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Abstract

In this paper, we evaluate determinants of several families of Hessenberg matrices having various subsequences of the Jacobsthal sequence as their nonzero entries. These identities may be written equivalently as formulas for certain linearly recurrent sequences expressed in terms of sums of products of Jacobsthal numbers with multinomial coefficients. Among the sequences that arise in this way include the Mersenne, Lucas and Jacobsthal-Lucas numbers as well as the squares of the Jacobsthal and Mersenne sequences. These results are extended to Hessenberg determinants involving sequences that are derived from two general families of linear second-order recurrences. Finally, combinatorial proofs are provided for several of our determinant results which make use of various correspondences between Jacobsthal tilings and certain restricted classes of binary words.

Keywords: Hessenberg matrix; Jacobsthal number; Jacobsthal-Lucas number; Mersenne number; Determinant; Trudi formula

MSC 2010 No.: 05A19, 11B39, 15B05

1. Introduction

The Jacobsthal numbers have several noteworthy properties and applications to various areas of mathematics such as number theory, graph theory, combinatorics and geometry (see, e.g., Barry (2016); Bruhn et al. (2015); Frey and Sellers (2000); Heubach (1999); Horadam (1988); Ramirez and Shattuck (2019); Yılmaz and Bozkurt (2012) and references contained therein). For instance, Akbulak and Öteleş (2014) and Öteleş et al. (2018) considered two n-square upper Hessenberg matrices one of which corresponds to the adjacency matrix of a directed pseudo graph and investigated relations between determinants and permanents of these Hessenberg matrices and sum formulas for the Jacobsthal sequence. Aktaş and Köse (2015) defined two upper Hessenberg matrices and then showed that the permanents of these matrices are Jacobsthal numbers. Köken and Bozkurt (2008) defined the n-square Jacobsthal matrix and using this matrix derived some properties of Jacobsthal numbers. Cılasun (2016) introduced a recurrence relation for the so-called multiple counting Jacobsthal sequences and showed their application to Fermat's little theorem.

Further related formulas were given by Daşdemir (2019), who extended the Jacobsthal numbers to terms with negative subscripts and presented many identities for new forms of these numbers. Cerin (2007) considered sums of squares of odd and even terms of the Jacobsthal sequence and sums of their products; these sums are related to products of appropriate Jacobsthal numbers and some integer sequences. Uygun (2017), by using Jacobsthal and Jacobsthal-Lucas matrix sequences, defined k-Jacobsthal and k-Jacobsthal-Lucas sequences depending upon a single parameter k and established a combinatorial representation. In Cook and Bacon (2013), the Jacobsthal recurrence is generalized to higher order recurrence relations and the main Jacobsthal identities are extended in this way. Deveci and Artun (2018) defined the adjacency-Jacobsthal numbers and obtained a combinatorial representation and sum formula by using the generating matrix of the sequence.

In Goy (2018), the first author considered determinants of some families of Toeplitz-Hessenberg matrices having various translates of the Jacobsthal numbers for the nonzero entries. By the Trudi formula, these determinant identities may be written equivalently as formulas involving sums of products of Jacobsthal numbers and multinomial coefficients. Here, some comparable results are provided within the framework of the generalized Trudi formula wherein the first column entries are modified in a certain way and combinatorial proofs are also given in several cases.

The organization of this paper is as follows. In the next section, we review some basic properties of Jacobsthal numbers and Hessenberg matrices. In the third section, we evaluate determinants of several families of Hessenberg matrices having various subsequences of the Jacobsthal sequence as their nonzero entries by an inductive approach. A comparable formula is also found for the companion sequence known as the Jacobsthal-Lucas numbers. Applying a generalization of the result of Trudi, one can rewrite these determinant formulas equivalently as identities expressing certain linearly recurrent sequences as sums of products of Jacobsthal numbers with multinomial coefficients. In the fourth section, we provide combinatorial proofs of several of our formulas which draw upon various relationships between Jacobsthal tilings and certain restricted classes of binary words. In the fifth section, we extend our results to sequences satisfying a more general recurrence with arbitrary initial conditions using a generating function approach. We remark that

some of the results in the third section were announced without proof in Goy (2020). Similar results for Fibonacci, Lucas, Pell, Catalan, tribonacci and tetranacci numbers have recently been obtained by the authors in Goy (2019), Goy and Shattuck (2019a), Goy and Shattuck (2019b), Goy and Shattuck (2020a), Goy and Shattuck (2020b), and Goy and Shattuck (2020c).

2. Preliminaries

We wish to consider the determinants of certain $n \times n$ Hessenberg matrices having Jacobsthal number entries. First recall (see, e.g., Horadam (1986)) that the Jacobsthal and Jacobsthal-Lucas sequences $(J_n)_{n\geq 0}$ and $(j_n)_{n\geq 0}$ are defined recursively by

$$J_0 = 0, \quad J_1 = 1, \quad J_n = J_{n-1} + 2J_{n-2}, \quad n \geq 2, \tag{1}$$

and

$$j_0 = 2, \quad j_1 = 1, \quad j_n = j_{n-1} + 2j_{n-2}, \quad n \geq 2. \tag{2}$$

Note that the two definitions differ only in the first initial condition in analogy with the Fibonacci and Lucas numbers. The main properties of these numbers are summarized in Koshy (2019), Chapter 44.

It follows from (1) and (2) that J_n and j_n at a specific point in the sequence may be calculated directly using the Binet-like formulas

$$J_n = \frac{2^n - (-1)^n}{3}, \quad n \geq 0, \tag{3}$$

and

$$j_n = 2^n + (-1)^n, \quad n \geq 0. \tag{4}$$

When n is even, $J_n = \frac{M_n}{3}$, where $n \geq 0$ and $M_n = 2^n - 1$ denotes the n-th Mersenne number, and when n is odd, $j_n = M_n$, $n \geq 1$.

These sequences occur in the On-Line Encyclopedia of Integer Sequences (Sloane (2020)) and their first few terms are as follows:

$$\{J_n\}_{n\geq 0} = \{0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683, 1365, 2731, 5461, 10923, \ldots\} : \quad A101045$$
$$\{j_n\}_{n\geq 0} = \{2, 1, 5, 7, 17, 31, 65, 127, 257, 511, 1025, 2047, 4097, 8191, 16385, \ldots\} : \quad A114551$$
$$\{M_n\}_{n\geq 0} = \{0, 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191, 16383, \ldots\} : \quad A000225$$

Consider now the $n \times n$ Hessenberg matrix having the form

$$H_n(a_0; a_1, a_2, \ldots, a_n) = \begin{bmatrix} k_1a_1 & a_0 & & & \\ k_2a_2 & a_1 & a_0 & & 0 \\ \vdots & \vdots & \ddots & \ddots & \\ k_{n-1}a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 \; a_0 \\ k_na_n & a_{n-1} & a_{n-2} & \cdots & a_2 \; a_1 \end{bmatrix}, \tag{5}$$

where $a_i \neq 0$ for at least one $i > 0$. In Muir (1960), p. 228, one finds the following general determinant formula for $H_n = H_n(a_0; a_1, a_2, \ldots, a_n)$:

$$\det(H_n) = \sum_{s_1+2s_2+\cdots+ns_n=n} \frac{(-a_0)^{n-|s|}}{|s|} \Big(\sum_{i=1}^{n} s_i k_i\Big) m_n(s) a_1^{s_1} a_2^{s_2} \cdots a_n^{s_n}, \qquad n \geq 1, \tag{6}$$

where $m_n(s) = \frac{(s_1+\cdots+s_n)!}{s_1!\cdots s_n!}$ and $|s| = s_1 + \cdots + s_n$ for an n-tuple $s = (s_1, \ldots, s_n)$ of non-negative integers.

Alternatively, a recurrence for $\det(H_n)$, which may be obtained by repeatedly expanding along the last column, is given by

$$\det(H_n) = (-a_0)^{n-1} k_n a_n + \sum_{i=1}^{n-1} (-a_0)^{i-1} a_i \det(H_{n-i}), \qquad n \geq 1. \tag{7}$$

Note that when $k_1 = \cdots = k_n = 1$ in (6), one gets the classical formula of Trudi. Thus, one may regard (6) as a generalized Trudi formula (Muir (1960), p. 214). If one takes $k_i = i$ for all i in (6), then

$$\det(H_n) = n(-a_0)^n \cdot \sum_{s_1+2s_2+\cdots+ns_n=n} \frac{m_n(s)}{|s|} \left(-\frac{a_1}{a_0}\right)^{s_1} \left(-\frac{a_2}{a_0}\right)^{s_2} \cdots \left(-\frac{a_n}{a_0}\right)^{s_n}. \tag{8}$$

Henceforth, we will be interested in the case when $k_i = i$ for all i in (5) and denote $\det\big(H_n(a_0; a_1, a_2, \ldots, a_n)\big)$ by $\det(a_0; a_1, a_2, \ldots, a_n)$ in this case for the sake of brevity.

3. Jacobsthal Determinant Identities

We have the following determinant formulas for Hessenberg matrices whose nonzero entries are given by the (untranslated) Jacobsthal and Jacobsthal-Lucas numbers.

Theorem 3.1.

