Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 27, 2021, No. 2, 54–63
DOI: 10.7546/nntdm.2021.27.2.54-63

Partial sum of the products of the Horadam numbers with subscripts in arithmetic progression

Kunle Adegoke[1], Robert Frontczak[2*] and Taras Goy[3]

[1] Department of Physics and Engineering Physics, Obafemi Awolowo University
220005 Ile-Ife, Nigeria
e-mail: adegoke00@gmail.com

[2] Landesbank Baden-Württemberg
70173 Stuttgart, Germany
e-mail: robert.frontczak@lbbw.de

[3] Faculty of Mathematics and Computer Science
Vasyl Stefanyk Precarpathian National University
76018 Ivano-Frankivsk, Ukraine
e-mail: taras.goy@pnu.edu.ua

Received: 8 February 2021 **Revised:** 3 May 2021 **Accepted:** 23 May 2021

Abstract: We evaluate the partial sum of the products of the terms of any two Horadam sequences with subscripts in arithmetic progression. Illustrative examples are drawn from six well-known Horadam sequences.

Keywords: Horadam sequence, Generating function, Fibonacci numbers, Lucas numbers, Pell numbers, Jacobsthal numbers, Pell–Lucas numbers, Jacobsthal–Lucas numbers.
2020 Mathematics Subject Classification: 11B37, 11B39.

1 Introduction

Our purpose in this paper is to evaluate the partial sum $\sum_{j=0}^{k} X^{(1)}_{rj+s} X^{(2)}_{mj+t} z^j$, where $(X^{(1)}_n)_{n\in\mathbb{Z}}$ and $(X^{(2)}_n)_{n\in\mathbb{Z}}$ are any two Horadam sequences, r, s, m, t and k are any integers and z is any complex variable. Our results are related to those from [2–5, 7–10].

*Statements and conclusions made in this paper by Robert Frontczak are entirely those of the author. They do not necessarily reflect the views of Landesbank Baden-Württemberg.

The Horadam sequence [6] $(w_n) = (w_n(a, b; p, q))$ is defined, for all integers, by the recurrence relation

$$w_0 = a,\ w_1 = b, \quad w_n = pw_{n-1} - qw_{n-2}\,, \quad n \geq 2,$$

with

$$w_{-n} = \frac{(ap - b)u_n - aqu_{n-1}}{q^n(bu_n - aqu_{n-1})} w_n$$

or, equivalently,

$$w_{-n} = q^{-n}(av_n - w_n)\,,$$

where a, b, p and q are arbitrary complex numbers, with $p \neq 0$ and $q \neq 0$; and $(u_n(p, q)) = (w_n(0, 1; p, q))$ and $(v_n(p, q)) = (w_n(2, p; p, q))$ are Lucas sequences of the first kind and of the second kind, respectively. The most well-known Lucas sequences are the Fibonacci sequence $(F_n) = (u_n(1, -1))$ and the sequence of Lucas numbers $(L_n) = (v_n(1, -1))$.

The Binet formulas for u_n, v_n and w_n in the non-degenerate case $p^2 \neq 4q$ are

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad v_n = \alpha^n + \beta^n, \qquad w_n = \frac{b - a\beta}{\alpha - \beta}\alpha^n + \frac{a\alpha - b}{\alpha - \beta}\beta^n\,,$$

where α and β are the distinct zeros of the characteristic polynomial $x^2 - px + q$ of the Horadam and Lucas sequences,

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2}, \qquad \beta = \frac{p - \sqrt{p^2 - 4q}}{2}\,.$$

The following power reduction formulas, which we require later, can be easily established by induction:

$$\alpha^n = u_n\alpha - qu_{n-1}, \qquad \beta^n = u_n\beta - qu_{n-1}\,. \tag{1}$$

Partial sum of Horadam numbers with subscripts in arithmetic progression for integers r, k and s and arbitrary z can be evaluated as [1, 10]

$$\sum_{j=0}^{k} w_{rj+s}z^j = \frac{q^r w_{rk+s}z^{k+2} - w_{rk+r+s}z^{k+1} - q^r w_{s-r}z + w_s}{q^r z^2 - v_r z + 1}\,. \tag{2}$$

