1,2$, the inequality

$$\left|D_s^r D_x^p u_{i0}(s,x,t)\right| \le C(t-s)^{-\frac{2r+p}{2}}\|\varphi\|, \tag{17}$$

(where r and p are positive integers for which $2r+p\le 2$) holds in each of the domains $0\le s<t\le T,\ x\in\mathbb{R}$.

Consider integral (15). If we assume that the density $V(\tau,t)$ is continuous for $\tau\in[s,t)$ and has a weak singularity with exponent $\ge -\frac{1}{2}$ when $\tau=t$, then the function $u_{i1}(s,x,t),\ i=1,2,$

is bounded and continuous in $0 \le s \le t \le T$, $x \in \mathbb{R}$, it satisfies equation (2) in the domain $(s,x) \in [0,t) \times (\mathbb{R} \setminus h(s))$ and the initial condition

$$\lim_{s \uparrow t} u_{i1}(s,x,t) = 0, \quad x \in \mathbb{R},\ i = 1,2. \tag{18}$$

An important property of the function u_{i1} is reflected in the so-called theorem on the jump of the co-normal derivative of the parabolic simple-layer potential (see, for instance, [5,8]). In the present paper this assertion is not used, and therefore we do not provide it.

1 Existence and uniqueness

We find a solution of (2)–(7) in the form of sum of potentials u_{i0} and u_{i1} with unknown densities $V_i(s,t)$, $i = 1,2$:

$$\begin{aligned} u(s,x,t) = &\int_{\mathbb{R}} G_i(s,x,t,y)\varphi(y)dy \\ &+ \int_s^t G_i(s,x,\tau,h(\tau))V_i(\tau,t)d\tau, \quad (s,x) \in \overline{S}_t^{(i)},\ i = 1,2. \end{aligned} \tag{19}$$

Using conjugation conditions (4), (5) and (8), we get the following system of Volterra integral equations of the first kind for $V_i(s,t)$:

$$\begin{aligned} &(\gamma(s)+1)\int_s^t G_i(s,h(s),\tau,h(\tau))V_i(\tau,t)d\tau \\ &-\sum_{j=1}^{2}\int_s^t V_j(\tau,t)d\tau \int_{D_{js}} G_j(s,y,\tau,h(\tau))\mu(s,dy) = \Phi_i(s,t), \quad i = 1,2, \end{aligned} \tag{20}$$

where

$$\Phi_i(s,t) = \sum_{j=1}^{2}\int_{D_{js}} u_{j0}(s,y,t)\mu(s,dy) - (\gamma(s)+1)u_{i0}(s,h(s),t), \quad i = 1,2.$$

Consider the function $\Phi_i(s,t)$ in (20). Let us prove that

$$\lim_{s\uparrow t} \Phi_i(s,t) = 0, \quad i = 1,2; \tag{21}$$

$$|\Phi_i(s,t) - \Phi_i(\widetilde{s},t)| \le C\|\varphi\|(t-s)^{-\frac{1+\alpha}{2}}(s-\widetilde{s})^{\frac{1+\alpha}{2}}, \quad \widetilde{s} < s. \tag{22}$$

Assertion (21) can be easily verified using property (16) of the Poisson potential u_{i0} and fitting condition (7):

$$\begin{aligned} \lim_{s\uparrow t} \Phi_i(s,t) &= \sum_{j=1}^{2}\int_{D_{jt}} \varphi(y)\mu(t,dy) - (\gamma(t)+1)\varphi(h(t)) \\ &= \int_{D_{1t}\cup D_{2t}} [\varphi(y) - \varphi(h(t))]\mu(t,dy) - \gamma(t)\varphi(h(t)) = 0. \end{aligned}$$

To prove inequality (22), we write the difference $\Phi_i(s,t) - \Phi_i(\widetilde{s},t)$ as a sum $I_1 + I_2 + I_3$, where

$$I_1 = \sum_{j=1}^{2} \int\limits_{D_{js}} [u_{j0}(s,y,t) - u_{j0}(\widetilde{s},y,t)]\mu(s,dy),$$
$$I_2 = (\gamma(\widetilde{s})+1)u_{i0}(\widetilde{s},h(\widetilde{s}),t) - (\gamma(s)+1)u_{i0}(s,h(s),t),$$
$$I_3 = \sum_{j=1}^{2} \Bigg(\int\limits_{D_{js}} u_{j0}(\widetilde{s},y,t)\mu(s,dy) - \int\limits_{D_{j\widetilde{s}}} u_{j0}(\widetilde{s},y,t)\mu(\widetilde{s},dy) \Bigg),$$

and study separately each term of this sum.

Since for $\widetilde{s} < s$

$$\begin{aligned}
&|u_{j0}(s,y,t) - u_{j0}(\widetilde{s},y,t)| \\
&= |u_{j0}(s,y,t) - u_{j0}(\widetilde{s},y,t)|^{\frac{1+\alpha}{2}} |u_{j0}(s,y,t) - u_{j0}(\widetilde{s},y,t)|^{\frac{1-\alpha}{2}} \\
&\le \left| \frac{\partial u_{j0}(\hat{s},y,t)}{\partial \hat{s}} \Big|_{\hat{s}=\widetilde{s}+\theta(s-\widetilde{s})} \cdot (s-\widetilde{s}) \right|^{\frac{1+\alpha}{2}} (|u_{j0}(s,y,t)| + |u_{j0}(\widetilde{s},y,t)|)^{\frac{1-\alpha}{2}} \\
&\le C\|\varphi\| \left[(t-\widetilde{s}-\theta(s-\widetilde{s}))^{-1}(s-\widetilde{s}) \right]^{\frac{1+\alpha}{2}} \le C\|\varphi\| [((t-s) \\
&+ (s-\widetilde{s})(1-\theta))^{-1}(s-\widetilde{s})]^{\frac{1+\alpha}{2}} \le C\|\varphi\|(t-s)^{-\frac{1+\alpha}{2}}(s-\widetilde{s})^{\frac{1+\alpha}{2}}, \quad 0<\theta<1,
\end{aligned}$$

inequality (22) holds for the term I_1. Recalling that the functions γ and h are Hölder continuous (see assumption V) and using previous considerations, we arrive at inequality (22) for I_2. For I_3 we have the estimate

$$|I_3| \le C\|\varphi\|(s-\widetilde{s})^{\frac{1+\alpha}{2}},$$

which is an obvious consequence of assumption IV. Thus,

$$|I_1 + I_2 + I_3| \le C\|\varphi\|(t-s)^{-\frac{1+\alpha}{2}}(s-\widetilde{s})^{\frac{1+\alpha}{2}}, \quad \widetilde{s} < s,$$

what had to be proved.

In order to regularize system of Volterra integral equations of the first kind (20), we apply to both sides of each of its equations the integro-differential operator $\mathcal{E}$, which acts by the rule

$$\mathcal{E}(s,t)\Phi_i = \sqrt{\frac{2}{\pi}} \frac{\partial}{\partial s} \int\limits_s^t (\rho-s)^{-\frac{1}{2}} \Phi_i(\rho,t)d\rho, \quad 0 \le s < t \le T,\ i=1,2. \tag{23}$$

Consider first the action of the operator $\mathcal{E}$ on the right hand side of the i-th equation of system (20), $i=1,2$.

In view of (21) and (22), for the function $\widehat{\Phi}_i(s,t) \equiv \mathcal{E}(s,t)\Phi_i$ we easily get the following formula:

$$\begin{aligned}
\widehat{\Phi}_i(s,t) = &\frac{1}{\sqrt{2\pi}} \int\limits_s^t (\rho-s)^{-\frac{3}{2}} [\Phi_i(\rho,t) - \Phi_i(s,t)]d\rho \\
&- \sqrt{\frac{2}{\pi}}(t-s)^{-\frac{1}{2}}\Phi_i(s,t), \quad i=1,2.
\end{aligned} \tag{24}$$

Besides, for the function $\widehat{\Phi}_i(s,t)$ in each domain of the form $0 \le s < t \le T$ the inequality

$$|\widehat{\Phi}_i(s,t)| \le C\|\varphi\|(t-s)^{-\frac{1}{2}} \tag{25}$$

holds.

Now, we apply the operator $\mathcal{E}$ to the left hand side of the i-th equation of system (20), $i=1,2$. As a result, we obtain the expression, which after changing the order of integration and using formulas (9), (10) can be represented in the form

$$-\frac{V_i(s,t)}{\sqrt{b_i(s,h(s))}} + \sqrt{\frac{2}{\pi}}\frac{\partial}{\partial s}\sum_{j=1}^{2}\int\limits_s^t N_{ij}(s,\tau)V_j(\tau,t)d\tau, \quad i=1,2, \tag{26}$$

where

$$\begin{aligned} N_{ii}(s,\tau) = \int\limits_s^\tau (\rho-s)^{-\frac{1}{2}}\Big[&(Z_{i0}(\rho,h(\rho),\tau,h(\tau)) - Z_{i0}(\rho,0,\tau,0)) + \gamma(\rho)G_i(\rho,h(\rho),\tau,h(\tau)) \\ &+ Z_{i1}(\rho,h(\rho),\tau,h(\tau)) - \int\limits_{D_{i\rho}} G_i(\rho,y,\tau,h(\tau))\mu(\rho,dy)\Big]d\rho, \quad i=j, \\ N_{ij}(s,\tau) = -\int\limits_s^\tau (\rho-s)^{-\frac{1}{2}}d\rho &\int\limits_{D_{j\rho}} G_j(\rho,y,\tau,h(\tau))\mu(\rho,dy), \quad i\ne j. \end{aligned}$$

To simplify the derivatives of integrals depending on parameters in expression (26), we show that

$$\lim_{s\uparrow\tau} N_{ij}(s,\tau) = 0. \tag{27}$$

In proving this fact, a certain complexity is only a study of the function

$$L_j(s,\tau) \equiv \int\limits_s^\tau (\rho-s)^{-\frac{1}{2}}d\rho\int\limits_{D_{j\rho}} Z_{j0}(\rho,y,\tau,h(\tau))\mu(\rho,dy),$$

which appears in the expression for $N_{ij}(s,\tau)$ immediately after we rewrite G_j according to formula (9). For all other terms in formula for $N_{ij}(s,\tau)$ the relation (27) is easily established by using the inequalities (12), (13) and condition V.

Write the function $L_j(s,\tau)$ as follows

$$L_j(s,\tau) = L_{j1}(s,\tau) + L_{j2}(s,\tau), \tag{28}$$

where

$$L_{j1}(s,\tau) = \frac{1}{\sqrt{2\pi b_j(\tau,h(\tau))}} \int\limits_s^\tau (\rho-s)^{-\frac{1}{2}}(\tau-\rho)^{-\frac{1}{2}} d\rho$$
$$\times \Big[\int\limits_{D_{j\rho}} \exp\Big\{ -\frac{(y-h(\tau))^2}{2b_j(\tau,h(\tau))(\tau-\rho)} \Big\} \mu(\rho,dy)$$
$$- \int\limits_{D_{js}} \exp\Big\{ -\frac{(y-h(\tau))^2}{2b_j(\tau,h(\tau))(\tau-\rho)} \Big\} \mu(s,dy) \Big],$$
$$L_{j2}(s,\tau) = \frac{1}{\sqrt{2\pi b_j(\tau,h(\tau))}} \int\limits_s^\tau (\rho-s)^{-\frac{1}{2}}(\tau-\rho)^{-\frac{1}{2}} d\rho$$
$$\times \int\limits_{D_{js}} \exp\Big\{ -\frac{(y-h(\tau))^2}{2b_j(\tau,h(\tau))(\tau-\rho)} \Big\} \mu(s,dy).$$

Since the functions $f_{\tau,\rho}(y) = \exp\left\{\frac{(y-h(\tau))^2}{2b_j(\tau,h(\tau))(\tau-\rho)}\right\}$ belong to $C_b(\mathbb{R})$ for all $0 \le s < \rho < \tau < t \le T$ and are bounded by 1 on this set, and since condition IV holds, we have

$$|L_{j1}(s,\tau)| \le C(\tau-s)^{\frac{1+\alpha}{2}}, \quad j=1,2. \tag{29}$$

Let us study the function $L_{j2}(s,\tau)$. Write it in the form

$$L_{j2}(s,\tau) = \frac{1}{\sqrt{2\pi b_j(\tau,h(\tau))}} \int\limits_{D_{js}} \left[\exp\left\{ -\frac{(y-h(\tau))^2}{2b_j(\tau,h(\tau))(\tau-s)} \right\} \right.$$
$$\left. - \exp\left\{ -\frac{(y-h(s))^2}{2b_j(\tau,h(\tau))(\tau-s)} \right\} \right] R_j(s,\tau,y)\mu(s,dy)$$
$$+ \frac{1}{\sqrt{2\pi b_j(\tau,h(\tau))}} \int\limits_{D_{js}} \exp\left\{ -\frac{(y-h(s))^2}{2b_j(\tau,h(\tau))(\tau-s)} \right\} R_j(s,\tau,y)\mu(s,dy), \tag{30}$$

where $R_j(s,\tau,y)$ denotes the integral

$$R_j(s,\tau,y) = \int\limits_s^\tau (\rho-s)^{-\frac{1}{2}}(\tau-\rho)^{-\frac{1}{2}} \exp\left\{ -\frac{(y-h(\tau))^2}{2b_j(\tau,h(\tau))(\tau-s)} \cdot \frac{\rho-s}{\tau-\rho} \right\} d\rho,$$

which after the substitution $z = \frac{\rho-s}{\tau-\rho}$ reduces to

$$R_j(s,\tau,y) = \int\limits_0^\infty z^{-\frac{1}{2}}(1+z)^{-1} \exp\left\{ -\frac{(y-h(\tau))^2}{2b_j(\tau,h(\tau))(\tau-s)} \cdot z \right\} dz,$$

and thus, satisfies the inequality

$$|R_j(s,\tau,y)| \le C. \tag{31}$$

Denote by $L_{j2}^{(1)}$ the first term in the right hand side of equality (30) and by $L_{j2}^{(2)}$ the second one.

If we express, using the Lagrange formula, the difference of exponents in the square brackets of the expression for $L_{j2}^{(1)}$ through the value of its derivative at the intermediate point $x = y - h(s) + \theta(h(s) - h(\tau))$, and then take this derivative, we get

$$L_{j2}^{(1)}(s,\tau) = \frac{1}{\sqrt{2\pi b_j(\tau,h(\tau))}} \int_{D_{js}} \frac{x}{b_j(\tau,h(\tau))(\tau - s)} \times \exp\left\{-\frac{x^2}{2b_j(\tau,h(\tau))(\tau-s)}\right\}(h(\tau)-h(s))R_j(s,\tau,y)\mu(s,dy).$$

From this equality and estimate (31) and condition V it follows that

$$|L_{j1}(s,\tau)| \leq C(\tau - s)^{\frac{\alpha}{2}}. \tag{32}$$

Then (31) implies

$$|L_{j2}(s,\tau)| \leq C\left(\mu\left(s, D_{js}^{\delta}\right) + \exp\left\{-\frac{\delta^2}{2B(\tau-s)}\right\}\right), \tag{33}$$

where $D_{js}^{\delta} = \{y \in D_{js} : \ |y - h(s)| < \delta\}$, δ is any positive number, B is the constant from I.

Combining (28)–(30), (32), (33), we conclude that

$$\lim_{s\uparrow\tau} L_j(s,\tau) = 0.$$

This completes the proof of (27).

With relation (27) in mind, we put the derivative under the integral sign in expression (26) and then equate this expression to (24). After elementary simplifications, we get the system of Volterra integral equations of the second kind, which is equivalent to (20)

$$V_i(s,t) = \sum_{j=1}^{2} \int_s^t K_{ij}(s,\tau)V_j(\tau,t)d\tau + \Psi_i(s,t), \quad i = 1,2, \tag{34}$$

where

$$\Psi_i(s,t) = -\sqrt{b_i(s,h(s))}\widehat{\Phi}_i(s,t),$$
$$K_{ij}(s,\tau) = \sqrt{\frac{2}{\pi}}\sqrt{b_i(s,h(s))} \cdot \frac{\partial}{\partial s}N_{ij}(s,\tau).$$

The function Ψ_i in (34) satisfies inequality (25), but kernels $K_{ij}(s,\tau)$ do not have the integrable singularity. For $K_{ij}(s,\tau)$ we can only get the estimate

$$K_{ij}(s,\tau) \leq C(\tau - s)^{-1}, \quad 0 \leq s < \tau < t \leq T. \tag{35}$$

Estimate (35) is caused by the integral

$$\int_{D_{js}^{\delta}} \frac{\partial Z_{j0}(s,y,\tau,h(\tau))}{\partial y}\mu(s,dy), \tag{36}$$

which is in the expression for the derivative of L_j

$$\frac{\partial}{\partial s}L_j(s,\tau)=\int\limits_s^\tau(\rho-s)^{-\frac{3}{2}}\Bigg[\int\limits_{D_{j\rho}}Z_{j0}(\rho,y,\tau,h(\tau))\mu(\rho,dy)$$

$$-\int\limits_{D_{js}}Z_{j0}(\rho,y,\tau,h(\tau))\mu(s,dy)\Bigg]d\rho$$

$$-\sqrt{\frac{\pi b_j(\tau,h(\tau))}{2}}\Bigg(\int\limits_{D_{js}^{(\delta)}}\frac{\partial Z_{j0}(s,y,\tau,h(\tau))}{\partial y}\mu(s,dy)$$

$$+\int\limits_{\mathbb{R}\setminus D_{js}^{(\delta)}}\frac{\partial Z_{j0}(s,y,\tau,h(\tau))}{\partial y}\mu(s,dy)\Bigg).$$

All other components of the expression for $K_{ij}(s,\tau)$ admit inequalities the right hand sides of which have the form $C(\delta)(\tau-s)^{-1+\frac{\alpha}{2}}$, where $C(\delta)$ is a positive constant depending on δ.

Despite the fact that the kernels $K_{ij}(s,\tau)$ do not have an integrable singularity, a solution of system of equations (34) exists and can be found by the ordinary method of successive approximations:

$$V_i(s,t)=\sum_{n=0}^{\infty}V_i^{(n)}(s,t),\quad 0\le s<t\le T,\ i=1,2,\tag{37}$$

where

$$V_i^{(0)}(s,t)=\Psi_i(s,t),$$
$$V_i^{(n)}(s,t)=\sum_{j=1}^{2}\int\limits_s^t K_{ij}(s,\tau)V_i^{(n-1)}(\tau,t)d\tau,\quad n=1,2,\ldots$$

The convergence of series (37) is the consequence of the following inequality, which is proved by induction according to the scheme applied in [9] in the study of system of equations (34) for the case when $h\equiv 0$:

$$\left|V_i^{(n)}(s,t)\right|\le C\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^{n}C_n^k a^{(n-k)}(m(\delta))^k,\quad n=0,1,\ldots,\tag{38}$$

where

$$a^{(k)}=\frac{\left(2c(\delta)T^{\frac{\alpha}{2}}\Gamma\left(\frac{\alpha}{2}\right)\right)^k\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+k\alpha}{2}\right)},\quad k=0,1,\ldots,n,$$
$$m(\delta)=\max_{s\in[0,T]}\mu(s,D_{1s}^{\delta}\cup D_{2s}^{\delta})<1\ (\text{for sufficiently small }\delta).$$

From inequality (38) it also follows that the function $V_i(s,t),\ i=1,2,$ admits the estimate

$$|V_i(s,t)|\le C\|\varphi\|(t-s)^{-\frac{1}{2}},\quad 0\le s<t\le T.\tag{39}$$

Thus, we have constructed the solution $u(s,x,t)$ of problem (2)–(7) of form (19), (37), which, in view of estimates (12), (13), (17), (39), belongs to the class $C^{1,2}(S_t^{(1)} \cup S_t^{(2)}) \cap C(\overline{S}_t)$ and satisfies the inequality

$$|u(s,x,t)| \leq C\|\varphi\|(t-s)^{-\frac{1}{2}}. \tag{40}$$

The assertion on the uniqueness of the constructed solution of problem (2)–(7) follows from the maximum principle [5].

The obtained result allows us to state the following theorem:

Theorem 1. *Let the conditions I–V hold. Then problem (2)–(7) has a unique solution belonging to* $C^{1,2}(S_t^{(1)} \cup S_t^{(2)}) \cap C(\overline{S}_t)$. *Besides, this solution admits representation (19), (37) and estimate (40).*

2 Construction of Feller semigroup

Denote by $C_0(\mathbb{R})$ the subspace of $C_b(\mathbb{R})$, which consists of all functions $\varphi \in C_b(\mathbb{R})$ for which the condition (7) holds. Since the subspace $C_0(\mathbb{R})$ is closed in $C_b(\mathbb{R})$, it is a Banach space.

We introduce the two-parameter family of linear operators $T_{st}:\ C_0(\mathbb{R}) \to C_0(\mathbb{R})$, $0 \leq s < t \leq T$, by the following rule:

$$T_{st}\varphi(x) = u(s,x,t,\varphi), \tag{41}$$

where $u(s,x,t,\varphi)$ is a solution of (2)–(7) with the function φ in (3).

Note that the operators T_{st} have the following properties in $C_0(\mathbb{R})$:

a) if a sequence of functions $\varphi_n \in C_0(\mathbb{R})$ is such that $\sup\limits_n \|\varphi_n\| < \infty$ and $\lim\limits_{n\to\infty} \varphi_n(x) = \varphi(x)$ for all $x \in \mathbb{R}$, then $\lim\limits_{n\to\infty} T_{st}\varphi_n(x) = T_{st}\varphi(x)$ for all $0 \leq s < t \leq T$, $x \in \mathbb{R}$;

b) the operators T_{st} are positivity preserving $(0 \leq s < t \leq T)$, i.e, $T_{st}\varphi \geq 0$ for every $\varphi \in C_0(\mathbb{R})$ such that $\varphi \geq 0$;

c) the operators T_{st} are contractive $(0 \leq s < t \leq T)$, i.e., they do not increase the norm of the element;

d) $T_{st} = T_{s\tau}T_{\tau t}$, $0 \leq s < \tau < t \leq T$ (the semigroup propery).

The proof of property a) is based on well known assertions of calculus on passage of the limit under the summation and integral signs (here this concerns series (37) and integrals on the right hand side of equality (19)). This property allows us to prove the next properties of the operator family T_{st}, without loss of generality, under the assumption that the function φ has a compact support.

Let us prove property b). Let $\varphi \in C_0(\mathbb{R})$ be a nonnegative function with a compact support. Denote by m the minimum of $T_{st}\varphi(x)$ in $(s,x) \in \overline{S}_t$. If we assume that $m < 0$, then from the minimum principle [5] it follows that the value m is attained only when $s \in (0,t)$ and $x = h(s)$. Fix $s_0 \in (0,t)$ for which $T_{s_0t}\varphi(h(s_0)) = m$. Then

$$\gamma(s_0)T_{s_0t}\varphi(h(s_0)) + \int\limits_{D_{1s_0}\cup D_{2s_0}} [T_{s_0t}\varphi(h(s_0)) - T_{s_0t}\varphi(y)]\mu(s_0,dy) < 0,$$

which contradicts (5). The contradiction we arrived at indicates that $m \geq 0$, what had to be proved.

The proof of property c) is similar to the proof of b).

The semigroup property of operators T_{st} is a consequence of the assertion on the uniqueness of the solution of problem (2)–(7). Indeed, to find $u(s,x,t) = T_{st}\varphi(x)$, provided $\lim_{s\uparrow t} u(s,x,t) = \varphi(x)$, one can solve the problem first in the time interval $[\tau, t]$, and then solve it in the time interval $[s,\tau]$ with that 'initial' function $u(\tau,x,t) = T_{\tau t}\varphi(x)$, which was obtained; in other words, $T_{st}\varphi(x) = T_{s\tau}(T_{\tau t}\varphi)(x),\ \varphi \in C_0(\mathbb{R})$, or $T_{st} = T_{s\tau}T_{\tau t}$.

Properties a)–d) of operators T_{st} imply the following assertion.

Theorem 2. *Let the conditions of Theorem 1 hold. Then the two-parameter family of operators T_{st}, $0 \leq s < t \leq T$, defined by (41), describes the inhomogeneous Feller process on the line $\mathbb{R}$, which coincides in D_{1s} and D_{2s} with given diffusion processes generated by operators $L_s^{(1)}$ and $L_s^{(2)}$ respectively, and its behavior at point $x = h(s)$ is determined by conjugation condition (5).*

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Копитко Б.І., Шевчук Р.В. *Про напівгрупу Феллера, породжену розв'язком нелокальної параболічної задачі спряження* // Карпатські матем. публ. — 2018. — Т.10, №2. — С. 333–345.

У статті розглядається задача побудови напівгрупи Феллера для одновимірного неоднорідного дифузійного процесу з мембраною, розташованою в точці, положення якої на числовій прямій визначається за допомогою заданої функції, що залежить від часової змінної. При цьому припускається, що у внутрішніх точках півпрямих, розділених між собою мембраною, шуканий процес має збігатися із заданими там звичайними дифузійними процесами, а його поведінка на спільній межі цих областей визначається заданою нелокальною умовою спряження типу Феллера-Вентцеля. Дану задачу ще називають задачею про склеювання двох дифузійних процесів на прямій.

З метою вивчення сформульованої проблеми в роботі застосовано аналітичні методи. Такий підхід дозволяє визначити шукану сім'ю операторів з допомогою розв'язку відповідної задачі спряження для лінійного параболічного рівняння другого порядку (оберненого рівняння Колмогорова) з розривними коефіцієнтами. Цей розв'язок побудовано методом граничних інтегральних рівнянь за припущення, що коефіцієнти рівняння задовольняють умову Гельдера з ненульовим показником, початкова функція є обмеженою і неперервною на всій числовій прямій, а параметри, які характеризують умову спряження Феллера-Вентцеля та крива, що визначає спільну межу областей, де задане рівняння, задовольняють умову Гельдера з показником більшим, ніж $\frac{1}{2}$.

Ключові слова і фрази: напівгрупа Феллера, дифузійний процес, параболічна задача спряження.

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OSYPCHUK M.M.

ON THE CROSSINGS NUMBER OF A HYPERPLANE BY A STABLE RANDOM PROCESS

The numbers of crossings of a hyperplane by discrete approximations for trajectories of an α-stable random process (with $1 < \alpha < 2$) and some processes related to it are investigated. We consider an α-stable process is killed with some intensity on the hyperplane and a pseudo-process that is formed from the α-stable process using its perturbation by a fractional derivative operator with a multiplier like a delta-function on the hyperplane. In each of these cases, the limit distribution of the crossing number of the hyperplane by some discret approximation of the process is related to the distribution of its local time on this hyperplane. Integral equations for characteristic functions of these distributions are constructed. Unique bounded solutions of these equations can be constructed by the method of successive approximations.

Key words and phrases: α-stable process, local time, pseudo-process.

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INTRODUCTION

Let $(x(t), \mathcal{M}_t, \mathbb{P}_x)$ denote a standard Markov process on $\mathbb{R}^d$ ($d \geq 1$). Consider a fixed hyperplane $S = \{x \in \mathbb{R}^d : (x, \nu) = r\}$, in $\mathbb{R}^d$ and two open sets

$$D_- = \{x \in \mathbb{R}^d : (x, \nu) < r\}, \quad D_+ = \{x \in \mathbb{R}^d : (x, \nu) > r\},$$

where $\nu \in \mathbb{R}^d$ is a given unit vector and $r \in \mathbb{R}$ is a given constant.

Our goal is to describe a changes number of the sets D_- and D_+ before a fixed time $t > 0$ by the trajectories of the process $(x(t))_{t\geq 0}$ started at fixed point $x \in \mathbb{R}^d$.

Consider for $m, n \in \mathbb{N}$ the random variable

$$\xi_m^{(n)} = \sum_{k=1}^{m} v\left(x\left(\frac{k-1}{n}\right), x\left(\frac{k}{n}\right)\right),$$

where $v(x, y) = \mathbb{1}_{D_-}(x)\mathbb{1}_{D_+}(y) + \mathbb{1}_{D_+}(x)\mathbb{1}_{D_-}(y)$.

The variable $\xi_{[nt]}^{(n)}$ equals to the number of crossings of the hyperplane S by the ordered set of points in $\mathbb{R}^d$: $x(0)$, $x(1/n)$,…,$x([nt]/n)$.

We are going to find out a sequence of normalizing multipliers $\{c_n : n \geq 1\}$ such that the limit distribution of the sequence $\{c_n \xi_{[nt]}^{(n)} : n \geq 1\}$ exists and to describe it. It is obvious that $c_n \to 0$, as $n \to \infty$.

The limit theorems of this type were initiated by I. I. Gikhman in connection with some problems of mathematical statistics. I. I. Gikhman considered sequences of one-dimensional Markov chains approaching a diffusion process with smooth local characteristics (see [1, 2]).

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1 SOME AUXILIARY RESULTS

We will use the following corollary of one A. V. Skorokhod's theorem (see [3, Th. 1]).

Lemma 1. *A limit distribution of the sequence of random variables $c_n\xi_{[nt]}^{(n)}$ exists if and only if a limit distribution exists for the variables $c_n\eta_{[nt]}^{(n)}$, where*

$$\eta_m^{(n)} = \sum_{k=1}^{m} v_n\left(x\left(\frac{k}{n}\right)\right), \quad v_n(x) = \mathbb{E}_x v\left(x(0), x\left(\frac{1}{n}\right)\right),$$

and these limit distributions coincide, if only they exist.

So, we will consider the random variables $c_n\eta_{[nt]}^{(n)}$.

For any fixed $t > 0$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$ we consider the characteristic function

$$u_n(t,x,\theta) = \mathbb{E}_x \exp\left\{i\theta c_n \eta_{[nt]}^{(n)}\right\}, \quad \theta \in \mathbb{R},$$

of the random variable $c_n\eta_{[nt]}^{(n)}$.

The next equation for the function $u_n(t,x,\theta)$

$$u_n(t,x,\theta) = 1 + n\int_0^{[nt]/n} d\tau \int_{\mathbb{R}^d} \left(1 - e^{-i\theta c_n v_n(y)}\right) u_n(\tau,y,\theta) g\left(\frac{[nt]-[n\tau]}{n}, x, y\right) dy \tag{1}$$

follows from the identity $\exp\left\{\sum_{k=1}^m a_k\right\} = 1 + \sum_{k=1}^m \left(1 - e^{-a_k}\right) \exp\left\{\sum_{j=k}^m a_j\right\}$, that holds true for each set of complex numbers $a_1, a_2, \ldots, a_m$ and each natural number m. Here the function $(g(t,x,y))_{t>0, x\in\mathbb{R}^d, y\in\mathbb{R}^d}$ denotes the transition probability density of the process $(x(t))_{t\geq 0}$.

If the transition probability density of the process $(x(t))_{t\geq 0}$ is given by the equality

$$g(t,x,y) = (2\pi)^{-d} \int_{\mathbb{R}^d} \exp\{i(\lambda, y-x) - ct|\lambda|^\alpha\}\, d\lambda, \quad t > 0,\ x \in \mathbb{R}^d,\ y \in \mathbb{R}^d,$$

for fixed parameters $c > 0$ and $\alpha \in (1,2]$, then the process $(x(t))_{t\geq 0}$ is called rotationally invariant α-stable random process. If $\alpha = 2$, this process is the Brownian motion. In this case, our problems have been addressed in many publications (see, for example, [4,5] and others). Therefore, we will not consider this case. So, we will further assume that $1 < \alpha < 2$, although most of our results remain correct also for $\alpha = 2$.

Consider the function $f(t,x) = \int_0^t d\tau \int_S g(\tau,x,y)\, d\sigma_y$. It is a W-function for the process $(x(t))_{t\geq 0}$ satisfying the inequality $f(t,x) \leq N\frac{\alpha}{\alpha-1} t^{1-1/\alpha}$. So, there exists a W-functional $(l_t)_{t\geq 0}$ of the process $(x(t))_{t\geq 0}$ such that $\mathbb{E}_x l_t = f(t,x)$ (see [8, Th. 6.6]). This functional is called the local time on S for the process $(x(t))_{t\geq 0}$.

