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PATTABIRAMAN K.[1,2]

INVERSE SUM INDEG COINDEX OF GRAPHS

The inverse sum indeg coindex $\overline{ISI}(G)$ of a simple connected graph G is defined as the sum of the terms $\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)}$ over all edges uv not in G, where $d_G(u)$ denotes the degree of a vertex u of G. In this paper, we present the upper bounds on inverse sum indeg coindex of edge corona product graph and Mycielskian graph. In addition, we obtain the exact value of both inverse sum indeg index and its coindex of a double graph.

Key words and phrases: inverse sum indeg index, edge corona graph, Mycielskian graph, double graph.

[1] Annamalai University, Annamalainagar, 608002, Tamil Nadu, India
[2] Department of Mathematics, Government Arts College (Autonomous), 612002, Kumbakonam, India
E-mail: pramank@gmail.com

INTRODUCTION

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum vertex degrees of G, respectively. A *topological index* or molecular descriptor of a graph is a parameter related to the graph; it does not depend on labeling or pictorial representation of the graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds. Several types of such indices exist, especially those based on vertex and edge distances.

Molecular descriptors, results of functions mapping molecule's chemical information into a number [16], have found applications in modeling many physicochemical properties in QSAR and QSPR studies [8, 6]. A particularly common type of molecular descriptors are those that are defined as functions of the structure of the underlying molecular graph, such as the Wiener index [18], the Zagreb indices [4], the Randić index [14] or the Balaban J-index [5]. Damir Vukicević and Marija Gasperov [17] observed that many of these descritors are defined simply as the sum of individual bond contributions.

Among the 148 discrete Adriatic indices studied in [17], whose predictive properties were evaluated against the benchmark datasets of the Internation Academy of Mathematical Chemistry [7], 20 indices were selected as significant predictors of physicochemical properties. In this connection, Sedlar et al. [15] studied the properties of the inverse sum indeg index, the descriptor that was selected in [17] as a significant predictor of total surface area of octane isomers and for which the extremal graphs obtained with the help of Math. Chem. have a particularly simple and elegant structure. The *inverse sum indeg index* is defined as

$$ISI(G) = \sum_{uv\in E(G)} \frac{1}{\frac{1}{d_G(u)} + \frac{1}{d_G(v)}} = \sum_{uv\in E(G)} \frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)}.$$

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The *first Zagreb index* $M_1(G)$ is the equal to the sum of the squares of the degrees of the vertices, and the *second Zagreb index* $M_2(G)$ is the equal to the sum of the products of the degrees of pairs of adjacent vertices, that is, $M_1(G) = \sum\limits_{u\in V(G)} d_G^2(u) = \sum\limits_{uv\in E(G)} (d_G(u) + d_G(v))$, $M_2(G) = \sum\limits_{uv\in E(G)} d_G(u)d_G(v)$, where $d_G(v)$ is a degree of a vertex v in G. For a connected graph G, the *harmonic index* $H(G)$ is defined as $H(G) = \sum\limits_{uv\in E(G)} \frac{2}{d_G(u)+d_G(v)}$.

The *first* and *second Zagreb coindices* are defined as $\overline{M}_1(G) = \sum\limits_{uv\notin E(G)} (d_G(u) + d_G(v))$, $\overline{M}_2(G) = \sum\limits_{uv\notin E(G)} d_G(u)d_G(v)$. Similarly, the *harmonic* coindex of G is defined as

$$\overline{H}(G) = \sum_{uv\notin E(G)} \frac{2}{d_G(u) + d_G(v)}.$$

Motivated by the invariants like Zagreb and harmonic indices, we proposed the another invariant *inverse sum indeg coindex* as

$$\overline{ISI}(G) = \sum_{uv\notin E(G)} \frac{d_G(u)d_G(v)}{d_G(u) + d_G(v)}.$$

Extremal values of inverse sum indeg index across several graph classes, including connected graphs, chemical graphs, trees and chemical trees were determined in [15]. The bounds of a descriptor are important information of a molecular graph in the sense that they establish the approximate range of the descriptor in terms of molecular structural parameters. In [2], some sharp bounds for the inverse sum indeg index of connected graphs are given. The inverse sum indeg index of some nanotubes is computed in [3]. Several upper and lower bounds on the inverse sum indeg index in terms of some molecular structural parameters and relate this index to various well-known molecular descriptors are presented in [12]. In this paper, we present the upper bounds on the inverse sum indeg coindex of edge corona product graph and Mycielskian graph. In addition, we obtain the exact value of both inverse sum indeg index and its coindex of double graph.