In particular,

$$\sum_{j=0}^{k} u_{rj+s}z^j = \frac{q^r u_{rk+s}z^{k+2} - u_{rk+r+s}z^{k+1} + q^s u_{r-s}z + u_s}{q^r z^2 - v_r z + 1}\,,$$

$$\sum_{j=0}^{k} v_{rj+s}z^j = \frac{q^r v_{rk+s}z^{k+2} - v_{rk+r+s}z^{k+1} - q^s v_{r-s}z + v_s}{q^r z^2 - v_r z + 1}\,.$$

The generating function of the Horadam sequence with subscripts in arithmetic progression for integers r and s is [1]

$$\sum_{j=0}^{\infty} w_{rj+s}z^j = \frac{-q^r w_{s-r}z + w_s}{q^r z^2 - v_r z + 1}\,.$$

In particular,

$$\sum_{j=0}^{\infty} u_{rj+s}z^j = \frac{q^s u_{r-s}z + u_s}{q^r z^2 - v_r z + 1}\,, \qquad \sum_{j=0}^{\infty} v_{rj+s}z^j = \frac{-q^s v_{r-s}z + v_s}{q^r z^2 - v_r z + 1}\,.$$

Further results on Horadam sequence can be found in the survey paper [13]. Properties of Lucas sequences can be found in [14, Chapter 1].

2 Main results

Theorem 2.1. *Let* $(X_n^{(1)}) = \big(w_n(X_0^{(1)}, X_1^{(1)}; p_1, q_1)\big)$ *and* $(X_n^{(2)}) = \big(w_n(X_0^{(2)}, X_1^{(2)}; p_2, q_2)\big)$ *be two non-degenerated Horadam sequences. Let*

$$\big\{(u_n^{(1)}) = (w_n(0,1;p_1,q_1)),\ (v_n^{(1)}) = (w_n(2,p_1;p_1,q_1))\big\},$$
$$\big\{(u_n^{(2)}) = (w_n(0,1;p_2,q_2)),\ (v_n^{(2)}) = (w_n(2,p_2;p_2,q_2))\big\}$$

be the respective Lucas sequences associated with $(X_n^{(1)})$ *and* $(X_n^{(2)})$*. Then*

$$\sum_{j=0}^{k} X_{rj+s}^{(1)} X_{mj+t}^{(2)} z^j = \frac{X_0^{(2)}(q_2EG + p_2FG + FH) + X_1^{(2)}(EH - FG)}{q_2G^2 + H^2 + p_2GH}\,,$$

where

$$E = q_1^r X_{rk+s}^{(1)} u_{mk+2m+t}^{(2)} z^{k+2} - X_{rk+r+s}^{(1)} u_{mk+m+t}^{(2)} z^{k+1} - q_1^r X_{s-r}^{(1)} u_{m+t}^{(2)} z + X_s^{(1)} u_t^{(2)}, \tag{3}$$

$$\begin{aligned} F = &- q_1^r X_{rk+s}^{(1)} u_{mk+2m+t-1}^{(2)} z^{k+2} + X_{rk+r+s}^{(1)} u_{mk+m+t-1}^{(2)} z^{k+1} \\ &+ q_1^r X_{s-r}^{(1)} u_{m+t-1}^{(2)} z - X_s^{(1)} u_{t-1}^{(2)}\,, \end{aligned} \tag{4}$$

$$G = q_1^r u_{2m}^{(2)} z^2 - v_r^{(1)} u_m^{(2)} z \tag{5}$$

and

$$H = -q_1^r q_2 u_{2m-1}^{(2)} z^2 + q_2 v_r^{(1)} u_{m-1}^{(2)} z + 1\,. \tag{6}$$