For $n \geq 1$, the following formulas hold:

$$\det(1; J_1, J_2, \ldots, J_n) = \begin{cases} M_n, & \text{if } n \text{ is odd,} \\ -M_{n/2}^2, & \text{if } n \text{ is even,} \end{cases} \tag{9}$$

$$\det(1; j_1, j_2, \ldots, j_n) = \begin{cases} M_n, & \text{if } n \text{ is odd,} \\ -9J_{n/2}^2, & \text{if } n \text{ is even.} \end{cases} \tag{10}$$

Proof:

To show (10), we proceed by induction on n, the $n = 1$ and $n = 2$ cases being clear. Let

$$v_n = \det(1; j_1, j_2, \ldots, j_n),$$

for $n \geq 1$. If $n \geq 3$ is odd, then by (7), we have

$$\begin{aligned}
v_n &= (-1)^{n-1} n j_n + \sum_{i=1}^{n-1} (-1)^{i-1} j_i v_{n-i} \\
&= (-1)^{n-1} n J_n - \sum_{i=1}^{\frac{n-1}{2}} j_{2i} M_{n-2i} - 9 \sum_{i=1}^{\frac{n-1}{2}} j_{2i-1} J_{(n-2i+1)/2}^2 \\
&= n\left(2^n - 1\right) - \sum_{i=1}^{\frac{n-1}{2}} \left(2^{2i} + 1\right)\left(2^{n-2i} - 1\right) - \sum_{i=1}^{\frac{n-1}{2}} \left(2^{2i-1} - 1\right)\left(2^{\frac{n-2i+1}{2}} - (-1)^{\frac{n-2i+1}{2}}\right)^2 \\
&= n\left(2^n - 1\right) - \left(\frac{n-1}{2} \cdot 2^n - \sum_{i=1}^{\frac{n-1}{2}} 2^{2i} + \sum_{i=1}^{\frac{n-1}{2}} 2^{n-2i} - \frac{n-1}{2} \right) \\
&\quad - \left(\frac{n-1}{2} \cdot 2^n - 2 \sum_{i=1}^{\frac{n-1}{2}} 2^{2i-1} (-2)^{\frac{n-2i+1}{2}} + \sum_{i=1}^{\frac{n-1}{2}} 2^{2i-1} - \sum_{i=1}^{\frac{n-1}{2}} 2^{n-2i+1} + 2 \sum_{i=1}^{\frac{n-1}{2}} (-2)^{\frac{n-2i+1}{2}} - \frac{n-1}{2} \right) \\
&= 2^n - 1 + \sum_{i=1}^{\frac{n-1}{2}} 2^{2i-1} + \sum_{i=1}^{\frac{n-1}{2}} 2^{n-2i} + (-2)^{\frac{n+1}{2}} \sum_{i=1}^{\frac{n-1}{2}} (-2)^i - 2 \sum_{i=1}^{\frac{n-1}{2}} (-2)^{\frac{n-2i+1}{2}} \\
&= 2^n - 1 + \sum_{i=1}^{\frac{n-1}{2}} 4^i + \left((-2)^{\frac{n+1}{2}} - 2\right) \sum_{i=1}^{\frac{n-1}{2}} (-2)^i \\
&= 2^n - 1 + \frac{4^{\frac{n+1}{2}} - 4}{3} - \frac{\left((-2)^{\frac{n+1}{2}} - 2\right)\left((-2)^{\frac{n+1}{2}} + 2\right)}{3} \\
&= 2^n - 1 + \frac{2^{n+1} - 4}{3} - \frac{2^{n+1} - 4}{3} = 2^n - 1 = M_n.
\end{aligned}$$

If n is even, then we have

$$\begin{aligned}
v_n &= (-1)^{n-1} n j_n + 9 \sum_{i=1}^{\frac{n}{2}-1} j_{2i} J_{(n-2i)/2}^2 + \sum_{i=1}^{\frac{n}{2}} j_{2i-1} M_{n-2i+1} \\
&= -n(2^n + 1) + \sum_{i=1}^{\frac{n}{2}-1} \left(2^{2i} + 1\right)\left(2^{\frac{n-2i}{2}} - (-1)^{\frac{n-2i}{2}}\right)^2 + \sum_{i=1}^{\frac{n}{2}} \left(2^{2i-1} - 1\right)\left(2^{n-2i+1} - 1\right).
\end{aligned}$$

Expanding the sums as in the odd case above and simplifying, we get

$$v_n = -2^n - 1 + 2(-2)^{\frac{n}{2}} = -\left(2^{\frac{n}{2}} - (-1)^{\frac{n}{2}}\right)^2 = -9 J_{n/2}^2,$$

which completes the induction and proof of (10). A similar argument may be given for (9). ∎

Let $L_n = F_{n+1} + F_{n-1}$ denote the n-th Lucas number, where F_n is the n-th Fibonacci number with initial conditions $F_0 = 0,\ F_1 = 1$. We have the following further determinant formulas involving various subsequences of the Jacobsthal numbers.

Theorem 3.2.

For $n \geq 1$, the following formulas hold:

$$\det(1; J_0, J_1, \ldots, J_{n-1}) = (-1)^n(L_n - j_n), \tag{11}$$

$$\det(-1; J_1, J_2, \ldots, J_n) = (-1)^{n+1} - 2^n + 2\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} 3^k, \tag{12}$$

$$\det(1; J_2, J_3, \ldots, J_{n+1}) = (-1)^{n-1} j_n, \tag{13}$$

$$\det(1; J_3, J_4, \ldots, J_{n+2}) = \begin{cases} M_{n+1}, & \text{if } n \text{ is odd,} \\ -1, & \text{if } n \text{ is even,} \end{cases} \tag{14}$$

$$\det(1; J_4, J_5, \ldots, J_{n+3}) = M_{n+1} - (-2)^n, \tag{15}$$

$$\det(1; J_2, J_4, \ldots, J_{2n}) = (-1)^{n-1} M_n^2, \tag{16}$$

$$\det(-1; J_2, J_4, \ldots, J_{2n}) = (3 + \sqrt{5})^n + (3 - \sqrt{5})^n - 4^n - 1, \tag{17}$$

$$\det(1; J_3, J_5, \ldots, J_{2n+1}) = (-1)^{n-1}(4^n - M_n), \tag{18}$$

$$\det(1; J_4, J_6, \ldots, J_{2n+2}) = (-1)^{n-1}(4^n + 1). \tag{19}$$

Proof:

These results can be shown by induction on n using (7). We demonstrate with identity (11). To establish the n-case of (11) from the m-cases for $m < n$, we must show

$$j_n - L_n = nJ_{n-1} + \sum_{i=1}^{n-1} J_{i-1}(L_{n-i} - j_{n-i}), \qquad n \geq 1, \tag{20}$$

upon multiplying through by $(-1)^{n-1}$. To prove (20), it suffices to show for $n \geq 1$,

$$\sum_{i=1}^{n-1} J_{i-1} L_{n-i} = J_{n+1} - L_n, \tag{21}$$

and

$$\sum_{i=1}^{n-1} J_{i-1} j_{n-i} = (n-2)J_{n-1}, \tag{22}$$

for then the right side of (20) would work out to

$$J_{n+1} + 2J_{n-1} - L_n = j_n - L_n,$$

as desired. Identities (21) and (22) can be established by induction on n. For (21), first note that it holds when $n = 1, 2$, so one may assume $n \geq 3$.

Then we have

$$\begin{aligned}\sum_{i=1}^{n-1} J_{i-1}L_{n-i} &= J_{n-2} + 3J_{n-3} + \sum_{i=1}^{n-3} J_{i-1}L_{n-i} \\ &= J_{n-2} + 3J_{n-3} + \sum_{i=1}^{n-3} J_{i-1}(L_{n-1-i} + L_{n-2-i}) \\ &= J_{n-2} + 2J_{n-3} + \sum_{i=1}^{n-2} J_{i-1}L_{n-1-i} + \sum_{i=1}^{n-3} J_{i-1}L_{n-2-i} \\ &= J_{n-2} + 2J_{n-3} + (J_n - L_{n-1}) + (J_{n-1} - L_{n-2}) \\ &= J_n + 2J_{n-1} - (L_{n-1} + L_{n-2}) = J_{n+1} - L_n,\end{aligned}$$

which completes the induction. A similar argument may be given for (22). ∎

In Section 4, combinatorial proofs are provided for identities (9), (13), (14), (16) and (18), and in the final section, some generalized determinant formulas are found.

We conclude this section with the following combinatorial identities involving sums of products of Jacobsthal numbers with multinomial coefficients which follow from combining formula (8) with Theorems 3.1 and 3.2 above.

Corollary 3.1.