1 EDGE CORONA

Hou and Shiu [5] introduced a kind of new graph operation, namely, edge corona product. The *edge corona product* $G \bullet H$ of G and H is defined as the graph obtained by taking one copy of G and $|E(G)|$ copies of H, and then joining two end vertices of the i^{th} edge of G to every vertex in the i^{th} copy of H. The computation for some of the topological indices of edge corona product are resently studied in [1, 13, 5].

Lemma 1 ([9]). *Let f be a convex function on the interval I and $x_1, x_2, \ldots, x_n \in I$. Then $f\left(\frac{x_1+x_2+\ldots+x_n}{n}\right) \le \frac{f(x_1)+f(x_2)+\ldots f(x_n)}{n}$, with equality if and only if $x_1 = x_2 = \ldots = x_n$.*

Theorem 1. *Let G_1 and G_2 be two graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively.*

Then

$$\begin{aligned}\overline{ISI}(G_1 \bullet G_2) &\leq (n_2+1)\overline{ISI}(G_1) + \frac{m_1}{4}\Big(\overline{ISI}(G_2) + 2\overline{H}(G_2) + \frac{\overline{M_2}(G_2)}{4} + \frac{\overline{M_1}(G_2)}{2} \\ &+ \frac{3n_2(n_2-1)}{8} - \frac{3m_2}{4}\Big) + \frac{n_2(n_2+1)}{4}\Big(2m_1n_1 - M_1(G)\Big) \\ &+ \frac{(m_2+n_2)(n_1^2-2m_1)}{2} + \frac{m_1(m_1-1)n_2^2(\Delta(G_2)+2)^2}{4(\delta(G_2)+2)}.\end{aligned}$$

Proof. Let x_{ij} be the jth vertex in the ith copy of H, $i \in \{1,2,\ldots,m_1\}$,$j \in \{1,2,\ldots,n_2\}$, and let y_k be the kth in G_1,$k \in \{1,2,\ldots,n_1\}$. Also let x_j be the jth vertex in G_2.

By the definition of edge corona of G_1 and G_2, for each vertex x_{ij}, we have $d_{G_1 \bullet G_2}(x_{ij}) = d_{G_2}(x_j) + 2$, and for every vertex y_k in G_1, $d_{G_1 \bullet G_2}(y_k) = d_{G_1}(y_k)n_2 + d_{G_1}(y_k) = (n_2+1)d_{G_1}(y_k)$.

Now, we consider the following four cases of nonadjacent vertex pairs in $G_1 \bullet G_2$.

Case 1: The nonadjacent vertex pairs $\{x_{ij}, x_{ih}\}$, $1 \leq i \leq m_1$,$1 \leq j < h \leq n_2$, and it is assumed that $x_jx_h \notin E(G_2)$.

$$\begin{aligned}C_1 &= \sum_{i=1}^{m_1} \sum_{x_{ij}x_{ih} \notin E(G_1 \bullet G_2)} \frac{d_{G_1 \bullet G_2}(x_{ij})d_{G_1 \bullet G_2}(x_{ih})}{d_{G_1 \bullet G_2}(x_{ij}) + d_{G_1 \bullet G_2}(x_{ih})} \\ &= \sum_{i=1}^{m_1} \sum_{x_jx_h \notin E(G_2)} \frac{(d_{G_2}(x_j)+2)(d_{G_2}(x_h)+2)}{d_{G_2}(x_j) + d_{G_2}(x_h) + 4}.\end{aligned}$$

By Lemma 1, we have $\frac{1}{d_{G_2}(x_j)+d_{G_2}(x_h)+4} \leq \frac{1}{4(d_{G_2}(x_j)+d_{G_2}(x_h))} + \frac{1}{16}$ with equality if and only if $d_{G_2}(x_j) + d_{G_2}(x_h) = 4$. Thus,

$$\begin{aligned}C_1 &\leq \frac{1}{4}\sum_{i=1}^{m_1} \sum_{x_jx_h \notin E(G_2)} \Big(\frac{(d_{G_2}(x_j)+2)(d_{G_2}(x_h)+2)}{d_{G_2}(x_j)+d_{G_2}(x_h)} + \frac{(d_{G_2}(x_j)+2)(d_{G_2}(x_h)+2)}{4}\Big) \\ &= \frac{1}{4}\sum_{i=1}^{m_1}\Big(\overline{ISI}(G_2) + 3\Big(\frac{n_2(n_2-1)}{2} - m_2\Big) + 2\overline{H}(G_2) + \frac{\overline{M_2}(G_2)}{4} + \frac{\overline{M_1}(G_2)}{2}\Big) \\ &= \frac{m_1}{4}\overline{ISI}(G_2) + \frac{m_1}{2}\overline{H}(G_2) + \frac{m_1}{16}\overline{M_2}(G_2) + \frac{m_1}{8}\overline{M_1}(G_2) + \frac{3m_1n_2(n_2-1)}{8} - \frac{3m_1m_2}{4}.\end{aligned}$$