Proof. Let α_i and β_i, $\alpha_i \neq \beta_i$, $i \in \{1;2\}$, be the zeros of $x^2 - p_i x + q_i$, the characteristic polynomial of the sequence $(u_n^{(i)})$. Then

$$X_n^{(i)} = A_i\alpha_i^n + B_i\beta_i^n\,, \qquad i \in \{1;2\},$$

where

$$A_i = \frac{X_1^{(i)} - X_0^{(i)}\beta_i}{\alpha_i - \beta_i}, \qquad B_i = \frac{X_0^{(i)}\alpha_i - X_1^{(i)}}{\alpha_i - \beta_i}\,.$$

In (2) make the identification $w_n \equiv X_n^{(1)}$, replace z with $\alpha_2^m z$ and multiply both sides by α_2^t to obtain

$$\begin{aligned} S_1 &= \sum_{j=0}^{k} X_{rj+s}^{(1)} \alpha_2^{mj+t} z^j \\ &= \frac{q_1^r X_{rk+s}^{(1)} \alpha_2^{mk+2m+t} z^{k+2} - X_{rk+r+s}^{(1)} \alpha_2^{mk+m+t} z^{k+1} - q_1^r X_{s-r}^{(1)} \alpha_2^{m+t} z + X_s^{(1)} \alpha_2^t}{q_1^r \alpha_2^{2m} z^2 - v_r^{(1)} \alpha_2^m z + 1}\,. \end{aligned} \tag{7}$$

Similarly,

$$\begin{aligned} S_2 &= \sum_{j=0}^{k} X_{rj+s}^{(1)} \beta_2^{mj+t} z^j \\ &= \frac{q_1^r X_{rk+s}^{(1)} \beta_2^{mk+2m+t} z^{k+2} - X_{rk+r+s}^{(1)} \beta_2^{mk+m+t} z^{k+1} - q_1^r X_{s-r}^{(1)} \beta_2^{m+t} z + X_s^{(1)} \beta_2^t}{q_1^r \beta_2^{2m} z^2 - v_r^{(1)} \beta_2^m z + 1}\,. \end{aligned} \tag{8}$$

Using the power reduction formulas (1), identities (7) and (8) can be written as

$$S_1 = \sum_{j=0}^{k} X_{rj+s}^{(1)} \alpha_2^{mj+t} z^j = \frac{E\alpha_2 + F}{G\alpha_2 + H} \tag{9}$$

and

$$S_2 = \sum_{j=0}^{k} X_{rj+s}^{(1)} \beta_2^{mj+t} z^j = \frac{E\beta_2 + F}{G\beta_2 + H}\,, \tag{10}$$

where E, F, G and H are as given in (3) – (6). Now

$$A_2 S_1 + B_2 S_2 = \sum_{j=0}^{k} X_{rj+s}^{(1)} (A_2 \alpha_2^{mj+t} + B_2 \beta_2^{mj+t}) z^j = \sum_{j=0}^{k} X_{rj+s}^{(1)} X_{mj+t}^{(2)} z^j\,.$$

But from (9) and (10) we have

$$A_2 S_1 + B_2 S_2 = A_2 \frac{E\alpha_2 + F}{G\alpha_2 + H} + B_2 \frac{E\beta_2 + F}{G\beta_2 + H}\,.$$

Thus,

$$\sum_{j=0}^{k} X_{rj+s}^{(1)} X_{mj+t}^{(2)} z^j = \frac{A_2(E\alpha_2 + F)(G\beta_2 + H) + B_2(G\alpha_2 + H)(E\beta_2 + F)}{(G\alpha_2 + H)(G\beta_2 + H)}\,,$$

from which the stated identity follows after multiplying out the right-hand side and some algebra. □

3 Examples

We will draw illustrations of Theorem 2.1 from six well-known second-order sequences, namely the Fibonacci, Lucas, Pell, Pell–Lucas, Jacobsthal, and Jacobsthal–Lucas numbers. First we give a quick review of the sequences.