For $n \geq 1$, the following formulas hold:

$$n \sum_{\sigma_n = n} \frac{(-1)^{|s|}}{|s|} m_n(s) J_1^{s_1} J_2^{s_2} \cdots J_n^{s_n} = \begin{cases} -M_n, & \text{if } n \text{ is odd,} \\ -M_{n/2}^2, & \text{if } n \text{ is even,} \end{cases}$$

$$n \sum_{\sigma_n = n} \frac{(-1)^{|s|}}{|s|} m_n(s) j_1^{s_1} j_2^{s_2} \cdots j_n^{s_n} = \begin{cases} -M_n, & \text{if } n \text{ is odd,} \\ -9J_{n/2}^2, & \text{if } n \text{ is even,} \end{cases}$$

$$n \sum_{\sigma_n = n} \frac{(-1)^{|s|}}{|s|} m_n(s) J_0^{s_1} J_1^{s_2} \cdots J_{n-1}^{s_n} = L_n - j_n,$$

$$n \sum_{\sigma_n = n} \frac{1}{|s|} m_n(s) J_1^{s_1} J_2^{s_2} \cdots J_n^{s_n} = (-1)^{n+1} - 2^n + 2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} 3^k,$$

$$n \sum_{\sigma_n = n} \frac{(-1)^{|s|}}{|s|} m_n(s) J_2^{s_1} J_3^{s_2} \cdots J_{n+1}^{s_n} = -j_n,$$

$$n \sum_{\sigma_n = n} \frac{(-1)^{|s|}}{|s|} m_n(s) J_3^{s_1} J_4^{s_2} \cdots J_{n+2}^{s_n} = \begin{cases} -M_{n+1}, & \text{if } n \text{ is odd,} \\ -1, & \text{if } n \text{ is even,} \end{cases}$$

$$n\sum_{\sigma_n=n}\frac{(-1)^{|s|}}{|s|}m_n(s)J_4^{s_1}J_5^{s_2}\cdots J_{n+3}^{s_n}=(-1)^nM_{n+1}-2^n,$$

$$n\sum_{\sigma_n=n}\frac{(-1)^{|s|}}{|s|}m_n(s)J_2^{s_1}J_4^{s_2}\cdots J_{2n}^{s_n}=-M_n^2,$$

$$n\sum_{\sigma_n=n}\frac{1}{|s|}m_n(s)J_2^{s_1}J_4^{s_2}\cdots J_{2n}^{s_n}=(3+\sqrt{5})^n+(3-\sqrt{5})^n-4^n-1,$$

$$n\sum_{\sigma_n=n}\frac{(-1)^{|s|}}{|s|}m_n(s)J_3^{s_1}J_5^{s_2}\cdots J_{2n+1}^{s_n}=M_n-4^n,$$

$$n\sum_{\sigma_n=n}\frac{(-1)^{|s|}}{|s|}m_n(s)J_4^{s_1}J_6^{s_2}\cdots J_{2n+2}^{s_n}=-4^n-1,$$

where $\sigma_n=s_1+2s_2+\cdots+ns_n$, $|s|=s_1+\cdots+s_n$, $m_n(s)=\frac{(s_1+\cdots+s_n)!}{s_1!\cdots s_n!}$ for $s=(s_1,\ldots,s_n)$ and the summation is over all s with non-negative integer components for which $\sigma_n=n$.

4. Combinatorial Proofs

Recall for an $n\times n$ matrix $A=(a_{i,j})$

$$\det(A)=\sum_{\sigma\in\mathcal{S}_n}(-1)^{\mathrm{sgn}(\sigma)}a_{1,\sigma(1)}a_{2,\sigma(2)}\cdots a_{n,\sigma(n)}, \tag{23}$$

where $\mathrm{sgn}(\sigma)$ denotes the sign of the permutation σ. Assume that σ is expressed in standard cycle form wherein the smallest element is first in each cycle and cycles are arranged from left to right in increasing order of first elements. If A is Hessenberg, then only σ in which every cycle comprises an interval of positive integers in increasing order can make a nonzero contribution to the expansion of $\det(A)$ in (23). Given the ordering of cycles, such $\sigma\in\mathcal{S}_n$ may be regarded as compositions of n.

If $a_0=1$ and $k_i=i$ for all i in the Hessenberg matrix H_n defined by (5), then each part of size i is assigned the weight a_i, with an initial part of size i receiving weight ia_i. The product of the weights of all the parts then gives the weight of a composition of n with the sign given by $(-1)^{n-m}$, where m denotes the number of parts. Thus, formula (23) implies $\det(A)$ gives the sum of the (signed) weights of all compositions of n, where the sign and weight are as stated.

We now recall a combinatorial interpretation for the Jacobsthal number implicit in Benjamin and Quinn (2003), Chapter 3, which will be made frequent use of. Consider coverings of the numbers $1,\ldots,n$, written in a row, by indistinguishable squares and dominos, where a square (domino) covers a single number (two adjacent numbers, respectively). Assume that the dominos come in one of two kinds, denoted by d and d', with squares denoted by s. Let $\mathcal{J}_n$ be the set of all such coverings of members of $[n]=\{1,\ldots,n\}$ if $n\geq 1$, with $\mathcal{J}_0$ representing the empty tiling of length zero. Members of $\mathcal{J}_n$ will be referred to as Jacobsthal n-tilings. Upon comparing recurrences and initial values, we have $|\mathcal{J}_n|=J_{n+1}$ for all $n\geq 0$.

Let $\mathcal{B}_n$ denote the set of words of length n in the alphabet $\{0,1\}$. Our proofs below will draw upon various correspondences between $\mathcal{J}_n$ and $\mathcal{B}_n$. In defining these correspondences, it is convenient to classify members of $\mathcal{B}_n$ according to the lengths of certain runs contained therein. Recall that a *run* within a word w is a maximal subsequence of consecutive equal entries of w. An *odd* (*even*) run will refer to one having an odd (even) number of entries. A run occurring at the very beginning (end) of a word will be referred to as being *initial* (*terminal*).

The following combinatorial lemma and its bijection will be used in subsequent proofs.

Lemma 4.1.

If $n \geq 1$, then $J_{n+1} + 2J_{n-1} = j_n$.

Proof:

We first consider the n odd case, where clearly we may assume $n \geq 3$. We define a bijection between $\mathcal{H}_n = \mathcal{J}_n \cup \mathcal{J}_{n-2} \cup \mathcal{J}'_{n-2}$ and $\mathcal{B}_n - \{1^n\}$, where $\mathcal{J}'_{n-2}$ denotes an identical copy of the set $\mathcal{J}_{n-2}$. We first encode $\lambda \in \mathcal{J}_n$ as follows. An initial s, d or d' corresponds to 0, 10 or 11, respectively. For each subsequent s encountered, we start a new run (i.e., put 1 if the current last letter is 0 and 0 if it is 1). For each subsequent d encountered, put 01 if the last letter is 0 and 10 if 1. If d' is encountered, put 00 if the last letter is 0 and 11 if 1. Let $f(\lambda)$ denote the resulting member of $\mathcal{B}_n$. Let $\mathcal{E}_n$ and $\mathcal{O}_n$ denote the subsets of $\mathcal{B}_n$ whose members have terminal run even or odd, respectively. Then one may verify that f defines a bijection between $\mathcal{J}_n$ and $\mathcal{O}_n - \{1^n\}$.

To complete the proof in the odd case, it suffices to define a bijection $g : \mathcal{J}_{n-2} \cup \mathcal{J}'_{n-2} \rightarrow \mathcal{E}_n$. Given $\rho \in \mathcal{J}_{n-2}$, we first apply f to ρ to obtain $\gamma = f(\rho) \in \mathcal{O}_{n-2} - \{1^{n-2}\}$. To γ, we append an extra copy of its final letter and then increase the length of the penultimate run by one by inserting the appropriate letter to obtain $g(\rho)$. (If $\gamma = 0^{n-2}$, then we take 10^{n-1} for $g(\rho)$, in which case $\rho = s(d')^{(n-3)/2}$.) Now let $\rho \in \mathcal{J}'_{n-2}$. In this case, we take $\gamma = f(\rho)$ and add two copies of its final letter to the end. Then we change all the letters within the final run of the current word to the other option except for the first letter, letting $g(\rho)$ denote the resulting word. (Note that $\gamma = 0^{n-2}$ in this case gives $g(\rho) = 01^{n-1}$.) One may verify that g is a bijection. Combining f and g then gives the desired bijection between $\mathcal{H}_n$ and $\mathcal{B}_n - \{1^n\}$.

Now assume $n \geq 2$ is even. In this case, we define a bijection between $\mathcal{H}_n$ and $\mathcal{B}_n \cup \{0^n\}$, where $\{0^n\}$ denotes an additional copy of the element 0^n. Note that in this case the range of f when applied to $\mathcal{J}_n$ yields all members of $\mathcal{O}_n$ as well as 1^n. Define $g : \mathcal{J}_{n-2} \cup \mathcal{J}'_{n-2} \rightarrow \mathcal{E}_n - \{1^n\}$ as before, noting that $\gamma = 1^{n-2}$ is now possible, in which case we let $g(\rho) = 0^n$ (where $\rho = (d')^{(n-2)/2}$). Since $\gamma = 1^{n-2}$ arises twice, the element 0^n has two pre-images under g (and is the only such element). Thus, combining f and g yields the desired bijection, which completes the proof. ∎

We now provide combinatorial proofs of formulas (13), (14), (9), (16) and (18).

Proof of (13)**:**

Let $\mathcal{M}_n$ denote the set of Jacobsthal n-tilings in which positions covered by squares or the right halves of dominos may be circled, the last position is circled, and some position to the left of and including the leftmost circled position is marked. Define the sign of $\rho \in \mathcal{M}_n$ by $(-1)^{n-\mu(\rho)}$, where $\mu(\rho)$ denotes the number of circled positions of ρ. Then, $\det(1; J_2, \ldots, J_{n+1})$ is seen to give the sum of the signs of all members of $\mathcal{M}_n$. Define an involution on $\mathcal{M}_n$ by either circling or removing the circle enclosing the position corresponding to the penultimate tile. Note that this operation is not defined on members of $\mathcal{M}_n$ containing one circled position (i.e., the final one) with the marked position being one that is covered by the final tile. Let $\mathcal{M}_n^*$ denote this subset of $\mathcal{M}_n$. Then, members of $\mathcal{M}_n^*$ each have sign $(-1)^{n-1}$ and

$$|\mathcal{M}_n^*| = J_n + 4J_{n-1} = J_{n+1} + 2J_{n-1},$$

upon considering whether a member of $\mathcal{M}_n^*$ ends in a square or a domino. Thus, by Lemma 4.1, we have that $\det(1; J_2, \ldots, J_{n+1})$ is given by

$$(-1)^{n-1}|\mathcal{M}_n^*| = (-1)^{n-1}2^n - 1 = (-1)^{n-1}j_n,$$

as desired. ■

Proof of (14)**:**

Given $1 \leq \ell \leq n$, let $\mathcal{P}_{n,\ell}$ denote the set of all possible vectors $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ such that the following conditions hold: (i) $\lambda_i \in \mathcal{J}_{n_i}$ where $n_i > 0$ for all i, (ii) $\sum_{i=1}^{\ell} n_i = n + \ell$ and (iii) one of the first $n_1 - 1$ positions within the tiling λ_1 is marked. Define the sign of $\lambda \in \mathcal{P}_{n,\ell}$ by $(-1)^{n-\ell}$ and let $\mathcal{P}_n = \cup_{\ell=1}^{n} \mathcal{P}_{n,\ell}$. Then, it is seen that $\det(1; J_3, \ldots, J_{n+2})$ gives the sum of the signs of all members of $\mathcal{P}_n$, where we may assume $n > 1$ henceforth.