Using the following representation of the functional $(l_t)_{t\geq 0}$:

$$l_t = \lim_{h\to 0+} \int_0^t d\tau \int_S g(h, x(\tau), y)\, d\sigma_y \text{ in mean-square},$$

and the Feynman-Kac formula, one can prove that the characteristic function of the random value l_t, that is $v(t,x,\theta) = \mathbb{E}_x \exp\{i\theta l_t\}$, satisfies the following equation

$$v(t,x,\theta) = 1 + i\theta \int_0^t d\tau \int_S g(t-\tau, x, y) v(\tau, y, \theta)\, d\sigma_y. \tag{2}$$

2 THE MAIN RESULTS

The first statement concerns to the rotationally invariant α-stable random process.

Theorem 1. *The limit distribution with respect to the measure $\mathbb{P}_x$ of the random variables sequence $n^{-1+1/\alpha}\zeta^{(n)}_{[nt]}$ for fixed $t>0$ and $x\in\mathbb{R}^d$ has the characteristic function $(u(t,x,\theta))_{\theta\in\mathbb{R}}$, which is the unique bounded solution of the integral equation*

$$u(t,x,\theta)=1+i\varkappa\theta\int_0^t d\tau\int_S g(t-\tau,x,y)u(\tau,y,\theta)\,d\sigma_y,$$

where $\varkappa=\frac{2c^{1/\alpha}}{\pi}\Gamma(1-1/\alpha)$. This distribution coincides with the distribution of the multiplied by $\varkappa$ local time on the hyperplane S of the process $(x(t))_{t\geq 0}$.

Next, let a continuous bounded function $(r(x))_{x\in S}$ with non-negative values be given. Consider the function $(G(t,x,y))_{t>0,x\in\mathbb{R}^d,y\in\mathbb{R}^d}$ which is a solution of to each one of the following equations

$$G(t,x,y)=g(t,x,y)-\int_0^t d\tau\int_S g(t-\tau,x,z)G(\tau,z,y)r(z)\,d\sigma_z,$$
$$G(t,x,y)=g(t,x,y)-\int_0^t d\tau\int_S G(t-\tau,x,z)g(\tau,z,y)r(z)\,d\sigma_z.$$

The function G is the transition probability density of the process $(x(t))_{t\geq 0}$ killed on the hyperplane S at some stopping time ζ (see [6]). The function $(r(x))_{x\in S}$ is the killing intensity of the process $(x(t))_{t\geq 0}$. It is clear that

$$\mathbb{P}_x(\{\zeta>t\})=\int_{\mathbb{R}^d}G(t,x,y)\,dy=1-\int_0^t d\tau\int_S G(\tau,x,y)r(y)\,d\sigma_y.$$

Theorem 2. *The limit distribution with respect to the measure $\mathbb{P}_x$ of the random variables sequence $n^{-1+1/\alpha}\zeta^{(n)}_{[nt]}$ for fixed $t>0$ and $x\in\mathbb{R}^d$ has the characteristic function $(u(t,x,\theta))_{\theta\in\mathbb{R}}$, which is the unique bounded solution of the integral equation*

$$u(t,x,\theta)=1+i\varkappa\theta\int_0^t d\tau\int_S G(t-\tau,x,y)u(\tau,y,\theta)\,d\sigma_y,$$

where $\varkappa=\frac{2c^{1/\alpha}}{\pi}\Gamma(1-1/\alpha)$. It is the distribution of the multiplied by $\varkappa$ local time on the hyperplane S for the process $(x(t))_{t\geq 0}$ killed at the stopping time ζ.

And the last, let a continuous bounded function $(q(x))_{x\in S}$ be given. Introduce an operator $\mathbf{B}_\nu$ determined by its symbol $(i|\xi|^{\alpha-2}(\xi,2c\nu))_{\xi\in\mathbb{R}^d}$. Define the function $(G(t,x,y))_{t>0,x\in\mathbb{R}^d,y\in\mathbb{R}^d}$ by the following formula

$$G(t,x,y)=g(t,x,y)+\int_0^t d\tau\int_S g(t-\tau,x,z)\mathbf{B}_\nu g(\tau,\cdot,y)(z)q(z)\,d\sigma_z.$$

This function is "a transition probability density" of some pseudo-process with a membrane on the hyperplane S (see [7]). The generator of this pseudo-process can be written in the following form: $\mathbf{A}+q(x)\delta_S(x)\mathbf{B}_\nu$, where $\mathbf{A}$ is the generator of the process $(x(t))_{t\geq 0}$ (that is a pseudo-differential operator whose symbol is given by the function $(-c|\xi|^\alpha)_{\xi\in\mathbb{R}^d}$).

Consider the function $(u(t,x,\theta))_{t\geq 0, x\in\mathbb{R}^d, \theta\in\mathbb{R}}$ defined by the equality

$$u(t,x,\theta) = \lim_{n\to\infty} \hat{\mathbb{E}}_x \exp\left\{i\theta n^{-1+1/\alpha}\eta^{(n)}_{[nt]}\right\} \overset{def}{=}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{k=1}^{[nt]} \exp\left\{i\theta n^{-1+1/\alpha}\hat{v}_n(x_k)\right\} G\left(\frac{1}{n}, x_{k-1}, x_k\right) dx_k,$$

where $x_0 = x$ and $\hat{v}_n(x) = \hat{\mathbb{E}}_x v\left(x(0), x\left(\frac{1}{n}\right)\right) \overset{def}{=} \int_{\mathbb{R}^d} v(x,y) G\left(\frac{1}{n}, x, y\right) dy$. This function is "the characteristic function" of the the random variables sequence $n^{-1+1/\alpha}\zeta^{(n)}_{[nt]}$ limit "distribution" for fixed $t>0$ and $x\in\mathbb{R}^d$.

Here we use quotes with notions that apply to the pseudo-process, similar to the ordinary random process. These notions must be understood in some special way described above.

Theorem 3. *The function $(u(t,x,\theta))_{\theta\in\mathbb{R}}$ for fixed $t>0$ and $x\in\mathbb{R}^d$ is the unique bounded solution of the integral equation*

$$u(t,x,\theta) = 1 + i\varkappa\theta \int_0^t d\tau \int_S g(t-\tau,x,y) u(\tau,y,\theta)(1-q^2(y))\, d\sigma_y,$$

where $\varkappa = \frac{2c^{1/\alpha}}{\pi}\Gamma(1-1/\alpha)$.

3 Proof of the main results

The proofs of these results are executed according to the same scheme. Consider the first result (i.e. it is for the rotationally invariant α-stable random process).

First of all, one can prove two technical lemmas. The first one prompts us that we must choose $c_n = n^{-1+1/\alpha}$. And the second one allows to pass from equation (1) to some simpler one.

Lemma 2. *Let the real-valued function $(\varphi(x))_{x\in\mathbb{R}^d}$ be such that $\sup\limits_{\rho\in\mathbb{R}} \int_{S_\rho} |\varphi(x)|\, d\sigma < \infty$, where $S_\rho = \{x\in\mathbb{R}^d : (x,\nu) = \rho\}$, and there exist the nontangentional limits $\varphi(x-)$ and $\varphi(x+)$ from the side of D_- and D_+ in each point $x\in S$.*

Then the following relation (with $\varkappa = \mathbb{E}_0|(x(1),\nu)| = \frac{2c^{1/\alpha}}{\pi}\Gamma(1-1/\alpha)$)

$$\lim_{n\to\infty} n^{1/\alpha} \int_{\mathbb{R}^d} v_n(x)\varphi(x)\, dx = \varkappa \int_S \frac{\varphi(y-)+\varphi(y+)}{2}\, d\sigma$$

holds true. In addition, the inequality $\left|n^{1/\alpha}\int_{\mathbb{R}^d} v_n(x)\varphi(x)\, dx\right| \leq \frac{\varkappa}{2} \sup\limits_{\rho\in\mathbb{R}} \int_{S_\rho} |\varphi(x)|\, d\sigma$ is fulfilled.

Let a measurable function $(\psi(t,x))_{t\geq 0, x\in\mathbb{R}^d}$ be such that $\sup\limits_{t\in[0,T], x\in\mathbb{R}^d} |\psi(t,x)| < \infty$ for any $T>0$. Consider its transformation Ψ_n for $n\in\mathbb{N}$ given by

$$\Psi_n(t,x) = n^{1/\alpha} \int_0^t d\tau \int_{\mathbb{R}^d} v_n(y)\psi(\tau,y) g(t-\tau,x,y)\, dy, \quad t>0,\ x\in\mathbb{R}^d.$$

Lemma 3. *For given numbers $\varepsilon > 0$, $L > 0$, $T > 0$, there exists a number $\delta > 0$ such that the inequality $|\Psi_n(t', x') - \Psi_n(t, x)| < \varepsilon$ is held for all $t \in [0, T]$, $t' \in [0, T]$, $x \in \mathbb{R}^d$, $x' \in \mathbb{R}^d$, $n \in \mathbb{N}$ and all measurable functions ψ with the property $\sup\limits_{t\in[0,T],x\in\mathbb{R}^d} |\psi(t, x)| \leq L$ if only the inequality $|t - t'| + |x - x'| < \delta$ is fulfilled.*

Next, using Lemma 3 one can easily prove that solutions of equation (1) for the characteristic function $u_n(t, x, \theta)$ of $n^{-1+1/\alpha}\eta^{(n)}_{[nt]}$ and solutions of the following equation

$$u^*_n(t, x, \theta) = 1 + i\theta n^{1/\alpha} \int_0^t d\tau \int_{\mathbb{R}^d} v_n(y) u^*_n(\tau, y, \theta) g(t - \tau, x, y)\, dy$$

satisfy the relation $\lim_{n\to\infty} \sup\limits_{x\in\mathbb{R}^d} \sup\limits_{0<t\leq T} \sup\limits_{\theta_1\leq\theta\leq\theta_2} |u_n(t, x, \theta) - u^*_n(t, x, \theta)| = 0$ for any $T > 0$, $\theta_k \in \mathbb{R}$ $(k = 1, 2)$, $\theta_1 < \theta_2$.

As the corollary of Lemma 2 one can say that the characteristic function $(u(t, x, \theta))_{\theta\in\mathbb{R}}$ (t and x are fixed) of the limit distribution with respect to the measure $\mathbb{P}_x$ for the sequence of the random variables $n^{-1+1/\alpha}\xi^{(n)}_{[nt]}$ (and $n^{-1+1/\alpha}\eta^{(n)}_{[nt]}$ also) satisfies the following equation

$$u(t, x, \theta) = 1 + i\theta\varkappa \int_0^t d\tau \int_S g(t - \tau, x, y) u(\tau, y, \theta)\, d\sigma_y. \tag{3}$$

A solution of equation (3) can be constructed by the method of successive approximations, that is we have $u(t, x, \theta) = \sum_{k=0}^{\infty} u^{(k)}(t, x, \theta)(i\theta\varkappa)^k$, where $u^{(0)}(t, x, \theta) \equiv 1$, $u^{(k)}(t, x, \theta) = \int_0^t d\tau \int_S g(t - \tau, x, y) u^{(k-1)}(\tau, y, \theta)\, d\sigma_y$.

This follows from the estimation $|u^{(k)}(t, x, \theta)| \leq C^k \frac{(\Gamma(\beta))^k}{\Gamma(1+k\beta)} t^{k\beta}$, getting by the induction, where $C > 0$ is some constant, $\beta = 1 - 1/\alpha$.

The solution of equation (3) is unique in the class of bounded functions, because the difference between each two solutions of equation (3) satisfies the following equation

$$w(t, x, \theta) = i\theta\varkappa \int_0^t d\tau \int_S g(t - \tau, x, y) w(\tau, y, \theta)\, d\sigma_y$$

and we have inequalities $|w(t, x, \theta)| \leq \frac{(C\theta\varkappa\Gamma(\beta))^k}{\Gamma(1+k\beta)} t^{k\beta}$ for each $k \in \mathbb{N}$.

Comparing equations (3) and (2) we get that the distribution of $\varkappa l_t$ and the limit distribution of $n^{-1+1/\alpha}\xi^{(n)}_{[nt]}$ (with respect to the measure $\mathbb{P}_x$) are equal.

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Досліджено числа перетинів гіперплощини дискретними наближеннями траекторій α-стійкого випадкового процесу ($1 < \alpha < 2$) та деяких пов'язаних з ним процесів. Розглядаються α-стійкий випадковий процес з убиванням з даною інтенсивністю на гіперплощині та псевдопроцес, утворений з α-стійкого випадкового процесу збуренням його оператором дробової похідної з множником типу дельта-функції на гіперплощині. В кожному з цих випадків граничний розподіл кількості перетинів гіперплощини деякою дискретною апроксимацією процесу пов'язаний з розподілом його локального часу на цій гіперплощині. Побудовані інтегральні рівняння для характеристичних функцій цих розподілів. Єдині обмежені розв'язки цих рівнянь можна одержати методом послідовних наближень.

Ключові слова і фрази: α-стабільний процес, локальний час, псевдо-процес.

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PROKIP V.M.

ON THE SIMILARITY OF MATRICES AB AND BA OVER A FIELD

Let A and B be n-by-n matrices over a field. The study of the relationship between the products of matrices AB and BA has a long history. It is well-known that AB and BA have equal characteristic polynomials (and, therefore, eigenvalues, traces, etc.). One beautiful result was obtained by H. Flanders in 1951. He determined the relationship between the elementary divisors of AB and BA, which can be treated as a criterion when two matrices C and D can be realized as $C = AB$ and $D = BA$. If one of the matrices (A or B) is invertible, then the matrices AB and BA are similar. If both A and B are singular then matrices AB and BA are not always similar. We give conditions under which matrices AB and BA are similar. The rank of matrices plays an important role in these investigations.

Key words and phrases: matrix, similarity, rank.

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1 INTRODUCTION

Let F be a field and let $M_{m,n}(\mathrm{F})$ denote the set of m-by-n matrices with entries from F. In what follows, $GL(n, \mathrm{F})$ the group of nonsingular matrices in $M_{n,n}(\mathrm{F})$, I_k is the identity $k \times k$ matrix, and $0_{m,n}$ is the zero $m \times n$ matrix.

Let $A, B \in M_{n,n}(\mathrm{F})$. It is well known that the characteristic polynomials of AB and BA are the same (see, for example, [6, 9, 10, 14]). If one of the matrices (A or B) is invertible, then the matrices AB and BA are similar. If both A and B are singular then matrices AB and BA are not always similar (see [6, Sec. 1.3]). It is clear that matrices AB and BA are similar if and only if the matrix polynomials $I_n\lambda - AB$ and $I_n\lambda - BA$ are equivalent. It is evident, if matrices A and B commute then AB and BA are similar.

Let $A \in M_{n,m}(\mathrm{F})$ and $B \in M_{m,n}(\mathrm{F})$. In paper [3], H. Flanders solved the problem of determining the relationship between the elementary divisors of AB and those of BA. Another proof of Flanders' theorem, with some generalizations, has been given in [11] (see also [1]). Robert C. Thompson [13] proposed a new proof of Flanders' theorem. It is obvious that some connection exists between the ranks of A and B and the intertwining of the elementary divisors of AB and BA. A constructive proof of Flanders' theorem was also given in [7]. Using the Weyr characteristic the relationship between the Jordan forms of the matrix products AB and BA for matrices A and B was given in [8]. Robert E. Hartwig [5] generalizes Flanders' result for matrices over a regular strongly-pi-regular ring. It will be observed that an extension of these results to rings would be valuable and interesting. The rank conditions under which matrices AB and BA are similar were proposed in [2, 3, 13].

Suppose that A and B are complex $n \times n$ matrices. The matrix AB is similar to BA if and only if $\operatorname{rank}(AB)^j = \operatorname{rank}(BA)^j$ for each $j = 1, 2, \ldots, n$ (see [6, Sec. 3]). If A is positive semidefinite matrix and B is normal matrix, in [4] it has been proved that AB and BA are

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similar. The smallest nonnegative integer k such that rank A^{k+1} = rank A^k, is the index for A and denoted by $\operatorname{Ind}(A)$. In [8] was proved that matrices AB and BA are similar if and only if $\operatorname{Ind}(AB) = \operatorname{Ind}(BA) = k$ and rank $(AB)^i$ = rank $(BA)^i$ for all $i = 1, 2, \ldots, k-1$.

In this note we investigate the following widely known question: Let $A, B \in M_{n,n}(\mathrm{F})$. When are matrices AB and BA similar? We give conditions in terms of rank matrices, under which matrices AB and BA are similar. If matrices AB and BA are similar we give their canonical form with respect to similarity.

2 MAIN RESULTS

Let $A, B \in M_{n,n}(\mathrm{F})$ be singular matrices and let rank $A = r$. We introduce the following notation for the matrices A and B. For A there exist matrices $U, V \in GL(n, \mathrm{F})$ such that

$$UAV = \begin{bmatrix} I_r & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix}.$$

Put $V^{-1}BU^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$, where $B_{11} \in M_{r,r}(\mathrm{F})$. It is easy to make sure that

$$UABU^{-1} = C = \begin{bmatrix} B_{11} & B_{12} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix} \tag{1}$$

and

$$V^{-1}BAV = D = \begin{bmatrix} B_{11} & 0_{r,n-r} \\ B_{21} & 0_{n-r,n-r} \end{bmatrix}. \tag{2}$$

We will use these notations to give the characterization of similarity of matrices AB and BA. Thus, AB and BA are similar if and only if the polynomial matrices $I_n\lambda - C$ and $I_n\lambda - D$ are equivalent, i.e. the Smith normal forms of these polynomial matrices are coincide.

In view of the above, we give the following description of similarity of the matrices AB and BA.

Theorem 1. *Let $A, B \in M_{n,n}(\mathrm{F})$ be singular matrices. If*

(a) rank B_{11} = rank $\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$ = rank $\begin{bmatrix} B_{11} & B_{12} \end{bmatrix}$*, or*

(b) $B_{11} = 0_{r,r}$ *and* rank B_{21} = rank B_{12}*, or*

(c) the matrix $\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & 0_{n-r,n-r} \end{bmatrix}$ *is symmetric,*

then matrices AB and BA are similar.

Proof. (a) Since rank B_{11} = rank $\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$ = rank $\begin{bmatrix} B_{11} & B_{12} \end{bmatrix}$, then the equations $XB_{11} = B_{21}$ and $B_{11}Y = B_{12}$ are solvable. Let matrices $X_1 \in M_{n-r,r}(\mathrm{F})$ and $Y_1 \in M_{r,n-r}(\mathrm{F})$ be the solutions to these equations respectively.

For matrix $T_1 = \begin{bmatrix} I_r & 0_{r,n-r} \\ -X_1 & I_{n-r} \end{bmatrix}$ we have

$$T_1 \begin{bmatrix} B_{11} & 0_{r,n-r} \\ B_{21} & 0_{n-r,n-r} \end{bmatrix} T_1^{-1} = \begin{bmatrix} B_{11} & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix}.$$

Similarly, for matrix $T_2 = \begin{bmatrix} I_r & -Y_1 \\ 0_{n-r,r} & I_{n-r} \end{bmatrix}$ we have

$$T_2^{-1} \begin{bmatrix} B_{11} & B_{12} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix} T_2 = \begin{bmatrix} B_{11} & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix}.$$

Hence, matrices $\begin{bmatrix} B_{11} & B_{12} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix}$ and $\begin{bmatrix} B_{11} & 0_{r,n-r} \\ B_{21} & 0_{n-r,n-r} \end{bmatrix}$ are similar. Thus, AB and BA are similar.

(b) Let $B_{11} = 0_{r,r}$ and $\operatorname{rank} B_{21} = \operatorname{rank} B_{12} = s$. For B_{12} there exist matrices $U_1 \in GL(r, \mathrm{F})$ and $V_1 \in GL(n-r, \mathrm{F})$ such that

$$U_1 B_{12} V_1 = \begin{bmatrix} 0_{s,n-r-s} & I_s \\ 0_{r-s,n-r-s} & 0_{r-s,s} \end{bmatrix}.$$

Thus, for the matrix $T_1 = \operatorname{diag}\left(U_1,\ V_1^{-1}\right)$ we have

$$T_1 \begin{bmatrix} 0_{r,r} & B_{12} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix} T_1^{-1} = \begin{bmatrix} 0_{s,n-s} & I_s \\ 0_{n-s,n-s} & 0_{n-s,s} \end{bmatrix}.$$

Similarly, for matrix B_{21} there exist $U_2 \in GL(n-r, \mathrm{F})$ and $V_2 \in GL(r, \mathrm{F})$ such that

$$U_2 B_{12} V_2 = \begin{bmatrix} 0_{n-r-s,s} & 0_{n-r-s,r-s} \\ I_s & 0_{s,r-s} \end{bmatrix}$$

and for the matrix $T_2 = \operatorname{diag}\left(V_2^{-1},\ U_2\right)$ we have

$$T_2 \begin{bmatrix} 0_{r,r} & B_{12} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix} T_2^{-1} = \begin{bmatrix} 0_{n-s,s} & 0_{n-s,n-s} \\ I_s & 0_{s,n-s} \end{bmatrix}.$$

It is obvious that matrices $\begin{bmatrix} 0_{r,r} & B_{12} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix}$ and $\begin{bmatrix} 0_{r,r} & 0_{r,n-r} \\ B_{21} & 0_{n-r,n-r} \end{bmatrix}$ are similar. Thus, AB and BA are similar.

(c) Matrix $\begin{bmatrix} B_{11} & B_{12} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix}$ and its transpose $\begin{bmatrix} B_{11} & B_{12} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix}^T$ are similar. Hence, we have

$$\begin{bmatrix} B_{11} & B_{12} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix}^T = \begin{bmatrix} B_{11}^T & 0_{r,n-r} \\ B_{12}^T & 0_{n-r,n-r} \end{bmatrix} = \begin{bmatrix} B_{11} & 0_{r,n-r} \\ B_{21} & 0_{n-r,n-r} \end{bmatrix}.$$

Thus, matrices AB and BA are similar. The proof of Theorem 1 is complete. □

From Theorem 1 we have the following statement.

Corollary 1. *Let $A, B \in M_{n,n}(\mathrm{F})$ be singular matrices. If $\det B_{11} \neq 0$ then matrices AB and BA are similar.*

Consider the following example.

Example. *Let* $\mathrm{F} = \mathbb{Q}$ *be the field of rational numbers and let*

$$A = \begin{bmatrix} 7 & -3 & -11 & 9 \\ 5 & -2 & -10 & 8 \\ -12 & 5 & 21 & -17 \\ 12 & -5 & -16 & 13 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} -23 & -18 & -2 & 16 \\ -55 & -43 & -5 & 38 \\ -65 & -52 & -4 & 48 \\ -80 & -64 & -5 & 59 \end{bmatrix}$$

be matrices over $\mathbb{Q}$. *For nonsingular matrices*

$$U = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix} \quad \textit{and} \quad V = \begin{bmatrix} 3 & 2 & 1 & 2 \\ 7 & 5 & 2 & 5 \\ 0 & 0 & 9 & 4 \\ 0 & 0 & 11 & 5 \end{bmatrix}$$

over $\mathbb{Q}$ *we have*

$$UAV = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0_{3,1} \\ 0_{3,1} & 0 \end{bmatrix}$$

and

$$V^{-1}BU^{-1} = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 1 \\ \hline 0 & 1 & 2 & 2 \end{array}\right] = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \textit{ where } B_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}.$$

Thus, $\operatorname{rank} B_{11} = \operatorname{rank} \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} B_{11} & B_{12} \end{bmatrix} = 2$. *By statement (a) of Theorem 1 matrices* AB *and* BA *are similar to the matrix* B_{11}.

Lemma 1. *Let* $A, B \in M_{n,n}(\mathrm{F})$ *be singular matrices. If* $\operatorname{rank} AB = \operatorname{rank} BA = 1$, *then* AB *and* BA *are similar.*

To prove the Lemma we need the following proposition (see also Chapter 2 in [6] and Theorem 1 in [12]).

Proposition 1. *Let* $C \in M_{n,n}(\mathrm{F})$ *be a matrix of rank one and* $\operatorname{tr} C = c$. *The matrix* C *is similar to one of the matrices*

$$D_1 = \operatorname{diag}(c, 0, \ldots, 0) \ \textit{if } c \neq 0$$

or

$$D_2 = \operatorname{diag}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ 0, \ \ldots, \ 0 \right) \ \textit{if } c = 0.$$

Proof. The proof of the Proposition is algorithmic. The matrix C we write in the form $C = \overline{p} \cdot \overline{q}$, where $\overline{p} \in M_{n,1}(\mathrm{F})$ and $\overline{q} \in M_{1,n}(\mathrm{F})$. For the vector $\overline{p}$ there exists a matrix $P \in GL(n, \mathrm{F})$ such that $P \cdot \overline{p} = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^T$. Then C is similar to a matrix of the form

$$PCP^{-1} = P\overline{p} \cdot \overline{q}P^{-1} = C_1 = \left[\begin{array}{c|ccc} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ \hline 0_{n-1,1} & & 0_{n-1,n-1} & \end{array}\right]. \tag{3}$$

It is clear that $\alpha_{11} = c$ is a trace of the matrix C.

Suppose, $c \neq 0$. For the matrix $T_1 = \left[\begin{array}{c|c} 1 & 0_{1,n-1} \\ \hline -\frac{\alpha_{12}}{c} & \\ \vdots & I_{n-1} \\ -\frac{\alpha_{1n}}{c} & \end{array}\right] \in GL(n, \mathrm{F})$ we have

$$T_1^{-1} C_1 T_1 = \operatorname{diag}(c, 0, \ldots, 0) = D_1.$$

Thus, if $\operatorname{tr} C = c \neq 0$, then matrices C and D_1 are similar.

Let $\operatorname{tr} C = 0$. From equality (3) it follows

$$C_1 = \left[\begin{array}{c|ccc} 0 & \alpha_{12} & \dots & \alpha_{1n} \\ \hline 0_{n-1,1} & & 0_{n-1,n-1} & \end{array}\right].$$

For elements $\{\alpha_{12}, \alpha_{13}, \ \dots, \ \alpha_{1n}\}$ there exists a matrix $T_0 \in GL(n-1, \mathrm{F})$ such that $\left[\ \alpha_{12}, \ \ \alpha_{13}, \ \ \dots, \ \ \alpha_n \ \right] T_0 = \left[\ 1 \ \ 0 \ \ \dots \ \ 0\ \right]$. Thus, for the matrix

$$T_2 = \left[\begin{array}{c|c} 1 & 0_{1,n-1} \\ \hline 0_{n-1,1} & T_0 \end{array}\right] \in GL(n, \mathrm{F})$$

we have

$$T_2^{-1} C_1 T_2 = \operatorname{diag}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ 0, \ \dots, \ 0 \right) = D_2.$$

Since $\operatorname{tr} C = 0$, matrices C and D_2 are similar. This completes the proof of the Proposition. □

Proof. Let $A, B \in M_{n,n}(\mathrm{F})$ be singular matrices and

$$\operatorname{rank} AB = \operatorname{rank} BA = 1.$$

Suppose $\operatorname{rank} B \geq \operatorname{rank} A = r$. Matrix AB is similar to the matrix

$$C = \begin{bmatrix} B_{11} & B_{12} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix},$$

where $B_{11} \in M_{r,r}(\mathrm{F})$ (see equalities (1) and (2)). Similarly BA is similar to the matrix

$$D = \begin{bmatrix} B_{11} & 0_{r,n-r} \\ B_{21} & 0_{n-r,n-r} \end{bmatrix}.$$

Thus, $\operatorname{tr} AB = \operatorname{tr} BA = \operatorname{tr} B_{11}$. Put $\operatorname{tr} B_{11} = c$.

Suppose $c \neq 0$. By Proposition 1 matrices AB and BA are similar to the matrix $D_1 = \operatorname{diag}(c, 0, \dots, 0)$.

If $c = 0$ then by Proposition matrices AB and BA are similar to the matrix $D_2 = \operatorname{diag}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ 0, \ \dots, \ 0 \right)$, which completes the proof of the Lemma. □

Corollary 2. *Let $A, B \in M_{n,n}(\mathrm{F})$ be singular matrices and $\operatorname{rank} A = 1$. If $AB \neq 0_{n,n}$ and $BA \neq 0_{n,n}$ then AB and BA are similar.*

Corollary 3. *Let $A, B \in M_{2,2}(\mathrm{F})$. If $AB \neq 0_{2,2}$ and $BA \neq 0_{2,2}$ then AB and BA are similar.*

Theorem 2. *Let $A, B \in M_{n,n}(\mathrm{F})$ and let $\operatorname{rank} A = 2$. If $\operatorname{rank} AB = \operatorname{rank} BA$ then AB and BA are similar.*

Proof. If $\operatorname{rank} AB = \operatorname{rank} BA = 1$ then by Lemma 1 matrices AB and BA are similar. Suppose $\operatorname{rank} AB = \operatorname{rank} BA = 2$. Matrix AB is similar to the matrix $C = \begin{bmatrix} B_{11} & B_{12} \\ 0_{n-2,2} & 0_{n-2,n-2} \end{bmatrix}$, where $B_{11} \in M_{2,2}(\mathrm{F})$ (see equalities (1) and (2)). Similarly, BA is similar to the matrix $D = \begin{bmatrix} B_{11} & 0_{2,n-2} \\ B_{21} & 0_{n-2,n-2} \end{bmatrix}$. Thus, $\operatorname{rank} \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} B_{11} & B_{12} \end{bmatrix} = 2.$

If $B_{11} = 0_{2,2}$ or $\det B_{11} \neq 0$ then by Theorem 1b or Corollary 1 respectively matrices AB and BA are similar. Let $\operatorname{rank} B_{11} = 1$ and let $\operatorname{tr} B_{11} \neq 0$. For B_{11} there exists a matrix $U_{11} \in GL(2, \mathrm{F})$ such that

$$U_{11} B_{11} U_{11}^{-1} = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix},$$

where $\alpha = \operatorname{tr} B_{11}$. For the matrix $T_{11} = \begin{bmatrix} U_{11} & 0_{2,n-2} \\ 0_{n-2,2} & I_{n-2} \end{bmatrix}$ we have

$$T_{11} C T_{11}^{-1} = C_{11} = \left[\begin{array}{cc|c} \alpha & 0 & \multirow{2}{*}{$\widetilde{B}_{12}$} \\ 0 & 0 & \\ \hline \multicolumn{2}{c|}{0_{n-2,1}} & 0_{n-2,n-1} \end{array} \right],$$

where $\widetilde{B}_{12} = B_{12} U_1^{-1}$. It is evident that $\operatorname{rank} C_{11} = 2$. It is easy to make sure that if $n = 3$ then $\widetilde{B}_{12} = \begin{bmatrix} c_{13} & c_{23} \end{bmatrix}^T$ and $c_{23} \neq 0$. For $\widetilde{B}_{12}$ the exists a matrix $U_{12} \in GL(n-2, \mathrm{F})$ such that

$$\widetilde{B}_{12} U_{12} = \begin{bmatrix} \alpha_1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{bmatrix}.$$

Thus, for the matrix $T_{12} = \begin{bmatrix} I_2 & 0_{2,n-2} \\ 0_{n-2,2} & U_{12} \end{bmatrix}$ we have

$$T_{12}^{-1} C_{11} T_{12} = C_{12} = \left[\begin{array}{cccc|c} \alpha & 0 & \alpha_1 & 0 & \multirow{2}{*}{$0_{2,n-4}$} \\ 0 & 0 & 0 & 1 & \\ \hline \multicolumn{4}{c|}{0_{n-2,4}} & 0_{n-2,n-4} \end{array} \right].$$

It is obvious that matrix C_{12} is similar to the matrix $C_{13} = \left[\begin{array}{ccc|c} \alpha & 0 & 0 & \multirow{2}{*}{$0_{2,n-3}$} \\ 0 & 0 & 1 & \\ \hline \multicolumn{3}{c|}{0_{n-2,4}} & 0_{n-2,n-3} \end{array} \right].$

It may be noted that matrices D and D^T are similar. Reasoning similarly we convince ourselves that the matrix $\begin{bmatrix} B_{11}^T & B_{21}^T \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix}$ is similar to the matrix C_{13}. Thus, in the case when $\operatorname{tr} B_{11} \neq 0$, matrices C and D are similar.