The Fibonacci numbers F_n and the Lucas numbers L_n are defined, for $n \in \mathbb{Z}$, as usual, through the recurrence $F_n = F_{n-1} + F_{n-2}$ ($n \geq 2$), $F_0 = 0$, $F_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ ($n \geq 2$), $L_0 = 2$, $L_1 = 1$, with $F_{-n} = (-1)^{n-1} F_n$ and $L_{-n} = (-1)^n L_n$. Exhaustive discussion of the properties of Fibonacci and Lucas numbers can be found in [11, 16].

The Jacobsthal numbers J_n and the Jacobsthal–Lucas numbers j_n are defined, for $n \in \mathbb{Z}$, through the recurrence relations $J_n = J_{n-1} + 2J_{n-2}$ ($n \geq 2$), $J_0 = 0$, $J_1 = 1$ and $j_n = j_{n-1} + 2j_{n-2}$ ($n \geq 2$), $j_0 = 2$, $j_1 = 1$, with $J_{-n} = (-1)^{n-1} 2^{-n} J_n$ and $j_{-n} = (-1)^n 2^{-n} j_n$. The entries A001045 and A014551 from [15] conclude good reference materials on the Jacobsthal and Jacobsthal–Lucas numbers, respectively.

The Pell numbers P_n and Pell–Lucas numbers Q_n are defined, for $n \in \mathbb{Z}$, through the recurrence relations $P_n = 2P_{n-1} + P_{n-2}$ ($n \geq 2$), $P_0 = 0$, $P_1 = 1$ and $Q_n = 2Q_{n-1} + Q_{n-2}$ ($n \geq 2$), $Q_0 = 2$, $Q_1 = 2$, with $P_{-n} = (-1)^{n-1} P_n$ and $Q_{-n} = (-1)^n Q_n$. [12] and [15] (entries A000129 and A002203) are useful source materials on Pell and Pell–Lucas numbers.

For reference, the first few values of the six sequences are given in Table 1 below.

n	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
F_n	5	-3	2	-1	1	0	1	1	2	3	5	8	13	21
L_n	-11	7	-4	3	-1	2	1	3	4	7	11	18	29	47
P_n	29	-12	5	-2	1	0	1	2	5	12	29	70	169	408
Q_n	-82	34	-14	6	-2	2	2	6	14	34	82	198	478	1154
J_n	$11/32$	$-5/16$	$3/8$	$-1/4$	$1/2$	0	1	1	3	5	11	21	43	85
j_n	$-31/32$	$17/16$	$-7/8$	$5/4$	$-1/2$	2	1	5	7	17	31	65	127	257

Table 1. Terms of F_n, L_n, P_n, Q_n, J_n and j_n

3.1 Sum of the products of Fibonacci numbers with subscripts in arithmetic progression

Let $(X_n^{(1)}) \equiv (F_n)$ and $(X_n^{(2)}) \equiv (F_n)$. Then $(u_n^{(1)}) = (F_n)$, $(v_n^{(1)}) = (L_n)$, $(u_n^{(2)}) = (F_n)$ and $(v_n^{(2)}) = (L_n)$. Thus $p_1 = p_2 = 1$, $q_1 = q_2 = -1$. We therefore have:

$$E = (-1)^r F_{rk+s} F_{mk+2m+t} z^{k+2} - F_{rk+r+s} F_{mk+m+t} z^{k+1} - (-1)^r F_{s-r} F_{m+t} z + F_s F_t\,, \tag{11}$$

$$\begin{aligned} F = {} & (-1)^r F_{rk+s} F_{mk+2m+t-1} z^{k+2} - F_{rk+r+s} F_{mk+m+t-1} z^{k+1} \\ & - (-1)^r F_{s-r} F_{m+t-1} z + F_s F_{t-1}\,, \end{aligned} \tag{12}$$

$$G = (-1)^r F_{2m} z^2 - L_r F_m z \tag{13}$$

and

$$H = (-1)^r F_{2m-1} z^2 - L_r F_{m-1} z + 1. \tag{14}$$

Theorem 3.1. *Let r, s, m, t and k be integers. Then*

$$\sum_{j=0}^{k} F_{rj+s} F_{mj+t} z^j = \frac{EH - FG}{H^2 - G^2 + GH}\,,$$

where E, F, G and H are as given in (11) – (14).