We define an involution on $\mathcal{P}_n$ upon considering several cases as follows. First, suppose that the final component λ_ℓ of λ ends in a square. If $\lambda \in \mathcal{P}_{n,\ell}$ where $\ell \geq 2$ and $\lambda_\ell = \rho s$, then replace λ_ℓ with the two components $\lambda_\ell = \rho,\ \lambda_{\ell+1} = ss$ assuming $|\rho| \geq 2$, and vice versa if the final component of λ is ss. Note that the preceding operation is also defined when $\ell = 1$, provided the penultimate position of λ_1 is not the marked one.

Now suppose λ_ℓ within $\lambda \in \mathcal{P}_{n,\ell}$ where $\ell \geq 1$ ends in a domino (of either kind). Assume further that at least one of the components λ_i of λ is not a tiling of length two consisting of a single d or d'. Let $t \in [\ell]$ denote the largest index i such that $\lambda_i \neq d, d'$. If $2 \leq t < \ell$ and $\lambda_t = \rho s$, then replace the $\lambda_t,\ \lambda_{t+1}$ components of λ where $\lambda_{t+1} = d, d'$ with the single component $\lambda_t = \rho d$ or $\rho d'$, whichever is appropriate, leaving all other components of λ unchanged. Perform the reverse operation of breaking the appropriate tiling into two subtilings if λ_t ends in a domino of either kind (with $t = \ell$ being permitted in this case). Note that these two operations may also be performed when $t = 1$, provided $\ell \geq 2$ concerning the former and the penultimate position within λ_1 is not marked concerning the latter.

So assume $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, where $\ell \geq 1$, $\lambda_1 = \beta d$ or $\beta d'$, the components $\lambda_2, \ldots, \lambda_\ell$ consist of single dominos and $\beta \neq \varnothing$, with the penultimate position of λ_1 marked. If β ends in s and $\ell > 1$, then replace this s with the same type of domino that comprises λ_2 and delete the λ_2 component from λ. On the other hand, if β ends in a domino, then replace this domino by an s and insert a second component consisting of a single domino of the same type. In both operations, the penultimate position within λ_1 is to remain marked in the resulting member of $\mathcal{P}_n$.

Let $\mathcal{P}_n^* \subseteq \mathcal{P}_n$ consist of those $\lambda \in \mathcal{P}_n$ having one of the following three forms: (i) $\lambda \in \mathcal{P}_{n,1}$, with λ_1 ending in s and having its penultimate position marked, (ii) $\lambda \in \mathcal{P}_{n,1}$, with λ_1 ending in sd or sd' and having its penultimate position marked, or (iii) $\lambda \in \mathcal{P}_{n,n}$, with $\lambda_i = d$ or d' for $1 \leq i \leq n$. Then, combining the three pairs of operations defined above is seen to yield a sign-changing involution of $\mathcal{P}_n - \mathcal{P}_n^*$. To complete the proof, we then must determine the sum of the signs of members of $\mathcal{P}_n^*$. Note that there are J_{n+1} and $2J_{n-1}$ possible λ in cases (i) and (ii) above, respectively, each of sign $(-1)^{n-1}$, whereas there are 2^n possible λ in (iii), each having a positive sign. By Lemma 4.1, we then have that $\mathcal{P}_n^*$ has signed cardinality

$$\begin{aligned}(-1)^{n-1}(J_{n+1} + 2J_{n-1}) + 2^n &= (-1)^{n-1}(2^n + (-1)^n) + 2^n \\ &= \begin{cases} 2^{n+1} - 1, & \text{if } n \text{ is odd,} \\ -1, & \text{if } n \text{ is even.} \end{cases}\end{aligned}$$

■

Proof of (9)**:**

Let $\mathcal{Q}_n$ denote the set of Jacobsthal n-tilings in which squares may be circled and ending in a circled square wherein some position to the left of and including the leftmost circled square is marked. Define the sign of $\rho \in \mathcal{Q}_n$ as -1 raised to n minus the number of circled squares of ρ. Then, $\det(1; J_1, \ldots, J_n)$ is seen to give the sum of the signs of all members of $\mathcal{Q}_n$. Define an involution on $\mathcal{Q}_n$ by either circling the rightmost non-terminal square or erasing the circle that encloses it. This operation is not defined on members of $\mathcal{Q}_n$ in which only one square is circled (i.e., the terminal one), with the terminal square the only square that occurs to the right of (possibly coinciding with) the marked position. Denote this excluded subset of $\mathcal{Q}_n$ by $\mathcal{Q}_n^*$. Note that each member of $\mathcal{Q}_n^*$ has sign $(-1)^{n-1}$. To complete the proof, we then must determine $|\mathcal{Q}_n^*|$, for which we consider cases based on the parity of n.

We first treat the n odd case, where we may assume $n \geq 5$. In what follows, we denote a domino that could either be d or d' by D. A domino whose left (right) half corresponds to the marked position will be denoted by D^* (D_*, respectively). A square corresponding to the marked position will be indicated by s^*.

Let $\mathcal{Q}_{n,i}^*$ for $1 \leq i \leq 7$ consist of those $\lambda \in \mathcal{Q}_n^*$ having the following respective forms, where $j, \ell \geq 0$:

$$
\begin{aligned}
(i)\ &\lambda = \lambda' D^* D^j s, \text{ where } \lambda' \neq \varnothing \text{ and } \lambda' \neq (d')^i \text{ for any } i \geq 1,\\
(ii)\ &\lambda = (d')^i D^* D^j s,\ i \geq 1,\\
(iii)\ &\lambda = D^j s^*,\\
(iv)\ &\lambda = D^* D^j s,\\
(v)\ &\lambda = \widetilde{\lambda} s D^j s^*,\\
(vi)\ &\lambda = \lambda' D_* D^j s, \text{ where } \lambda' \text{ is as in } (i),\\
(vii)\ &\lambda = (d')^\ell D_* D^j s.
\end{aligned}
$$

Note that $\widetilde{\lambda}$ in (*v*) is nonempty, by parity.

Now let $\mathcal{B}_{n,i}$ for $1 \leq i \leq 7$ denote the subset of $\mathcal{B}_n$ whose members satisfy the following respective properties P_i:

P_1 : ends in an odd run with at least three odd runs altogether, with the penultimate run even if the final run is of length one,

P_2 : single odd run at end of length ≥ 3, preceded by one or more even runs,

P_3 : single odd run of length one at end,

P_4 : single odd run of the form 1^ℓ followed by even runs (possibly none),

P_5 : ends in one or more even runs, preceded by an odd run (but not a single odd run of the form 1^ℓ comprising all of the remaining letters),

P_6 : has at least three odd runs altogether, with the last run of length one, penultimate run odd and the third rightmost odd run not initial,

P_7 : same as P_6, but containing exactly three odd runs with the leftmost odd run initial.

One may verify that $\mathcal{Q}_n^* = \cup_{i=1}^7 \mathcal{Q}_{n,i}^*$ and $\mathcal{B}_n - \{0^n\} = \cup_{i=1}^7 \mathcal{B}_{n,i}$, with both unions being disjoint.

Suppose $D^* D^j$ in (*i*), (*ii*), (*iv*) and D^j if $j \geq 1$ in (*iii*), (*v*)–(*vii*) are expressed as

$$(d')^{n_0} d^{n_1} (d')^{n_2} \cdots (D)^{n_k},$$

where D depends upon the parity of k, $n_0 \geq 0$ and $n_j \geq 1$ for $1 \leq j \leq k$ if $k \geq 1$ and $n_0 \geq 1$ if $k = 0$. Let f denote the bijection defined on $\mathcal{J}_n$ from the proof of Lemma 4.1 above. Recall that the range of f is given by $\mathcal{O}_n - \{1^n\}$ when n is odd and by $\mathcal{O}_n \cup \{1^n\}$ when n is even. By the *run profile* of $\rho \in \mathcal{B}_n$, we mean the composition of n obtained by considering the lengths of all the runs of ρ.

We now define bijections $f_i : \mathcal{Q}^*_{n,i} \mapsto \mathcal{B}_{n,i}$ for $1 \leq i \leq 7$ as follows.