Let us now consider the case when $\operatorname{rank} B_{11} = 1$ and $\operatorname{tr} B_{11} = 0$. For B_{11} there exists a matrix $V_{11} \in GL(2, \mathrm{F})$ such that

$$V_{11} B_{11} V_{11}^{-1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

For the matrix $S_{11} = \begin{bmatrix} V_{11} & 0_{2,n-2} \\ 0_{n-2,2} & I_{n-2} \end{bmatrix}$ we have

$$S_{11} C S_{11}^{-1} = C_{21} = \left[\begin{array}{cc|c} 0 & 1 & \multirow{2}{*}{$\widehat{B}_{12}$} \\ 0 & 0 & \\ \hline \multicolumn{2}{c|}{0_{n-2,1}} & 0_{n-2,n-1} \end{array} \right], \quad \text{where} \quad \widehat{B}_{12} = B_{12} V_{11}^{-1}.$$

Obviously that $\operatorname{rank} C_{21} = 2$. We note, if $n = 3$ then $\widehat{B}_{12} = \begin{bmatrix} c_{13} & c_{23} \end{bmatrix}^T$ and $c_{23} \neq 0$.

For $\widehat{B}_{12}$ the exists a matrix $V_{12} \in GL(n-2, \mathrm{F})$ such that

$$\widehat{B}_{12} V_{12} = \begin{bmatrix} \beta_1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{bmatrix}.$$

Thus, for the matrix $S_{12} = \begin{bmatrix} I_2 & 0_{2,n-2} \\ 0_{n-2,2} & V_{12} \end{bmatrix}$ we have

$$S_{12}^{-1} C_{21} S_{12} = C_{22} = \left[\begin{array}{cccc|c} 0 & 1 & \beta_1 & 0 & \multirow{2}{*}{$0_{2,n-4}$} \\ 0 & 0 & 0 & 1 & \\ \hline \multicolumn{4}{c|}{0_{n-2,4}} & 0_{n-2,n-4} \end{array}\right].$$

It is evident that matrix C_{22} is similar to the matrix $C_{23} = \left[\begin{array}{ccc|c} 0 & 1 & 0 & \multirow{2}{*}{$0_{2,n-3}$} \\ 0 & 0 & 1 & \\ \hline \multicolumn{3}{c|}{0_{n-2,4}} & 0_{n-2,n-3} \end{array}\right].$

Reasoning similarly, we can prove that matrix $\begin{bmatrix} B_{11}^T & B_{21}^T \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix}$ is similar to the matrix C_{23}. Thus in the case when $\operatorname{tr} B_{11} = 0$ matrices C and D are similar.

So, we have that matrices AB and BA are similar and the proof of Theorem 2 is complete. $\square$

From Theorem 2 we have the following statement.

Corollary 4. *Let $A, B \in M_{3,3}(\mathrm{F})$. If $\operatorname{rank} AB = \operatorname{rank} BA$ then matrices AB and BA are similar.*

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Прокіп В.М. *Про подібність матриць AB і BA над полем* // Карпатські матем. публ. — 2018. — Т.10, №2. — С. 352–359.

Нехай A і B — $n \times n$ матриці над полем. Вивчення зв'язків між добутками матриць AB і BA має давню історію. Загальновідомо, що матриці AB та BA мають однакові характеристичні многочлени (отже, власні значення, сліди тощо). Один вагомий результат був отриманий Х. Фландерсом у 1951 році. Він вказав зв'язок між елементарними дільниками AB та BA, який можна розглядати як критерій, коли дві матриці C і D можуть бути зображені у вигляді добутків $C = AB$ і $D = BA$. Якщо одна з матриць (A або B) є неособливою, то матриці AB і BA подібні. Якщо ж A і B особливі матриці, то матриці AB і BA не завжди подібні. В статті наведено умови, за яких матриці AB і BA подібні. Поняття рангу відіграє важливу роль у цих дослідженнях.

Ключові слова і фрази: матриця, подібність, ранг.

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GREEN-RVACHEV'S QUASI-FUNCTION METHOD FOR CONSTRUCTING TWO-SIDED APPROXIMATIONS TO POSITIVE SOLUTION OF NONLINEAR BOUNDARY VALUE PROBLEMS

A homogeneous Dirichlet problem for a semilinear elliptic equations with the Laplace operator and Helmholtz operator is investigated. To construct the two-sided approximations to a positive solution of this boundary value problem the transition to an equivalent nonlinear integral equation (with the help of the Green-Rvachev's quasi-function) with its subsequent analysis by methods of the theory of semi-ordered spaces is used. The work and efficiency of the developed method are demonstrated by a computational experiment for a test problem with exponential nonlinearity.

Key words and phrases: positive solution, semilinear elliptic boundary value problem, heterotone operator, two-sided approach, Green-Rvachev's quasi-function.

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INTRODUCTION

Let us consider the problem of finding a positive solution of a semilinear elliptic equation with a homogeneous Dirichlet condition:

$$\mathfrak{L}u = f(\mathbf{x}, u), \quad \mathbf{x} \in \Omega, \tag{1}$$

$$u(\mathbf{x}) > 0, \quad \mathbf{x} \in \Omega, \tag{2}$$

$$u|_{\partial\Omega} = 0, \tag{3}$$

where $\mathfrak{L}u \equiv -\Delta u$ or $\mathfrak{L}u \equiv -\Delta u + \kappa^2 u$, Ω is a bounded Jordan-measurable domain from $\mathbb{R}^2$ or $\mathbb{R}^3$ with piecewise smooth boundary $\partial\Omega$ ($\bar{\Omega} = \Omega \cup \partial\Omega$), Δ is the Laplace operator, $\mathbf{x} = (x_1, x_2)$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$, if $\Omega \subset \mathbb{R}^2$, and $\mathbf{x} = (x_1, x_2, x_3)$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$, if $\Omega \subset \mathbb{R}^3$.

Let us assume that the function $f(\mathbf{x}, u)$ is continuous and positive for $\mathbf{x} \in \bar{\Omega}$, $u > 0$.

The problem (1)–(3) is often encountered in the mathematical modeling of nonlinear stationary processes considered in thermophysics, electromagnetism, biology, chemical kinetics, etc. [11]. In this case, the condition of positivity (2) naturally arises from the meaning of the function u in a particular applied field. It is convenient to carry out the analysis of the problem by the methods of the theory of nonlinear operators in semi-ordered spaces [1, 5, 9, 10], passing to the equivalent Hammerstein integral equation with the help of the Green's functions method. In this case, it is possible to construct a two-sided iteration process to the desired solution [4, 14]. But, the practical application of this approach has certain limitations due to

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the fact that the analytic expression of the Green's function must be known. It is possible to get rid of this restriction, if the corresponding Green-Rvachev's quasi-function [12, 14] will be used instead of Green's function.

The purpose of the paper is to develop the iterative methods for solving the boundary value problem (1)–(3), which have a two-sided nature of convergence to the desired solution and are not tied to the presence of a known Green's function. Two-sided approximate methods for solving nonlinear operator equations based on the theory of nonlinear operators in semi-ordered spaces were developed in [3, 6, 13, etc.]. This paper continues the research begun in [4, 14], and extends them to the areas of arbitrary geometry and elliptic equations with the Helmholtz operator.

1 Construction of an equivalent integral equation

To construct an integral equation that is equivalent to the problem (1)–(3), let us use the Green-Rvachev's quasi-function [12, 14].

Let the boundary $\partial\Omega$ of the domain Ω consists of a finite number of pieces of lines $\sigma_i(\mathbf{x}) = 0$, $i = 1, 2, \ldots, r$, where each $\sigma_i(\mathbf{x})$ is an elementary function. Then with the help of the R-functions method [12] one can construct in the form of a single analytic expression an elementary function $\omega(\mathbf{x})$, which describes the geometry of the domain Ω, that is:

a) $\omega(\mathbf{x}) > 0$ in Ω;

b) $\omega(\mathbf{x}) = 0$ on $\partial\Omega$;

c) $|\nabla\omega(\mathbf{x})| \neq 0$ on $\partial\Omega$.

Also, the function $\omega(\mathbf{x})$ can have certain properties of differentiation due to the use of various sufficiently complete systems of R-functions [12].

Definition 1. *Let $g_m(r)$ be a fundamental solution of the equation $\mathfrak{L}u = 0$ in $\mathbb{R}^m$. The Green-Rvachev's quasi-function of the first boundary value problem for the operator $\mathfrak{L}$ in $\mathbb{R}^m$ is the function*

$$Q_m(\mathbf{x}, \mathbf{s}) = g_m(r) - \tilde{g}_m(\mathbf{x}, \mathbf{s}), \tag{4}$$

where $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{s} = (s_1, \ldots, s_m)$, $r = |\mathbf{x} - \mathbf{s}| = \sqrt{\sum\limits_{i=1}^{m} (x_i - s_i)^2}$,

$$\tilde{g}_m(\mathbf{x}, \mathbf{s}) = g_m\left(\sqrt{r^2 + 4\omega(\mathbf{x})\omega(\mathbf{s})}\right),$$

$\omega(\mathbf{x})$ *is the function that describes the geometry of the domain* Ω.

Let us note [12] that for the case when $\mathfrak{L}u \equiv -\Delta u$, Ω is a ball of radius R in $\mathbb{R}^m$, and $\omega(\mathbf{x}) = \frac{1}{2R}(R^2 - x_1^2 - \cdots - x_m^2)$, the Green-Rvachev's quasi-function (4) turns into the exact Green's function of the first boundary value problem for the Laplace operator considered in a ball Ω.

The fundamental solutions of the Laplace equation $-\Delta u = 0$ in $\mathbb{R}^2$ and $\mathbb{R}^3$ have the form

$$g_2(r) = \frac{1}{2\pi} \ln \frac{1}{r}, \quad g_3(r) = \frac{1}{4\pi} \cdot \frac{1}{r},$$

consequently, the Green-Rvachev's quasi-function of the first boundary value problem for the operator $-\Delta$ acquires the form

$$Q_2(\mathbf{x},\mathbf{s}) = \frac{1}{2\pi}\ln\sqrt{1+\frac{4\omega(\mathbf{x})\omega(\mathbf{s})}{r^2}} \text{ in } \mathbb{R}^2, \tag{5}$$

$$Q_3(\mathbf{x},\mathbf{s}) = \frac{1}{4\pi}\cdot\frac{\sqrt{r^2+4\omega(\mathbf{x})\omega(\mathbf{s})}-r}{r\sqrt{r^2+4\omega(\mathbf{x})\omega(\mathbf{s})}} \text{ in } \mathbb{R}^3. \tag{6}$$

For the Helmholtz equation $-\Delta u + \kappa^2 u = 0$ in $\mathbb{R}^2$ and $\mathbb{R}^3$ the fundamental solutions have the form

$$g_2(r) = \frac{1}{2\pi}K_0(\kappa r), \quad g_3(r) = \frac{1}{4\pi r}e^{-\kappa r},$$

where $K_0(z)$ is modified Bessel function of the second kind, and the Green-Rvachev's quasi-function of the first boundary value problem for the operator $-\Delta + \kappa^2$ acquires the form

$$Q_2(\mathbf{x},\mathbf{s}) = \frac{1}{2\pi}\left(K_0(\kappa r) - K_0\left(\kappa\sqrt{r^2+4\omega(\mathbf{x})\omega(\mathbf{s})}\right)\right) \text{ in } \mathbb{R}^2, \tag{7}$$

$$Q_3(\mathbf{x},\mathbf{s}) = \frac{1}{4\pi}\cdot\frac{\sqrt{r^2+4\omega(\mathbf{x})\omega(\mathbf{s})}e^{-\kappa r} - re^{-\kappa\sqrt{r^2+4\omega(\mathbf{x})\omega(\mathbf{s})}}}{r\sqrt{r^2+4\omega(\mathbf{x})\omega(\mathbf{s})}} \text{ in } \mathbb{R}^3. \tag{8}$$

From (5)–(8) and Definition 1 the following lemma on the properties of the Green-Rvachev's quasi-function follows.

Lemma 1. *The Green-Rvachev's quasi-function* (4) *has the following properties:*

a) $Q_m(\mathbf{x},\mathbf{s}) = 0$ on $\partial\Omega$;

b) is a symmetric function: $Q_m(\mathbf{x},\mathbf{s}) = Q_m(\mathbf{s},\mathbf{x})$;

c) has the same feature for $\mathbf{x} = \mathbf{s}$ as the usual Green's function;

d) is positive in the domain Ω: $Q_m(\mathbf{x},\mathbf{s}) > 0$, $\mathbf{x},\mathbf{s} \in \Omega$, $\mathbf{x} \neq \mathbf{s}$.

For the function $u \in C^2(\Omega)\bigcap C^1(\bar{\Omega})$ such that $\mathfrak{L}u \in L_2(\Omega)$, the following integral representation [7,8] holds:

$$u(\mathbf{x}) = \int\limits_{\partial\Omega}\left[g(\mathbf{x},\mathbf{s})\frac{\partial u(\mathbf{s})}{\partial \mathbf{n_s}} - u(\mathbf{s})\frac{\partial g(\mathbf{x},\mathbf{s})}{\partial \mathbf{n_s}}\right]d_{\mathbf{s}}\sigma + \int\limits_{\Omega} g(\mathbf{x},\mathbf{s})\mathfrak{L}_{\mathbf{s}}u(\mathbf{s})d\mathbf{s}, \quad \mathbf{x}\in\Omega, \tag{9}$$

and for the functions $u, \tilde{g} \in C^2(\bar{\Omega})$ the second Green's formula [7]

$$0 = -\int\limits_{\Omega}[\tilde{g}(\mathbf{s})\mathfrak{L}_{\mathbf{s}}u(\mathbf{s}) - u(\mathbf{s})\mathfrak{L}_{\mathbf{s}}\tilde{g}(\mathbf{s})]d\mathbf{s} + \int\limits_{\partial\Omega}\left[u(\mathbf{s})\frac{\partial\tilde{g}(\mathbf{s})}{\partial \mathbf{n_s}} - \tilde{g}(\mathbf{s})\frac{\partial u(\mathbf{s})}{\partial \mathbf{n_s}}\right]d_{\mathbf{s}}\sigma \tag{10}$$

holds.

In formulas (9), (10) $\mathbf{n_s}$ is the outer to $\partial\Omega$ normal in the variables $\mathbf{s}$, $d_{\mathbf{s}}\sigma$ means that the integration for $\mathbf{s}$ is along $\partial\Omega$, $\mathfrak{L}_{\mathbf{s}}u \equiv -\Delta_{\mathbf{s}}u$ or $\mathfrak{L}_{\mathbf{s}}u \equiv -\Delta_{\mathbf{s}}u + \kappa^2 u$, $\Delta_{\mathbf{s}} = \frac{\partial^2}{\partial s_1^2} + \frac{\partial^2}{\partial s_2^2}$, if $\Omega \subset \mathbb{R}^2$, and $\Delta_{\mathbf{s}} = \frac{\partial^2}{\partial s_1^2} + \frac{\partial^2}{\partial s_2^2} + \frac{\partial^2}{\partial s_3^2}$, if $\Omega \subset \mathbb{R}^3$.

Let u be a classical solution of the problem (1)–(3), and let us choose the function $\tilde{g}$ in (10) as in Definition 1. Adding the equalities (9) and (10), taking into account (4), we obtain

$$u(\mathbf{x}) = \int_{\Omega} \mathfrak{L}_{\mathbf{s}}\tilde{g}(\mathbf{x},\mathbf{s}) \cdot u(\mathbf{s})d\mathbf{s} + \int_{\Omega} Q_m(\mathbf{x},\mathbf{s}) \cdot \mathfrak{L}_{\mathbf{s}}u(\mathbf{s})d\mathbf{s}$$
$$+ \int_{\partial\Omega} \left[Q_m(\mathbf{x},\mathbf{s})\frac{\partial u(\mathbf{s})}{\partial \mathbf{n}_{\mathbf{s}}} - u(\mathbf{s})\frac{\partial Q_m(\mathbf{x},\mathbf{s})}{\partial \mathbf{n}_{\mathbf{s}}} \right] d_{\mathbf{s}}\sigma.$$

Then, taking into account that $Q_m(\mathbf{x},\mathbf{s}) = 0$ and $u(\mathbf{x}) = 0$ on $\partial\Omega$, and making allowance for the equation (1), we finally obtain the integral equation for the function u in the form

$$u(\mathbf{x}) = \int_{\Omega} K_m(\mathbf{x},\mathbf{s})u(\mathbf{s})d\mathbf{s} + \int_{\Omega} Q_m(\mathbf{x},\mathbf{s})f(\mathbf{s},u(\mathbf{s}))d\mathbf{s}, \tag{11}$$

where $K_m(\mathbf{x},\mathbf{s}) = \mathfrak{L}_{\mathbf{s}}\tilde{g}(\mathbf{x},\mathbf{s})$.

The nonlinear integral equation (11) can be written in the form of Urysohn equation

$$u(\mathbf{x}) = \int_{\Omega} P(\mathbf{x},\mathbf{s},u(\mathbf{s}))d\mathbf{s},$$

where $P(\mathbf{x},\mathbf{s},u(\mathbf{s})) = K_m(\mathbf{x},\mathbf{s})u(\mathbf{s}) + Q_m(\mathbf{x},\mathbf{s})f(\mathbf{s},u(\mathbf{s}))$.

If the boundary value problem (1)–(3) has a classical solution, then it also satisfies the equation (11). If the classical solution of the problem does not exist, then the equation (11) can be used to introduce the concept of a generalized solution of the boundary value problem (1)–(3).

The equation (11) will be considered in a Banach space $C(\bar{\Omega})$ of the functions continuous in $\bar{\Omega}$. The norm in $C(\bar{\Omega})$ is entered by the rule $\|u\| = \max\limits_{\mathbf{x}\in\bar{\Omega}} |u(\mathbf{x})|$. Let us select in $C(\bar{\Omega})$ the cone $\mathcal{K}_+ = \{u \in C(\bar{\Omega}) : u(\mathbf{x}) \geq 0, \mathbf{x} \in \bar{\Omega}\}$ of non-negative functions. Note that the cone $\mathcal{K}_+$ in $C(\bar{\Omega})$ is normal (and even acute). With the help of the cone $\mathcal{K}_+$ in the space $C(\bar{\Omega})$ let us introduce a semiordering by the rule:

$$\text{for } u, v \in C(\bar{\Omega}) \quad u \leqslant v, \text{ if } v - u \in \mathcal{K}_+,$$

that is,

$$u \leqslant v, \quad \text{if } u(\mathbf{x}) \leq v(\mathbf{x}) \text{ for all } \mathbf{x} \in \bar{\Omega}.$$

Definition 2. *By a solution (generalized) of the boundary value problem* (1)–(3) *will be meant a function $u^* \in \mathcal{K}_+$, which is a solution of the integral equation* (11).

2 CONSTRUCTION OF A PROCESS OF TWO-SIDED APPROXIMATIONS

Let us construct a process of two-sided approximations for finding the solution of the integral equation (11) (and consequently, the solution of the boundary value problem (1)–(3) using the methods of the theory of nonlinear operators in semi-ordered spaces [1,5,9,10].

Let us introduce a nonlinear operator T acting in $C(\bar{\Omega})$ by the rule

$$T(u)(\mathbf{x}) = \int_{\Omega} P(\mathbf{x},\mathbf{s},u(\mathbf{s}))d\mathbf{s}. \tag{12}$$

Let us denote

$$K_m^+(\mathbf{x},\mathbf{s}) = \max\{0, K_m(\mathbf{x},\mathbf{s})\}, \quad K_m^-(\mathbf{x},\mathbf{s}) = \max\{0, -K_m(\mathbf{x},\mathbf{s})\}.$$

Then $K_m^+(\mathbf{x},\mathbf{s}) \geq 0$, $K_m^-(\mathbf{x},\mathbf{s}) \geq 0$ for $\mathbf{x},\mathbf{s} \in \Omega$ ($\mathbf{x} \neq \mathbf{s}$),

$$K_m(\mathbf{x},\mathbf{s}) = K_m^+(\mathbf{x},\mathbf{s}) - K_m^-(\mathbf{x},\mathbf{s}), \quad |K_m(\mathbf{x},\mathbf{s})| = K_m^+(\mathbf{x},\mathbf{s}) + K_m^-(\mathbf{x},\mathbf{s}),$$

and operator T of the form (12) will be written in the form

$$T(u)(\mathbf{x}) = \int\limits_\Omega K_m^+(\mathbf{x},\mathbf{s})u(\mathbf{s})d\mathbf{s} - \int\limits_\Omega K_m^-(\mathbf{x},\mathbf{s})u(\mathbf{s})d\mathbf{s} + \int\limits_\Omega Q_m(\mathbf{x},\mathbf{s})f(\mathbf{s},u(\mathbf{s}))d\mathbf{s}. \tag{13}$$

Suppose that the function $f(\mathbf{x},u)$ allows a diagonal representation $f(\mathbf{x},u) = \hat{f}(\mathbf{x},u,u)$, besides, continuous on the sets of variables $\mathbf{x}$, v, w non-negative function $\hat{f}(\mathbf{x},v,w)$ monotonically increases with respect to v and monotonically decreases with respect to w for all $\mathbf{x} \in \Omega$. Then the operator T of the form (13) will be heterotone with the companion operator

$$\hat{T}(v,w)(\mathbf{x}) = \int\limits_\Omega K_m^+(\mathbf{x},\mathbf{s})v(\mathbf{s})d\mathbf{s} - \int\limits_\Omega K_m^-(\mathbf{x},\mathbf{s})w(\mathbf{s})d\mathbf{s} + \int\limits_\Omega Q_m(\mathbf{x},\mathbf{s})\hat{f}(\mathbf{s},v(\mathbf{s}),w(\mathbf{s}))d\mathbf{s}. \tag{14}$$

Operators T and $\hat{T}$ are completely continuous.

Note that for the case when the function $f(\mathbf{x},u)$ increases monotonically with respect to u for all $\mathbf{x} \in \Omega$ we can choose $\hat{f}(\mathbf{x},v,w) = f(\mathbf{x},v)$, and if it decreases monotonically with respect to u for all $\mathbf{x} \in \Omega$ we can set $\hat{f}(\mathbf{x},v,w) = f(\mathbf{x},w)$.

In the cone $\mathcal{K}_+$ let us select a strongly invariant cone segment $< v^0, w^0 >$ by conditions $\hat{T}(v^0,w^0) \geq v^0$, $\hat{T}(w^0,v^0) \leq w^0$, which for the operator $\hat{T}$ that is defined by (14) will have the form:

$$\begin{aligned} &\int\limits_\Omega K_m^+(\mathbf{x},\mathbf{s})v^0(\mathbf{s})d\mathbf{s} - \int\limits_\Omega K_m^-(\mathbf{x},\mathbf{s})w^0(\mathbf{s})d\mathbf{s} \\ &+ \int\limits_\Omega Q_m(\mathbf{x},\mathbf{s})\hat{f}(\mathbf{s},v^0(\mathbf{s}),w^0(\mathbf{s}))d\mathbf{s} \geq v^0(\mathbf{x}) \text{ for all } \mathbf{x} \in \bar{\Omega}, \end{aligned} \tag{15}$$

$$\begin{aligned} &\int\limits_\Omega K_m^+(\mathbf{x},\mathbf{s})w^0(\mathbf{s})d\mathbf{s} - \int\limits_\Omega K_m^-(\mathbf{x},\mathbf{s})v^0(\mathbf{s})d\mathbf{s} \\ &+ \int\limits_\Omega Q_m(\mathbf{x},\mathbf{s})\hat{f}(\mathbf{s},w^0(\mathbf{s}),v^0(\mathbf{s}))d\mathbf{s} \leq w^0(\mathbf{x}) \text{ for all } \mathbf{x} \in \bar{\Omega}. \end{aligned} \tag{16}$$

Let us form an iterative process by the scheme $v^{(k+1)} = \hat{T}(v^{(k)}, w^{(k)})$, $w^{(k+1)} = \hat{T}(w^{(k)}, v^{(k)})$, $k = 0,1,2,\ldots$ ($v^{(0)} = v^0$, $w^{(0)} = w^0$):

$$\begin{aligned} v^{(k+1)}(\mathbf{x}) = &\int\limits_\Omega K_m^+(\mathbf{x},\mathbf{s})v^{(k)}(\mathbf{s})d\mathbf{s} - \int\limits_\Omega K_m^-(\mathbf{x},\mathbf{s})w^{(k)}(\mathbf{s})d\mathbf{s} \\ &+ \int\limits_\Omega Q_m(\mathbf{x},\mathbf{s})\hat{f}(\mathbf{s},v^{(k)}(\mathbf{s}),w^{(k)}(\mathbf{s}))d\mathbf{s}, \quad k = 0,1,2,\ldots, \end{aligned} \tag{17}$$

$$w^{(k+1)}(\mathbf{x}) = \int\limits_{\Omega} K_m^+(\mathbf{x},\mathbf{s}) w^{(k)}(\mathbf{s}) d\mathbf{s} - \int\limits_{\Omega} K_m^-(\mathbf{x},\mathbf{s}) v^{(k)}(\mathbf{s}) d\mathbf{s} + \int\limits_{\Omega} Q_m(\mathbf{x},\mathbf{s}) \hat{f}(\mathbf{s}, w^{(k)}(\mathbf{s}), v^{(k)}(\mathbf{s})) d\mathbf{s}, \quad k = 0,1,2,\ldots, \tag{18}$$

$$v^{(0)}(\mathbf{x}) = v^0(\mathbf{x}), \quad w^{(0)}(\mathbf{x}) = w^0(\mathbf{x}). \tag{19}$$

Since the cone segment $< v^0, w^0 >$ is strongly invariant for the heterotone operator T for which the operator $\hat{T}$ is a companion one, it follows that the sequence $\{v^{(k)}(\mathbf{x})\}$ does not decrease with respect to the cone $\mathcal{K}_+$, and the sequence $\{w^{(k)}(\mathbf{x})\}$ does not increase with respect to the cone $\mathcal{K}_+$. Besides, the cone $\mathcal{K}_+$ is normal and the operator $\hat{T}$ is completely continuous, so the boundaries $v^*(\mathbf{x})$ and $w^*(\mathbf{x})$ of these sequences exist. Thus, the chain of inequalities holds:

$$v^0 = v^{(0)} \leqslant v^{(1)} \leqslant \cdots \leqslant v^{(k)} \leqslant \cdots \leqslant v^* \leqslant w^* \leqslant \cdots \leqslant w^{(k)} \leqslant \cdots \leqslant w^{(1)} \leqslant w^{(0)} = w^0.$$

There are two possible cases: $v^* < w^*$ and $v^* = w^*$. In the second case, $u^* := v^* = w^*$ is the unique on $< v^0, w^0 >$ fixed point of the operator T, that is, it is the unique on $< v^0, w^0 >$ solution of the boundary value problem (1)–(3).

The functions $v^*(\mathbf{x})$ and $w^*(\mathbf{x})$ are a solution of the system of equations $v = \hat{T}(v,w)$, $w = \hat{T}(w,v)$, which in the considered case has the form

$$v(\mathbf{x}) = \int\limits_{\Omega} K_m^+(\mathbf{x},\mathbf{s}) v(\mathbf{s}) d\mathbf{s} - \int\limits_{\Omega} K_m^-(\mathbf{x},\mathbf{s}) w(\mathbf{s}) d\mathbf{s} + \int\limits_{\Omega} Q_m(\mathbf{x},\mathbf{s}) \hat{f}(\mathbf{s}, v(\mathbf{s}), w(\mathbf{s})) d\mathbf{s}, \tag{20}$$

$$w(\mathbf{x}) = \int\limits_{\Omega} K_m^+(\mathbf{x},\mathbf{s}) w(\mathbf{s}) d\mathbf{s} - \int\limits_{\Omega} K_m^-(\mathbf{x},\mathbf{s}) v(\mathbf{s}) d\mathbf{s} + \int\limits_{\Omega} Q_m(\mathbf{x},\mathbf{s}) \hat{f}(\mathbf{s}, w(\mathbf{s}), v(\mathbf{s})) d\mathbf{s}. \tag{21}$$

The equality $v^* = w^*$ will hold if the system (20), (21) does not have on $< v^0, w^0 >$ such solutions that $v \neq w$ [9,10].

Thus, such a theorem is true.

Theorem 1. *Let $< v^0, w^0 >$ be a strongly invariant cone segment for the heterotone operator T of the form* (13) *with the companion operator $\hat{T}$ of the form* (14) *and the system of equations* (20), (21) *does not have on $< v^0, w^0 >$ solutions such that $v \neq w$. Then the iterative process* (17)–(19) *converges in the norm of the space $C(\bar{\Omega})$ to the unique on $< v^0, w^0 >$ continuous positive solution u^* of the boundary value problem* (1)–(3)*, and a chain of inequalities holds:*

$$v^0 = v^{(0)} \leqslant v^{(1)} \leqslant \cdots \leqslant v^{(k)} \leqslant \cdots \leqslant u^* \leqslant \cdots \leqslant w^{(k)} \leqslant \cdots \leqslant w^{(1)} \leqslant w^{(0)} = w^0. \tag{22}$$

Note that the chain of inequalities (22) characterizes the iterative process (17)–(19) as a method of two-sided approximations.

From the chain of inequalities (22) it follows that each of the cone segments $< v^{(k)}, w^{(k)} >$, $k = 0,1,2,\ldots,$ is strongly invariant for the heterotone operator T of the form (13) with the companion operator $\hat{T}$ of the form (14).

Let us determine the conditions for the existence of a unique positive solution of the boundary value problem (1)–(3) and two-sided convergence of the successive approximations (17)–(19) to it, by clarifying the conditions under which the system of equations (20), (21) does not have on some of the strongly invariant cone segments $< v^{(k)}, w^{(k)} >$, $k = 0, 1, 2, \ldots$, such solutions that $v \neq w$.

First we use such a condition [9, 10]: if

$$\hat{T}(v+u, w-u) \ngeq \hat{T}(v,w) + u, \tag{23}$$

where $u > \theta$, $v < w$, $v, w, v+u, w-u \in < v^0, w^0 >$, then the system $v = \hat{T}(v,w)$, $w = \hat{T}(w,v)$ does not have solutions on $< v^0, w^0 >$, such that $v \neq w$.

Theorem 2. *Let $< v^0, w^0 >$ be a strongly invariant cone segment for the heterotone operator T of the form* (13) *with the companion operator $\hat{T}$ of the form* (14) *and the following condition holds: for any numbers v, w, u such, that $0 < v < w$, $0 < u < w$, and for all $\mathbf{x} \in \Omega$ the following inequality is satisfied:*

$$\hat{f}(\mathbf{x}, v+u, w-u) < \hat{f}(\mathbf{x}, v, w) + \frac{u}{M+M_1},$$

where

$$M = \max_{\mathbf{x}\in\bar{\Omega}} \int\limits_{\Omega} Q_m(\mathbf{x},\mathbf{s})d\mathbf{s}, \quad M_1 = \max_{\mathbf{x}\in\bar{\Omega}} \int\limits_{\Omega} [K_m^+(\mathbf{x},\mathbf{s}) + K_m^-(\mathbf{x},\mathbf{s})]d\mathbf{s}. \tag{24}$$

Then, the iterative process (17)–(19) *bilaterally converges in the norm of the space $C(\bar{\Omega})$ to the unique on $< v^0, w^0 >$ continuous positive solution u^* of the boundary value problem* (1)–(3).