Case 2: The nonadjacent vertex pairs $\{y_k, y_s\}$,$1 \leq k < s \leq n_1$ and it is assumed that $y_ky_s \notin E(G_1)$. Thus,

$$\begin{aligned}C_2 &= \sum_{y_ky_s \notin E(G_1 \bullet G_2)} \frac{d_{G_1 \bullet G_2}(y_k)d_{G_1 \bullet G_2}(y_s)}{d_{G_1 \bullet G_2}(y_k) + d_{G_1 \bullet G_2}(y_s)} = \sum_{y_ky_s \notin E(G_1)} \frac{(n_2+1)^2 d_{G_1}(y_k)d_{G_1}(y_s)}{(n_2+1)(d_{G_1}(y_k)+d_{G_1}(y_s))} \\ &= (n_2+1)\sum_{y_ky_s \notin E(G_1)} \frac{d_{G_1}(y_k)d_{G_1}(y_s)}{d_{G_1}(y_k)+d_{G_1}(y_s)} = (n_2+1)\overline{ISI}(G_1).\end{aligned}$$

Case 3: The nonadjacent vertex pairs $\{x_{ij}, y_k\}$, $1 \leq i \leq m_1$,$1 \leq j \leq n_2$, $1 \leq k \leq n_1$, and it is assumed that the ith edge e_i $1 \leq i \leq m_1$ in G_1 does not pass through y_k.

Note that each vertex y_k is adjacent to all vertices of $d_{G_1}(y_k)$ copies of G_2, that is, each y_k is not adjacent to any vertex of $m_1 - d_{G_1}(y_k)$ copies of G_2. Hence

$$C_3 = \sum_{k=1}^{n_1}(n_1 - d_{G_1}(y_k)) \sum_{j=1}^{n_2} \frac{(n_2+1)(d_{G_2}(x_j)+2)d_{G_1}(y_k)}{d_{G_2}(x_j) + 2 + (n_2+1)d_{G_1}(y_k)}$$

By Lemma 1, we obtain $\frac{1}{d_{G_2}(x_j)+2+(n_2+1)d_{G_1}(y_k)} \leq \frac{1}{4(d_{G_2}(x_j)+2)} + \frac{1}{4(n_2+1)d_{G_1}(y_k)}$. Thus,

$$\begin{aligned}
C_3 &\leq \frac{1}{4}\sum_{k=1}^{n_1}(n_1 - d_{G_1}(y_k))\sum_{j=1}^{n_2}\Big(\frac{(n_2+1)(d_{G_2}(x_j)+2)d_{G_1}(y_k)}{d_{G_2}(x_j)+2} + \frac{(n_2+1)(d_{G_2}(x_j)+2)d_{G_1}(y_k)}{(n_2+1)d_{G_1}(y_k)}\Big) \\
&= \frac{1}{4}\sum_{k=1}^{n_1}(n_1 - d_G(y_k))\sum_{j=1}^{n_2}\Big((n_2+1)d_{G_1}(y_k) + (d_{G_2}(x_j)+2)\Big) \\
&= \frac{1}{4}\sum_{k=1}^{n_1}(n_1 - d_{G_1}(y_k))\Big(n_2(n_2+1)d_{G_1}(y_k) + 2m_2 + 2n_2\Big) \\
&= \frac{n_2(n_2+1)}{4}\Big(2m_1n_1 - M_1(G)\Big) + \frac{(m_2+n_2)(n_1^2 - 2m_1)}{2}.
\end{aligned}$$

Case 4: The nonadjacent vertex pairs $\{x_{ij}, x_{\ell h}\}$, $1 \leq i < \ell \leq m_1$,$1 \leq j,h \leq n_2$.