(*a*) $f_1(\lambda) = f(\lambda')\gamma,$ where γ has run profile $(2n_1, \ldots, 2n_k, 2n_0 + 1)$,

(*b*) $f_2(\lambda) = \gamma,$ where γ has profile $(2n_0, \ldots, 2n_k, 2i + 1)$ and ending in 0 if $n_0 \geq 1$ and profile $(2n_1, \ldots, 2n_k, 2i + 1)$ and ending in 1 if $n_0 = 0$,

(*c*) $f_3(\lambda) = \gamma,$ where $\gamma = 0^{2n_0}1^{2n_1}0^{2n_2} \cdots a^{2n_k}(1 - a)^1$ and $a \equiv k \pmod 2$,

(*d*) $f_4(\lambda) = \gamma,$ where $\gamma = 1^{2n_0+1}0^{2n_1}1^{2n_2} \cdots a^{2n_k}$ and $a \equiv k + 1 \pmod 2$,

(*e*) $f_5(\lambda) = f(\widetilde{\lambda})\gamma,$ where γ has profile $(2n_0 + 2, 2n_1, \ldots, 2n_k)$,

(*f*) $f_6(\lambda) = g(\lambda')\gamma,$ where γ has profile $(2n_1, \ldots, 2n_k, 2n_0 + 1, 1),$ if $D_* = d_*,$ and $h(\lambda')\gamma,$ if $D_* = d'_*$,

(*g*) $f_7(\lambda) = \gamma,$ where γ has profile $(2\ell + 1, 2n_1, \ldots, 2n_k, 2n_0 + 1, 1)$ and the first letter of γ is determined by D_*,

where $g(\lambda')$ in part (*f*) is obtained from λ' by adding a run of length one to the beginning of $f(\lambda')$ and $h(\lambda')$ is obtained by increasing the initial run of $f(\lambda')$ by one. Note that the words $f(\lambda')$ and γ in part (*a*) are understood to be concatenated such that the first letter of γ starts a new run and similarly for γ in (*e*) and (*f*). Also, $n_0 = k = 0$ is possible in parts (*e*)–(*g*), which corresponds to $j = 0$ in (*v*)–(*vii*) above. One may verify that the f_i, $1 \leq i \leq 7$, are indeed bijections and hence

$$|\mathcal{Q}^*_n| = |\mathcal{B}_n - \{0^n\}| = 2^n - 1,$$

if n is odd, as desired.

Now assume $n \geq 4$ is even. In this case, we partition $\mathcal{Q}^*_n$ into subsets $\mathcal{Q}^*_{n,i}$ for $1 \leq i \leq 6$ whose respective forms are given as follows where $j \geq 0$:

$$\begin{aligned}
(i)\ & \lambda = \lambda' D^* D^j s,\\
(ii)\ & \lambda = (d')^i s D^j s^*,\ i \geq 1,\\
(iii)\ & \lambda = s D^j s^*,\\
(iv)\ & \lambda = \widetilde{\lambda} s D^j s^*,\ \text{where } \widetilde{\lambda} \neq \varnothing \text{ and } \widetilde{\lambda} \neq (d')^i \text{ for any } i \geq 1,\\
(v)\ & \lambda = s(d')^\ell D_* D^j s,\ \text{where } 0 \leq \ell \leq (n-4)/2,\\
(vi)\ & \lambda = \widehat{\lambda} D_* D^j s,\ \text{where } \widehat{\lambda} \neq s(d')^\ell \text{ for any } \ell \geq 0.
\end{aligned}$$

Note that λ' in (*i*) and $\widehat{\lambda}$ in (*vi*) are nonempty, by parity.

Let $\mathcal{B}_{n,i}$ for $1 \leq i \leq 6$ consist of those members of $\mathcal{B}_n$ satisfying the following respective properties P_i:

P_1 : ends in an odd run with at least two odd runs altogether, where if the terminal run is of length one, it is preceded by an even run, and where it is not possible for a word to have exactly two odd runs if the first odd run is initial and of the form 1^r for some $r \geq 1$,

P_2 : consists of one or more even runs, followed by an odd run, followed by a 1-run, where the first letter is 0,

P_3 : consists exclusively of even runs and starting with 0,

P_4 : ends in one or more even runs, preceded by an odd run,

P_5 : has run profile $(1, a_1, b_1, \ldots, b_k, a_2, 1)$, where a_1, a_2 are odd, $k \geq 0$ and any b_i are even,

P_6 : has last run of length one, preceded by an odd run, with at least four runs altogether, but not of the form in P_5.

One may verify that the $\mathcal{B}_{n,i}$ for $1 \leq i \leq 6$ partition the set $\mathcal{B}_n - \mathcal{B}'_n$, where $\mathcal{B}'_n \subseteq \mathcal{B}_n$ consists of those binary words satisfying one of the following:

(I) satisfies P_2 above, but with first letter 1,

(*II*) satisfies P_3 above, but with first letter 1,

(*III*) contains exactly two odd runs, occurring at the very beginning and end, with initial run 1^r for some $r \geq 1$ and terminal run of length ≥ 3,

(*IV*) same as in (III), but with terminal run of length one and preceded by at least one even run,

(*V*) has the form $0^{n-1}1$ or $1^{n-1}0$.

Recalling that there are 2^{m-1} compositions of a positive integer m, and making use of halving arguments, it is seen that cases (*I*), (*III*) and (*IV*) above each yield $2^{\frac{n}{2}-1} - 1$ members of $\mathcal{B}'_n$, whereas (*II*) gives rise to $2^{\frac{n}{2}-1}$ members. Thus, we have

$$|\mathcal{B}'_n| = 3(2^{\frac{n}{2}-1} - 1) + 2^{\frac{n}{2}-1} + 2 = 2^{\frac{n}{2}+1} - 1,$$

whence

$$|\mathcal{B}_n - \mathcal{B}'_n| = 2^n - \left(2^{\frac{n}{2}+1} - 1\right) = \left(2^{\frac{n}{2}} - 1\right)^2.$$

Assume that D^*D^j in (*i*) and D^j in (*ii*)–(*vi*) if $j \geq 1$ are given sequentially as above in the odd case by $(d')^{n_0} d^{n_1} (d')^{n_2} \cdots D^k$, where we take $k = n_0 = 0$ if $j = 0$ in (*ii*)–(*vi*). To complete the proof in the even case, it suffices to define bijections $f_i : \mathcal{Q}^*_{n,i} \to \mathcal{B}_{n,i}$ for $1 \leq i \leq 6$. These mappings are given as follows:

(a) use f_1 from the n odd case,

(b) $f_2(\lambda) = 0^{2n_0+2}1^{2n_1}0^{2n_2} \cdots a^{2n_k}(1-a)^{2i-1}a^1$ with $a \equiv k \pmod 2$,

(c) $f_3(\lambda) = 0^{2n_0+2}1^{2n_1}0^{2n_2} \cdots a^{2n_k}$ with $a \equiv k \pmod 2$,

(d) $f_4(\lambda) = f(\widetilde{\lambda})\gamma$, where γ has same run profile as in (c),

(e) $f_5(\lambda) = \gamma$, where γ has profile $(1, 2\ell+1, 2n_1, \ldots, 2n_k, 2n_0+1, 1)$ and first letter is determined by D_*,

(f) use f_6 from the n odd case.

One may verify that f_i in each case yields a bijection between the respective sets, as desired. ∎

Remark 4.1.

Upon considering the number i of dominos occurring in the run directly prior to the terminal square within a member of $\mathcal{Q}_n^*$, one obtains with a combinatorial proof the following identity for $n \geq 1$:

$$\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} 2^i(2i+1)J_{n-2i-1} = \begin{cases} M_n - n2^{(n-1)/2}, & \text{if } n \text{ is odd,} \\ M_{n/2}^2, & \text{if } n \text{ is even.} \end{cases}$$

A comparable proof to the preceding one for (9) may be given for (16), but it is a simpler matter to define a one-to-one correspondence between survivors of a certain involution and the set $\mathcal{Q}_{2n}^*$ from the even case of the preceding proof.

Proof of (16)**:**

Let $\mathcal{R}_n$ denote the set of Jacobsthal $(2n)$-tilings in which squares corresponding to even-numbered positions may be circled and ending in a circled square, wherein an odd-numbered position to the left of the leftmost circled square is marked. Define the sign of $\rho \in \mathcal{R}_n$ by -1 raised to n minus the number of circled squares of ρ. Then $\det(1; J_2, \ldots, J_{2n})$ is seen to give the sum of the signs of all members of $\mathcal{R}_n$. Define an involution on $\mathcal{R}_n$ by either circling or removing the circle enclosing the rightmost non-terminal square corresponding to an even position. This mapping is not defined for $\lambda \in \mathcal{R}_n$ containing only one circled square and of the form $\lambda = \lambda' D^i s D^j s$, where $i, j \geq 0$, λ' ends in a square if nonempty and some odd position within the section $D^i s D^j$ is marked. Denote this subset of $\mathcal{R}_n$ by $\mathcal{R}_n^*$. Each member of $\mathcal{R}_n^*$ has sign $(-1)^{n-1}$ and so to complete the proof, it suffices to define a bijection α between $\mathcal{R}_n^*$ and the set $\mathcal{Q}_{2n}^*$ from the even case of the preceding proof.