Proof. Let u and $w - v$ be such functions from $\mathcal{K}_+ \backslash \{\theta\}$ (θ is a zero element of $C(\bar{\Omega})$), that $v, w, v+u, w-u \in < v^0, w^0 >$. Then $u(\mathbf{x}) \geq 0$ in $\bar{\Omega}$ and $u|_{\partial\Omega} = 0$. So, if the function $u(\mathbf{x})$ gets the maximum value at the point $\mathbf{x}_0$, then $\mathbf{x}_0 \in \Omega$. Thus,

$$\begin{aligned}
&\hat{T}(v+u, w-u)(\mathbf{x}_0) \\
&= \int\limits_{\Omega} K_m^+(\mathbf{x}_0,\mathbf{s})[v(\mathbf{s}) + u(\mathbf{s})]d\mathbf{s} - \int\limits_{\Omega} K_m^-(\mathbf{x}_0,\mathbf{s})[w(\mathbf{s}) - u(\mathbf{s})]d\mathbf{s} \\
&+ \int\limits_{\Omega} Q_m(\mathbf{x}_0,\mathbf{s})\hat{f}(\mathbf{x}, v(\mathbf{s}) + u(\mathbf{s}), w(\mathbf{s}) - u(\mathbf{s}))d\mathbf{s} \\
&< \int\limits_{\Omega} K_m^+(\mathbf{x}_0,\mathbf{s})v(\mathbf{s})d\mathbf{s} - \int\limits_{\Omega} K_m^-(\mathbf{x}_0,\mathbf{s})w(\mathbf{s})d\mathbf{s} + \int\limits_{\Omega} [K_m^+(\mathbf{x}_0,\mathbf{s}) + K_m^-(\mathbf{x}_0,\mathbf{s})]u(\mathbf{s})d\mathbf{s} \\
&+ \int\limits_{\Omega} Q_m(\mathbf{x}_0,\mathbf{s}) \left[\hat{f}(\mathbf{s}, v(\mathbf{s}), w(\mathbf{s})) + \frac{u(\mathbf{s})}{M+M_1}\right] d\mathbf{s} \\
&\leq \int\limits_{\Omega} K_m^+(\mathbf{x}_0,\mathbf{s})v(\mathbf{s})d\mathbf{s} - \int\limits_{\Omega} K_m^-(\mathbf{x}_0,\mathbf{s})w(\mathbf{s})d\mathbf{s} + \int\limits_{\Omega} Q_m(\mathbf{x}_0,\mathbf{s})\hat{f}(\mathbf{s}, v(\mathbf{s}), w(\mathbf{s}))d\mathbf{s} \\
&+ \frac{1}{M+M_1}\max_{\mathbf{x}\in\bar{\Omega}} u(\mathbf{x}) \left[\int\limits_{\Omega} [K_m^+(\mathbf{x}_0,\mathbf{s}) + K_m^-(\mathbf{x}_0,\mathbf{s})]d\mathbf{s} + \int\limits_{\Omega} Q_m(\mathbf{x}_0,\mathbf{s})d\mathbf{s}\right] \\
&\leq \int\limits_{\Omega} K_m^+(\mathbf{x}_0,\mathbf{s})v(\mathbf{s})d\mathbf{s} - \int\limits_{\Omega} K_m^-(\mathbf{x}_0,\mathbf{s})w(\mathbf{s})d\mathbf{s} + \int\limits_{\Omega} Q_m(\mathbf{x}_0,\mathbf{s})\hat{f}(\mathbf{s}, v(\mathbf{s}), w(\mathbf{s}))d\mathbf{s} + u(\mathbf{x}_0) \\
&= \hat{T}(v,w)(\mathbf{x}_0) + u(\mathbf{x}_0),
\end{aligned}$$

that is, the condition (23) holds, consequently, the system of equations (20), (21) does not have on $< v^0, w^0 >$ such solutions that $v \neq w$ and the Theorem 1 is valid. □

Another condition that ensures equality $v^* = w^*$ is the existence of $\gamma \in (0;1)$ such that

$$\left\|\hat{T}(v,w) - \hat{T}(w,v)\right\| \le \gamma \left\|v - w\right\|$$

for all $v, w \in < v^0, w^0 >$ [2].

Let there exists a number $L > 0$, that the function $\hat{f}(\mathbf{x}, v, w)$ for all numbers v, w such that $0 < v, w < M_0$, where $M_0 = \max\limits_{\mathbf{x}\in\bar{\Omega}} w^0(\mathbf{x})$, and for all $\mathbf{x} \in \Omega$ satisfies the inequality

$$\left|\hat{f}(\mathbf{x}, w, v) - \hat{f}(\mathbf{x}, v, w)\right| \le L\left|w - v\right|. \tag{25}$$

Let us consider the difference $\hat{T}(v,w)(\mathbf{x}) - \hat{T}(w,v)(\mathbf{x})$:

$$\begin{aligned}\hat{T}(w,v)(\mathbf{x}) - \hat{T}(v,w)(\mathbf{x}) = &\int\limits_{\Omega} [K_m^+(\mathbf{x},\mathbf{s}) + K_m^-(\mathbf{x},\mathbf{s})][w(\mathbf{s}) - v(\mathbf{s})]d\mathbf{s} \\ &+ \int\limits_{\Omega} Q_m(\mathbf{x},\mathbf{s})[\hat{f}(\mathbf{s}, w(\mathbf{s}), v(\mathbf{s})) - \hat{f}(\mathbf{s}, v(\mathbf{s}), w(\mathbf{s}))]d\mathbf{s}.\end{aligned}$$

Then, taking into account the inequality (25), we obtain an estimate

$$\begin{aligned}\left\|\hat{T}(w,v) - \hat{T}(v,w)\right\| &= \max_{\mathbf{x}\in\bar{\Omega}} \left|\hat{T}(w,v)(\mathbf{x}) - \hat{T}(v,w)(\mathbf{x})\right| \\ &\le (M_1 + LM)\cdot\max_{\mathbf{x}\in\bar{\Omega}} \left|w(\mathbf{x}) - v(\mathbf{x})\right| = (M_1 + LM)\left\|w - v\right\|,\end{aligned}$$

where constants M and M_1 are defined by equalities (24).

Thus,

$$\left\|\hat{T}(w,v) - \hat{T}(v,w)\right\| \le \gamma\left\|w - v\right\|,$$

where $\gamma = M_1 + LM$.

Then the equality $v^* = w^*$ will be held, if $\gamma = M_1 + LM < 1$, and the following theorem holds.

Theorem 3. *Let $< v^0, w^0 >$ be a strongly invariant cone segment for the heterotone operator T of the form* (13) *with the companion operator $\hat{T}$ of the form* (14) *and the condition* (25) *holds, besides, $\gamma = M_1 + LM < 1$, where the constants M and M_1 are defined by the equalities* (24)*. Then, the iterative process* (17)–(19) *bilaterally converges in the norm of the space $C(\bar{\Omega})$ to the unique on $< v^0, w^0 >$ continuous positive solution u^* of the boundary value problem* (1)–(3).

If the k-th iteration have been performed, then as an approximate solution of the boundary value problem (1)–(3) the function

$$u^{(k)}(\mathbf{x}) = \frac{w^{(k)}(\mathbf{x}) + v^{(k)}(\mathbf{x})}{2} \tag{26}$$

is accepted.

Then for an approximate solution (26) there will be convenient a posteriori estimate of the error:

$$\left\|u^* - u^{(k)}\right\| \le \frac{1}{2}\max_{\mathbf{x}\in\bar{\Omega}} (w^{(k)}(\mathbf{x}) - v^{(k)}(\mathbf{x})). \tag{27}$$

The presence of an estimation of the form (27) is an unconditional advantage of the constructed two-sided iterative process.

If the accuracy $\varepsilon > 0$ is given, then the iterative process should be carried out until the inequality

$$\max_{\mathbf{x}\in\bar{\Omega}} (w^{(k)}(\mathbf{x}) - v^{(k)}(\mathbf{x})) < 2\varepsilon$$

will be satisfied and then with an accuracy ε it can be expected that $u^*(\mathbf{x}) \approx u^{(k)}(\mathbf{x})$.

If the conditions of Theorem 3 are satisfied, then an a priori estimate of the error will be:

$$\left\| u^* - u^{(k)} \right\| \leq \frac{\gamma^k}{2} \max_{\mathbf{x}\in\bar{\Omega}} (w^0(\mathbf{x}) - v^0(\mathbf{x})).$$

Then from the inequality

$$\frac{\gamma^k}{2} \max_{\mathbf{x}\in\bar{\Omega}} (w^0(\mathbf{x}) - v^0(\mathbf{x})) < \varepsilon$$

we obtained that to achieve the accuracy ε it is necessary to do

$$k_0(\varepsilon) = \left[\frac{\ln \frac{\max\limits_{\mathbf{x}\in\bar{\Omega}} (w^0(\mathbf{x}) - v^0(\mathbf{x}))}{2\varepsilon}}{\ln \frac{1}{M_1 + LM}} \right] + 1 \tag{28}$$

iterations, where the square brackets denote the integer part of the number.

3 Numerical experiments

The construction of two-sided approximations to the solution of the boundary value problem (1)–(3) will be demonstrated on the problem with exponential nonlinearities:

$$\mathfrak{L}u = e^u + 2e^{-u}, \quad \mathbf{x} \in \Omega, \tag{29}$$

$$u(\mathbf{x}) > 0, \quad \mathbf{x} \in \Omega, \tag{30}$$

$$u|_{\partial\Omega} = 0, \tag{31}$$

where $\mathfrak{L}u \equiv -\Delta u$ or $\mathfrak{L}u \equiv -\Delta u + u$, $\Omega = \{\mathbf{x} = (x_1, x_2) : 0 < x_1, x_2 < 1\}$.

The function $f(\mathbf{x}, u) = e^u + 2e^{-u}$ is positive and continuous with respect to the set of variables, if $u > 0$, and it allows a diagonal representation with the help of function

$$\hat{f}(\mathbf{x}, v, w) = e^v + 2e^{-w}.$$

The problem (29)–(31) is replaced by an equivalent integral equation

$$u(\mathbf{x}) = \int\limits_{\Omega} K_2(\mathbf{x}, \mathbf{s}) u(\mathbf{s}) d\mathbf{s} + \int\limits_{\Omega} Q_2(\mathbf{x}, \mathbf{s}) [e^{u(\mathbf{s})} + 2e^{-u(\mathbf{s})}] d\mathbf{s}, \tag{32}$$

where $Q_2(\mathbf{x}, \mathbf{s})$ is determined by the formula (5), if $\mathfrak{L}u \equiv -\Delta u$, and is determined by the formula (7), if $\mathfrak{L}u \equiv -\Delta u + u$, $K_2(\mathbf{x}, \mathbf{s}) = -\frac{\partial^2}{\partial s_1^2} \tilde{g}_2(\mathbf{x}, \mathbf{s}) - \frac{\partial^2}{\partial s_2^2} \tilde{g}_2(\mathbf{x}, \mathbf{s})$,

$$\tilde{g}_2(\mathbf{x}, \mathbf{s}) = \frac{1}{2\pi} \ln \frac{1}{\sqrt{r^2 + 4\omega(\mathbf{x})\omega(\mathbf{s})}}, \text{ if } \mathfrak{L}u \equiv -\Delta u,$$

$$\tilde{g}_2(\mathbf{x}, \mathbf{s}) = \frac{1}{2\pi} K_0 \left(\kappa \sqrt{r^2 + 4\omega(\mathbf{x})\omega(\mathbf{s})} \right), \text{ if } \mathfrak{L}u \equiv -\Delta u + u,$$

$$\omega(\mathbf{x}) = [x_1(1-x_1)]\wedge_0[x_2(1-x_2)] \equiv x_1(1-x_1) + x_2(1-x_2) - \sqrt{x_1^2(1-x_1)^2 + x_2^2(1-x_2)^2}.$$

With the equation (32) let us associate a heterotone operator

$$T(u)(\mathbf{x}) = \int\limits_{\Omega} K_2(\mathbf{x},\mathbf{s})u(\mathbf{s})d\mathbf{s} + \int\limits_{\Omega} Q_2(\mathbf{x},\mathbf{s})[e^{u(\mathbf{s})} + 2e^{-u(\mathbf{s})}]d\mathbf{s}, \tag{33}$$

for which the companion operator has the form

$$\hat{T}(v,w)(\mathbf{x}) = \int\limits_{\Omega} K_2^+(\mathbf{x},\mathbf{s})v(\mathbf{s})d\mathbf{s} - \int\limits_{\Omega} K_2^-(\mathbf{x},\mathbf{s})w(\mathbf{s})d\mathbf{s} + \int\limits_{\Omega} Q_2(\mathbf{x},\mathbf{s})[e^{v(\mathbf{s})} + 2e^{-w(\mathbf{s})}]d\mathbf{s},$$

where

$$K_2^+(\mathbf{x},\mathbf{s}) = \max\{0, K_2(\mathbf{x},\mathbf{s})\}, \quad K_2^-(\mathbf{x},\mathbf{s}) = \max\{0, -K_2(\mathbf{x},\mathbf{s})\}.$$

For the operator T of the form (33) a strongly invariant cone segment will be sought in the form $< v^0, w^0 >$, where $v^0(\mathbf{x}) = \alpha u_0(\mathbf{x})$, $w^0(\mathbf{x}) = \beta u_0(\mathbf{x})$, $0 < \alpha < \beta$, and

$$u_0(\mathbf{x}) = \int\limits_{\Omega} Q_2(\mathbf{x},\mathbf{s})d\mathbf{s}.$$

For the chosen functions v^0, w^0 the system of inequalities (15), (16) leads to the next system of inequalities for determining the constants α, β: for all $\mathbf{x} \in \bar{\Omega}$

$$\alpha \int\limits_{\Omega} K_2^+(\mathbf{x},\mathbf{s})u_0(\mathbf{s})d\mathbf{s} - \beta \int\limits_{\Omega} K_2^-(\mathbf{x},\mathbf{s})u_0(\mathbf{s})d\mathbf{s} + \int\limits_{\Omega} Q_2(\mathbf{x},\mathbf{s})[e^{\alpha u_0(\mathbf{s})} + 2e^{-\beta u_0(\mathbf{s})}]d\mathbf{s} \geq \alpha u_0(\mathbf{x}),$$

$$\beta \int\limits_{\Omega} K_2^+(\mathbf{x},\mathbf{s})u_0(\mathbf{s})d\mathbf{s} - \alpha \int\limits_{\Omega} K_2^-(\mathbf{x},\mathbf{s})u_0(\mathbf{s})d\mathbf{s} + \int\limits_{\Omega} Q_2(\mathbf{x},\mathbf{s})[e^{\beta u_0(\mathbf{s})} + 2e^{-\alpha u_0(\mathbf{s})}]d\mathbf{s} \leq \beta u_0(\mathbf{x}).$$

If $0 < v, w < M_0$, where $M_0 = \beta \max\limits_{x \in \bar{\Omega}} u_0(\mathbf{x})$, then

$$\left|\hat{f}(\mathbf{x}, v, w) - \hat{f}(\mathbf{x}, w, v)\right| = \left|(e^v + 2e^{-w}) - (e^w + 2e^{-v})\right| \leq (2 + e^{M_0})\,|v - w|\,.$$

For the considering problem the iterative process (17)–(19) has the form

$$\begin{aligned} v^{(k+1)}(\mathbf{x}) = &\int\limits_{\Omega} K_2^+(\mathbf{x},\mathbf{s})v^{(k)}(\mathbf{s})d\mathbf{s} - \int\limits_{\Omega} K_2^-(\mathbf{x},\mathbf{s})w^{(k)}(\mathbf{s})d\mathbf{s} \\ &+ \int\limits_{\Omega} Q_2(\mathbf{x},\mathbf{s})[e^{v^{(k)}(\mathbf{s})} + 2e^{-w^{(k)}(\mathbf{s})}]d\mathbf{s} \end{aligned} \tag{34}$$

$$\begin{aligned} w^{(k+1)}(\mathbf{x}) = &\int\limits_{\Omega} K_2^+(\mathbf{x},\mathbf{s})w^{(k)}(\mathbf{s})d\mathbf{s} - \int\limits_{\Omega} K_2^-(\mathbf{x},\mathbf{s})v^{(k)}(\mathbf{s})d\mathbf{s} \\ &+ \int\limits_{\Omega} Q_2(\mathbf{x},\mathbf{s})[e^{w^{(k)}(\mathbf{s})} + 2e^{-v^{(k)}(\mathbf{s})}]d\mathbf{s}, \end{aligned} \tag{35}$$

$$k = 0, 1, 2, \ldots,$$

$$v^{(0)}(\mathbf{x}) = \alpha u_0(\mathbf{x}), \quad w^{(0)}(\mathbf{x}) = \beta u_0(\mathbf{x}). \tag{36}$$

For the problem (29)–(31), if $\mathfrak{L}u \equiv -\Delta u$, it was found, that the system of inequalities for determining the constants α, β is satisfied by the values $\alpha = 2.5$, $\beta = 5.8$. Further we find

$$M = \max_{\mathbf{x}\in\bar{\Omega}} \int_{\Omega} Q_2(\mathbf{x},\mathbf{s})d\mathbf{s} = 0.04093, \quad M_1 = \max_{\mathbf{x}\in\bar{\Omega}} \int_{\Omega} [K_2^+(\mathbf{x},\mathbf{s}) + K_2^-(\mathbf{x},\mathbf{s})]d\mathbf{s} = 0.70819,$$

$$M_0 = \beta \max_{x\in\bar{\Omega}} u_0(\mathbf{x}) = 0.23740, \quad L = 2 + e^{M_0} = 3.26795, \quad \gamma = M_1 + LM = 0.842.$$

Thus, $\gamma < 1$ and by Theorem 3, the successive approximations that are formed by the scheme (34)–(36) bilaterally converge to the solution of the problem (29)–(31) for $\mathfrak{L}u \equiv -\Delta u$.

Let us choose $\varepsilon = 10^{-4}$. Then, in accordance with (28), to achieve this accuracy, it is necessary to make $k_0(\varepsilon) = \left[\dfrac{\ln \frac{(\beta-\alpha)M}{2\varepsilon}}{\ln \frac{1}{\gamma}}\right] + 1 = 38$ iterations. In fact, the accuracy $\varepsilon = 10^{-4}$ was achieved at the eleventh iteration. As one can see, the theoretical error estimate turned out to be greatly overestimated. As an approximate solution of the problem (29)–(31) for $\mathfrak{L}u \equiv -\Delta u$, the function $u^{(11)}(\mathbf{x}) = \frac{v^{(11)}(\mathbf{x})+w^{(11)}(\mathbf{x})}{2}$ will be accepted.

Iteration number k	0	1	2	3	4	5
$\varepsilon^{(k)}$	$0.67 \cdot 10^{-1}$	$0.39 \cdot 10^{-1}$	$0.22 \cdot 10^{-1}$	$0.22 \cdot 10^{-1}$	$0.65 \cdot 10^{-2}$	$0.35 \cdot 10^{-2}$
Iteration number k	6	7	8	9	10	11
$\varepsilon^{(k)}$	$0.19 \cdot 10^{-2}$	$0.10 \cdot 10^{-2}$	$0.57 \cdot 10^{-3}$	$0.31 \cdot 10^{-3}$	$0.17 \cdot 10^{-3}$	$0.91 \cdot 10^{-4}$

Table 1. The values of the estimate $\varepsilon^{(k)}$ of the approximate solution error for $\mathfrak{L}u \equiv -\Delta u$

Table 1 gives the data how the estimate $\varepsilon^{(k)} = \max_{\mathbf{x}\in\bar{\Omega}} \frac{1}{2}(w^{(k)}(\mathbf{x}) - v^{(k)}(\mathbf{x}))$ of the norm of the error $\left\|u^* - u^{(k)}\right\|$ of the approximate solution $u^{(k)}(\mathbf{x})$ varies depending on the iteration number k, $k = 0, 1, \ldots, 11$. It was found that $\left\|u^{(11)}\right\| = 0.2130$.

Figure 1 shows the graph of the cross-sections of the upper $w^{(k)}(\mathbf{x})$ and the lower $v^{(k)}(\mathbf{x})$ approximations at $x_2 = 0.5$ for $k = 0, 2, 6, 8$. Figures 2, 3 show the surface of the approximate solution $u^{(11)}(\mathbf{x})$ and its contour lines (with the step 0.02) respectively. Considering the relationship $\frac{\varepsilon^{(k+1)}}{\varepsilon^{(k)}}$, $k = 0, 1, \ldots, 11$, according to the Table 1, it was received that $\frac{\varepsilon^{(k+1)}}{\varepsilon^{(k)}} \approx 0.543$, that indicates the geometric rate of convergence of the iterative sequence with the corresponding index. Let us note that the convergence exponent turned out to be less than the exponent γ estimated in accordance with Theorem 3.

Let us now consider the problem (29)–(31) for $\mathfrak{L}u \equiv -\Delta u + u$. It was found that the system of inequalities to determine the constants α, β is satisfied by the values $\alpha = 0.1$, $\beta = 5.3$. Further we find

$$M = \max_{\mathbf{x}\in\bar{\Omega}} \int_{\Omega} Q_2(\mathbf{x},\mathbf{s})d\mathbf{s} = 0.03760,$$

$$M_1 = \max_{\mathbf{x}\in\bar{\Omega}} \int_{\Omega} [K_2^+(\mathbf{x},\mathbf{s}) + K_2^-(\mathbf{x},\mathbf{s})]d\mathbf{s} = 0.60410,$$

$$M_0 = \beta \max_{x\in\bar{\Omega}} u_0(\mathbf{x}) = 0.19929, \quad L = 2 + e^{M_0} = 3.22053, \quad \gamma = M_1 + LM = 0.725.$$

Thus, $\gamma < 1$ and by Theorem 3, the successive approximations that are formed by the scheme (34)–(36) bilaterally converge to the solution of the problem (29)–(31) for $\mathfrak{L}u \equiv -\Delta u + u$.

Let us choose $\varepsilon = 10^{-4}$. Then, in accordance with (28), to achieve this accuracy, it is necessary to make $k_0(\varepsilon) = \left[\frac{\ln\frac{(\beta-\alpha)M}{2\varepsilon}}{\ln\frac{1}{\gamma}}\right] + 1 = 22$ iterations. In fact, the accuracy $\varepsilon = 10^{-4}$ was achieved at the tenth iteration. As one can see, the theoretical error estimate turned out to be greatly overestimated too. As an approximate solution of the problem (29)–(31) for $\mathfrak{L}u \equiv -\Delta u + u$, the function $u^{(10)}(\mathbf{x}) = \frac{v^{(10)}(\mathbf{x})+w^{(10)}(\mathbf{x})}{2}$ will be accepted.

Iteration number k	0	1	2	3	4	5
$\varepsilon^{(k)}$	$0.98 \cdot 10^{-1}$	$0.49 \cdot 10^{-1}$	$0.24 \cdot 10^{-1}$	$0.12 \cdot 10^{-1}$	$0.58 \cdot 10^{-2}$	$0.28 \cdot 10^{-2}$
Iteration number k	6	7	8	9	10	
$\varepsilon^{(k)}$	$0.14 \cdot 10^{-2}$	$0.67 \cdot 10^{-3}$	$0.33 \cdot 10^{-3}$	$0.16 \cdot 10^{-3}$	$0.78 \cdot 10^{-4}$	

Table 2. The values of the estimate $\varepsilon^{(k)}$ of the approximate solution error for $\mathfrak{L}u \equiv -\Delta u + u$

Table 2 gives the data how the estimate $\varepsilon^{(k)} = \max\limits_{\mathbf{x}\in\bar{\Omega}} \frac{1}{2}(w^{(k)}(\mathbf{x}) - v^{(k)}(\mathbf{x}))$ of the norm of the error $\left\|u^* - u^{(k)}\right\|$ of the approximate solution $u^{(k)}(\mathbf{x})$ varies depending on the iteration number k, $k = 0, 1, \ldots, 10$. It was found that $\left\|u^{(10)}\right\| = 0.1742$. We can see, the norm of the approximate solution in the transition to the equation with $\mathfrak{L}u \equiv -\Delta u + u$ has decreased.

Figure 4 shows the graph of the cross-sections of the upper $w^{(k)}(\mathbf{x})$ and the lower $v^{(k)}(\mathbf{x})$ approximations at $x_2 = 0.5$ for $k = 0, 2, 6, 8$. Figures 5, 6 show the surface of the approximate solution $u^{(10)}(\mathbf{x})$ and its contour lines (with the step 0.02) respectively. Considering the relationship $\frac{\varepsilon^{(k+1)}}{\varepsilon^{(k)}}$, $k = 0, 1, \ldots, 10$, according to the Table 2, it was received that $\frac{\varepsilon^{(k+1)}}{\varepsilon^{(k)}} \approx 0.488$, that indicates the geometric rate of convergence of the iterative sequence with the corresponding index. Let us note that for case $\mathfrak{L}u \equiv -\Delta u + u$ the convergence exponent turned out to be less than the exponent γ estimated in accordance with Theorem 3.

4 CONCLUSIONS

In the paper a method of two-sided approximations of the solution of the homogeneous Dirichlet problem for a semilinear elliptic equation with Laplace operator $-\Delta$ and Helmholtz operator $-\Delta + \kappa^2$ is proposed on the basis of the Green-Rvachev's quasi-function method. A computational experiment carried out for two equations with heterotone exponential nonlinearity demonstrated the possibilities and effectiveness of the method. The proposed approach to the numerical solution of semilinear elliptic equations can be used in solving various applied problems, the mathematical model of which is the problem (1)–(3). The proposed method is more universal than the existing methods, and it allows to solve the boundary problem in domains of arbitrary geometry, provided that this domain can be described by the R-function method.

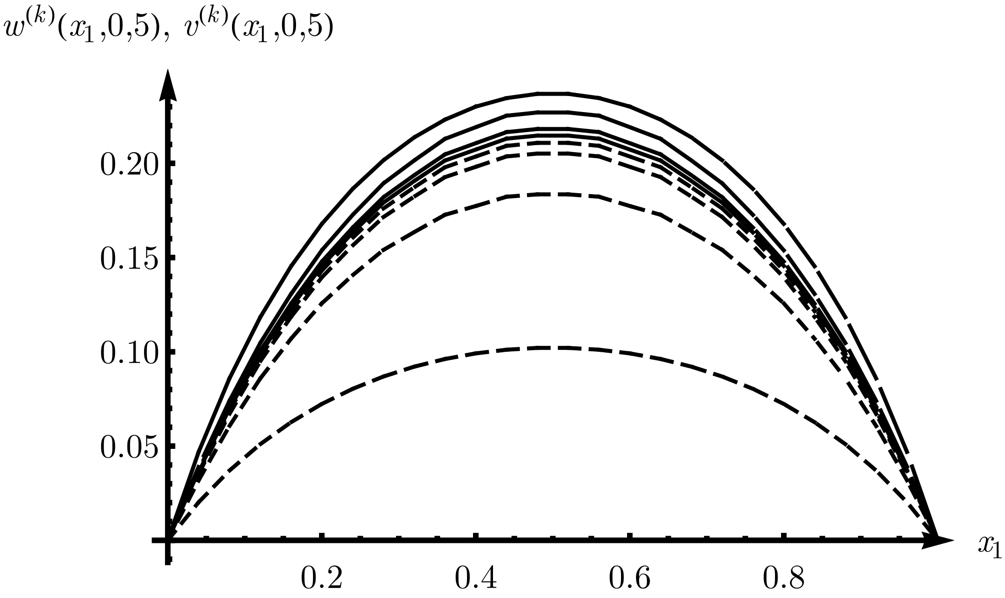

Figure 1. Graph of the cross-sections of upper and lower approximations $w^{(k)}(x_1, 0.5)$, $v^{(k)}(x_1, 0.5)$, $k = 0, 2, 6, 8$, for $\mathfrak{L}u \equiv -\Delta u$

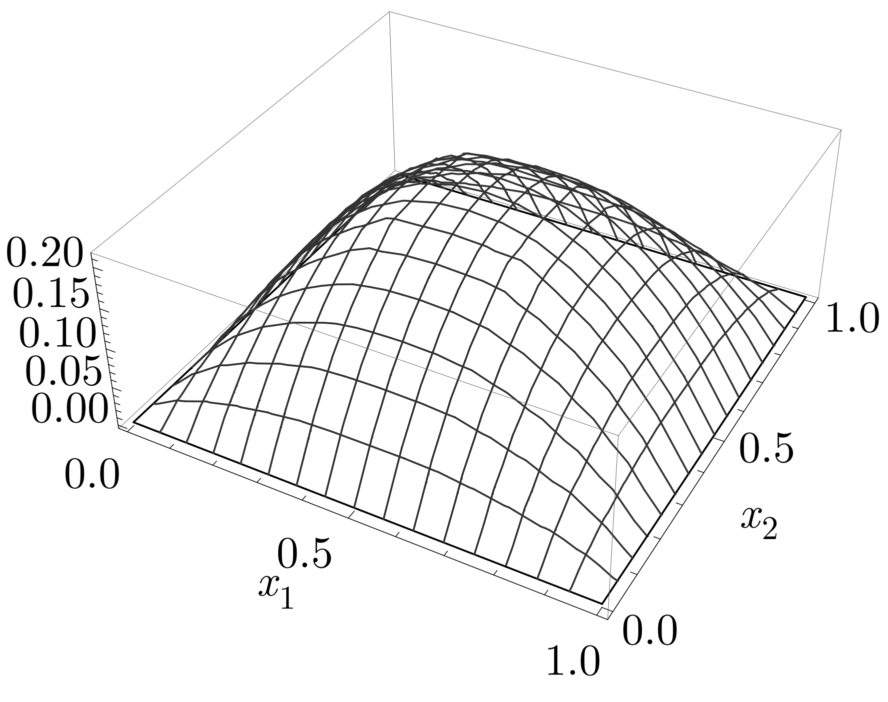

Figure 2. Surface of the approximate solution $u^{(11)}(\mathbf{x})$ for $\mathfrak{L}u \equiv -\Delta u$

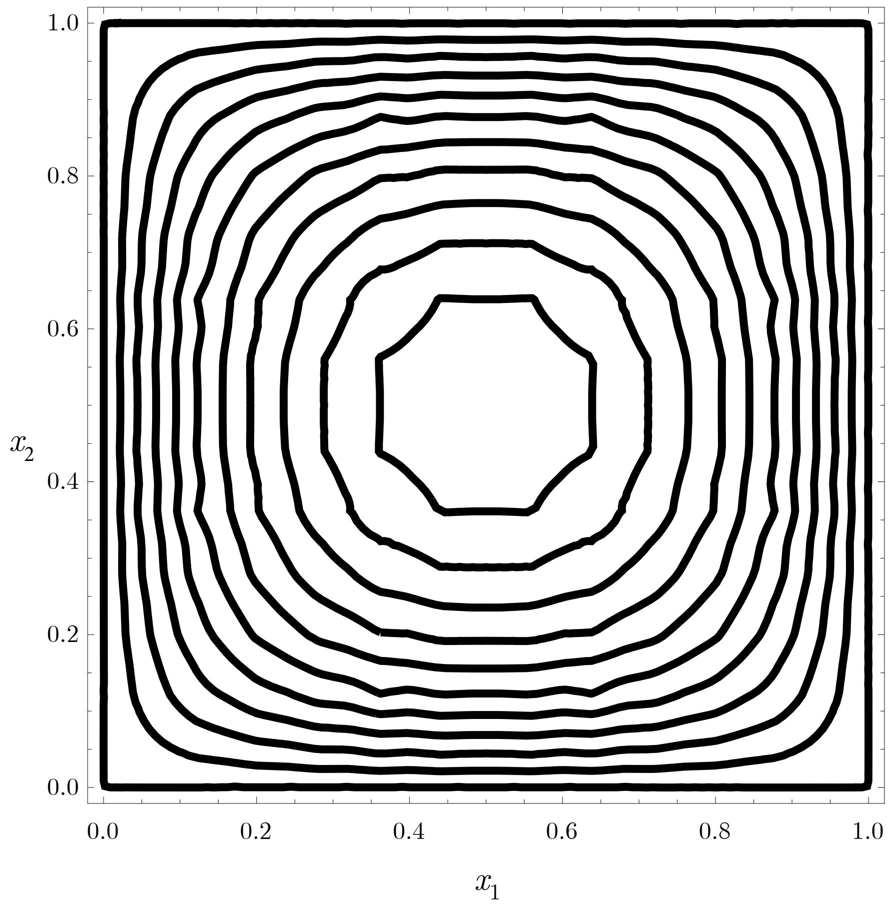

Figure 3. Contour lines of the approximate solution $u^{(11)}(\mathbf{x})$ for $\mathfrak{L}u \equiv -\Delta u$