In particular, we have

$$\begin{aligned} \sum_{j=0}^{k} F_j^2 z^j &= \frac{(F_k F_{k+2} z^{k+2} + F_{k+1}^2 z^{k+1} - z)(1 - z^2) + F_k F_{k+1} z^k (z + z^2)^2}{(z + z^2)^2 - (1 - z^2)^2 + (z + z^2)(1 - z^2)}\,, \\ &= \frac{(F_k F_{k+2} z^{k+2} + F_{k+1}^2 z^{k+1} - z)(z^2 - 1) - F_k F_{k+1} z^k (z + z^2)^2}{(1 - 3z + z^2)(z + 1)^2}\,, \end{aligned} \tag{15}$$

which at $z = 1$ gives the classical result

$$\sum_{j=0}^{k} F_j^2 = F_k F_{k+1}\,,$$

and from which we also get the generating function of the squares of Fibonacci numbers

$$\sum_{j=0}^{\infty} F_j^2 z^j = \frac{z - z^2}{1 - 2z - 2z^2 + z^3}\,.$$

Evaluation of (15) at $z=-1$, with the aid of L'Hospital rule gives

$$\sum_{j=0}^{k}(-1)^{j}F_j^2=-\frac{2}{5}(k+1)+(-1)^{k}\frac{F_{k+1}L_k}{5}\,.$$

3.2 Sum of the products of Fibonacci and Lucas numbers with subscripts in arithmetic progression

Let $(X_n^{(1)})\equiv(F_n)$ and $(X_n^{(2)})\equiv(L_n)$. Then $(u_n^{(1)})=(F_n)$, $(v_n^{(1)})=(L_n)$, $(u_n^{(2)})=(F_n)$ and $(v_n^{(2)})=(L_n)$. Thus $p_1=p_2=1$, $q_1=q_2=-1$. We have

$$E=(-1)^{r}F_{rk+s}F_{mk+2m+t}z^{k+2}-F_{rk+r+s}F_{mk+m+t}z^{k+1}-(-1)^{r}F_{s-r}F_{m+t}z+F_sF_t\,, \tag{16}$$

$$\begin{aligned}F&=(-1)^{r}F_{rk+s}F_{mk+2m+t-1}z^{k+2}-F_{rk+r+s}F_{mk+m+t-1}z^{k+1}\\&\quad-(-1)^{r}F_{s-r}F_{m+t-1}z+F_sF_{t-1}\,,\end{aligned} \tag{17}$$

$$G=(-1)^{r}F_{2m}z^{2}-L_rF_mz \tag{18}$$

and

$$H=(-1)^{r}F_{2m-1}z^{2}-L_rF_{m-1}z+1\,. \tag{19}$$

Theorem 3.2. *Let r, s, m, t and k be integers. Then*

$$\sum_{j=0}^{k}F_{rj+s}L_{mj+t}z^{j}=\frac{2(FH-EG)+EH+FG}{H^2-G^2+GH}\,,$$

where E, F, G and H are as given in (16) – (19).

In particular, we have

$$\sum_{j=0}^{k}F_jL_jz^{j}=\frac{A+B}{(1-z^2)^2-(z+z^2)^2-(z+z^2)(1-z^2)}\,, \tag{20}$$

where

$$A=-2\big(F_kF_{k+1}z^{k+1}(1+z)(1-z^2)+(F_kF_{k+2}z^{k+2}+F_{k+1}^2z^{k+1}-z)(z+z^2)\big),$$
$$B=(-F_kF_{k+2}z^{k+2}-F_{k+1}^2z^{k+1}+z)(1-z^2)+F_kF_{k+1}z^{k+2}(1+z)^2.$$