Since only odd positions within members of $\mathcal{R}_n$ may be marked, we may regard some tile within the section $D^i s D^j$ of λ as being marked. If the square in $D^i s D^j$ within λ is marked, then let $\alpha(\lambda)$ be the member of $\mathcal{Q}_{2n}^*$ having the same sequence of tiles as λ, but with the final square marked. If a domino in D^j is marked, then let $\alpha(\lambda)$ be the same as λ except that the position corresponding to the left half of this domino is marked (instead of the entire tile). Finally, if a domino in D^i is marked, say the ℓ-th, where $1 \leq \ell \leq i$, then replace the section $D^i s D^j$ within λ by $D^{i-\ell} s D^{\ell+j}$ and mark the position covered by the right half of the ℓ-th domino within $D^{\ell+j}$ to obtain $\alpha(\lambda)$, where all other tiles of λ remain unchanged. Combining the three cases above yields all possible members of $\mathcal{Q}_{2n}^*$ and implies α is the desired bijection. ∎

Similar to the argument for (16), the combinatorial proof for (18) may be shortened by defining a near bijection between the set $\mathcal{R}_n^*$ above and survivors of a certain involution.

Proof of (18)**:**

Let $\mathcal{T}_n$ denote the set of Jacobsthal $(2n)$-tilings in which pieces terminating in even-numbered positions may be circled and ending in a circled piece wherein an odd-numbered position to the

left of (possibly including the left half of) the circled piece is marked. Let the sign of $\rho \in \mathcal{T}_n$ be given by -1 raised to n minus the number of circled pieces of ρ. Then, $\det(1; J_3, \ldots, J_{2n+1})$ gives the sum of the signs of all members of $\mathcal{T}_n$. Consider the rightmost piece ending at $2i$ for some $i < n$ within a member of $\mathcal{T}_n$ and either circling that piece or removing the circle that encloses it. This operation is seen to define a sign-changing involution of $\mathcal{T}_n - \mathcal{T}_n^*$, where $\mathcal{T}_n^* \subseteq \mathcal{T}_n$ consists of those tilings ρ containing only a single circled piece and of one of the following two forms: (i) $\rho = \rho' s D^i s$, where $i \geq 0$ and one of the pieces within sD^i is marked (i.e., covers the marked position), or (ii) $\rho = \rho' D$, where the final D is marked.

To complete the proof, we must determine $|\mathcal{T}_n^*|$. To do so, we define a near bijection β between $\mathcal{R}_n^*$ and $\mathcal{T}_n^*$, where $\mathcal{R}_n^*$ is the set of survivors of the involution used in the proof of (16) above. Let $\lambda = \lambda' D^i s D^j s \in \mathcal{R}_n^*$, where $i, j \geq 0$, λ' ends in a square if nonempty and some piece within the section $D^i s D^j$ is marked. If the s or a D within D^j is marked, then let $\beta(\lambda) = \lambda$, which yields all members of $\mathcal{T}_n^*$ of the form (i) above. On the other hand, if some D within the run D^i is marked, then let $D^i = D^{i_1} \widetilde{D} D^{i_2}$, where $i_1, i_2 \geq 0$ and $\widetilde{D}$ denotes the marked domino. Let $\beta(\lambda)$ in this case be given by $\beta(\lambda) = \lambda' D^{i_1} s D^{i_2} s D^j \widetilde{D}$. Note that this yields in a one-to-one fashion all members of $\mathcal{T}_n^*$ of the form (ii) above except for those containing no squares, of which there are 2^n. Combining the two cases of β above then implies

$$|\mathcal{T}_n^*| = |\mathcal{R}_n^*| + 2^n = (2^n - 1)^2 + 2^n = 4^n - 2^n + 1,$$

as desired. ∎

Remark 4.2.

Determining the cardinalities of the survivor sets $\mathcal{R}_n^*$ and $\mathcal{T}_n^*$ in a different way in the proofs of (16) and (18) above leads to the following further Jacobsthal identities for $n \geq 1$:

$$\sum_{i=1}^{n-1} 2^{i-1} i^2 J_{2n-2i} = M_n^2 - 2^{n-1} n^2,$$

and

$$\sum_{i=1}^{n} 2^{i-1} i J_{2n-2i+1} = J_{2n+1} - 2^n.$$

5. General Determinant Formulas

In this section, we establish some general formulas involving determinants of Hessenberg matrices whose nonzero entries are given by a certain second-order recurrence. Let w_n be defined recursively by

$$w_n = \left(2^\ell + (-1)^\ell\right) w_{n-1} - (-2)^\ell w_{n-2}, \qquad n \geq 2, \tag{24}$$

with $w_0 = a$ and $w_1 = b$, where ℓ is a positive integer and a and b are arbitrary. Note when $\ell = 1$ that J_n corresponds to the $a = 0, b = 1$ case and j_n to the $a = 2, b = 1$ case of w_n. We have the following formula for determinants involving w_n.

Theorem 5.1.

If $n \geq 1$, then

$$(-1)^n \det(a_0; w_1, w_2, \ldots, w_n) = \alpha_1^n + \alpha_2^n - \left(2^\ell a_0\right)^n - \left((-1)^\ell a_0\right)^n, \tag{25}$$

where α_1 and α_2 are given by

$$\alpha_1, \alpha_2 = \frac{\left(2^\ell + (-1)^\ell\right)a_0 - b \pm \sqrt{\left(\left(2^\ell + (-1)^\ell\right)a_0 - b\right)^2 - 4(-2)^\ell a_0(a_0 - a)}}{2}.$$

Proof:

We compute the generating function

$$f(x) = \sum_{n\geq 1} \det(a_0; w_1, \ldots, w_n)x^n.$$

First, note that recurrence (7) when $k_n = n$ may be rewritten equivalently in terms of generating functions as

$$f(x) = -\frac{h(-a_0 x)}{a_0 + g(-a_0 x)}, \tag{26}$$

where $g(x) = \sum_{n\geq 1} w_n x^n$ and $h(x) = \sum_{n\geq 1} n w_n x^n = x\frac{d}{dx}g(x)$. To find $g(x)$, first note that w_n is given explicitly by

$$w_n = \frac{1}{2^\ell - (-1)^\ell}\left(\left(b - (-1)^\ell a\right)2^{\ell n} + \left(2^\ell a - b\right)(-1)^{\ell n}\right), \qquad n \geq 0.$$

Therefore, we get

$$\begin{aligned} g(x) &= \frac{1}{2^\ell - (-1)^\ell}\sum_{n\geq 1}\left(\left(b - (-1)^\ell a\right)2^{\ell n} + \left(2^\ell a - b\right)(-1)^{\ell n}\right)x^n \\ &= \frac{1}{2^\ell - (-1)^\ell}\left(\frac{\left(b - (-1)^\ell a\right)2^\ell x}{1 - 2^\ell x} + \frac{\left(2^\ell a - b\right)(-1)^\ell x}{1 - (-1)^\ell x}\right) \\ &= \frac{x\left(b - (-2)^\ell a x\right)}{\left(1 - 2^\ell x\right)\left(1 - (-1)^\ell x\right)}, \end{aligned}$$

and thus

$$\begin{aligned} h(x) &= x\left(\frac{b - 2a(-2)^\ell x}{\left(1 - 2^\ell x\right)\left(1 - (-1)^\ell x\right)} + \frac{2^\ell x\left(b - a(-2)^\ell x\right)}{\left(1 - 2^\ell x\right)^2\left(1 - (-1)^\ell x\right)} + \frac{(-1)^\ell x\left(b - a(-2)^\ell x\right)}{\left(1 - 2^\ell x\right)\left(1 - (-1)^\ell x\right)^2}\right) \\ &= \frac{x\left(b + (-2)^{\ell+1}a x + \left((-4)^\ell a + 2^\ell a - (-2)^\ell b\right)x^2\right)}{\left(1 - 2^\ell x\right)^2\left(1 - (-1)^\ell x\right)^2}. \end{aligned}$$

By (26), we then have

$$f(x) = \frac{x\left(b - (-2)^{\ell+1}a a_0 x + \left((-4)^\ell a + 2^\ell a - (-2)^\ell b\right)a_0^2 x^2\right)}{\left(1 + 2^\ell a_0 x\right)\left(1 + (-1)^\ell a_0 x\right)\left(1 + \left(2^\ell a_0 + (-1)^\ell a_0 - b\right)x + (-2)^\ell a_0(a_0 - a)x^2\right)}. \tag{27}$$

By partial fractions, Equation (27) may be written as

$$f(x) = \frac{A}{1+2^\ell a_0 x} + \frac{B}{1+(-1)^\ell a_0 x} + \frac{C}{1+\alpha_1 x} + \frac{D}{1+\alpha_2 x}, \tag{28}$$

for some constants A, B, C, D, where α_1 and α_2 are as given above and are assumed for now to be distinct. Clearing fractions in (28), and taking $x = -\frac{1}{2^\ell a_0}$, implies

$$\begin{aligned} A &= -\frac{\frac{1}{2^\ell a_0}\left(b + 2a(-1)^{\ell+1} + a(-1)^\ell + a\left(\frac{1}{2}\right)^\ell - b\left(-\frac{1}{2}\right)^\ell\right)}{\left(1-\left(-\frac{1}{2}\right)^\ell\right)\left(1-\left(1+\left(-\frac{1}{2}\right)^\ell - \frac{b}{2^\ell a_0}\right) + \left(-\frac{1}{2}\right)^\ell\left(1-\frac{a}{a_0}\right)\right)} \\ &= -\frac{b - a(-1)^\ell - \left(-\frac{1}{2}\right)^\ell\left(b - a(-1)^\ell\right)}{\left(1-\left(-\frac{1}{2}\right)^\ell\right)\left(b - a(-1)^\ell\right)} = -1. \end{aligned}$$