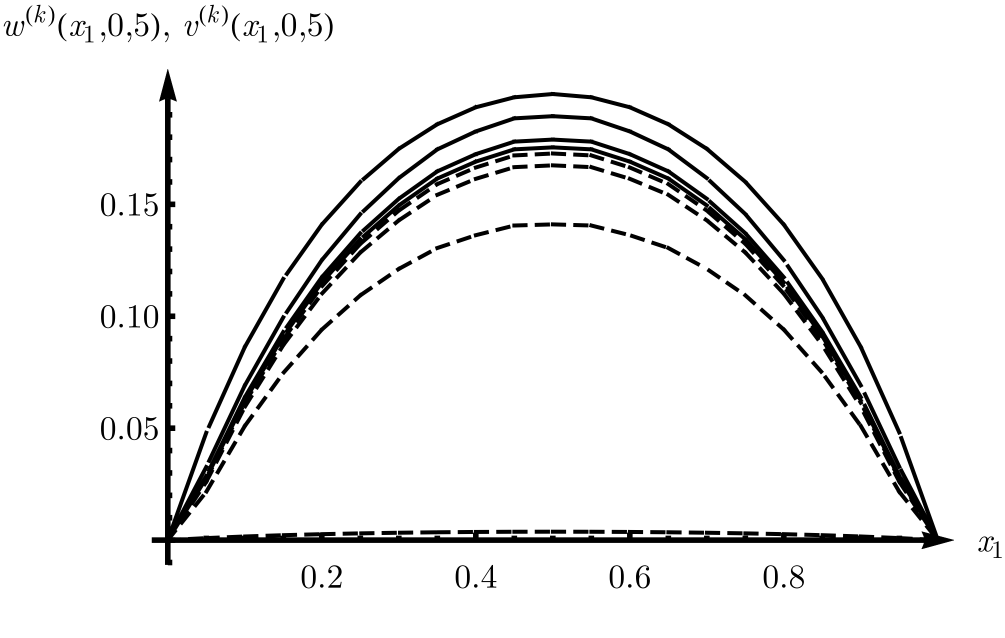

Figure 4. Graph of the cross-sections of upper and lower approximations $w^{(k)}(x_1, 0.5)$, $v^{(k)}(x_1, 0.5)$, $k = 0, 2, 6, 8$, for $\mathfrak{L}u \equiv -\Delta u + u$

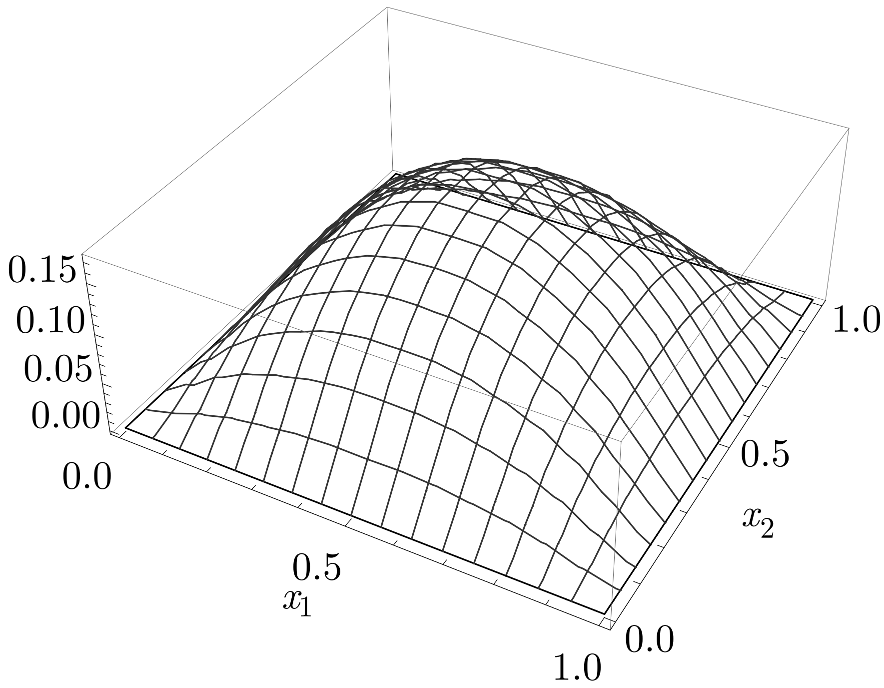

Figure 5. Surface of the approximate solution $u^{(10)}(\mathbf{x})$ for $\mathfrak{L}u \equiv -\Delta u + u$

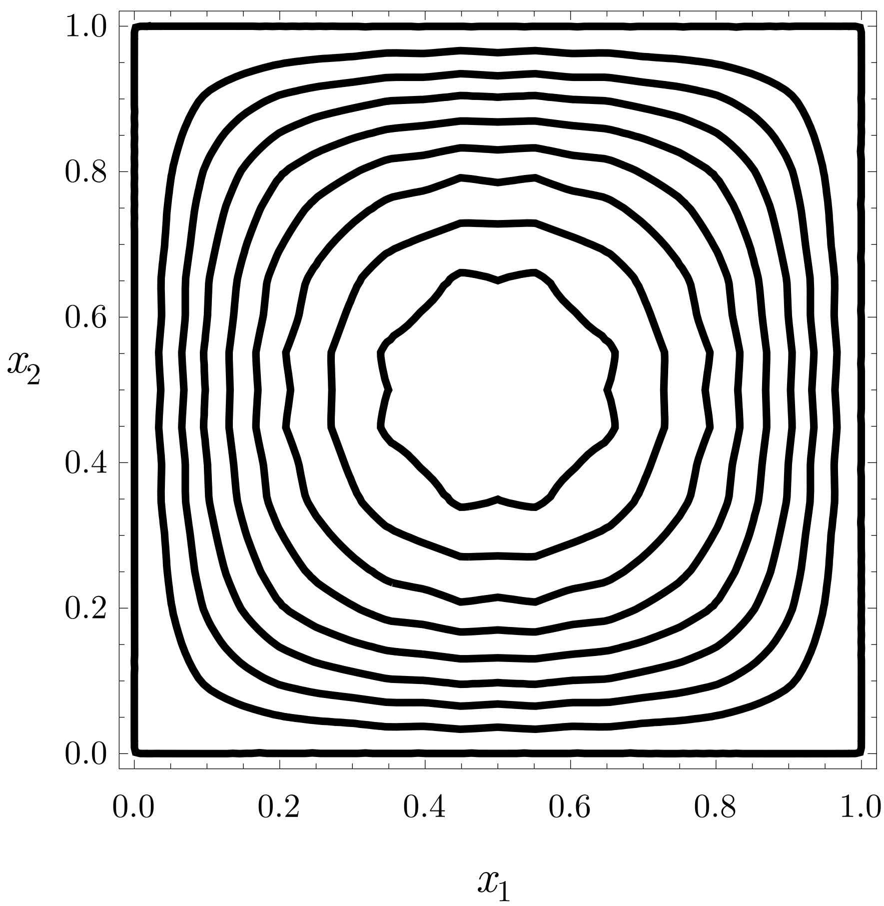

Figure 6. Contour lines of the approximate solution $u^{(10)}(\mathbf{x})$ for $\mathfrak{L}u \equiv -\Delta u + u$

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Сидоров М.В. *Метод квазіфункцій Гріна-Рвачова побудови двобічних наближень до додатного розв'язку нелінійних крайових задач* // Карпатські матем. публ. — 2018. — Т.10, №2. — С. 360–375.

Розглядається однорідна задача Діріхле для напівлінійних еліптичних рівнянь з оператором Лапласа та оператором Гельмгольця. Для побудови двобічних наближень до додатного розв'язку цієї крайової задачі використовується перехід за допомогою квазіфункції Гріна-Рвачова до еквівалентного нелінійного інтегрального рівняння з подальшим його аналізом методами теорії напівупорядкованих просторів. Робота і ефективність розробленого метода продемонстрована обчислювальним експериментом для тестової задачі з експоненціальною нелінійністю.

Ключові слова і фрази: додатний розв'язок, напівлінійна еліптична крайова задача, гетеротонний оператор, двобічні наближення, квазіфункція Гріна-Рвачова.

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SLIMANE A.

SPACES GENERATED BY THE CONE OF SUBLINEAR OPERATORS

This paper deals with a study on classes of non linear operators. Let $SL(X,Y)$ be the set of all sublinear operators between two Riesz spaces X and Y. It is a convex cone of the space $H(X,Y)$ of all positively homogeneous operators. In this paper we study some spaces generated by this cone, therefore we study several properties, which are well known in the theory of Riesz spaces, like order continuity, order boundedness etc. Finally, we try to generalise the concept of adjoint operator. First, by using the analytic form of Hahn-Banach theorem, we adapt the notion of adjoint operator to the category of positively homogeneous operators. Then we apply it to the class of operators generated by the sublinear operators.

Key words and phrases: Riesz space, Banach lattice, homogeneous operator, sublinear operator, order continuous operator.

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INTRODUCTION

The theory of Riesz spaces plays an important role in several branches of mathematics, in particular in the geometry of Banach spaces and the theory of linear operators where the notion of Banach lattice play a central role. In this work we generalize some vector lattice properties to the category of sublinear operators i.e., positively homogenous and subadditive. The set obtained is not a Banach space but a positive convex cone. Hence, this paper deals with the extension of this set and their properties. The paper is organized as follows.

In Section 1 we recall some basic definitions and properties of Riesz spaces, we also recall the notion of sublinear operators between a vector space X and a Riesz space Y.

In Section 2 we introduce the spaces spanned by different cones of sublinear operators. In other hand we present some principal notions concerning the theory of Riesz spaces like order continuity, order ideal, and we apply these notions on these spaces.

In Section 3 we introduce the adjoint of positively homogeneous operator. We first establish the following result.

Let u be in $\mathcal{L}(X,Y)$. Then the bounded adjoint operator u^* of u can be extended to a bounded linear operator $\widetilde{u}^*$ belongs to $\mathcal{L}(H^*(Y),H^*(X))$ such that $\widetilde{u}^* = u^*$ on Y^* and $\|\widetilde{u}^*\| = \|u^*\| = \|u\|$, where $H^*(Y)$ is the space of all bounded positively homogeneous functionals on Y, Y^* is the topological dual space of Y and $\mathcal{L}(X,Y)$ is the Banach space of all bounded linear operators from X into Y. Finally we adapt the existence theorem of bounded adjoint linear operator to the category of positively homogeneous operators as follows.

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2010 *Mathematics Subject Classification:* 46B42, 46B40, 47A05, 47B60, 47B65.

Let X, Y be two Banach spaces and $T \in \mathcal{H}(X,Y)$. Then, $T_h^* \in \mathcal{L}(H^*(Y), H^*(X))$ such that $\|T\| = \|T_h^*\|$, where T_h^* denotes the adjoint of T and $\mathcal{H}(X,Y)$ is the Banach space of all bounded positively homogeneous operators from X into Y.

1 Preliminaries

In this section, we introduce some terminology concerning Riesz spaces and Banach lattices. These spaces are well known. For more details, the interested reader can consult, for example, the references [2, 4–6]. But for our convenience, we include some recalls. We also introduce the class of positively homogeneous operators.

Let X be a real vector space. Then X is called a Riesz space (or vector lattice) if it is an ordered vector space with the additional property that the supremum of every nonempty finite subset of X exists in X. We denote the supremum of the set $\{x, y\}$ by $\sup\{x, y\}$ or $x \vee y$. Similarly, $\inf\{x, y\}$ or $x \wedge y$ denote the infimum of the set $\{x, y\}$.

Let X be a Riesz space. The subset $X^+ = \{x \in E : x \geq 0\}$ is called the positive cone of X (which is salient, i.e. $X^+ \cap (-X^+) = \{0\}$) and the elements of X^+ are called the positive elements of X.

Let X be a Riesz space, equipped with a norm. The norm in X is called a Riesz norm if

$$|x| \leq |y| \Longrightarrow \|x\| \leq \|y\|,$$

where $|x| = \sup\{x, -x\}$. Denote $x^+ = \sup\{x, 0\}$, $x^- = \sup\{-x, 0\}$. Then obviously we have $x = x^+ - x^-$ and $|x| = x^+ + x^-$. Note that this implies that for any $x \in X$, the elements x and $|x|$ have the same norm. A Riesz space X equipped with a Riesz norm, is called a normed Riesz space. If the norm is complete, X is called a Banach lattice. The convex cone X^+ is norm closed. A complete Banach lattice is a Banach lattice such that every order bounded set in X has a supremum.

By a Riesz subspace (or a vector sublattice) of a Riesz space X we mean a linear subspace E of X so that $\sup\{x, y\}$ belongs to E whenever $x, y \in E$. A vector subspace E of a Riesz space X is said to be an order ideal or simply ideal whenever $|x| \leq |y|$ and $y \in E$ imply $x \in E$.

A non-empty subset D is said to be upwards directed (respectively downwards directed) if for all $x_1, x_2 \in D$ there is $x_3 \in D$ such that $x_1 \vee x_2 \leq x_3$ (respectively $x_1 \wedge x_2 \geq x_3$), if $\sup D = x$ exists and D upwards directed (respectively $\inf D = y$ exists and D downwards directed) we shall write $D \uparrow x$ (respectively $D \downarrow y$).

Definition. *Let X be a vector space and Y be a Riesz space. An operator $T : X \longrightarrow Y$ is*

1- positively homogeneous if for all x in X and λ in $\mathbb{R}_+$ we have

$$T(\lambda x) = \lambda T(x),$$

2- subadditive if for all x, y in X we have

$$T(x + y) \leq T(x) + T(y).$$

The operator T is sublinear if it is positively homogeneous and subadditive. The operator T is said to be superlinear if T is positively homogeneous and superadditive (i.e. $T(x + y) \geq T(x) + T(y)$ for all x, y in X). We have for all x in X

$$-T(-x) \leq T(x). \tag{1}$$

We denote by $H(X,Y)$ (respectively $SL(X,Y)$) the real vector space of all positively homogeneous (the set of all sublinear) operators from X into Y, equipped with the natural order inducted by Y, i.e.

$$T \leq S \quad \text{if} \quad T(x) \leq S(x), \quad \forall\, x \in X.$$

The set $SL(X,Y)$ is a pointed convex cone of $H(X,Y)$ which is not salient.

Let T be in $SL(X,Y)$. We will denote by ∇T the subdifferential of T, which is the set of all linear operators $u : X \longrightarrow Y$ such that $u(x) \leq T(x)$ for all x in X. We know (see, for example, [1]), that ∇T is not empty if Y is a complete Banach lattice and $T(x) = \sup\{u(x) : u \in \nabla T\}$, moreover, the supremum is attained. If Y is simply a Banach lattice, then ∇T is empty in general (see [3]).

If X is a Banach space and Y is a Banach lattice, then we will denote by $\mathcal{SL}(X,Y)$ the set of all bounded (= continuous) sublinear operators from X into Y and by $\mathcal{L}(X,Y)$ the Banach space of all bounded linear operators from X into Y. Let T be in $SL(X,Y)$. We have (see [1]), that T is bounded if and only if u is bounded for all u in ∇T. The set $\mathcal{SL}(X,Y)$ (respectively the space $\mathcal{L}(X,Y)$) is a subset (respectively a subspace) of the space $\mathcal{H}(X,Y)$ of all homogeneous bounded operators from X into Y. The space $\mathcal{H}(X,Y)$ is normed by the standard norm

$$\|T\| = \sup_{\|x\| \leq 1} \|T(x)\|.$$

2 Spaces Spanned by Sublinear Operators

Let X be a vector space and Y be a Riesz space. We denote by

$$\triangle SL(X,Y) = SL(X,Y) - SL(X,Y)$$

the subspace of $H(X,Y)$ spanned by $SL(X,Y)$, i.e.

$$\triangle SL(X,Y) = \{T - S : T, S \in SL(X,Y)\}.$$

We denote by $\triangle\mathcal{SL}(X,Y)$ the subspace of all bounded operators in $\triangle SL(X,Y)$.

Proposition 1. *Let X be a vector space and Y be a Riesz space. Then $H(X,Y)$ is a Riesz space. If in addition X is a Banach space and Y is a Banach lattice, then $\mathcal{H}(X,Y)$ is also a Banach lattice.*

Proof. It is sufficent to endow the vector space $H(X,Y)$ with the partial order induced by Y. It is clear that $H(X,Y)$ is a Riesz space with respect to this order. Suppose now X be a Banach space and Y be a Banach lattice. Let $(T_n)_n \subset H(X,Y)$ be a Cauchy sequence, then $\lim\limits_{n\to+\infty} \|T_{n+p} - T_n\| = 0$ implies that $\lim\limits_{n\to+\infty} \|T_{n+p}(x) - T_n(x)\| = 0$ for all x in X.

As Y is a Banach space there is $T(x) \in Y$ such that $\lim\limits_{n\to+\infty} T_n(x) = T(x)$. Since $T_n(\alpha x) = \alpha T_n(x)$ for all α in $\mathbb{R}_+$ and all x in X we have $T(\alpha x) = \lim\limits_{n\to+\infty} T_n(\alpha x) = \lim\limits_{n\to+\infty} \alpha T_n(x) = \alpha T(x)$ for all α in $\mathbb{R}_+$ and all x in X. Thus, T is positively homogeneous. The operator T is clearly bounded and hence $\mathcal{H}(X,Y)$ is a Banach space. Let now $T, S \in \mathcal{H}(X,Y)$ such that $|T| \leq |S|$ then $\|T(x)\| \leq \|S(x)\|$ for all x in X, so $\|T\| \leq \|S\|$ and $\mathcal{H}(X,Y)$ is a Banach lattice. $\square$

Proposition 2. *Let X be a vector space and Y be a Riesz space. Then*

(a) the space $\triangle SL(X,Y)$ is a Riesz subspace of $H(X,Y)$;

(b) if X is a normed space and Y be a normed Riesz space, then $\triangle\mathcal{SL}(X,Y)$ is a normed Riesz space.

Proof. (a) The space $\triangle SL(X,Y)$, which is included in $H(X,Y)$, is partially ordered by the natural order inducted by Y. Consider T, S in $\triangle SL(X,Y)$. Then, there are T_1, T_2, S_1, S_2 in $SL(X,Y)$ such that

$$T = T_1 - T_2, S = S_1 - S_2.$$

For all x in X we define $T \vee S$ by

$$(T \vee S)(x) = T(x) \vee S(x).$$

Using for x, y, z in X the identity $x \vee y + z = (x+z) \vee (y+z)$, we obtain

$$\begin{aligned}(T \vee S)(x) &= (T_1 - T_2)(x) \vee (S_1 - S_2)(x)\\ &= (T_1 + S_2)(x) \vee (S_1 + T_2)(x) - (T_2 + S_2)(x) = \widetilde{T}(x) - \widetilde{S}(x)\end{aligned}$$

with $\widetilde{T}, \widetilde{S} \in SL(X,Y)$, where

$$\widetilde{T} = (T_1 + S_2) \vee (S_1 + T_2) \quad \text{and} \quad \widetilde{S} = T_2 + S_2.$$

(b) It is clear that $\triangle\mathcal{SL}(X,Y)$ is a normed Riesz space with the norm induced by the standard norm of $\mathcal{H}(X,Y)$ on $\triangle\mathcal{SL}(X,Y)$, i.e. by the norm $\|T\|_{\triangle\mathcal{SL}(X,Y)} = \sup\limits_{\|x\|\le 1} \|T(x)\|$. ☐

Proposition 3. *Let X be a vector space and Y be a Dedekind complete Riesz space. Then $H(X,Y)$ is also a Dedekind complete Riesz space.*

Proof. Let $M \subset H(X,Y)$ be a nonempty subset, which is upper bounded. Then there is $S \in H(X,Y)$ such that for all $T \in M$ we have $T \le S$, that is for all $T \in M$ and all $x \in X$ we have $T(x) \le S(x)$. This implies that for all $x \in X$ the set $\{T(x) : T \in M\}$ is upper bounded by $S(x) \in Y$. Since Y is a Dedekind complete Riesz space, the supremum of $\{T(x) : T \in M\}$ exists in Y. We can put now $R(x) = \sup\{T(x) : T \in M\}$. It is clear that R is a positively homogeneous operator. ☐

Remark 1. *For all $T = P - Q$ in $\triangle SL(X,Y)$ there is $\varphi_T \in SL(X,Y)$ and $\overline{\varphi}_T$ super linear (i.e. $-\overline{\varphi}_T$ sublinear) such that $\overline{\varphi}_T \le T \le \varphi_T$ and $\varphi_T(-x) = \varphi_{-T}(x)$ (respectively $\overline{\varphi}_T(-x) = \overline{\varphi}_{-T}(x)$) for all x in X. It suffices to define φ_T, $\overline{\varphi}_T$ by*

$$\varphi_T(x) = P(x) + Q(-x), \quad \overline{\varphi}_T(x) = -P(-x) - Q(x)$$

and use the inequality (1).

Definition 1. *Let $T \in \triangle SL(X,Y)$ be an operator between two Riesz spaces. The operator T is said to be order bounded if T carries order bounded subsets of X to order bounded subsets of Y.*

Definition 2. *Let $T \in \triangle SL(X,Y)$ be an order bounded operator. Then T is said to be*

(1) order continuous if for any downwards directed set D in E having infimum the null element (i.e. $D \downarrow 0$) we have $\inf(|T(x)|, x \in D) = 0$ in Y;

(2) σ-order continuous if for all $x_n \downarrow 0$ in X we have in Y

$$\inf(|T(x_n)|, n \geq 0) = 0.$$

We denote by

$$\begin{aligned}\triangle SL_b(X,Y) &= \{T \in \triangle SL(X,Y), T \text{ order bounded}\},\\ \triangle SL_{co}(X,Y) &= \{T \in \triangle SL(X,Y), T \text{ order continuous}\}.\end{aligned}$$

It should be clear that all these collections are real vector spaces under the usual pointwise algebraic operations.

Proposition 4. *The set $\triangle SL_b(X,Y)$ is a Riesz subspace of $\triangle SL(X,Y)$.*

Proof. Consider T_1, T_2 in $\triangle SL_b(X,Y)$, (α,β) in $\mathbb{R}^2$ and $\alpha \leq x \leq \beta$. Then

$$|(\alpha T_1 + \beta T_2)(x)| \leq |\alpha||T_1(x)| + |\beta||T_2(x)| \leq |\alpha|c_1 + |\beta|c_2 = c.$$

This implies that $\alpha T_1 + \beta T_2 \in \triangle SL_b(X,Y)$ and hence $T_1 \vee T_2 \in \triangle SL_b(X,Y)$ because $T_1 \vee T_2 = \frac{1}{2}(T_1 + T_2 + |T_1 - T_2|)$. Consequently, $\triangle SL_b(X,Y)$ is a Riesz subspace of the Riesz space $\triangle SL(X,Y)$. $\square$

3 THE ADJOINT OF POSITIVELY HOMOGENEOUS OPERATORS

Definition 3. *Let X, Y be two Riesz spaces. Put*

$$\triangle_r SL(X,Y) = \{T_1 - T_2 : \quad T_1, T_2 \in (SL(X,Y))^+\} \subset \triangle SL(X,Y).$$

A sublinear operator $T \in SL(X,Y)$ is said to be regular if $T \in \triangle_r SL(X,Y)$.

We denote by

$$\begin{aligned}SL_i(X,Y) &= \{T \in SL(X,Y) : T \text{ increasing}\},\\ \triangle SL_i(X,Y) &= \{T_1 - T_2 : \quad T_1, T_2 \in SL_i(X,Y)\}\\ &= SL_i(X,Y) - SL_i(X,Y),\\ L_i(X,Y) &= \{T \in L(X,Y) : T\text{increasing}\},\\ \triangle L_i(X,Y) &= \{T_1 - T_2 : \quad T_1, T_2 \in L_i(X,Y)\}\\ &= L_i(X,Y) - L_i(X,Y),\end{aligned}$$

and we put $X_i' = \triangle L_i(X,\mathbb{R}), X_{i,s}' = \triangle SL_i(X,\mathbb{R})$.

Proposition 5. *The spaces $\triangle_r SL(X,Y), \triangle SL_i(X,Y)$ are Riesz subspaces of $\triangle SL(X,Y)$.*

Proof. The set $\triangle_r SL(X,Y)$ is a subspace of $\triangle SL(X,Y)$. Further, if $T_1, T_2 \in \triangle_r SL(X,Y)$, then there is $P_1, Q_1, P_2, Q_2 \in (SL(X,Y))^+$ such that $T_1 = P_1 - Q_1$ and $T_2 = P_2 - Q_2$. We have $T_1 \vee T_2 = (P_1 + Q_2) \vee (P_2 + Q_1) - (Q_1 + Q_2)$, which is in $\triangle_r SL(X,Y)$ because

$$(P_1 + Q_2) \vee (P_2 + Q_1), (Q_1 + Q_2) \in (SL(X,Y))^+.$$

The same for $\triangle SL_i(X,Y)$. □

Proposition 6. *The spaces $\triangle_r SL(X,Y), \triangle SL_i(X,Y)$ are Riesz subspaces of $\triangle SL(X,Y)$.*

Proof. The set $\triangle_r SL(X,Y)$ is a subspace of $\triangle SL(X,Y)$. Further, if $T_1, T_2 \in \triangle_r SL(X,Y)$, then there is $P_1, Q_1, P_2, Q_2 \in (SL(X,Y))^+$ such that $T_1 = P_1 - Q_1$ and $T_2 = P_2 - Q_2$. We have $T_1 \vee T_2 = (P_1 + Q_2) \vee (P_2 + Q_1) - (Q_1 + Q_2)$, which is in $\triangle_r SL(X,Y)$ because

$$(P_1 + Q_2) \vee (P_2 + Q_1), (Q_1 + Q_2) \in (SL(X,Y))^+.$$

The same for $\triangle SL_i(X,Y)$. □

Remark 2. *1) Any linear operator is a regular sublinear operator. Indeed, if $u \in L(X,Y)$, then $u = u^+ - u^-$ with $u^+(x) = 0 \vee u(x)$, $u^-(x) = 0 \vee (-u(x))$, which are positive sublinear operators.*

2) The existence of the regular sublinear operators (not linear) is assured by the fact that if $T \in SL(X,Y)$ such that $|T| \in SL(X,Y)$, then T is regular

$$T = T^+ - T^- = 2T^+ - |T| \quad (2T^+, |T| \in (SL(X,Y))^+).$$

As example, consider $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha > \beta$ and $T : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$T(x) = \begin{cases} \alpha x, & \text{if } x \geq 0, \\ \beta x, & \text{if } x < 0. \end{cases}$$

Then T is sublinear ($T(x) = (\alpha x) \vee (\beta x)$) and $|T|$ also because

$$|T|(x) = |T(x)| = (\alpha x) \vee (-\beta x).$$

Lemma 1 ([6, Lemma 21.3]). *Let E be an ordered vector space, and let A, B be two subsets of E such that* $\inf A = x_0$, $\inf B = y_0$. *Then*

$$x_0 + y_0 = \inf(A + B) = \inf\{a + b \text{ such that } a \in A, b \in B\}.$$

Proposition 7. *Let X, Y be two Riesz spaces. Put*

$$SL_o(X,Y) = \{T \in SL_i(X,Y) \text{ such that } T \text{ order continuous}\},$$
$$\triangle SL_o(X,Y) = SL_o(X,Y) - SL_o(X,Y).$$

Then

(a) the set $SL_o(X,Y)$ is a convex cone;

(b) the space $\triangle SL_o(X,Y) \subset \triangle SL_{co}(X,Y)$ is an order ideal.

Proof. (a) Let $D \downarrow 0$, and $p, q \in SL_o(X, Y)$, then $(p+q)(D)$ is upwards directed such that $(p+q)(D) \downarrow 0$. Indeed, if $x_1, x_2 \in D$, then there is $x_3 \in D$ such that $x_3 \leq x_1$ and $x_3 \leq x_2$. This implies that $(p+q)(x_3) \in (p+q)(D)$. Thus

$$(p+q)(x_3) \leq (p+q)(x_1) \text{ and } (p+q)(x_3) \leq (p+q)(x_2).$$

Let h be the infimum of $(p+q)(D)$, then for all $x_1, x_2 \in D$ there is $x_3 \in D$ such that

$$h \leq (p+q)(x_3) \leq p(x_1) + q(x_2) \text{ for all } x_1, x_2 \in D.$$

We have

$$\begin{aligned} h &\leq \inf\{p(x_1) + q(x_2), x_1, x_2 \in D\} \\ &\leq \inf\{p(x_1), x_1 \in D\} + \inf\{q(x_2), x_2 \in D\ \} \\ &\leq \inf\{|p(x_1)|, x_1 \in D\} + \inf\{|q(x_2)|, x_2 \in D\} \leq 0. \end{aligned}$$

Consequently,

$$\begin{aligned} \inf\{|(p+q)(x)|,\ x \in D\} &\leq \inf\{|p(x)| + |q(x)|,\ x \in D\} \\ &\leq \inf\{p(x) + q(x),\ x \in D\} \leq 0. \end{aligned}$$

It is clear that $\lambda p \in SL_o\,(X, Y)$ for all $\lambda \in \mathbb{R}^+$ and all $p \in SL_o(X, Y)$. Furthermore

$$\begin{aligned} \inf\{|(p \vee q)(x)|,\ x \in D\} &= \inf\{(p \vee q)(x),\ x \in D\} \\ &\leq \inf\{(p+q)(x),\ x \in D\} \leq 0. \end{aligned}$$

(b) Let $T \in \triangle SL_o(X, Y)$. Then $T = p - q$ with $p, q \in SL_o(X, Y)$. Let $D \downarrow 0$. We have

$$|p-q|(x) \leq |p(x)| + |q(x)| \leq p(x) + q(x) \text{ for all } x \in D.$$

So,

$$\inf\{|(p-q)(x)|,\ x \in D\} \leq \inf\{(p+q)(x),\ x \in D\} \leq 0.$$

Consequently, $T \in \triangle SL_{co}(X, Y)$.
Let now $D \downarrow 0$. Assume that $|T| \leq |S|$, $S \in \triangle SL_o(X, Y)$, then

$$\inf\{|T|(x),\ x \in D\} \leq \inf\{|S|(x),\ x \in D\} \leq 0.$$

This ends the proof. $\square$

In the sequel, we extend the notion of adjoint operator on some spaces defined above. Let X be a Banach space and Y be a Banach lattice. Put

$$\begin{aligned} X' &= L(X, \mathbb{R}), \\ X^* &= \mathcal{L}(X, \mathbb{R}), \\ X'_\Delta &= \triangle SL(X, \mathbb{R}), \\ X^*_\Delta &= \triangle \mathcal{SL}(X, \mathbb{R}), \\ H'(X) &= H(X, \mathbb{R}), \\ H^*(X) &= \mathcal{H}(X, \mathbb{R}). \end{aligned}$$

We have $X' \subset X'_\Delta \subset H'(X)$ and $X^* \subset X^*_\Delta \subset H^*(X)$.