Two special evaluations are

$$\sum_{j=0}^{k}F_jL_j=F_{2k+1}-1\,,$$

which is also a classical result and

$$\sum_{j=0}^{k}\frac{F_jL_j}{2^j}=\frac{2F_{k+1}^2+F_kF_{k+3}}{2^k}-2\,.$$

Applying L'Hospital rule twice to the right hand side of (20) at $z=-1$ and making use of Cassini's identity, we find

$$\sum_{j=0}^{k}(-1)^{j}F_jL_j=(-1)^{k}F_kF_{k+1}\,.$$

3.3 Sum of the products of Fibonacci and Pell numbers with subscripts in arithmetic progression

Let $(X_n^{(1)}) \equiv (F_n)$ and $(X_n^{(2)}) \equiv (P_n)$. Then $(u_n^{(1)}) = (F_n)$, $(v_n^{(1)}) = (L_n)$, $(u_n^{(2)}) = (P_n)$ and $(v_n^{(2)}) = (Q_n)$. Thus, $p_1 = 1$, $q_1 = -1$, $p_2 = 2$ and $q_2 = -1$. We therefore have

$$E = (-1)^r F_{rk+s} P_{mk+2m+t} z^{k+2} - F_{rk+r+s} P_{mk+m+t} z^{k+1} - (-1)^r F_{s-r} P_{m+t} z + F_s P_t \,, \tag{21}$$

$$\begin{aligned} F = {} & (-1)^r F_{rk+s} P_{mk+2m+t-1} z^{k+2} - F_{rk+r+s} P_{mk+m+t-1} z^{k+1} \\ & - (-1)^r F_{s-r} P_{m+t-1} z + F_s P_{t-1} \,, \end{aligned} \tag{22}$$

$$G = (-1)^r P_{2m} z^2 - L_r P_m z \tag{23}$$

and

$$H = (-1)^r P_{2m-1} z^2 - L_r P_{m-1} z + 1 \,. \tag{24}$$

Theorem 3.3. *Let r, s, m, t and k be integers. Then*

$$\sum_{j=0}^{k} F_{rj+s} P_{mj+t} z^j = \frac{EH - FG}{H^2 - G^2 + 2GH} \,,$$

where E, F, G and H are as given in (21) – (24).

In particular, we have

$$\begin{aligned} \sum_{j=0}^{k} F_j P_j z^j = {} & \frac{F_k P_k z^{k+4} + (F_{k+1} P_{k-1} - F_k P_{k+1}) z^{k+3}}{z^4 - 2z^3 - 7z^2 - 2z + 1} \\ & - \frac{(F_k P_{k+2} + F_{k+1} P_k) z^{k+2} + F_{k+1} P_{k+1} z^{k+1} + z^3 - z}{z^4 - 2z^3 - 7z^2 - 2z + 1} \,, \end{aligned} \tag{25}$$

of which we can mention the special values

$$\sum_{j=0}^{k} F_j P_j = \frac{F_k P_{k+1} + P_k F_{k+1}}{3}$$

and

$$\sum_{j=0}^{k} (-1)^j F_j P_j = (-1)^k (P_{k+1} F_k - F_{k+1} P_k) \,.$$

Note that from (25), it follows that the generating function of the product of Fibonacci numbers and Pell numbers is

$$\sum_{j=0}^{\infty} F_j P_j z^j = \frac{z - z^3}{1 - 2z - 7z^2 - 2z^3 + z^4} \,.$$

3.4 Sum of the products of Fibonacci and Jacobsthal numbers with subscripts in arithmetic progression

Let $(X_n^{(1)}) \equiv (F_n)$ and $(X_n^{(2)}) \equiv (J_n)$. Then $(u_n^{(1)}) = (F_n)$, $(v_n^{(1)}) = (L_n)$, $(u_n^{(2)}) = (J_n)$ and $(v_n^{(2)}) = (j_n)$. Thus $p_1 = 1$, $q_1 = -1$, $p_2 = 1$ and $q_2 = -2$. We therefore have

$$E = (-1)^r F_{rk+s} J_{mk+2m+t} z^{k+2} - F_{rk+r+s} J_{mk+m+t} z^{k+1} - (-1)^r F_{s-r} J_{m+t} z + F_s J_t\,, \tag{26}$$