Similarly, taking $x = \frac{(-1)^{\ell+1}}{a_0}$ implies $B = -1$. To find C and D at this point, it is simplest to substitute $A = B = -1$ into (28), clear fractions and equate coefficients of x^0 and x^3 on both sides of the resulting equation. This gives $C + D = 2$ and

$$\left(a(-4)^\ell + a2^\ell - b(-2)^\ell\right)a_0^2 + (-2)^\ell\left(2^\ell + (-1)^\ell\right)a_0^2(a_0 - a) = (\alpha_1 D + \alpha_2 C)(-2)^\ell a_0^2,$$

i.e.,

$$\begin{aligned} (-4)^\ell a_0 + 2^\ell a_0 - b(-2)^\ell = &\left(\frac{C+D}{2}\right)\left(2^\ell a_0 + (-1)^\ell a_0 - b\right)(-2)^\ell \\ &+ \left(\frac{D-C}{2}\right)(-2)^\ell a_0^2\sqrt{\left(2^\ell a_0 + (-1)^\ell a_0 - b\right)^2 - 4(-2)^\ell a_0(a_0 - a)}. \end{aligned}$$

Since $C + D = 2$, the latter equation implies $D - C = 0$, i.e., $C = D = 1$. Therefore, we have

$$f(x) = \frac{1}{1+\alpha_1 x} + \frac{1}{1+\alpha_2 x} - \frac{1}{1+2^\ell a_0 x} - \frac{1}{1+(-1)^\ell a_0 x}. \tag{29}$$

On the other hand, if $\alpha_1 = \alpha_2$, then

$$f(x) = \frac{A}{1+2^\ell a_0 x} + \frac{B}{1+(-1)^\ell a_0 x} + \frac{C}{1+\alpha_1 x} + \frac{D}{(1+\alpha_1 x)^2},$$

where

$$\alpha_1 = \frac{\left(2^\ell + (-1)^\ell\right)a_0 - b}{2}.$$

Then $A = B = -1$ as before and clearing fractions leads to $C + D = 2$ and

$$\begin{aligned} &\left(a(-4)^\ell + a2^\ell - b(-2)^\ell\right)a_0^2 + (-2)^\ell\left(2^\ell + (-1)^\ell\right)a_0^2(a_0 - a) \\ &\qquad = \alpha_1(-2)^\ell a_0^2 C = \frac{(-2)^\ell a_0^2 C}{2}\left(2^\ell a_0 + (-1)^\ell a_0 - b\right). \end{aligned}$$

The last equation implies $C = 2$ and hence $D = 0$. Thus formula (29) is seen to hold also in the case when $\alpha_1 = \alpha_2$. Extracting the coefficient of x^n in (29) then yields (25). ∎

For example, taking $a = 0$ and $a_0 = b = \ell = 1$ in (25) gives

$$(-1)^n \det(1; J_1, J_2, \ldots, J_n) = (\sqrt{2})^n + (-\sqrt{2})^n - 2^n - (-1)^n,$$

which may be rewritten as (9). Similarly, taking $a = 2$ and $a_0 = b = \ell = 1$ in (25) gives

$$(-1)^n \det(1; j_1, j_2, \ldots, j_n) = (\sqrt{2}i)^n + (-\sqrt{2}i)^n - 2^n - (-1)^n,$$

where $i = \sqrt{-1}$, which implies (10). The formulas in Theorem 3.2 above may also be deduced in a similar fashion.

The preceding result may be specialized to subsequences of J_n whose subscripts form an arithmetic progression as follows. Let $\ell \geq 1$ and c be fixed integers. Note first that the sequence $w_n = J_{n\ell+c}$ satisfies recurrence (24) for all *integers* n. To realize this in the case when $c = 0$ or $c = 1$, first note that the sequence $J_{n\ell+c}$ for such c satisfies (24) for all $n \geq 2$, upon using the explicit formula for J_n. It also satisfies (24) for all integers $n \leq 1$, upon observing $J_{-n} = \frac{(-1)^{n-1}J_n}{2^n}$ for $n \geq 0$, where it is understood that J_m for negative indices m are obtained by applying $J_{n-2} = \frac{1}{2}(J_n - J_{n-1})$ repeatedly for $n = 1, 0, -1, \ldots$. Once it is established that $J_{n\ell+c}$ satisfies (24) if $c = 0, 1$, the case for general c follows from using $J_n = J_{n-1} + 2J_{n-2}$ to prove it for subsequently larger (and smaller) c. Note that the initial term J_c when $c < 0$ may be obtained alternatively from (3), as it is seen to hold also when n is negative.

Thus, taking $a = J_c$ and $b = J_{\ell+c}$ in Theorem 5.1 yields a comparable closed-form expression for $\det(a_0; J_{\ell+c}, J_{2\ell+c}, \ldots, J_{n\ell+c})$ for all $n \geq 1$. For example, if $c = 0$, then we get

$$(-1)^n \det(1; J_\ell, J_{2\ell}, \ldots, J_{n\ell}) = \lambda_1^n + \lambda_2^n - 2^{\ell n} - (-1)^{\ell n}, \tag{30}$$

where λ_1, λ_2 are given by

$$\lambda_1, \lambda_2 = \frac{2^\ell + (-1)^\ell - J_\ell \pm \sqrt{(2^\ell + (-1)^\ell - J_\ell)^2 - 4(-2)^\ell}}{2}.$$

Letting $\ell = 1$ in (30) gives (9). Upon taking $c = 1$, a similar formula can be given for $\det(1; J_{\ell+1}, J_{2\ell+1}, \ldots, J_{n\ell+1})$. If $c = -\ell$, one gets

$$(-1)^n \det(1; 0, J_\ell, \ldots, J_{(n-1)\ell}) = \theta_1^n + \theta_2^n - 2^{\ell n} - (-1)^{\ell n}, \tag{31}$$

where θ_1, θ_2 are given by

$$\theta_1, \theta_2 = \frac{2^\ell + (-1)^\ell \pm \sqrt{4^\ell + (-2)^{\ell+1} - 4J_\ell + 1}}{2}.$$

Recalling $j_n = 2^n + (-1)^n$ and $L_n = \rho^n + (-1/\rho)^n$, where $\rho = \frac{1+\sqrt{5}}{2}$, it is seen that the $\ell = 1$ case of (31) corresponds to (11) above.

We conclude with a further general result. Let v_n denote a generalized Jacobsthal sequence defined by $v_n = v_{n-1} + 2v_{n-2}$ if $n \geq 2$, with $v_0 = a$ and $v_1 = b$, where a and b are arbitrary. Consider the Hessenberg matrix H_n defined by (5) with $k_i = i$, wherein a_i for $i \geq 1$ corresponds to a subsequence of v_n whose subscripts are evenly spaced and two adjacent terms of the original sequence v_n are specified (rather than of the subsequence as discussed above). Then, there is the following explicit formula for $\det(H_n)$ in this case.

Theorem 5.2.

Let $\ell \geq 1$ and c be integers. Then

$$(-1)^n \det(a_0; v_{\ell+c}, v_{2\ell+c}, \ldots, v_{n\ell+c}) = \beta_1^n + \beta_2^n - (2^\ell a_0)^n - \left((-1)^\ell a_0\right)^n, \qquad n \geq 1, \tag{32}$$

where β_1, β_2 are the zeros of the polynomial

$$x^2 + \frac{1}{3}\left((a+b)2^{\ell+c} + (2a-b)(-1)^{\ell+c} - 3\left(2^\ell + (-1)^\ell\right)a_0\right)x - \frac{(-2)^\ell a_0}{3}\left((a+b)2^c + (2a-b)(-1)^c - 3a_0\right).$$

Proof:

First note that the initial conditions imply v_n is given explicitly by

$$v_n = \frac{a+b}{3}\cdot 2^n + \frac{2a-b}{3}\cdot(-1)^n, \qquad n \in \mathbb{Z}.$$

Let $g(x) = \sum_{n\geq 1} v_{n\ell+c}x^n$ and $h(x) = \sum_{n\geq 1} n v_{n\ell+c}x^n = x\frac{d}{dx}g(x)$. Then we have

$$\begin{aligned} g(x) &= \frac{a+b}{3}\sum_{n\geq 1} 2^{n\ell+c}x^n + \frac{2a-b}{3}\sum_{n\geq 1}(-1)^{n\ell+c}x^n \\ &= \frac{a+b}{3}\cdot\frac{2^{\ell+c}x}{1-2^\ell x} + \frac{2a-b}{3}\cdot\frac{(-1)^{\ell+c}x}{1-(-1)^\ell x} \\ &= \frac{\left((a+b)2^{\ell+c} + (2a-b)(-1)^{\ell+c}\right)x - \left((a+b)(-1)^\ell 2^{\ell+c} + (2a-b)(-1)^{\ell+c}2^\ell\right)x^2}{3\left(1-2^\ell x\right)\left(1-(-1)^\ell x\right)}, \end{aligned}$$

and

$$\begin{aligned} h(x) &= \frac{a+b}{3}\cdot\frac{2^{\ell+c}x}{\left(1-2^\ell x\right)^2} + \frac{2a-b}{3}\cdot\frac{(-1)^{\ell+c}x}{\left(1-(-1)^\ell x\right)^2} \\ &= \frac{(a+b)2^{\ell+c}x\left(1-(-1)^\ell x\right)^2 + (2a-b)(-1)^{\ell+c}x\left(1-2^\ell x\right)^2}{3\left(1-2^\ell x\right)^2\left(1-(-1)^\ell x\right)^2}. \end{aligned}$$