Theorem 1. *Let X, Y be two Riesz spaces and u be in $L(X,Y)$. Then there exists an $\widetilde{u}'$ in $L(H'(Y), H'(X))$ such that $\widetilde{u}' = u'$ on Y' and $\widetilde{u}'(\varphi) \leq |\varphi \circ u|$ for all $\varphi \in H'(Y)$, where u' is the adjoint operator of u.*

Proof. Let u be in $L(X,Y)$, the adjoint operator of u is defined by

$$u' : Y' \longrightarrow X' \subset H'(X)$$

such that

$$u'(\varphi) = \varphi \circ u \text{ for all } \varphi \in Y'.$$

Let now $P \in SL(H'(Y), H'(X))$ be defined by

$$P(\varphi) = |\varphi \circ u|.$$

We have

$$u'(\varphi) = \varphi \circ u \leq |\varphi \circ u| = P(\varphi) \text{ for all } \varphi \in Y'.$$

By the Hahn-Banach theorem (the analytic form), there is $\widetilde{u}' \in L(H'(Y), H'(X))$ such that $\widetilde{u}' = u'$ on Y' and

$$\widetilde{u}'(\varphi) \leq P(\varphi) \leq |\varphi \circ u|$$

for all $\varphi \in H'(Y)$ and this completes the proof. □

Theorem 2. *Let X, Y be two Banach spaces and u be in $\mathcal{L}(X,Y)$. Then there exists an $\widetilde{u}'$ in $\mathcal{L}(H^*(Y), H^*(X))$ such that $\widetilde{u}' = u^*$ on Y^* and $\|\widetilde{u}'\| = \|u^*\| = \|u\|$. In this case $\widetilde{u}'$ is denoted by $\widetilde{u}^*$.*

Proof. Let u be in $\mathcal{L}(X,Y)$. By Theorem 1 there is $\widetilde{u}'$ in $L(H'(Y), H'(X))$ such that $\widetilde{u}' = u^*$ on Y' and $\widetilde{u}'(\varphi) \leq |\varphi \circ u|$ for all $\varphi \in H'(Y)$. On the other hand, because $\widetilde{u}'(\varphi) \leq |\varphi \circ u|$ we obtain $|\widetilde{u}'(\varphi)| \leq |\varphi \circ u|$ and hence for all $\varphi \in H^*(Y)$

$$\|\widetilde{u}'(\varphi)\| \leq \|\varphi \circ u\| \leq \|u\|\|\varphi\|.$$

So, $\widetilde{u}' \in \mathcal{L}(H^*(Y), H^*(X))$. It remains to show that $\|\widetilde{u}'\| = \|u\|$. Since $\|\widetilde{u}'(\varphi)\| \leq \|u\|\|\varphi\|$, we conclude that $\|\widetilde{u}'\| \leq \|u\|$. For the converse inequality, we know that $\|u^*\| = \|u\|$, hence

$$\begin{aligned}
\|u\| = \|u^*\| &= \sup_{\varphi \in B_{Y^*}} \|u^*(\varphi)\| \\
&= \sup_{\varphi \in B_{Y^*}} \|\widetilde{u}'(\varphi)\| \text{ (because } \widetilde{u}'_{/Y^*} = u^*) \\
&\leq \sup_{\varphi \in B_{H^*(Y)}} \|\widetilde{u}'(\varphi)\| \text{ (because } B_{Y^*} \subset B_{H^*(Y)}) \\
&= \|\widetilde{u}'\|
\end{aligned}$$

and then the theorem is proved. □

Now, we extend the notion of adjoint operator to positively homogeneous operators.

Definition 4. *Let X, Y be two Riesz spaces and $T \in H(X,Y)$. We define the adjoint of T by*

$$\begin{aligned}
T'_h : H'(Y) &\longrightarrow H'(X) \\
\varphi &\longmapsto T'_h(\varphi) = \varphi \circ T
\end{aligned}$$

such that $T'_h(\varphi)(x) = \varphi \circ T(x)$.

Proposition 8. *Let X, Y be two Banach spaces and $T \in \mathcal{H}(X,Y)$. Then $T'_h \in \mathcal{L}(H^*(Y), H^*(X))$ such that $\|T\| = \|T'_h\|$. In this case T'_h is denoted by T^*_h.*

Proof. Consider T in $\mathcal{H}(X,Y)$. We have for all $\varphi \in H^*(Y)$

$$\|T'_h(\varphi)\| = \|\varphi \circ T\| \leq \|\varphi\|\|T\|.$$

So, $T'_h \in \mathcal{L}(H^*(Y), H^*(X))$. To show that $\|T\| = \|T'_h\|$, we first consider the mapping $i : x \in X \longmapsto i(x) \in H^{**}(X)$ such that

$$\begin{aligned} i(x) : H^*(X) &\longrightarrow \mathbb{R}, \\ \varphi &\longmapsto (i(x), \varphi) = \langle \varphi, x \rangle. \end{aligned}$$

Then i is such that $\|i(x)\| = \|x\|$ for all $x \in X$. Indeed,

$$\begin{aligned} \|i(x)\| &= \sup_{\varphi \in B_{H^*(X)}} \|(i(x), \varphi)\| \\ &= \sup_{\varphi \in B_{H^*(X)}} \|\langle \varphi, x \rangle\| \\ &\leq \|x\|. \end{aligned}$$

Conversely

$$\begin{aligned} \|x\| &= \sup_{\xi \in B_{X^*}} \|\langle \xi, x \rangle\| \leq \sup_{\varphi \in B_{H^*(X)}} \|\langle \varphi, x \rangle\| \text{ (because } B_{X^*} \subset B_{H^*(X)}) \\ &\leq \sup_{\varphi \in B_{H^*(X)}} \|(i(x), \varphi)\| \leq \|i(x)\|. \end{aligned}$$

Finally, we have

$$\begin{aligned} \|T'_h\| = \sup_{\varphi \in B_{H^*(Y)}} \|T'_h(\varphi)\| &= \sup_{\varphi \in B_{H^*(Y)}} \|\varphi \circ T\| \\ &= \sup_{\varphi \in B_{H^*(Y)}} (\sup_{x \in B_X} \|\langle \varphi \circ T, x \rangle\|) \\ &= \sup_{\varphi \in B_{H^*(Y)}} (\sup_{x \in B_X} \|\langle \varphi, T(x) \rangle\|) \\ &= \sup_{x \in B_X} (\sup_{\varphi \in B_{H^*(Y)}} \|\langle \varphi, T(x) \rangle\|) \\ &= \sup_{x \in B_X} (\sup_{\varphi \in B_{H^*(Y)}} \|(i(T(x)), \varphi)\|) \\ &= \sup_{x \in B_X} \|i(T(x))\| \\ &= \sup_{x \in B_X} \|T(x)\| = \|T\|. \end{aligned}$$

This completes the proof. □

Definition 5. *Let X, Y be two Riesz spaces. Consider $T \in \triangle SL(X,Y)$ with $T = P - Q$. We define a linear operator on $Y'_{i,s}$ denoted T'_i by*

$$\begin{aligned} &T'_i : Y'_{i,s} \longrightarrow X'_\Delta, \\ &T_1 - T_2 \longmapsto T'_i(T_1 - T_2) = T_1 \circ P + T_2 \circ Q - (T_1 \circ Q + T_2 \circ P). \end{aligned}$$

Note that this operator is well defined. Indeed, if $S \in Y'_{i,s}$ such that $S = S_1 - S_2 = S_3 - S_4$, then

$$\begin{aligned} T'_i(S_1 - S_2) &= S_1 \circ P + S_2 \circ Q - (S_1 \circ Q + S_2 \circ P) \\ &= (S_1 - S_2) \circ P - (S_1 - S_2) \circ Q \\ &= (S_3 - S_4) \circ P - (S_3 - S_4) \circ Q = T'_i(S_3 - S_4). \end{aligned}$$

Proposition 9. *Let X, Y be two Riesz spaces, then there is $\widetilde{T}'_i$ in $L(H'(Y), H'(X))$ such that $\widetilde{T}'_i = T'_i$ on $Y'_{i,s}$.*

Proof. We define a sublinear operator $S : H'(Y) \longrightarrow H'(X)$ by

$$S(\varphi) = |\varphi \circ P| + |\varphi \circ Q|.$$

For all $\varphi = \varphi_1 - \varphi_2 \in Y'_{i,s}$ we have

$$\begin{aligned} T'_i(\varphi) = T'_i(\varphi_1 - \varphi_2) = \varphi_1 \circ P + \varphi_2 \circ Q - (\varphi_1 \circ Q + \varphi_2 \circ P) &= (\varphi_1 - \varphi_2) \circ P - (\varphi_1 - \varphi_2) \circ Q \\ \leq |(\varphi_1 - \varphi_2) \circ P| + |(\varphi_1 - \varphi_2) \circ Q| &= S(\varphi). \end{aligned}$$

The Hahn-Banach theorem implies that T'_i can be extended to a linear operator $\widetilde{T}'_i \in L(H'(Y), H'(X))$ such that $\widetilde{T}'_i(\varphi) \leq S(\varphi)$ for all $\varphi \in H'(Y)$. □

Remark 3. *If $T \in L(X, Y)$, then we have $\widetilde{T}' = T^*_h$ on Y', where T'_h denote the operator defined in Definition 4. If $T \in \triangle SL(X, Y)$, then we have $\widetilde{T}'_i = T'_h$ on Y'_i.*

Proposition 10. *Let X, Y be two Riesz spaces and T be in $(SL(X, Y))^+$. Then the following properties are satisfied.*

(1) We have $|T|'_i \leq |T'_i|$.

(2) The restriction of T'_i to $SL_i(Y, \mathbb{R})$ verifies $|T'_i| = |T|'_i$.

Proof. (1) Let $T \in (SL(X, Y))^+$ and $\varphi \in Y'_{i,s}$, then there is $\varphi_1, \varphi_2 \in SL_i(X, Y)$ such that $\varphi = \varphi_1 - \varphi_2$ and

$$\begin{aligned} |T'_i|(\varphi) = |T'(\varphi)| = |\varphi_1 \circ T - \varphi_2 \circ T| &\geq \varphi_1 \circ T - \varphi_2 \circ T \\ &\geq \varphi_1 \circ |T| - \varphi_2 \circ |T| \geq |T|'_i(\varphi). \end{aligned}$$

(2) Let $T \in (SL(X, Y))^+$ and $\varphi \in SL_i(Y, \mathbb{R})$ we have

$$\begin{aligned} |T'_i|(\varphi) &= |T'(\varphi)| = |\varphi(T)| = \varphi(T) \text{ (because } \varphi \uparrow \text{ and } T \geq 0) \\ &= \varphi(|T|) = |T|'_i(\varphi) \end{aligned}$$

and this completes the proof. □

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Слімане А. *Простори, породжені конусом сублінійних операторів* // Карпатські матем. публ. — 2018. — Т.10, №2. — С. 376–386.

У цій статті досліджуються деякі класи нелінійних операторів. Нехай $SL(X, Y)$ — множина всіх сублінійних операторів між двома просторами Ріса X та Y. Це є опуклий конус в просторі $H(X, Y)$ всіх позитивно однорідних операторів. У цій статті досліджено деякі простори, породжені цим конусом, зокрема ми досліджуємо деякі властивості, які добре відомі в теорії просторів Ріса, такі як порядкова неперервність, порядкова обмеженість та ін. Насамкінець, ми пробуємо узагальнити концепцію спряженого оператора. Спочатку, використовуючи аналітичну форму теореми Гана-Банаха, ми пристосовуємо поняття спряженого оператора до категорії позитивно однорідних операторів, а потім застосовуємо його до класу операторів, породжених сублінійними операторами.

Ключові слова і фрази: простір Ріса, банахова ґратка, однорідний оператор, сублінійний оператор, порядково неперервний оператор.

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ON AN APPROACH TO THE CONSTRUCTION OF THE FRIEDRICHS AND NEUMANN-KREIN EXTENSIONS OF NONNEGATIVE LINEAR RELATIONS

Let L_0 be a closed linear nonnegative (probably, positively defined) relation ("multivalued operator") in a complex Hilbert space H. In terms of the so called boundary value spaces (boundary triples) and corresponding Weyl functions and Kochubei-Strauss characteristic ones, the Friedrichs (hard) and Neumann-Krein (soft) extensions of L_0 are constructed.

It should be noted that every nonnegative linear relation L_0 in a Hilbert space H has two extremal nonnegative selfadjoint extensions: the Friedrichs extension L_F and the Neumann-Krein extension L_K, satisfying the following property:

$$(\forall \varepsilon > 0)(L_F + \varepsilon 1)^{-1} \leq (\widetilde{L} + \varepsilon 1)^{-1} \leq (L_K + \varepsilon 1)^{-1}$$

in the set of all nonnegative selfadjoint subspace extensions $\widetilde{L}$ of L_0.

The boundary triple approach to the extension theory was initiated by F. S. Rofe-Beketov, M. L. and V. I. Gorbachuk, A. N. Kochubei, V. A. Mikhailets, V. O. Dercach, M. N. Malamud, Yu. M. Arlinskii and other mathematicians.

In addition, it is showed that the construction of the mentioned extensions may be realized in a more simple way under the assumption that initial relation is a positively defined one.

Key words and phrases: Hilbert space, relation, operator, extension, boundary value space.

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INTRODUCTION

Beginning with the work by R. Arens [2], the efforts of many authors were directed at the studying of linear relations (multivalued operators), in particular, at the investigations concerning the extension theory of the linear relations in Hilbert space (see, e.g., [4,5,8,9]). A number of problems arising in the mentioned theory have been solved in terms of the so called boundary value spaces (boundary triples) and corresponding Weyl functions (see Definitions 1, 2 and [3,6,7,10,11]).

Let $\oplus$ and $\ominus$ be the symbols of orthogonal sum and orthogonal complement, respectively. Explain that under (closed) linear relation in H, where H is a fixed complex Hilbert space equipped with the inner product $(\cdot|\cdot)$ and norm $\|\cdot\|$, we understand a (closed) linear manifold in $H^2 \overset{def}{=} H \oplus H$ and that in the theory of linear relations every linear operator is identified with its graph. Each such relation T has the inverse $T^{-1} \overset{def}{=} \{(y', y) \in H^2 \,|\, (y, y') \in T\}$ and the adjoint $T^* = H^2 \ominus JT \; (= J(H^2 \ominus T))$, where $\forall\, h_1, h_2 \in H \quad J(h_1, h_2) \overset{def}{=} (-ih_2, ih_1)$. This

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circumstance (the inverse and adjoint existence) makes the theory of linear relations extremely useful in the study of various problems.

Remind that a linear relation S in H is said to be nonnegative (in symbols $S \geq 0$) if for all $(y, y') \in S \ \ (y'|y) \geq 0$, positively defined (in symbols $S \gg 0$) if, in addition,

$$\inf S \stackrel{def}{=} \inf \left\{ \left(u'|u\right) \mid \left(u, u'\right) \in S, \ \|u\| = 1 \right\} > 0,$$

and selfadjoint if $S = S^*$.

In this paper the role of initial object is played by a closed linear nonnegative relation L_0 in H. It is known [5] that there exist selfadjoint extensions (probably, subspace ones) L_F and L_K of L_0 satisfying the following property:

selfadjoint extension L_1 of L_0 is nonnegative iff for any $\varepsilon > 0$

$$\forall y \in H \quad \left((L_F + \varepsilon 1_H)^{-1} y|y \right) \leq \left((L_1 + \varepsilon 1_H)^{-1} y|y \right) \leq \left((L_K + \varepsilon 1_H)^{-1} y|y \right). \tag{1}$$

In the case when L_0 is a densely defined operator, this fact was proved by M. Krein [14].

The extensions L_F and L_K are called the Friedrichs and Neumann-Krein extensions of L_0, respectively. If L_0 is a positively defined, the first of the inequalities (1) holds under $\varepsilon = 0$, too.

The aim of this article is to construct the mentioned extensions in the terms of boundary value spaces and corresponding Weyl functions. We widely use the results exposed in [1,3,6, 7,16,19], but our approach is different from ones of these papers. In particular, we (as in our previous articles [17] and [18]) deal with Cayley transforms $U(\lambda)$ of Weyl functions (Strauss-Kochubei characteristic functions in the sence of [13] and [20]). But the papers are mentioned above devoted to the investigation of $U(\lambda)$ under $\mathrm{Im}\lambda \neq 0$, while we are interested to consider the behaviour of $U(\lambda)$ in the case when $\lambda \in \mathcal{R}$, first of all in the situations as $\lambda \to -0$ and $\lambda \to -\infty$.

1 NOTATIONS AND PRELIMINARY RESULTS

Through this paper we use the following notations:

$D(T), \ R(T), \ \ker T$ are, respectively, the domain, range, and kernel of a (linear) relation (in partial, operator) T;

$D(T) = \{y \in H | \, (\exists y' \in H) : \ (y, y') \in T\}; \ \ R(T) = \{y' \in H | \, (\exists y \in H) : \ (y, y') \in T\};$

$\ker T = \{y \in H | \, (y, 0) \in T\};$

if $\lambda \in \mathbb{C}$ then $T - \lambda = \{(y, y' - \lambda y) \, | \, (y, y') \in T\}$, and so

$\ker(T - \lambda) = \{y \in H | \, (y, 0) \in T - \lambda\} \ (= \{y \in H | \, (y, \lambda y) \in T\});$

$\overset{\wedge}{\ker}(T - \lambda) = \{(y, \lambda y : y \in \ker(T - \lambda)\};$

$\rho(T) = \{\lambda \in \mathbb{C} | \ker(T - \lambda) = \{0\}, \ R(T - \lambda) = H\}$ (the resolvent set of T);

1_X is the identity in X.

If X, Y are Hilbert spaces then $(\cdot|\cdot)_X$ is the symbol of scalar product in X, $\mathcal{B}(X, Y)$ is the set of linear bounded operators $A : X \to Y$ such that $D(A) = X$; $\mathcal{B}(X) \stackrel{def}{=} \mathcal{B}(X, X)$.

If $A_i : X \to Y_i \, (i = 1, 2)$ are linear operators then the notation $A = A_1 \oplus A_2$ means that $Ax = \begin{pmatrix} A_1 x \\ A_2 x \end{pmatrix}$ for every $x \in X$. Let $s - \lim$ denotes the strong limit.

Under L_0 we understand the linear relation described in the Introduction, and $L \stackrel{def}{=} L_0^*$.

Definition 1. *Let $\mathcal{H}$ be a Hilbert space and $\Gamma_1, \Gamma_2 \in \mathcal{B}(L, \mathcal{H})$. The triple $(\mathcal{H}, \Gamma_1, \Gamma_2)$ is called the boundary value space (BVS) for the linear relation L_0 if*

$$R(\Gamma_1 \oplus \Gamma_2) = \mathcal{H} \oplus \mathcal{H}, \quad \ker(\Gamma_1 \oplus \Gamma_2) = L_0$$

and for any $\hat{y} = (y, y')$, $\hat{z} = (z, z') \in L$ we have

$$(y'|z) - (y|z') = (\Gamma_1 \hat{y}|\Gamma_2 \hat{z})_{\mathcal{H}} - (\Gamma_2 \hat{y}|\Gamma_1 \hat{z})_{\mathcal{H}}.$$

Through the paper we suppose that (the selfadjoint) relation $L_2 \stackrel{def}{=} \ker \Gamma_2$ is nonnegative, and so $\forall \lambda < \inf L_2$ the following operators are correctly defined:

$$L_\lambda = (L_2 - \lambda)^{-1} \in \mathcal{B}(H),\ \hat{L}_\lambda = \begin{pmatrix} L_\lambda \\ 1_H + \lambda L_\lambda \end{pmatrix} \in \mathcal{B}\left(H, H^2\right), \tilde{L}_\lambda = (L_\lambda,\ 1_H + \lambda L_\lambda) \in \mathcal{B}\left(H^2, H\right),$$

i.e. $\forall y \in H \ \ \hat{L}_\lambda y = \begin{pmatrix} L_\lambda y \\ y + \lambda L_\lambda y \end{pmatrix}$, $\forall \hat{y} = (y, y') \in H^2 \ \tilde{L}_\lambda \hat{y} = L_\lambda y + (y' + \lambda L_\lambda y')$ (it is easy to see that $R(\hat{L}_\lambda) = L_2$ and $\hat{L}_\lambda^* = \tilde{L}_\lambda$). Put

$$Z_\lambda = \left(\Gamma_1 \hat{L}_\lambda\right)^*, \quad \hat{Z}_\lambda = \begin{pmatrix} Z_\lambda \\ \lambda Z_\lambda \end{pmatrix}.$$

Definition 2. *A $\mathcal{B}(\mathcal{H})$-valued function*

$$M(\lambda) = \Gamma_1 \hat{Z}_\lambda \ \ (\lambda < \inf L_2)$$

is called the Weyl function of the relation L_0 corresponding to its boundary value space $(\mathcal{H}, \Gamma_1, \Gamma_2)$.

Note that $M(\lambda) = M(\lambda)^*$.

Remark 1. *The notion of BVS had been introduced at first in [12] under the assumption that L_0 is a densely defined symmetric operator having equal defect numbers. In [16] this notion was extended onto the case of nondensely defined Hermitian operators. The conception of Weyl function corresponding to a given BVS was appeared in [6] and had found its development in many papers (see, for example, [7, 10, 11] and references therein). It is easy to see that Definition 2 is equivalent to suitable defintions from the mentioned articles. It becomes clear after analyzing the results of the monograph [15] (see also [17] and [18]).*

Theorem 1. *For arbitrary $\lambda, \mu \in (-\infty, \inf L_2)$ $M(\lambda) - M(\mu) = (\lambda - \mu) Z_\lambda^* Z_\mu \left(= (\lambda - \mu)\, Z_\mu^* Z_\lambda\right)$, in particular, $\mu < \lambda$ implies $M(\lambda) - M(\mu) \gg 0$. Hence for any $z < \inf L_2$ there exist*

$$s - \lim_{\lambda \to -0} (M(\lambda) - M(z))^{-1} \stackrel{def}{=} R_0\, (\geq 0)\,,$$
$$s - \lim_{\lambda \to -\infty} (M(\lambda) - M(z))^{-1} \stackrel{def}{=} R_{-\infty}\, (\leq 0)\,.$$

Theorem 2. *Let $L_A = \ker(A_1 \Gamma_1 + A_2 \Gamma_2)$, where $A_1, A_2 \in \mathcal{B}(\mathcal{H})$ and*

$$A_\lambda \stackrel{def}{=} A_1 M(\lambda) + A_2 \ \ (\lambda < \inf L_2)\,.$$

If $A_\lambda^{-1} \in \mathcal{B}(\mathcal{H})$, then $\lambda \in \rho(L_A)$ and

$$(L_A - \lambda)^{-1} = (L_2 - \lambda)^{-1} - Z_\lambda A_\lambda^{-1} A_1 Z_\lambda^*. \tag{2}$$

Theorem 3. *The linear relation L_1 is a selfadjoint extension of L_0 iff there exists a unitary operator $K \in \mathcal{B}(\mathcal{H})$ such that $L_1 = \ker\left[(K - 1_{\mathcal{H}})\,\Gamma_1 + i\,(K + 1_{\mathcal{H}})\,\Gamma_2\right]$.*

Put

$$L^{(\lambda)} = L_0 \dot{+} \overset{\wedge}{\ker}\,(L - \lambda) \quad (\lambda < \inf L_2)\,. \tag{3}$$

Theorem 4. $L^{(\lambda)} = \ker\left(\Gamma_1 - M(\lambda)\Gamma_2\right)$.

Theorem 5. *Suppose that $z < \inf L_2$, $\lambda < \inf L_2$ and $z \neq \lambda$. Then $L^{(\lambda)}$ is a selfadjoint relation and $z \in \rho(L^{(\lambda)})$. Moreover,*

$$(L_F - z)^{-1} = s - \lim_{\lambda \to -\infty}\left(L^{(\lambda)} - z\right)^{-1}, \quad (L_K - z)^{-1} = s - \lim_{\lambda \to -0}\left(L^{(\lambda)} - z\right)^{-1}.$$

Remark 2. *The results mentioned in Theorems 1–5 above are well known or are immediate consequences of such ones (see, e. g., [1,3,5,7,9,16]).*

2 Main results

Let λ and z be as above. Before formulating the main results let us introduce the following (defined on $\rho(L_2)$) operator-functions by setting

$$\begin{gathered} R(\lambda) = (M(\lambda) - M(z))^{-1}, \quad \Omega_{\pm}(\lambda) = (M(\lambda) \pm i)\,R(\lambda), \\ U(\lambda) = (M(\lambda) - i)\,(M(\lambda) + i)^{-1}. \end{gathered} \tag{4}$$

It is easily to check by calculation that

$$U(\lambda) = \Omega_-(\lambda)\Omega_+^{-1}(\lambda), \tag{5}$$

$$\Omega_{\pm}(\lambda) = 1_{\mathcal{H}} + (M(z) \pm i)\,R(\lambda), \tag{6}$$

$$\Omega_{\pm}^{-1}(\lambda) = 1_{\mathcal{H}} - (M(z) \pm i)\,(M(\lambda) \pm i)^{-1}. \tag{7}$$

Lemma 1.

$$L^{(\lambda)} = \left\{\hat{y} \in L \mid (U(\lambda) - 1_{\mathcal{H}})\,\Gamma_1\hat{y} + i\,(U(\lambda) + 1_{\mathcal{H}})\,\Gamma_2\hat{y} = 0\right\}. \tag{8}$$

Proof. It is clear that (4) yields

$$(U(\lambda) - 1_{\mathcal{H}})\,M(\lambda) = -i\,(U(\lambda) + 1_{\mathcal{H}})\,. \tag{9}$$

Let us denote (temporarily) the relation from the right side of (8) by $L^{[\lambda]}$. Taking into account (9) we obtain the following:

$$\hat{y} \in L^{(\lambda)} \Rightarrow \Gamma_1\hat{y} - M(\lambda)\Gamma_2\hat{y} = 0 \Rightarrow (U(\lambda) - 1_{\mathcal{H}})\,\Gamma_1\hat{y} + i\,(U(\lambda) + 1_{\mathcal{H}})\,\Gamma_2\hat{y} = 0 \Rightarrow y \in L^{[\lambda]}.$$

Thus $L^{(\lambda)} \subset L^{[\lambda]}$. But $L^{(\lambda)}$, $L^{[\lambda]}$ are selfadjoint relations (see Theorem 3), therefore $L^{(\lambda)} = L^{[\lambda]}$. □

Lemma 2. *Let B and R be selfadjoint operators from $\mathcal{B}(\mathcal{H})$ and*

$$\Omega_{\pm} \overset{def}{=} 1_{\mathcal{H}} + BR \pm iR.$$

Then $\Omega_{\pm}^{-1} \in \mathcal{B}(\mathcal{H})$.

Proof. One can readily check by calculations that

$$\begin{pmatrix} B-i & -\Omega_- \\ -(B+i) & \Omega_+ \end{pmatrix}\begin{pmatrix} \Omega_-^* & \Omega_+^* \\ B+i & B-i \end{pmatrix} = \begin{pmatrix} \Omega_-^* & \Omega_+^* \\ B+i & B-i \end{pmatrix}\begin{pmatrix} B-i & -\Omega_- \\ -(B+i) & \Omega_+ \end{pmatrix} = -2i1_{\mathcal{H}\oplus\mathcal{H}},$$

in particular

$$\Omega_-^*\Omega_- = \Omega_+^*\Omega_+, \tag{10}$$

$$\Omega_-^*\left(B-i\right) - \Omega_+^*\left(B+i\right) = -2i1_{\mathcal{H}}, \tag{11}$$

$$\left(B-i\right)\Omega_+^* = \Omega_-\left(B-i\right), \quad \left(B+i\right)\Omega_-^* = \Omega_+\left(B+i\right). \tag{12}$$

It follows from (10) that $\|\Omega_- h\| = \|\Omega_+ h\|$ for each $h \in \mathcal{H}$. This yields that there exists an isometry $K : R(\Omega_-) \to R(\Omega_+)$ such that $\Omega_+ = K\Omega_-$, consequently there exist $K_+, K_- \in \in \mathcal{B}(\mathcal{H})$, satisfying the equalities $\Omega_-^* = \Omega_+^* K_+,\ \Omega_+^* = \Omega_-^* K_-$. Thus $R\left(\Omega_-^*\right) = R\left(\Omega_+^*\right)$. Taking into account (11) we see that $R\left(\Omega_-^*\right) + R\left(\Omega_+^*\right) = \mathcal{H}$, therefore

$$R\left(\Omega_-^*\right) = R\left(\Omega_+^*\right) = \mathcal{H}. \tag{13}$$

The equalities (13) imply

$$\ker\Omega_+ = \ker\Omega_- = \{0\}. \tag{14}$$

In view of (12) and (14) we obtain $\ker\Omega_-^* = \ker\Omega_+^* = \{0\}$. To complete the proof it is sufficient to apply (13). □

Proposition 1. *There exist the unitary operators $U_{-\infty}$, $U_0 \in \mathcal{B}(\mathcal{H})$ defined as follows:*

$$U_{-\infty} = s - \lim_{\lambda\to-\infty} U(\lambda), \quad U_0 = s - \lim_{\lambda\to-0} U(\lambda). \tag{15}$$

Moreover,

$$U_{-\infty} = \left(1_{\mathcal{H}} + (M(z)-i)R_{-\infty}\right)\left(1_{\mathcal{H}} + (M(z)+i)R_{-\infty}\right)^{-1}, \tag{16}$$

$$U_0 = \left(1_{\mathcal{H}} + (M(z)-i)R_0\right)\left(1_{\mathcal{H}} + (M(z)+i)R_0\right)^{-1}, \tag{17}$$

where $R_{-\infty}$ and R_0 are as in the Theorem 1.

Proof. It follows from Theorem 1, from (6) and from Lemma 2, applied to the operators $1_{\mathcal{H}} + BR \pm iR$ with $B = M(z),\ R = R_{-\infty}$, that $s - \lim\limits_{\lambda\to-\infty} \Omega_\pm(\lambda) = 1_{\mathcal{H}} + \left(M(z) \pm i\right) R_{-\infty}$ and the operators in the right side of the latter equality are invertible in $\mathcal{B}(\mathcal{H})$. Further, in view of (7) we obtain $\left\|\Omega_+^{-1}(\lambda)\right\| \le 1 + \|M(z)+i\| \cdot \left\|(M(\lambda)+i)^{-1}\right\|$.

On the other hand, using the elementary properties of the resolvent of a selfadjoint operator we conclude that for each $\lambda < \inf L_2$ $\left\|(M(\lambda)+i)^{-1}\right\| \le 1$. Thus the family

$$\left\{\Omega_+^{-1}(\lambda) \mid -\infty < \lambda < \inf L_2\right\}$$

is uniformly bounded in $\mathcal{B}(\mathcal{H})$, therefore

$$s - \lim_{\lambda\to-\infty}\Omega_+^{-1}(\lambda)\ \left(= s - \lim_{\lambda\to-\infty}\Omega_+(\lambda)\right)^{-1} = \left(1_{\mathcal{H}} + \left(M(z)+i\right)R_{-\infty}\right)^{-1}.$$

Whence using (5) we conclude that there exists the first limit in (15) and the equality (16) holds. Similar arguments show that there exists the second limit in (15) and the equality (17) holds.

Finally, taking into account (15) and the invertibility in $\mathcal{B}(\mathcal{H})$ of the operators in right sides of (16)–(17), we conclude that the unitarity of $U(\lambda)$ under $\lambda < \inf L_2$ yields the unitarity of $U_{-\infty}$ and U_0. □

Theorem 6.

$$L_F = \{\hat{y} \in L \,|\, (U_{-\infty} - 1_{\mathcal{H}})\,\Gamma_1\hat{y} + i\,(U_{-\infty} + 1_{\mathcal{H}})\,\Gamma_2\hat{y} = 0\}\,, \tag{18}$$

$$L_K = \{\hat{y} \in L \,|\, (U_0 - 1_{\mathcal{H}})\,\Gamma_1\hat{y} + i\,(U_0 + 1_{\mathcal{H}})\,\Gamma_2\hat{y} = 0\}\,, \tag{19}$$

where $U_{-\infty}$ and U_0 are defined according to (15).

Proof. Applying (2) under $A_1 = 1_{\mathcal{H}},\ A_2 = -M(\lambda)$ and Theorem 4 we obtain

$$\left(L^{(\lambda)} - z\right)^{-1} = (L_2 - z)^{-1} - Z_z\,(M(z) - M(\lambda))^{-1}\,Z_\lambda^* \quad (\lambda, z < \inf L_2,\ z \neq \lambda)$$

(recall that $L^{(\lambda)}$ is defined by (3)). The latter equality together Theorem 1 and Theorem 5 implies

$$(L_F - z)^{-1} = (L_2 - z)^{-1} + Z_z R_{-\infty} Z_z^*, \quad (L_K - z)^{-1} = (L_2 - z)^{-1} + Z_z R_0 Z_z^*. \tag{20}$$

On the other hand, Theorem 3 shows that there exists an unitary operator $K \in \mathcal{B}(\mathcal{H})$ such that $L_1 = \ker\left[(K - 1_{\mathcal{H}})\,\Gamma_1 + i\,(K + 1_{\mathcal{H}})\,\Gamma_2\right].$

Applying Theorem 2 under $A_1 = (K - 1_{\mathcal{H}})\,,\ A_2 = i\,(K + 1_{\mathcal{H}})$ we conclude that

$$(L_F - z)^{-1} = (L_2 - z)^{-1} - Z_z\left[(K - 1_{\mathcal{H}})\,M(z) + i\,(K + 1_{\mathcal{H}})\right]^{-1}(K - 1_{\mathcal{H}})\,Z_z^*. \tag{21}$$

Comparing (20) and (21) we see that

$$\left[(K - 1_{\mathcal{H}})\,M(z) + i\,(K + 1_{\mathcal{H}})\right]^{-1}(K - 1_{\mathcal{H}}) + R_{-\infty} = 0,$$

i. e. (multiplying this identity from left by the expression contained in square brackets)

$$K\left[1_{\mathcal{H}} + M(z)R_{-\infty} + iR_{-\infty}\right] = 1_{\mathcal{H}} + M(z)R_{-\infty} - iR_{-\infty}.$$

Whence using (16) we obtain $K = U_{-\infty}$. The relation (18) is proved. The proof of relation (19) is analogous. □

The construction of Friedrichs and Neumann-Krein extensions of L_0 may be realized in a more simple way in the case when L_2 (and hence L_0) is a positively defined relation. Before considering this case note that the Theorem 5 implies

$$L_0 \gg 0 \Rightarrow L_F^{-1} = s - \lim_{\lambda \to -\infty}\left(L^{(\lambda)}\right)^{-1}. \tag{22}$$

Further, put

$$B \overset{def}{=} s - \lim_{\lambda \to -\infty}\,(M(\lambda) - M(0))^{-1}. \tag{23}$$

It follows from the Theorem 1 that the limit in (23) exists. Moreover, $B \in \mathcal{B}(\mathcal{H})$ and $B \leq 0$.