$$\begin{aligned} F = &(-1)^r 2F_{rk+s} J_{mk+2m+t-1} z^{k+2} - 2F_{rk+r+s} J_{mk+m+t-1} z^{k+1} \\ &- (-1)^r 2F_{s-r} J_{m+t-1} z + 2F_s J_{t-1}\,, \end{aligned} \tag{27}$$

$$G = (-1)^r J_{2m} z^2 - L_r J_m z \tag{28}$$

and

$$H = (-1)^r 2J_{2m-1} z^2 - 2L_r J_{m-1} z + 1\,. \tag{29}$$

Theorem 3.4. *Let r, s, m, n and k be integers. Then*

$$\sum_{j=0}^{k} F_{rj+s} J_{mj+t} z^j = \frac{EH - FG}{H^2 - 2G^2 + GH}\,,$$

where E, F, G and H are as given in (26) – (29).

In particular, we have

$$\begin{aligned} \sum_{j=0}^{k} F_j J_j z^j = &\frac{4F_k J_k z^{k+4} + 2(2F_{k+1} J_{k-1} - F_k J_{k+1}) z^{k+3}}{4z^4 - 2z^3 - 7z^2 - z + 1} \\ &- \frac{(F_k J_{k+2} + 2F_{k+1} J_k) z^{k+2} + F_{k+1} J_{k+1} z^{k+1} + 2z^3 - z}{4z^4 - 2z^3 - 7z^2 - z + 1}\,, \end{aligned} \tag{30}$$

giving the special values

$$\sum_{j=0}^{k} F_j J_j = \frac{F_k (J_{k+3} - 4J_k) + F_{k+1} (J_{k+2} - 4J_{k-1})}{5}$$

and

$$\sum_{j=0}^{k} (-1)^j F_j J_j = (-1)^k (F_k J_{k+2} - F_{k+1} J_{k+1}) + 1\,.$$

From (30) we obtain the generating function of the product of Fibonacci and Jacobsthal numbers

$$\sum_{j=0}^{\infty} F_j J_j z^j = \frac{z - 2z^3}{1 - z - 7z^2 - 2z^3 + 4z^4}\,.$$

3.5 Sum of the products of Pell and Jacobsthal numbers with subscripts in arithmetic progression

Let $(X_n^{(1)}) \equiv (P_n)$ and $(X_n^{(2)}) \equiv (J_n)$. Then $(u_n^{(1)}) = (P_n)$, $(v_n^{(1)}) = (Q_n)$, $(u_n^{(2)}) = (J_n)$ and $(v_n^{(2)}) = (j_n)$. Thus $p_1 = 2$, $q_1 = -1$, $p_2 = 1$ and $q_2 = -2$. We therefore have

$$E = (-1)^r P_{rk+s} J_{mk+2m+t} z^{k+2} - P_{rk+r+s} J_{mk+m+t} z^{k+1} - (-1)^r P_{s-r} J_{m+t} z + P_s J_t\,, \tag{31}$$

$$
\begin{aligned}
F = &(-1)^r 2P_{rk+s}J_{mk+2m+t-1}z^{k+2} - 2P_{rk+r+s}J_{mk+m+t-1}z^{k+1} \\
&- (-1)^r 2P_{s-r}J_{m+t-1}z + 2P_sJ_{t-1}\,,
\end{aligned} \tag{32}
$$

$$
G = (-1)^r J_{2m}z^2 - Q_rJ_mz\,, \tag{33}
$$

and

$$
H = (-1)^r 2J_{2m-1}z^2 - 2Q_rJ_{m-1}z + 1\,. \tag{34}
$$

Theorem 3.5. *Let r, s, m, t and k be integers. Then*

$$
\sum_{j=0}^{k} P_{rj+s}J_{mj+t}z^j = \frac{EH - FG}{H^2 - 2G^2 + GH}\,,
$$

where E, F, G and H are given in (31) – (34).