Let $f(x) = \sum_{n\geq 1}\det(a_0; v_{\ell+c}, v_{2\ell+c}, \ldots, v_{n\ell+c})x^n$. By (26), we have

$$f(x) = \frac{\frac{x}{3}u(x)}{\left(1+2^\ell a_0 x\right)\left(1+(-1)^\ell a_0 x\right)v(x)},$$

where $u(x)$ and $v(x)$ are given by

$$u(x) = (a+b)2^{\ell+c}\left(1+(-1)^\ell a_0 x\right)^2 + (2a-b)(-1)^{\ell+c}\left(1+2^\ell a_0 x\right)^2,$$

and

$$v(x) = \left(1+2^\ell a_0 x\right)\left(1+(-1)^\ell a_0 x\right) - \frac{1}{3}\left((a+b)2^{\ell+c} + (2a-b)(-1)^{\ell+c}\right)x - \frac{1}{3}\left((a+b)(-1)^\ell 2^{\ell+c} + (2a-b)(-1)^{\ell+c}2^\ell\right)a_0 x^2.$$

Upon proceeding as before using partial fractions (the details of which are left to the reader), we have

$$f(x) = \frac{1}{1+\beta_1 x} + \frac{1}{1+\beta_2 x} - \frac{1}{1+2^\ell a_0 x} - \frac{1}{1+(-1)^\ell a_0 x},$$

where β_1 and β_2 are as given and not necessarily distinct. Extracting the coefficient of x^n in the last expression completes the proof. ∎

For example, taking $a = \ell = 2$, $a_0 = b = 1$ and $c = 0$ in Theorem 5.2 gives

$$(-1)^n \det(1; j_2, j_4, \ldots, j_{2n}) = 2^n + (-2)^n - 4^n - 1,$$

which may be written as

$$\det(1; j_2, j_4, \ldots, j_{2n}) = \begin{cases} 4^n + 1, & \text{if } n \text{ is odd,} \\ -M_n^2, & \text{if } n \text{ is even.} \end{cases}$$

Assuming all the same values of the parameters as before but instead taking $a = 0$ gives

$$(-1)^n \det(1; J_2, J_4, \ldots, J_{2n}) = 2^{n+1} - 4^n - 1,$$

which implies formula (16) above.

6. Conclusion

In this paper, we have evaluated the determinants of several Hessenberg matrices of the form (5) in which $k_i = i$ and whose nonzero entries correspond to a subsequence of the Jacobsthal numbers. As a consequence, some new connections are made between the Jacobsthal and other second-order linearly recurrent sequences, such as the Mersenne, Lucas and Jacobsthal-Lucas numbers. By the generalized Trudi formula, these determinant identities may be viewed explicitly as sums of products of multinomial coefficients with certain translates of the Jacobsthal sequence.

Combinatorial proofs are given for several of these identities which make use of sign-changing involutions and the definition of the determinant as a signed sum over the set of permutations of a given length. In the process, we found some one-to-one correspondences between various subsets of the binary words and Jacobsthal tilings of a given length; in particular, the explicit formula for j_n is afforded a bijective explanation which accounts for the $(-1)^n$ term. Finally, our results for Jacobsthal determinants are extended to sequences satisfying a more general recurrence and/or initial condition. As a consequence, one obtains comparable formulas for determinants involving Jacobsthal and Jacobsthal-Lucas subsequences whose indices form an arbitrary arithmetic progression. In future work, one might consider determinants of matrices given by (5) wherein k_i is a different sequence or assumes a more general form (i.e., k_i itself may be defined by a general recurrence). Further, one could consider additional classes of recurrent sequences for the a_i in conjunction with the case $k_i = i$ in (5).

REFERENCES

Akbulak, M. and Öteleş, A. (2014). On the sums of Pell and Jacobsthal numbers by matrix method, Bull. Iranian Math. Soc., Vol. 40, No. 4, pp. 1017–1025.

Aktaş, İ. and Köse, H. (2015). Hessenberg matrices and the Pell-Lucas and Jacobsthal numbers, Int. J. Pure Appl. Math., Vol. 101, No. 3, pp. 425–432. https://doi.org/10.12732/ijpam.v101i3.11

Barry, P. (2016). Jacobsthal decompositions of Pascal's triangle, ternary trees, and alternating sign matrices, J. Integer Seq., Vol. 19, Article 16.3.5.

Benjamin, A.T. and Quinn, J.J. (2003). *Proofs that Really Count: The Art of Combinatorial Proof*, Mathematical Association of America, Washington, DC.

Bruhn, H., Gellert, L. and Günther, J. (2015). Jacobsthal numbers in generalised Petersen graphs, Electron. Notes Discrete Math., Vol. 49, pp. 465–472. https://doi.org/10.1016/j.endm.2015.06.065

Cerin, Z. (2007). Sums of squares and products of Jacobsthal numbers, J. Integer Seq., Vol. 10, Article 07.2.5.

Cılasun, M.H. (2016). Generalized multiple counting Jacobsthal sequences of Fermat pseudo-primes, J. Integer Seq., Vol. 19, Article 16.2.3.

Cook, C.K. and Bacon, M.R. (2013). Some identities for Jacobsthal and Jacobsthal-Lucas numbers satisfying higher order recurrence relations, Ann. Math. Inform., Vol. 41, pp. 27–39.

Daşdemir, A. (2019). Mersenne, Jacobsthal, and Jacobsthal-Lucas numbers with negative subscripts, Acta Math. Univ. Comenianae, Vol. 88, No. 1, pp. 145–156.

Deveci, Ö. and Artun, G. (2018). On the adjacency-Jacobsthal numbers, Comm. Algebra, Vol. 47, No. 11, pp. 4520–4532. http://doi.org/10.1080/00927872.2018.1541464

Frey, D.D. and Sellers, J.A. (2000). Jacobsthal numbers and alternating sign matrices, J. Integer Seq., Vol. 3, Article 0.

Goy, T. (2018). On determinants and permanents of some Toeplitz-Hessenberg matrices whose entries are Jacobsthal numbers, Eurasian Math. J., Vol. 9, No. 4, pp. 61–67. https://doi.org/10.32523/2077-9879-2018-9-4-61-67

Goy, T. (2019). Pell numbers identities from Toeplitz-Hessenberg determinants, Novi Sad J. Math., Vol. 49, No. 2, pp. 87–94. https://doi.org/10.30755/NSJOM.08406

Goy, T. (2020). Jacobsthal number identities using the generalized Brioschi formula, In: Proc. of 8th Intern. Conf. "Mathematics in Modern Technical University" (Kyiv, December, 27–28, 2019), pp. 13–17.

Goy, T. and Shattuck, M. (2019a). Determinant formulas of some Toeplitz-Hessenberg matrices with Catalan entries, Proc. Indian Acad. Sci. Math. Sci., Vol. 129, Article 46. https://doi.org/10.1007/s12044-019-0513-9

Goy, T. and Shattuck, M. (2019b). Fibonacci and Lucas identities using Toeplitz-Hessenberg matrices, Appl. Appl. Math., Vol. 14, No. 2, pp. 699–715.

Goy, T. and Shattuck, M. (2020a). Determinant identities for Toeplitz-Hessenberg matrices with tribonacci number entries, Trans. Comb., Vol. 9, No. 3, pp. 89–109. https://doi.org/10.22108/TOC.2020.116257.1631

Goy, T. and Shattuck, M. (2020b). Fibonacci-Lucas identities and the generalized Trudi formula, Notes Number Theory Discrete Math., Vol. 26, No. 3, pp. 203–217. https://doi.org/10.7546/nntdm.2020.26.3.203-217

Goy, T. and Shattuck, M. (2020c). Some Toeplitz-Hessenberg determinant identities for the tetranacci numbers, J. Integer Seq., Vol. 23, Article 20.6.8.

Heubach, S. (1999). Tiling an m-by-n area with squares of size up to k-by-k $(m \leq 5)$, Congr. Numer., Vol. 140, pp. 43–64.

Horadam, A.F. (1986). Jacobsthal representation numbers, Fibonacci Quart., Vol. 34, No. 1, pp. 40–54.

Horadam, A.F. (1988). Jacobsthal and Pell curves, Fibonacci Quart., Vol. 34, No. 1, pp. 77–83.

Köken, F. and Bozkurt, D. (2008). On the Jacobsthal numbers by matrix methods, Int. J. Contemp. Math. Sci., Vol. 34, No. 3, pp. 605–614.

Koshy, T. (2019). *Fibonacci and Lucas Numbers with Applications, Vol. 2*, Wiley, Hoboken, New Jersey.

Muir, T. (1960). *The Theory of Determinants in the Historical Order of Development, Vol. 3*, Dover Publications, New York.

Öteleş, A., Karatas, Z.Y. and Zangana, D.O.M. (2018). Jacobsthal numbers and associated Hessenberg matrices, J. Integer Seq., Vol. 21, Article 18.2.5.

Ramirez, J.L. and Shattuck, M. (2019). Generalized Jacobsthal numbers and restricted k-ary words, Pure Math. Appl. (PU.M.A.), Vol. 28, No. 1, pp. 91–108. http://doi.org/10.1515/puma-2015-0034

Sloane, N.J.A., ed. (2020). *The On-Line Encyclopedia of Integer Sequences*, available electronically at https://oeis.org.

Uygun, Ş. (2017). The combinatorial representation of Jacobsthal and Jacobsthal-Lucas matrix sequences, Ars Combin., Vol. 135, pp. 83–92.

Yılmaz, F. and Bozkurt, D. (2012). The adjacency matrix of one type of directed graph and the Jacobsthal numbers and their determinantal representation, J. Appl. Math., Vol. 2012, Article ID 423163. http://doi.org/10.1155/2012/423163