Theorem 7. *Assume that $L_2 \gg 0$ and put*

$$\gamma_1\hat{y} = \Gamma_1\hat{y} - M(0)\Gamma_2\hat{y}, \tag{24}$$

$$\gamma_2\hat{y} = \Gamma_2\hat{y} - B\gamma_1\hat{y} \equiv -B\Gamma_1\hat{y} + (1_{\mathcal{H}} + BM(0))\,\Gamma_2\hat{y}, \tag{25}$$

where $\hat{y}$ runs through L and B is defined according to (23)*. Then* *i)* $(\mathcal{H}, \gamma_1, \gamma_2)$ *is a BVS for* L_0*;*

ii) $L_F = \ker \gamma_2 \equiv \{\hat{y} \in L \mid \gamma_2 \hat{y} = 0\}$;

iii) $L_K = \ker \gamma_1 \equiv \{\hat{y} \in L \mid \gamma_1 \hat{y} = 0\}$.

Proof. Since $L_K = L_0 \dot{+} \widehat{\ker} L$ (see [5] and [3, Prop. 3.2.1]) the statement iii) is an immediate consequence of (3) and Theorem 4 under $\lambda = 0$. Further, thinking as in the proof of Theorem 6 we obtain

$$\left(L^{(\lambda)}\right)^{-1} = L_2^{-1} + Z_0 \left(M(\lambda) - M(0)\right)^{-1} Z_0^* \quad (\lambda < 0),$$

$$\tilde{L}^{-1} = L_2^{-1} + Z_0 \left[-BM(0) + (1_{\mathcal{H}} + BM(0))\right]^{-1} BZ_0^* = L_2^{-1} + Z_0 BZ_0^*,$$

where $\tilde{L} = \ker \gamma_2$. So, item *ii)* follows from (22) and (23).

Furthermore, (24), (25) may be written in the following form:

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 1_{\mathcal{H}} & -M(0) \\ -B & 1_{\mathcal{H}} + BM(0) \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix}. \tag{26}$$

It is clear that the matrix operator in the right side of (26) is invertible in $\mathcal{B}(H \oplus H)$ and

$$\begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} = \begin{pmatrix} 1_{\mathcal{H}} + M(0)B & M(0) \\ B & 1_{\mathcal{H}} \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}.$$

Moreover, the equality

$$\begin{pmatrix} 1_{\mathcal{H}} & -M(0) \\ -B & 1_{\mathcal{H}} + BM(0) \end{pmatrix} \begin{pmatrix} 0 & 1_{\mathcal{H}} \\ -1_{\mathcal{H}} & 0 \end{pmatrix} \begin{pmatrix} 1_{\mathcal{H}} & -B \\ -M(0) & 1_{\mathcal{H}} + BM(0) \end{pmatrix} = \begin{pmatrix} 0 & 1_{\mathcal{H}} \\ -1_{\mathcal{H}} & 0 \end{pmatrix}$$

implies that for any $\hat{y}, \ \hat{z} \in L$ $(\Gamma_1 \hat{y} | \Gamma_2 \hat{z})_{\mathcal{H}} - (\Gamma_2 \hat{y} | \Gamma_1 \hat{z})_{\mathcal{H}} = (\gamma_1 \hat{y} | \gamma_2 \hat{z})_{\mathcal{H}} - (\gamma_2 \hat{y} | \gamma_1 \hat{z})_{\mathcal{H}}$. Hence (see [15] for the details) $(\mathcal{H}, \gamma_1, \gamma_2)$ is a boundary value space for L_0.

□

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Сторож О.Г. *Про один підхід до побудови розширень Фрідріхса та Неймана-Крейна невід'ємного лінійного відношення* // Карпатські матем. публ. — 2018. — Т.10, №2. — С. 387–394.

Нехай L_0 — замкнене лінійне невід'ємне (можливо, додатно визначене) відношення ("багатозначний оператор") у комплексному гільбертовому просторі H. У термінах так званих просторів граничних значень (граничних трійок) і віповідних функцій Вейля та характеристичних функцій Кочубея-Штрауса побудовано розширення Фрідріхса (жорстке розширення) та Неймана-Крейна (м'яке розширення) відношення L_0.

Зазначимо, що кожне невід'ємне лінійне відношення L_0 у гільбертовому просторі H має два екстремальні невід'ємні самоспряжені розширення: розширення Фрідріхса L_F та розширення Неймана-Крейна L_K, які володіють такою властивістю:

$$(\forall \varepsilon > 0)(L_F + \varepsilon 1)^{-1} \leq (\widetilde{L} + \varepsilon 1)^{-1} \leq (L_K + \varepsilon 1)^{-1}$$

на множині всіх невід'ємних самоспряжених розширень-відношень $\widetilde{L}$ відношення L_0.

Розвивається підхід, заснований на понятті граничної трійки. Цей підхід був започаткований Ф. С. Рофе-Бекетовим, М. Л. Горбачуком та В. І. Горбачук, А. Н. Кочубеєм, В. А. Михайлецем, В. О. Деркачем, М. Н. Маламудом, Ю. М. Арлінським та іншими математиками.

Показано, що побудова згаданих розширень може бути реалізованою простішим шляхом у випадку, коли відношення L_0 є додатно визначеним.

Ключові слова і фрази: гільбертів простір, відношення, оператор, розширення, простір граничних значень.

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SYMMETRIC $*$-POLYNOMIALS ON $\mathbb{C}^n$

$*$-Polynomials are natural generalizations of usual polynomials between complex vector spaces. A $*$-polynomial is a function between complex vector spaces X and Y, which is a sum of so-called (p,q)-polynomials. In turn, for nonnegative integers p and q, a (p,q)-polynomial is a function between X and Y, which is the restriction to the diagonal of some mapping, acting from the Cartesian power X^{p+q} to Y, which is linear with respect to every of its first p arguments, antilinear with respect to every of its last q arguments and invariant with respect to permutations of its first p arguments and last q arguments separately.

In this work we construct formulas for recovering of (p,q)-polynomial components of $*$-polynomials, acting between complex vector spaces X and Y, by the values of $*$-polynomials. We use these formulas for investigations of $*$-polynomials, acting from the n-dimensional complex vector space $\mathbb{C}^n$ to $\mathbb{C}$, which are symmetric, that is, invariant with respect to permutations of coordinates of its argument. We show that every symmetric $*$-polynomial, acting from $\mathbb{C}^n$ to $\mathbb{C}$, can be represented as an algebraic combination of some "elementary" symmetric $*$-polynomials.

Results of the paper can be used for investigations of algebras, generated by symmetric $*$-polynomials, acting from $\mathbb{C}^n$ to $\mathbb{C}$.

Key words and phrases: (p,q)-polynomial, $*$-polynomial, symmetric $*$-polynomial.

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INTRODUCTION AND PRELIMINARIES

$*$-Polynomials (see definition below), acting between complex vector spaces X and Y, were studied in [4–6]. If X has a symmetric structure, like a symmetric basis, it is natural to consider $*$-polynomials, which are invariant (symmetric) with respect to a group of operators, acting on X, which preserve this structure.

Symmetric (invariant) analytic functions of several complex variables with respect to a group of operators on the n-dimensional complex vector space $\mathbb{C}^n$ were investigated by many authors (see, e. g., [1–3]).

In this work we consider symmetric (see definition below) $*$-polynomials, acting from $\mathbb{C}^n$ to $\mathbb{C}$. We investigate the structure of such $*$-polynomials and show that every symmetric $*$-polynomial, acting from $\mathbb{C}^n$ to $\mathbb{C}$, can be represented as an algebraic combination of some "elementary" symmetric $*$-polynomials. Also we establish the general result, which gives us the method of recovering of components of a $*$-polynomial by the values of this $*$-polynomial.

Let $\mathbb{N}$ be the set of all positive integers and $\mathbb{Z}_+$ be the set of all nonnegative integers. Let X and Y be complex vector spaces. A mapping $A : X^{p+q} \to Y$, where $p,q \in \mathbb{Z}_+$ are such that $p \neq 0$ or $q \neq 0$, is called a (p,q)-*linear mapping*, if A is linear with respect to every

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of first p arguments and it is antilinear with respect to every of last q arguments. A (p,q)-linear mapping, which is invariant with respect to permutations of its first p arguments and last q arguments separately, is called (p,q)-*symmetric*. A mapping $P : X \to Y$ is called a (p,q)-*polynomial* if there exists a (p,q)-symmetric (p,q)-linear mapping $A_P : X^{p+q} \to Y$ such that P is the restriction to the diagonal of A_P, i.e.

$$P(x) = A_P(\underbrace{x, \ldots, x}_{p+q})$$

for every $x \in X$. The mapping A_P is called the (p,q)-symmetric (p,q)-linear mapping, *associated* with P. Note that

$$P(x_1 + \ldots + x_m) = \sum_{\substack{\mu_1+\ldots+\mu_m=p \\ \mu_1,\ldots,\mu_m \in \mathbb{Z}_+}} \sum_{\substack{\nu_1+\ldots+\nu_m=p \\ \nu_1,\ldots,\nu_m \in \mathbb{Z}_+}} \frac{p!}{\mu_1! \ldots \mu_m!} \frac{q!}{\nu_1! \ldots \nu_m!} \times A_P(\underbrace{x_1, \ldots, x_1}_{\mu_1}, \ldots, \underbrace{x_m, \ldots, x_m}_{\mu_m}, \underbrace{x_1, \ldots, x_1}_{\nu_1}, \ldots, \underbrace{x_m, \ldots, x_m}_{\nu_m}), \quad (1)$$

for every $x_1, \ldots, x_m \in X$. Also note that

$$P(\lambda x) = \lambda^p \bar{\lambda}^q P(x) \quad (2)$$

for every $x \in X$ and $\lambda \in \mathbb{C}$.

For convenience, we define $(0,0)$-polynomials from X to Y as constant mappings.

A mapping $P : X \to Y$ is called a $*$-*polynomial* if it can be represented in the form

$$P = \sum_{k=0}^{K} \sum_{j=0}^{k} P_{j,k-j}, \quad (3)$$

where $K \in \mathbb{Z}_+$ and $P_{j,k-j}$ is a $(j,k-j)$-polynomial for every $k \in \{0, \ldots, K\}$ and $j \in \{0, \ldots, k\}$. Let $\deg P$ be the maximal number $k \in \mathbb{Z}_+$, for which there exists $j \in \{0, \ldots, k\}$ such that $P_{j,k-j} \not\equiv 0$.

A $*$-polynomial $P : \mathbb{C}^n \to \mathbb{C}$, where $n \in \mathbb{N}$, is called *symmetric* if

$$P((z_1, \ldots, z_n)) = P((z_{\sigma(1)}, \ldots, z_{\sigma(n)}))$$

for every $(z_1, \ldots, z_n) \in \mathbb{C}^n$ and for every bijection $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

For every $\gamma = (\gamma_1, \gamma_2) \in \mathbb{Z}_+^2$ let us define a (γ_1, γ_2)-polynomial $H_\gamma^{(n)} : \mathbb{C}^n \to \mathbb{C}$ by

$$H_\gamma^{(n)}(z) = \sum_{m=1}^{n} z_m^{\gamma_1} \bar{z}_m^{\gamma_2}, \quad (4)$$

where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Note that $H_\gamma^{(n)}$ is symmetric.

A mapping $f : S \to \mathbb{C}$, where S is an arbitrary set, is called an *algebraic combination* of mappings $f_1, \ldots, f_k : S \to \mathbb{C}$ if there exists a polynomial $Q : \mathbb{C}^k \to \mathbb{C}$ such that

$$f(x) = Q(f_1(x), \ldots, f_k(x))$$

for every $x \in S$.

In this work we show that every symmetric $*$-polynomial, acting from $\mathbb{C}^n$ to $\mathbb{C}$, can be represented as an algebraic combination of $*$-polynomials $H_\gamma^{(n)}$, defined by (4).

1 THE MAIN RESULT

Let us prove formulas for recovering of (p,q)-polynomials by the values of a $*$-polynomial. For complex numbers $t_1, \ldots, t_m$, let $V_{t_1,\ldots,t_m}$ be the Vandermonde matrix:

$$V_{t_1,\ldots,t_m} := \begin{pmatrix} 1 & t_1 & t_1^2 & \ldots & t_1^{m-1} \\ 1 & t_2 & t_2^2 & \ldots & t_2^{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_m & t_m^2 & \ldots & t_m^{m-1} \end{pmatrix}.$$

It is well-known that

$$\det(V_{t_1,\ldots,t_m}) = \prod_{1\le j<s\le m} (t_s - t_j).$$

If all the numbers $t_1, \ldots, t_m$ are distinct, then $\det(V_{t_1,\ldots,t_m}) \neq 0$.

Proposition 1. *Let $P : X \to Y$ be a $*$-polynomial of the form* (3)*, where X and Y are complex vector spaces. Let $\lambda_0, \ldots, \lambda_K$ be distinct real numbers. Then*

$$\sum_{j=0}^{k} P_{j,k-j}(x) = \sum_{s=0}^{K} w_{ks} P(\lambda_s x)$$

for every $k \in \{0, \ldots, K\}$ and $x \in X$, where w_{ks} are elements of the matrix $W = (w_{ks})_{k,s=\overline{0,K}}$, which is the inverse matrix of the Vandermonde matrix $V_{\lambda_0,\ldots,\lambda_K}$.

Proof. Let $x \in X$. For every $s \in \{0, \ldots, K\}$, by (3),

$$P(\lambda_s x) = \sum_{k=0}^{K} \sum_{j=0}^{k} P_{j,k-j}(\lambda_s x).$$

By (2), taking into account that λ_s is real,

$$P_{j,k-j}(\lambda_s x) = \lambda_s^j \bar{\lambda}_s^{k-j} P_{j,k-j}(x) = \lambda_s^j \lambda_s^{k-j} P_{j,k-j}(x) = \lambda_s^k P_{j,k-j}(x).$$

Therefore, for every $s \in \{0, \ldots, K\}$,

$$P(\lambda_s x) = \sum_{k=0}^{K} \lambda_s^k \sum_{j=0}^{k} P_{j,k-j}(x).$$

Thus, we have the vector equality

$$\big(P(\lambda_0 x), \ldots, P(\lambda_K x)\big)^T = V_{\lambda_0,\ldots,\lambda_K} \big(P_{0,0}(x), \textstyle\sum_{j=0}^{1} P_{j,1-j}(x), \ldots, \sum_{j=0}^{K} P_{j,K-j}(x)\big)^T.$$

Since $\lambda_0, \ldots, \lambda_K$ are distinct, it follows that $\det(V_{\lambda_0,\ldots,\lambda_K}) \neq 0$. Consequently, $V_{\lambda_0,\ldots,\lambda_K}$ is invertible. Let

$$W = (w_{ks})_{k,s=\overline{0,K}} := V^{-1}_{\lambda_0,\ldots,\lambda_K}.$$

Then

$$\big(P_{0,0}(x), \textstyle\sum_{j=0}^{1} P_{j,1-j}(x), \ldots, \sum_{j=0}^{K} P_{j,K-j}(x)\big)^T = W\big(P(\lambda_0 x), \ldots, P(\lambda_K x)\big)^T.$$

Therefore,

$$\sum_{j=0}^{k} P_{j,k-j}(x) = \sum_{s=0}^{K} w_{ks} P(\lambda_s x)$$

for every $k \in \{0, \ldots, K\}$. $\square$

Proposition 2. *Let $k \in \mathbb{Z}_+$ and $P_{j,k-j} : X \to Y$ be a $(j,k-j)$-polynomial for every $j \in \{0,\ldots,k\}$, where X and Y are complex vector spaces. Let $\varepsilon_0,\ldots,\varepsilon_k$ be complex numbers such that $\varepsilon_0^2,\ldots,\varepsilon_k^2$ are distinct and $|\varepsilon_0| = \ldots = |\varepsilon_k| = 1$. Then*

$$P_{j,k-j}(x) = \sum_{l=0}^{k} u_{jl}\varepsilon_l^k \sum_{j=0}^{k} P_{j,k-j}(\varepsilon_l x)$$

for every $j \in \{0,\ldots,k\}$ and $x \in X$, where u_{jl} are elements of the matrix $U = (u_{jl})_{j,l=\overline{0,K}}$, which is the inverse matrix of the Vandermonde matrix $V_{\varepsilon_0^2,\ldots,\varepsilon_K^2}$.

Proof. Let $x \in X$. For every $j,l \in \{0,\ldots,k\}$, by (2), $P_{j,k-j}(\varepsilon_l x) = \varepsilon_l^j \bar{\varepsilon}_l^{k-j} P_{j,k-j}(x)$. Since $|\varepsilon_l| = 1$, it follows that $\bar{\varepsilon}_l^{k-j} = \varepsilon_l^{j-k}$. Therefore,$P_{j,k-j}(\varepsilon_l x) = \varepsilon_l^{2j-k} P_{j,k-j}(x)$.

Consequently,

$$\varepsilon_l^k \sum_{j=0}^{k} P_{j,k-j}(\varepsilon_l x) = \sum_{j=0}^{k} \varepsilon_l^{2j} P_{j,k-j}(x)$$

for every $l \in \{0,\ldots,k\}$. Thus, we have the vector equality

$$\big(\varepsilon_0^k \textstyle\sum_{j=0}^{k} P_{j,k-j}(\varepsilon_0 x),\ldots,\varepsilon_k^k \sum_{j=0}^{k} P_{j,k-j}(\varepsilon_k x)\big)^T = V_{\varepsilon_0^2,\ldots,\varepsilon_k^2}\big(P_{0,k}(x), P_{1,k-1}(x),\ldots,P_{k,0}(x)\big)^T.$$

Since $\varepsilon_0^2,\ldots,\varepsilon_k^2$ are distinct, it follows that $\det(V_{\varepsilon_0^2,\ldots,\varepsilon_k^2}) \neq 0$. Consequently, $V_{\varepsilon_0^2,\ldots,\varepsilon_k^2}$ is invertible. Let

$$U = (u_{jl})_{j,l=\overline{0,k}} := V^{-1}_{\varepsilon_0^2,\ldots,\varepsilon_k^2}.$$

Then

$$\big(P_{0,k}(x), P_{1,k-1}(x),\ldots,P_{k,0}(x)\big)^T = U\big(\varepsilon_0^k \textstyle\sum_{j=0}^{k} P_{j,k-j}(\varepsilon_0 x),\ldots,\varepsilon_k^k \sum_{j=0}^{k} P_{j,k-j}(\varepsilon_k x)\big)^T.$$

Therefore,

$$P_{j,k-j}(x) = \sum_{l=0}^{k} u_{jl}\varepsilon_l^k \sum_{j=0}^{k} P_{j,k-j}(\varepsilon_l x)$$

for every $j \in \{0,\ldots,k\}$. $\square$

Let us consider $*$-polynomials on $\mathbb{C}^n$.

Lemma 1. *Every $*$-polynomial $P : \mathbb{C}^n \to \mathbb{C}$ can be uniquely represented in the form*

$$P(z) = \sum_{k=0}^{K}\sum_{j=0}^{k} \sum_{\substack{\mu_1+\ldots+\mu_n=j \\ \mu_1,\ldots,\mu_n\in\mathbb{Z}_+}} \sum_{\substack{\nu_1+\ldots+\nu_n=k-j \\ \nu_1,\ldots,\nu_n\in\mathbb{Z}_+}} \alpha_{\mu_1,\ldots,\mu_n,\nu_1,\ldots,\nu_n} z_1^{\mu_1}\ldots z_n^{\mu_n}\bar{z}_1^{\nu_1}\ldots\bar{z}_n^{\nu_n}, \tag{5}$$

where $z = (z_1,\ldots,z_n) \in \mathbb{C}^n$, $K = \deg P$ and $\alpha_{\mu_1,\ldots,\mu_n,\nu_1,\ldots,\nu_n} \in \mathbb{C}$.

Proof. Let $P : \mathbb{C}^n \to \mathbb{C}$ be a $*$-polynomial of the form (3). If $K = 0$, then $P = P_{0,0}$, where $P_{0,0} \in \mathbb{C}$. Thus, in this case, we have the representation of P in the form (5). Consider the case $K > 0$. Every $z = (z_1,\ldots,z_n) \in \mathbb{C}^n$ can be represented as $z = \sum_{m=1}^{n} z_m e_m$, where

$$e_m = (\underbrace{0,\ldots,0}_{m-1}, 1, \underbrace{0,\ldots,0}_{n-m})$$

for every $m \in \{1, \dots, n\}$. Therefore, by (1),

$$P(z) = P_{0,0} + \sum_{k=0}^{K} \sum_{j=0}^{k} \sum_{\substack{\mu_1+\dots+\mu_n=j \\ \mu_1,\dots,\mu_n \in \mathbb{Z}_+}} \sum_{\substack{\nu_1+\dots+\nu_n=k-j \\ \nu_1,\dots,\nu_n \in \mathbb{Z}_+}} \frac{j!}{\mu_1! \dots \mu_n!} \frac{(k-j)!}{\nu_1! \dots \nu_n!} z_1^{\mu_1} \dots z_n^{\mu_n} \bar{z}_1^{\nu_1} \dots \bar{z}_n^{\nu_n}$$
$$\times A_{P_{j,k-j}}(\underbrace{e_1, \dots, e_1}_{\mu_1}, \dots, \underbrace{e_n, \dots, e_n}_{\mu_n}, \underbrace{e_1, \dots, e_1}_{\nu_1}, \dots, \underbrace{e_n, \dots, e_n}_{\nu_n}),$$

where $A_{P_{j,k-j}}$ is the $(j, k-j)$-symmetric $(j, k-j)$-linear mapping, associated with the $(j, k-j)$-polynomial $P_{j,k-j}$. Let $\alpha_{0,\dots,0} = P_{0,0}$ and

$$\alpha_{\mu_1,\dots,\mu_n,\nu_1,\dots,\nu_n} = \frac{j!}{\mu_1! \dots \mu_n!} \frac{(k-j)!}{\nu_1! \dots \nu_n!}$$
$$\times A_{P_{j,k-j}}(\underbrace{e_1, \dots, e_1}_{\mu_1}, \dots, \underbrace{e_n, \dots, e_n}_{\mu_n}, \underbrace{e_1, \dots, e_1}_{\nu_1}, \dots, \underbrace{e_n, \dots, e_n}_{\nu_n})$$

for $\mu_1, \dots, \mu_n, \nu_1, \dots, \nu_n \in \mathbb{Z}_+$ such that $1 \le \mu_1 + \dots + \mu_n + \nu_1 + \dots + \nu_n \le K$. Then

$$P(z) = \sum_{k=0}^{K} \sum_{j=0}^{k} \sum_{\substack{\mu_1+\dots+\mu_n=j \\ \mu_1,\dots,\mu_n \in \mathbb{Z}_+}} \sum_{\substack{\nu_1+\dots+\nu_n=k-j \\ \nu_1,\dots,\nu_n \in \mathbb{Z}_+}} \alpha_{\mu_1,\dots,\mu_n,\nu_1,\dots,\nu_n} z_1^{\mu_1} \dots z_n^{\mu_n} \bar{z}_1^{\nu_1} \dots \bar{z}_n^{\nu_n}.$$

□

Theorem 1. *Every symmetric $*$-polynomial $P : \mathbb{C}^n \to \mathbb{C}$ can be represented as an algebraic combination of $*$-polynomials $H_\gamma^{(n)}$, where $\gamma = (\gamma_1, \gamma_2) \in \mathbb{Z}_+^2$ are such that $\gamma_1 + \gamma_2 \le \deg P$.*

Proof. We proceed by induction on n. In the case $n = 1$ for $z = z_1 \in \mathbb{C}$, by Lemma 1, we have

$$P(z) = \sum_{k=0}^{\deg P} \sum_{j=0}^{k} \alpha_{j,k-j} z_1^j \bar{z}_1^{k-j} = \sum_{k=0}^{\deg P} \sum_{j=0}^{k} \alpha_{j,k-j} H_{(j,k-j)}^{(1)}(z).$$

Suppose the statement holds for $n-1$ and prove it for n. Let $P : \mathbb{C}^n \to \mathbb{C}$ be a symmetric $*$-polynomial and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. Then $P(z)$ can be represented in the form

$$P(z) = \sum_{k=0}^{K} \sum_{j=0}^{k} z_n^j \bar{z}_n^{k-j} r_{j,k-j}((z_1, \dots, z_{n-1})),$$

where $K = \deg P$ and $r_{j,k-j} : \mathbb{C}^{n-1} \to \mathbb{C}$ are $*$-polynomials. Let us show that $*$-polynomials $r_{j,k-j}$ are symmetric. For fixed $z_1, \dots, z_{n-1} \in \mathbb{C}$, the mapping $R : z_n \mapsto P((z_1, \dots, z_n))$ is a $*$-polynomial, acting from $\mathbb{C}$ to $\mathbb{C}$. Let $\lambda_0, \dots, \lambda_K$ be distinct real numbers. Then, by Proposition 1,

$$\sum_{j=0}^{k} z_n^j \bar{z}_n^{k-j} r_{j,k-j}((z_1, \dots, z_{n-1})) = \sum_{s=0}^{K} w_{ks} R(\lambda_s z_n) \tag{6}$$

for every $k \in \{0, \dots, K\}$. For $k \in \{0, \dots, K\}$, let $\varepsilon_0, \dots, \varepsilon_k$ be complex numbers such that $\varepsilon_0^2, \dots, \varepsilon_k^2$ are distinct and $|\varepsilon_0| = \dots = |\varepsilon_k| = 1$. Then, by Proposition 2,

$$z_n^j \bar{z}_n^{k-j} r_{j,k-j}((z_1, \dots, z_{n-1})) = \sum_{l=0}^{k} u_{jl} \varepsilon_l^k \sum_{j=0}^{k} (\varepsilon_l z_n)^j (\bar{\varepsilon}_l \bar{z}_n)^{k-j} r_{j,k-j}((z_1, \dots, z_{n-1})) \tag{7}$$

for every $j \in \{0, \ldots, k\}$. By (6) and (7),

$$z_n^j \bar{z}_n^{k-j} r_{j,k-j}((z_1, \ldots, z_{n-1})) = \sum_{l=0}^{k} u_{jl} \varepsilon_l^k \sum_{s=0}^{K} w_{ks} R(\lambda_s \varepsilon_l z_n)$$

for every $k \in \{0, \ldots, K\}$ and $j \in \{0, \ldots, k\}$. Let $z_n = 1$. Then

$$r_{j,k-j}((z_1, \ldots, z_{n-1})) = \sum_{l=0}^{k} u_{jl} \varepsilon_l^k \sum_{s=0}^{K} w_{ks} R(\lambda_s \varepsilon_l) = \sum_{l=0}^{k} u_{jl} \varepsilon_l^k \sum_{s=0}^{K} w_{ks} P((z_1, \ldots, z_{n-1}, \lambda_s \varepsilon_l)). \quad (8)$$

Let $\sigma : \{1, \ldots, n-1\} \to \{1, \ldots, n-1\}$ be a bijection. Then, by (8) and by the symmetry of P,

$$\begin{aligned} r_{j,k-j}((z_{\sigma(1)}, \ldots, z_{\sigma(n-1)})) &= \sum_{l=0}^{k} u_{jl} \varepsilon_l^k \sum_{s=0}^{K} w_{ks} P((z_{\sigma(1)}, \ldots, z_{\sigma(n-1)}, \lambda_s \varepsilon_l)) \\ &= \sum_{l=0}^{k} u_{jl} \varepsilon_l^k \sum_{s=0}^{K} w_{ks} P((z_1, \ldots, z_{n-1}, \lambda_s \varepsilon_l)) = r_{j,k-j}((z_1, \ldots, z_{n-1})). \end{aligned}$$

Thus, $r_{j,k-j}$ is symmetric for every $k \in \{0, \ldots, K\}$ and $j \in \{0, \ldots, k\}$. By the induction hypothesis, every $*$-polynomial $r_{j,k-j}$ can be represented as an algebraic combination of $*$-polynomials $H_\gamma^{(n-1)}$. Since

$$H_\gamma^{(n-1)}((z_1, \ldots, z_{n-1})) = H_\gamma^{(n)}((z_1, \ldots, z_n)) - z_n^{\gamma_1} \bar{z}_n^{\gamma_2}$$

for every $\gamma = (\gamma_1, \gamma_2) \in \mathbb{Z}_+^2$, it follows that P can be represented as an algebraic combination of $*$-polynomials $H_\gamma^{(n)}$ and $*$-polynomials, defined by $\mathbb{C}^n \ni (z_1, \ldots, z_n) \mapsto z_n^{\gamma_1} \bar{z}_n^{\gamma_2} \in \mathbb{C}$, where $\gamma = (\gamma_1, \gamma_2) \in \mathbb{Z}_+^2$. Therefore,

$$P(z) = \sum_{k=0}^{K} \sum_{j=0}^{k} z_n^j \bar{z}_n^{k-j} Q_{j,k-j}(z),$$

where $Q_{j,k-j}$ is an algebraic combination of $*$-polynomials $H_\gamma^{(n)}$ for every $k \in \{0, \ldots, K\}$ and $j \in \{0, \ldots, k\}$. Since $*$-polynomials $H_\gamma^{(n)}$ are symmetric, it follows that $*$-polynomials $Q_{j,k-j}$ are symmetric. Since $*$-polynomials P and $Q_{j,k-j}$ are symmetric, it follows that

$$P(z) = \sum_{k=0}^{K} \sum_{j=0}^{k} z_m^j \bar{z}_m^{k-j} Q_{j,k-j}(z),$$

for every $m \in \{1, \ldots, n\}$. Therefore,

$$\sum_{m=1}^{n} P(z) = \sum_{m=1}^{n} \sum_{k=0}^{K} \sum_{j=0}^{k} z_m^j \bar{z}_m^{k-j} Q_{j,k-j}(z),$$

that is,

$$nP(z) = \sum_{k=0}^{K} \sum_{j=0}^{k} \sum_{m=1}^{n} z_m^j \bar{z}_m^{k-j} Q_{j,k-j}(z).$$

Thus,

$$P(z) = \frac{1}{n} \sum_{k=0}^{K} \sum_{j=0}^{k} H_{(j,k-j)}^{(n)}(z) Q_{j,k-j}(z).$$

Hence, P is an algebraic combination of $*$-polynomials $H_\gamma^{(n)}$. This completes the proof. $\square$

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Василишин Т.В. *Симетричні $*$-поліноми на $\mathbb{C}^n$* // Карпатські матем. публ. — 2018. — Т.10, №2. — С. 395–401.

Поняття $*$-полінома є природним узагальненням поняття полінома між комплексними векторними просторами. $*$-Поліном — це функція між комплексними векторними просторами X та Y, яка є сумою так званих (p,q)-поліномів. В свою чергу, для невід'ємних цілих чисел p і q, (p,q)-поліном — це функція між просторами X та Y, яка є звуженням на діагональ деякого відображення, що діє з декартового степеня X^{p+q} в Y, яке є лінійним відносно кожного зі своїх перших p аргументів, антилінійним відносно кожного зі своїх останніх q аргументів і інваріантним відносно перестановок окремо перших p аргументів і останніх q агрументів.