In particular, we have

$$
\begin{aligned}
\sum_{j=0}^{k} P_jJ_jz^j = &\frac{(-P_kJ_{k+2}z^{k+2} - P_{k+1}J_{k+1}z^{k+1} + z)(1 - 2z^2)}{(4z^2 + 4z - 1)(z^2 - 2z - 1)} \\
&- \frac{2(P_kJ_{k+1}z^{k+2} + P_{k+1}J_kz^{k+1})(2z + z^2)}{(4z^2 + 4z - 1)(z^2 - 2z - 1)}\,,
\end{aligned} \tag{35}
$$

from which we get the special values

$$
\sum_{j=0}^{k} P_jJ_j = \frac{3}{7}(P_kJ_{k+1} + J_kP_{k+1}) - \frac{1}{14}(P_kJ_{k+2} + J_{k+1}P_{k+1}) + \frac{1}{14},
$$

and

$$
\sum_{j=0}^{k} (-1)^j P_jJ_j = \frac{(-1)^k}{2}(P_{k+1}J_{k+2} - P_kJ_{k+3}) - \frac{1}{2}\,.
$$

From (35), we obtain the generating function of the product of Pell and Jacobsthal numbers as follows

$$
\sum_{j=0}^{\infty} P_jJ_jz^j = \frac{z - 2z^3}{(4z^2 + 4z - 1)(z^2 - 2z - 1)}\,.
$$

4 Conclusion

In this paper, we have derived an expression for the partial sum of the products of two arbitrary Horadam sequences with subscripts in arithmetic progression. Illustrative examples were drawn from six well-known Horadam sequences. Some more ideas for future work were stated implicitly in the text.

Acknowledgements

The authors would like to thank the referees for their valuable comments.

References

[1] Adegoke, K. (2018). Partial sums and generating functions for power of second order sequences with indices in arithmetic progression. Preprint. Available online at: `https://arxiv.org/abs/1904.09916`.

[2] Belbachir, H., & Bencherif, F. (2013). Sums of products of generalized Fibonacci and Lucas numbers. *Ars Combinatoria*, 110, 33–43.

[3] Čerin, Z. (2015). Formulas for quadratic sums that involve generalized Fibonacci and Lucas numbers. *Rad Hrvatske Akademije Znanosti i Umjetnosti, Matematičke Znanosti*, 19(523), 1–12.

[4] Čerin, Z. (2009). On sum of products of Horadam numbers. *Kyungpook Mathematical Journal*, 49, 483–492.

[5] Čerin, Z. (2009) Sum of products of generalized Fibonacci and Lucas numbers. *Demonstratio Mathematica*, 42, 247–258.

[6] Horadam, A. F. (1965). Basic properties of a certain generalized sequence of numbers. *The Fibonacci Quarterly*, 3(3), 161–176.

[7] Khan, M. A., & Kwong, H. (2014). On sums of products of Fibonacci-type recurrences. *The Fibonacci Quarterly*, 52(2), 20–26.

[8] Kiliç, E., Ömür, N., & Ulutaş, Y. T. (2011). Some finite sums involving generalized Fibonacci and Lucas numbers. *Discrete Dynamics in Nature and Society*, 2011(1), Article ID 284261.

[9] Kiliç, E., & Prodinger, H. (2015). Sums of products of generalized Fibonacci and Lucas numbers. *Acta Mathematica Hungarica*, 145(1), 17–25.

[10] Kiliç, E., & Stănică, P. (2013). General approach in computing sums of products of binary sequences. *Hacettepe Journal of Mathematics and Statistics*, 42(1), 1–7.

[11] Koshy, T. (2017). *Fibonacci and Lucas Numbers with Applications*, Wiley, New York.

[12] Koshy, T. (2014). *Pell and Pell–Lucas Numbers with Applications*, Springer, New York.

[13] Larcombe, P. J. (2017). Horadam sequences: A survey update and extension. *Bulletin of the Institute of Combinatorics and its Applications*, 80, 99–118.

[14] Ribenboim, P. (2000). *My Numbers, My Friends*, Springer, New York.

[15] Sloane, N. J. A. (ed.). The On-Line Encyclopedia of Integer Sequences. Available online at: `https://oeis.org`.

[16] Vajda, S. (2008). *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications*, Dover, New York.