В даній роботі побудовано формули для знаходження (p,q)-поліноміальних компонентів $*$-поліномів, які діють між комплексними векторними просторами X та Y, за значеннями цих $*$-поліномів. Цей результат використано для дослідження $*$-поліномів, які діють з n-вимірного комплексного векторного простору $\mathbb{C}^n$ в $\mathbb{C}$, які є симетричними, тобто, інваріантними відносно перестановок координат їхнього аргумента. Показано, що кожен симетричний $*$-поліном, який діє з $\mathbb{C}^n$ в $\mathbb{C}$, можна подати у вигляді алгебраїчної комбінації деяких "елементарних" симетричних $*$-поліномів.

Результати даної роботи можуть бути використані для дослідження алгебр, породжених симетричними $*$-поліномами, які діють з $\mathbb{C}^n$ в $\mathbb{C}$.

Ключові слова і фрази: (p,q)-поліном, $*$-поліном, симетричний $*$-поліном.

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ZABAVSKY B.V., ROMANIV O.M.

COMMUTATIVE BEZOUT DOMAINS IN WHICH ANY NONZERO PRIME IDEAL IS CONTAINED IN A FINITE SET OF MAXIMAL IDEALS

We investigate commutative Bezout domains in which any nonzero prime ideal is contained in a finite set of maximal ideals. In particular, we have described the class of such rings, which are elementary divisor rings. A ring R is called an elementary divisor ring if every matrix over R has a canonical diagonal reduction (we say that a matrix A over R has a canonical diagonal reduction if for the matrix A there exist invertible matrices P and Q of appropriate sizes and a diagonal matrix $D = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r, 0, \ldots, 0)$ such that $PAQ = D$ and $R\varepsilon_i \subseteq R\varepsilon_{i+1}$ for every $1 \leq i \leq r-1$). We proved that a commutative Bezout domain R in which any nonzero prime ideal is contained in a finite set of maximal ideals and for any nonzero element $a \in R$ the ideal aR a decomposed into a product $aR = Q_1 \ldots Q_n$, where Q_i $(i = 1, \ldots, n)$ are pairwise comaximal ideals and $\mathrm{rad}\, Q_i \in \mathrm{spec}\, R$, is an elementary divisor ring.

Key words and phrases: Bezout domain, elementary divisor ring, adequate ring, ring of stable range, valuation ring, prime ideal, maximal ideal, comaximal ideal.

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INTRODUCTION

The classical notion of an elementary divisor ring was first introduced by I. Kaplansky [5]. Among the well-known classes of rings, a special place is occupied by adequate rings introduced by Helmer [3]. Henriksen proved that in an adequate ring any nonzero prime ideal is contained in a unique maximal ideal, i.e. an adequate ring is a PM^*-ring [4]. Larsen, Lewis and Shores [6] raised the question: is it true that every commutative Bezout domain, in which any non-zero prime ideal is contained in a unique maximal ideal, is an adequate ring? In [1], an example is given for a commutative PM^* Bezout domain that is not adequate, but when is an elementary divisor ring. Gatalevych and Zabavsky proved that a commutative Bezout domain, in which any nonzero prime ideal is contained in a unique maximal ideal (PM^*-ring), is an elementary divisor ring [9]. While investigating Bezout rings with the Noetherian spectrum [2], the authors encountered examples of commutative Bezout domains, in which any nonzero prime ideal is contained in a finite set of maximal ideals. An obvious example of such a ring is an adequate ring. In this paper, the existence and properties of such rings are established.

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We introduce the necessary definitions and facts.

All rings considered will be commutative with identity. A ring is *a Bezout ring*, if every its finitely generated ideal is principal. Let $\mathrm{GL}_n(R)$ be the group (*the general linear group*) of all invertible $(n \times n)$-matrices over the ring R. We say that matrices A and B over a ring R are *equivalent* if there exist invertible matrices P and Q of appropriate sizes such that $B = PAQ$. The fact that matrices A and B are equivalent is denoted by $A \sim B$. If for a matrix A there exists a diagonal matrix $D = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r, 0, \ldots, 0)$ such that $A \sim D$ and $R\varepsilon_i \subseteq R\varepsilon_{i+1}$ for every i then we say that the matrix A has *a canonical diagonal reduction*. A ring R is called *an elementary divisor ring* if every matrix over R has a canonical diagonal reduction.

Let I be an ideal of a ring R. The *radical of an ideal I*, denoted by $\mathrm{rad}\, I$ or $\sqrt{I}$, is defined as

$$\mathrm{rad}\, I = \{\, a \in R \mid a^n \in I \text{ for some } n \in \mathbb{N} \,\}.$$

Obviously, $\mathrm{rad}\, I = \bigcap\limits_{P \in \mathrm{spec}\, I} P$ where $\mathrm{spec}\, I$ denotes the set of all the prime ideals of the ring R containing the ideal I (the spectrum of the ideal I). Note that $\mathrm{rad}\, I$ can be defined differently, namely $\mathrm{rad}\, I = \bigcap\limits_{P \in \mathrm{minspec}\, I} P$, where $\mathrm{minspec}\, I$ is the set of minimal ideals of the ideal I, i.e. proper prime ideals of $\mathrm{spec}\, I$, not containing prime ideals from $\mathrm{spec}\, I$.

Two ideals I, J of a ring R are said to be *comaximal* if $x + y = 1$ for some $x \in I$ and $y \in J$.

1 SECTION WITH RESULTS

Let R be a commutative domain, $\mathrm{mspec}\, R$ be a set of all maximal ideals of the ring R, M be any maximal ideal of the ring R ($M \in \mathrm{mspec}\, R$). Let us denote by R_M the localization of the ring R with respect to the multiplicatively closed set $S = R \backslash M$. Note that if R is a commutative Bezout domain, then R_M is a local Bezout domain for any maximal ideal $M \in \mathrm{mspec} R$. And since a local Bezout domain is a valuation ring, i.e. a ring in which the set of ideals is linearly ordered with respect to ideal inclusion, we obtain such a result.

Proposition 1. *Let R be a commutative Bezout domain. For any maximal ideal $M \in \mathrm{mspec}\, R$, the set of the prime ideals of R, contained in M, is linearly ordered with respect to inclusion.*

The Proposition 1 shows that $\mathrm{spec}\, R$ is a tree [1].

Let us consider the case of the commutative Bezout domain R in which the set $\mathrm{minspec}\, R$ is finite for any nonzero element $a \in R$.

Theorem 1. *Let R be a commutative Bezout domain, a be a nonzero element R such that $\mathrm{minspec}\, aR$ is a finite and any prime ideal of $\mathrm{spec}\, aR$ is contained in a unique maximal ideal. Then the factor ring R/aR is the direct sum of valuation rings.*

Proof. Let $P_1, P_2, \ldots, P_n \in \mathrm{minspec}\, aR$. We consider the factor ring $\overline{R} = R/aR$. We denote $\overline{P}_i = P_i/aR$, where $P_i \in \mathrm{minspec}\, aR$, $i = 1, 2, \ldots, n$. Note that $\overline{P}_i \in \mathrm{minspec}\, \overline{R}$ are all minimal prime ideals of the ring $\overline{R}$. Moreover, by Proposition 1, the ideals $\overline{P}_i$ are comaximal in $\overline{R}$. Obviously, $\mathrm{rad}\, \overline{R} = \bigcap\limits_{i=1}^{n} \overline{P}_i$, and by the Chinese remainder theorem we have

$$\overline{R}/\mathrm{rad}\, \overline{R} \cong \overline{R}/\overline{P}_1 \oplus \overline{R}/\overline{P}_2 \oplus \ldots \oplus \overline{R}/\overline{P}_n.$$

Since any prime ideal of $\operatorname{spec} aR$ is contained in a unique maximal ideal, $\overline{R}/\overline{P}_i$ are valuation rings. Moreover, there exist pairwise orthogonal idempotents $\overline{\overline{e}}_1, \ldots, \overline{\overline{e}}_n$, where $\overline{\overline{e}}_i \in \overline{R}/\overline{P}_i$ such that $\overline{\overline{e}}_1 + \ldots + \overline{\overline{e}}_n = \overline{1}$. Then, by lifting the idempotent $\overline{\overline{e}}_i$ modulo $\operatorname{rad} \overline{R}$ to pairwise orthogonal idempotents $\overline{e}_1, \ldots, \overline{e}_n \in \overline{R}$ we find that $1 - (e_1 \ldots + e_n)$ is an idempotent and $1 - (e_1 + \ldots + e_n) \in \operatorname{rad} \overline{R}$, which is possible only if it is zero. Therefore,

$$\overline{R} = \overline{e}_1\overline{R} \oplus \overline{e}_2\overline{R} \oplus \cdots \oplus \overline{e}_n\overline{R}$$

and each $\overline{e}_i\overline{R}$ is a homomorphic image of $\overline{R}$, i.e. a commutative Bezout ring. Since any prime ideal of $\overline{R}$ is contained in a unique maximal ideal, $\overline{e}_i\overline{R}$ is a valuation ring. □

A minor modification of the proof of Theorem 1 gives us the following result.

Theorem 2. *Let R be a commutative Bezout domain in which any nonzero prime ideal is contained in a finite set of maximal ideals. Then for any nonzero element $a \in R$ such that the set* $\operatorname{minspec} aR$ *is finite, the factor ring $\overline{R} = R/aR$ is a direct sum of semilocal rings.*

Proof. According to the notations from Theorem 1 and its proof, we have

$$\overline{R} = \overline{e}_1\overline{R} \oplus \overline{e}_2\overline{R} \oplus \ldots \oplus \overline{e}_n\overline{R}.$$

Since any prime ideal of the ring $\overline{R}$ is contained in a finite set of maximal ideals, $\overline{e}_i\overline{R}$ is a semilocal ring. □

Obviously, if a commutative ring R is a direct sum of valuation rings R_i, then R is a commutative Bezout ring. Let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ be any elements of R, where $a_i, b_i \in R_i$, $i = 1, 2, \ldots, n$. Since R_i is a valuation ring, $a_i = r_i s_i$, where $r_i R + b_i R = R$ and $s_i' R_i + b_i R_i \neq R_i$ for any non invertible divisor s_i' of the element s_i. If $r = (r_1, \ldots, r_n)$, $s = (s_1, \ldots, s_n)$ then obviously $a = rs$, $rR + bR = R$. For each i such that s_i' is a non invertible divisor of $s_i \in R_i$, we have $s_i R_i + b_i R_i \neq R_i$. Hence $s'R + bR \neq R$, i.e. a is an adequate element.

Recall the definitions.

Definition 1. *An element a of a commutative ring R is called* adequate*, if for every element $b \in R$ one can find elements $r, s \in R$ such that:*

1) $a = rs$;

2) $rR + bR = R$;

3) $s'R + bR \neq R$ for any $s' \in R$ such that $sR \subset s'R \neq R$.

The most trivial examples of adequate elements are units, atoms in a ring, and also square-free elements [8].

A ring R is said to be *everywhere adequate* if any element of R is adequate.

Note that, as shown above, in the case of a commutative ring, which is a direct sum of valuation rings, any element of the ring (in particular zero) is adequate, i.e. this ring is everywhere adequate. Moreover, by [10], this ring is clean, i.e. a ring in which any element is the sum of an idempotent and an invertible element.

Definition 2. *A ring R is called a ring of stable range 1 if for every $a, b \in R$ such that $aR + bR = R$ there exists an element $t \in R$ such that $(a + bt)R = R$.*

Definition 3. *An nonzero element a of a ring R is called an element of almost stable range 1 if the quotient-ring R/aR is a ring of stable range 1.*

Any ring of stable range 1 is a ring of almost stable 1 (see [7]). But not every element of stable range 1 is an element of almost stable range 1. For example, let e be a nonzero idempotent of a commutative ring R and $eR + aR = R$. Then $ex + ay = 1$ for some elements $x, y \in R$ and $(1 - e)ex + (1 - e)ay = 1 - e$, so $e + a(1 - e)y = 1$. And we have that e is an element of stable range 1 for any commutative ring. However if you consider the ring $R = \mathbb{Z} \times \mathbb{Z}$ and the element $e = (1, 0) \in R$ then, as shown above, e is an element of stable range 1, by $R/eR \cong \mathbb{Z}$, and e is not element of almost stable range 1. Moreover, if R is a commutative principal ideal domain (i.e. ring of integers), which is not of stable range 1, then every nonzero element of R is an element of almost stable range 1.

Definition 4. *A commutative ring in which every nonzero element is an element of almost stable range 1 is called a ring of* almost stable range 1.

The first example of a ring of almost stable range 1 is a ring of stable range 1. Also, every commutative principal ideal ring which is not a ring of stable range 1 (for example, the ring of integers) is a ring of almost stable range 1 which is not a ring of stable range 1.

We note that the semilocal ring is an example of a ring of stable range 1. Moreover, the direct sum of rings of stable range 1 is a ring of stable range 1. As a result, we obtain the result from the previous theorems.

Theorem 3. *Let R be a commutative Bezout domain, a be a nonzero element R such that the set* $\operatorname{minspec} aR$ *is finite and any prime ideal of* $\operatorname{spec} aR$ *is contained in a unique maximal ideal. Then the factor ring R/aR is everywhere adequate if and only if R/aR is a direct sum of a valuation rings.*

Proof. Since R be a commutative Bezout domain, a be a nonzero element R such that the set $\operatorname{minspec} aR$ is finite and any prime ideal of $\operatorname{spec} aR$ is contained in a unique maximal ideal, factor ring R/aR is a semilocal ring. By [6] proof the semilocal ring R is everywhere adequate if and only if R is a direct sum of a valuation rings. □

Theorem 4. *Let R be a commutative Bezout domain and a be a nonzero element of R such that the set* $\operatorname{minspec} aR$ *is finite, and any nonzero prime ideal* $\operatorname{spec} aR$ *is contained in a finite set of maximal ideals. Then a is an element of almost stable range 1.*

The proof of the Theorem 4 is similar to the proof of the Theorem 3.

Proposition 2 ([2])**.** *Let R be a commutative Bezout domain in which any nonzero prime ideal is contained in a finite set of maximal ideals. Then the following properties are equivalent:*

1) *for any nonzero element $a \in R$ there exists a representation $aR = Q_1 \dots Q_n$, where $Q_1, \dots, Q_n$ are pairwise commaximal ideals such that* $\operatorname{rad} Q_i$ *is a prime ideal;*

2) $\operatorname{minspec} aR$ *is finite.*

As a result of Proposition 2 and Theorem 4 we obtain the following results.

Theorem 5. *Let R be a commutative Bezout domain in which any nonzero prime ideal is contained in a finite set of maximal ideals and for any nonzero element $a \in R$ there exists a representation $aR = Q_1 \dots Q_n$, where $Q_1, \dots, Q_n$ are pairwise comaximal ideals such that* $\operatorname{rad} Q_i \in \operatorname{spec} R$. *Then R is a ring of almost stable range 1.*

Proof. Since R be a commutative Bezout domain in which any nonzero prime ideal is contained in a finite set of maximal ideals and for any nonzero element $a \in R$ there exists a representation $aR = Q_1 \dots Q_n$, where $Q_1, \dots, Q_n$ are pairwise comaximal ideals such that $\operatorname{rad} Q_i \in \operatorname{spec} R$, $\operatorname{minspec} aR$ is finite. By Theorem 4, a is an element of almost stable range 1. Then R is a ring of almost stable range 1. □

Since a commutative Bezout ring of almost stable range 1 is an elementary divisor ring [7], as a result, we obtain the following.

Theorem 6. *Let R be a commutative Bezout domain in which any nonzero prime ideal is contained in a finite set of maximal ideals and for any nonzero element $a \in R$ let the ideal aR is decomposed into a product $aR = Q_1 \dots Q_n$, where Q_i $(i = 1, \dots, n)$ are pairwise comaximal ideals and* $\operatorname{rad} Q_i \in \operatorname{spec} R$. *Then R is an elementary divisor ring.*

Open Question. Is it true that every commutative Bezout domain in which any non-zero prime ideal is contained in a finite set of maximal ideals is an elementary divisor ring?

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Забавський Б.В., Романів О.М. *Комутативні області Безу, в яких довільний ненульовий простий ідеал міститься у скінченній множині максимальних ідеалів* // Карпатські матем. публ. — 2018. — Т.10, №2. — С. 402–407.

Досліджуються комутативні області Безу, яких довільний ненульовий простий ідеал міститься в скінченній множині максимальних ідеалів. Зокрема описано клас таких кілець, які є кільцями елементарних дільників. Кільце R називається кільцем елементарних дільників, якщо кожна матриця над R володіє канонічною діагональною редукцією (матриця A володіє канонічною діагональною редукцією, якщо існує така діагональна матриця $D = \operatorname{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r, 0, \ldots, 0)$, що матриці A та D еквівалентні і $R\varepsilon_i \subseteq R\varepsilon_{i+1}$ для кожного $1 \leq i \leq r-1$). Зокрема, ми довели, що комутативна область Безу R, в якій кожен ненульовий простий ідеал міститься в скінченній множині максимальних ідеалів і для довільного елемента $a \in R$ ідеал aR розкладається в добуток $aR = Q_1 \ldots Q_n$, де Q_i $(i = 1, \ldots, n)$ є попарно комаксимальними ідеалами і $\operatorname{rad} Q_i \in \operatorname{spec} R$, є кільцем елементарних дільників.

Ключові слова і фрази: кільце Безу, кільце елементарних дільників, адекватне кільце, кільце стабільного рангу, кільце нормування, простий ідеал, максимальний ідеал, комаксимальний ідеал.

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THE CONVERGENCE CLASSES FOR ANALYTIC FUNCTIONS IN THE REINHARDT DOMAINS

Let L^0 be the class of positive increasing on $[1,+\infty)$ functions l such that $l((1+o(1))x) = (1+o(1))l(x)$ $(x\to+\infty)$. We assume that α is a concave function such that $\alpha(e^x)\in L^0$ and function $\beta\in L^0$ such that $\int_1^{+\infty}\alpha(x)/\beta(x)dx<+\infty$. In the article it is proved the following theorem: If $f(z)=\sum_{\|n\|=0}^{+\infty}a_nz_n$, $z\in\mathbb{C}^p$, is analytic function in the bounded Reinhard domain $G\subset\mathbb{C}^p$, then the condition $\int\limits_{R_0}^{1}\frac{\alpha(\ln^+M_G(R,f))}{(1-R)^2\beta(1/(1-R))}dR<+\infty$, $M_G(R,f)=\sup\{|F(Rz)|\colon z\in G\}$, yields that

$$\sum_{k=0}^{+\infty}(\alpha(k)-\alpha(k-1))\beta_1\left(k/\ln^+|A_k|\right)<+\infty,\quad \beta_1(x)=\int\limits_x^{+\infty}\frac{dt}{\beta(t)},\quad A_k=\max\{|a_n|\colon\|n\|=k\}.$$

Key words and phrases: analytic function, Reinhardt domain, convergence class.

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1 INTRODUCTION

We denote by $\mathcal{A}^p(G)$, $p\in\mathbb{N}$, the class of analytic functions f in $G\subset\mathbb{C}^p$, represented by power series of the form

$$f(z)=f(z_1,\ldots,z_p)=\sum_{\|n\|=0}^{+\infty}a_nz^n,\quad z=(z_1,\ldots,z_p),\tag{1}$$

with the domain of convergence G, where $z^n=z_1^{n_1}\ldots z_p^{n_p}$, $n=(n_1,\ldots,n_p)\in\mathbb{Z}_+^p$, $\|n\|=\sum_{j=1}^p n_j$; $\mathcal{E}^p:=\mathcal{A}^p(\mathbb{C}^p)$ is the class of entire functions in several variables (i.e., analytic functions in $\mathbb{C}^p$). From the one hand, it is well-known that every analytic function f in the complete Reinhardt domain G with center at $z=0$ can be represented in G by the series of form (1). On the other hand, the domain of convergence of each series of form (1) is the logarithmically-convex complete Reinhardt domain with center $z=0$.

We say that a domain $G\subset\mathbb{C}^p$ is the complete Reinhardt domain if:

a) $z=(z_1,\ldots,z_p)\in G\Longrightarrow(\forall R=(R_1,\ldots,R_p)\in[0,1]^p)\colon Rz=(R_1z_1,\ldots,R_pz_p)\in G$ (a complete domain);

b) $(z_1,\ldots,z_p)\in G\Longrightarrow(\forall(\theta_1,\ldots,\theta_p)\in\mathbb{R}^p)\colon(z_1e^{i\theta_1},\ldots,z_pe^{i\theta_p})\in G$ (a multiple-circular domain).

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The Reinhardt domain G is called logarithmically-convex if the image of the set $G^* = \{z \in G\colon z_1 \cdot \ldots \cdot z_p \neq 0\}$ under the mapping $Ln\colon z \to Ln(z) = (\ln|z_1|, \ldots, \ln|z_p|)$ is a convex set in the space $\mathbb{R}^p$.

In one complex variable ($p = 1$), a logarithmically-convex Reinhardt domain is a disc.

The following complete Reinhardt domains ($p \geq 2$) are considered most frequently:

$$C_p(R) := \{z \in \mathbb{C}^p\colon |z_1| < R_1, \ldots, |z_p| < R_p\}, \quad R = (R_1, \ldots, R_p) \in (0, +\infty)^p, \quad \text{(polydisk)},$$
$$\mathbb{B}_p(r) := \{z \in \mathbb{C}^p\colon |z| := \sqrt{|z_1|^2 + \ldots + |z_p|^2} < r\} \quad \text{(ball)},$$
$$\Pi_p(r) := \{z \in \mathbb{C}^p\colon |z_1| + \ldots + |z_p| < r\}, \quad r > 0.$$

Remark 1. *$C_p(R) \subset G$ for every $w = (w_1, \ldots, w_p) \in G$ and $R = (|w_1|, \ldots, |w_p|)$. In particular, $C_p(rw) \subset G_r$ for every $w = (w_1, \ldots, w_p) \in G$.*

The domains $C_p(re_1)$, $e_1 = (1, \ldots, 1) \in \mathbb{R}^p$, $\mathbb{B}_p(r)$, $\Pi_p(r)$ $(r > 0)$ are the logarithmically-convex complete Reinhardt domains. But, for example, the complete Reinhardt domain

$$G = \{z = (z_1, z_p)\colon |z_1| < 1, |z_2| < 2\} \cup \{z = (z_1, z_p)\colon |z_1| < 2, |z_2| < 1\}$$

is not logarithmically-convex.

For a domain G and any $R \in (0,1)$ we denote $G_R = R \cdot G := \{Rz\colon z \in G\}$, and for a function $f \in \mathcal{A}^p(G)$ of the form (1) set

$$M_G(R, f) = \max\{|f(z)| : z \in \overline{G}_R\}, \quad \mu_G(R, f) = \max\{|a_n z^n|\colon z \in \overline{G}_R,\ n \in \mathbb{Z}_+^p\},$$
$$d_G(n) = \max\{|z^n|\colon z \in G\}.$$

Note, that $d_G(n) = 1$ in the case $G = C_p(e_1)$.

Let us denote by L the class of positive increasing on $[0, +\infty)$ functions, and by L^0 the class of functions $\alpha \in L$ such that $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ $(x \to +\infty)$.

For $\alpha \in L$ and $\beta \in L$ we consider the following convergence classes of integrals (in one variable definition see in [1])

$$\int_{R_0}^{1} \frac{\alpha(\ln^+ M_G(R, f))}{(1 - R)^2 \beta(1/(1 - R))} d\,R < +\infty, \tag{2}$$

$$\int_{R_0}^{1} \frac{\alpha(\ln^+ \mu_G(R, f))}{(1 - R)^2 \beta(1/(1 - R))} d\,R < +\infty. \tag{3}$$

By $\mathcal{E}^p_{\alpha\beta}$ and $\underline{\mathcal{E}}^p_{\alpha\beta}$ we denote the classes of entire functions $f \in \mathcal{E}^p$ for which conditions (2) and (3) are fulfilled, respectively.

We prove the following theorem.

Theorem 1. *Let α be a concave function on $[x_0, +\infty)$, $\alpha(e^x) \in L^0$, and a function $\beta \in L^0$ satisfies the conditions $x\beta'(x)/\beta(x) - 2 \geq h > 0$ on $[x_0, +\infty)$ and $\int_{x_0}^{+\infty} \frac{\alpha(x)}{\beta(x)} dx < +\infty$. In order that the function $f \in \mathcal{E}^p(G)$ of form (1) belongs to the class $\mathcal{E}^p_{\alpha\beta}(G)$, it is necessary that*

$$\sum_{k=0}^{+\infty} (\alpha(k) - \alpha(k-1))\beta_1\left(\frac{k}{\ln^+ |A_k|}\right) < +\infty, \quad \beta_1(x) = \int_x^{+\infty} \frac{dt}{\beta(t)}, \tag{4}$$

where $A_k := \max\{|a_n| : \|n\| = k\}$.

2 THE PROOF OF THE MAIN RESULT

Proposition 1. *For each function $f \in \mathcal{E}^p(G)$ and any functions $\alpha, \beta \in L_2$ we have the following implication $f \in \mathcal{E}^p_{\alpha\beta}(G) \Longrightarrow F \in \underline{\mathcal{E}}^p_{\alpha\beta}$.*

The statement of this propositions follows from Proposition 2.

Proposition 2. *For every function $f \in \mathcal{E}^p(G)$ and for any $r \in (0,1)$*

$$\mu_G(r,F) = \max\{B_k r^k \colon k \geq 0\} \leq M_G(r,f) \leq c\frac{1}{(1-r)^{p+1}}\mu_G\Big(\frac{1+r}{2}, F\Big),$$

where $c = c(p) < +\infty$.

Lemma 1 ([2]). *Let $r \in (0,1)$, $\tau \in \mathbb{C}^p$, $k \geq 0$,*

$$B_k = \max\{|a_n| d_G(n) \colon \|n\| = k\}, \quad P_k(\tau) = \sum_{\|n\|=k} a_n \tau^n, \quad F_1(r) = \sum_{k=0}^{+\infty} M_G(1, P_k) r^k.$$

Then

$$B_k \leq M_G(1, P_k) \leq B_k (k+1)^p, \quad \mu_{F_1}(r) = \max\{M_G(1, P_k) r^k \colon\ k \geq 0\} \leq M_G(r,f) \leq F_1(r).$$

Proof of Proposition 2. By Lemma 1,

$$\begin{aligned}\mu_G(r,F) &= \max\{\big|a_n z^n\big| \colon z \in \overline{G}_r, n \in \mathbb{Z}^p_+\} = \max\{\big|a_n\big| \max\{\big|z^n\big| \colon z \in \overline{G}_r, \} \colon n \in Z^p_+\} \\ &= \max\{\big|a_n\big| d_G(n) r^k \colon n \in \mathbb{Z}^p_+, \|n\| = k \geq 0\} \\ &= \max\Big\{ \max\big\{\big|a_n\big| d_G(n) \colon n \in \mathbb{Z}^p_+, \|n\| = k\big\} r^k \colon k \geq 0\Big\} = \max\{B_k r^k \colon k \geq 0\} \\ &\leq \max\{M_G(1, P_k) r^k \colon\ k \geq 0\} = \mu_{F_1}(r) \leq M_G(r,f).\end{aligned}$$

On the other hand,

$$\begin{aligned}M_G(r,f) &\leq \sum_{k=0}^{+\infty} \sum_{\|n\|=k} |a_n| \max\{\big|z^n\big| \colon z \in \overline{G}_r\} = \sum_{k=0}^{+\infty} \Big(\frac{2r}{1+r}\Big)^k \sum_{\|n\|=k} |a_n| \max\{\big|z^n\big| \colon z \in \overline{G}_{\frac{1+r}{2}}\} \\ &\leq \mu_G\Big(\frac{1+r}{2}, F\Big) \sum_{k=0}^{+\infty} \Big(\frac{2r}{1+r}\Big)^k (k+1)^p \leq c\frac{1}{(1-r)^p}\mu_G\Big(\frac{1+r}{2}, F\Big),\ c = c(p) < +\infty.\end{aligned}$$

□

The proof of Theorem 1. Let

$$F_2(R) = \sum_{k=0}^{+\infty} B_k R^k, \quad F_3(R) = \sum_{k=0}^{+\infty} A_k R^k, \quad R \in (0,1).$$

From Remark 1 it follows

$$\begin{aligned}A_k R^k &= \max\{|a_n| \colon \|n\| = k\} R^k = \max\{|a_n| \max\{|z|^n \colon z \in \overline{C}_p(e_1)\} \colon \|n\| = k\} R^k \\ &= \max\{|a_n| \max\{|z|^n \colon z \in \overline{C}_p(Re_1)\} \colon \|n\| = k\} \max\{|a_n| \max\{|z|^n \colon z \in \overline{G}_R\} \colon \|n\| = k\} \\ &= \max\{|a_n| \max\{|z|^n \colon z \in \overline{G}\} \colon \|n\| = k\} R^k = \max\{|a_n| d_G(n) \colon \|n\| = k\} R^k = B_k R^k.\end{aligned}$$

Therefore, $\mu_{F_3}(R) \leq \mu_{F_2}(R) = \mu_G(R,F)$, $R \in (0,1)$.

Hence, by Proposition 2

$$f \in \mathcal{E}^p_{\alpha\beta}(G) \Longrightarrow \int\limits_{R_0}^{1} \frac{\alpha(\ln^+ \mu_{F_3}(R))}{(1-R)^2 \beta(1/(1-R))} d\,R < +\infty.$$

Thus, from Theorem 2 in [1] it follows that for the function F_3 condition (4) holds. □

In the case $\alpha(x) \equiv x$, $\beta(x) = e^{px}$, $p > 0$, we obtain the following converse statement to Theorem 1.

Theorem 2. *Let* $f \in \mathcal{E}^p(G)$ *of form* (1) *with* $G = C_p(e_1)$, $A_k = \max\{|a_n|\colon \|n\|\} = k \geq 0$. *If* $A_k/A_{k+1} \nearrow 1$ *as* $k_0 \leq k \uparrow +\infty$ *and*

$$\sum_{k=1}^{+\infty} \Big(\frac{\ln^+ A_k}{k}\Big)^2 \exp\Big\{-\frac{pk}{\ln^+ A_k}\Big\} < +\infty,$$

then

$$\int\limits_{R_0}^{1} \frac{\ln^+ M_G(R,F)}{(1-R)^2 \exp\{p/(1-R)\}} dR < +\infty.$$

From Lemma 1 we obtain the following statement (see also proof of Proposition 2).

Lemma 2. *For* $R \in (0,1)$

$$\mu_{F_2}(R) \leq \mu_{F_1}(R) \leq c(p)\frac{1}{(1-R)^p}\mu_{F_2}\Big(\frac{1+R}{2}\Big).$$

Then $A_k = B_k$. The statement of Theorem 2 follows from Theorem 6 in [1] in a similar way as in the proof of Theorem 1 we use Theorem 2 from [1].

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Нехай L^0 — клас додатних неспадних на $[1,+\infty)$ функцій l таких, що $l((1+o(1))x) = (1+o(1))l(x)$ $(x \to +\infty)$. Припустимо, що α — вгнута функція така, що $\alpha(e^x) \in L^0$, а функція $\beta \in L^0$ така, що $\int_1^{+\infty} \alpha(x)/\beta(x)dx < +\infty$. У статті доведено теорему: якщо $f(z) = \sum_{\|n\|=0}^{+\infty} a_n z_n$, $z \in \mathbb{C}^p$, — аналітична в обмеженій області Рейнгарда $G \subset \mathbb{C}^p$ функція, то з того, що виконується умова $\int\limits_{R_0}^{1} \frac{\alpha(\ln^+ M_G(R,f))}{(1-R)^2\beta(1/(1-R))} d\,R < +\infty$, $M_G(R,f) = \sup\{|F(Rz)|\colon z \in G\}$, випливає, що

$$\sum_{k=0}^{+\infty} (\alpha(k) - \alpha(k-1))\beta_1\left(k/\ln^+ |A_k|\right) < +\infty, \quad \beta_1(x) = \int\limits_{x}^{+\infty} \frac{dt}{\beta(t)}, \quad A_k = \max\{|a_n|\colon \|n\| = k\}.$$

Ключові слова і фрази: аналітична функція, область Рейнгарда, клас збіжності.