$$C_4 = \sum_{x_{ij}x_{\ell h}\notin E(G_1\bullet G_2)} \frac{d_{G_1\bullet G_2}(x_{ij})d_{G_1\bullet G_2}(x_{\ell h})}{d_{G_1\bullet G_2}(x_{ij}) + d_{G_1\bullet G_2}(x_{\ell h})} = \frac{m_1(m_1-1)}{2}\sum_{j=1}^{n_2}\sum_{h=1}^{n_2}\frac{(d_{G_2}(x_j)+2)(d_{G_2}(x_h)+2)}{d_{G_2}(x_j)+d_{G_2}(x_h)+4}$$

Since for any vertex $x_j \in V(G_2)$, $\delta(G_2) \leq d_{G_2}(x_j) \leq \Delta(G_2)$. Hence

$$C_4 \leq \frac{m_1(m_1-1)n_2^2(\Delta(G_2)+2)^2}{4(\delta(G_2)+2)}.$$

From the above four cases of nonadjacent vertex pairs, we can obtain the desired result. This completes the proof. □

1.1 Mycieskian graph

In a search for triangle-free graphs with arbitrarily large chromatic number, Mycielski [8] developded an interesting graph transformation as follows: Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *Mycielskian graph*$\mu(G)$ of G contains G itself as an isomorphic subgraph, together with $n+1$ additional vertices: a vertex u_i corresponding to each vertex v_i of G, and another vertex w. Each vertex u_i is connected by an edge to w, so that these vertices form a subgraph in the form of a star $K_{1,n}$. Some topological indices of Mycielskian graph were computed in [10, 11].

Lemma 2. *Let G be a connected graph on n vertices and m edges. Then for each $i \in \{1,\ldots,n\}$, we have $d_{\mu(G)}(v_i) = 2d_G(v_i)$, $d_{\mu(G)}(u_i) = d_G(v_i)+1$ and $d_{\mu(G)}(w) = n$.*

By the definition of Mycielskian graph, for each edge v_iv_j of G, the Mycielskian graph includes two edges, u_iv_j and v_iu_j. Now we find the upper bound for inverse sum indeg coindex of Mycielskian graph.

Theorem 2. *Let G be a graph on n vertices and m edges. Then*

$$\begin{aligned}
\overline{ISI}(\mu(G)) &\leq \frac{n(n-1)-2m+16}{8}\overline{ISI}(G) + \frac{1}{4}\Big(\frac{n(n-1)}{2} - m\Big)\Big(\frac{\overline{M_2}(G)}{2} + \frac{\overline{M_1}(G)}{2} \\
&+ \frac{\overline{H}(G)}{2} + \frac{3n(n-1)}{4} - \frac{3m}{2}\Big) + \frac{m}{4}\Big(ISI(G) + \frac{M_2(G)}{2} + \frac{H(G)}{2} + \frac{3m}{2}\Big) \\
&+ \frac{m+4}{4}M_1(G) + \Big(\frac{n(n-1)}{2} - m\Big)\frac{2\Delta(G)(\Delta(G)+1)}{3\delta G+1} + \frac{7m}{3} + \frac{n(3n+5)}{12}.
\end{aligned}$$

Proof. Let $V(\mu(G)) = \{v_1, \dots, v_n\}$ and let $V(\mu(G)) = \{v_1, \dots, v_n, u_1, \dots, u_n, w\}$. By the structure of Mycielskian graph, if $v_iv_j \notin E(G)$, then $v_iu_j \notin E(G)$, and $v_ju_i \notin E(G)$.

Now we consider the following cases of nonadjacent vertex pairs in $\mu(G)$.

Case 1: The nonadjacent vertex pairs $\{v_i, v_j\}$ in $\mu(G)$.

$$
\begin{aligned}
C_1 &= \sum_{v_iv_j \notin E(\mu(G))} \frac{d_{\mu(G)}(v_i)d_{\mu(G)}(v_j)}{d_{\mu(G)}(v_i) + d_{\mu(G)}(v_j)} = \sum_{v_iv_j \notin E(G)} \frac{4d_G(v_i)d_G(v_j)}{2d_G(v_i) + 2d_G(v_j)}, \qquad \text{by Lemma 2} \\
&= 2 \sum_{v_iv_j \notin E(G)} \frac{d_G(v_i)d_G(v_j)}{d_G(v_i) + d_G(v_j)} = 2\overline{ISI}(G).
\end{aligned}
$$

Case 2: The nonadjacent vertex pairs $\{u_i, u_j\}$ in $\mu(G)$.

Case 2.1: $u_iu_j \notin E(\mu(G))$ and $v_iv_j \notin E(G)$.

$$
C_2' = \sum_{u_iu_j \notin E(\mu(G))} \frac{d_{\mu(G)}(u_i)d_{\mu(G)}(u_j)}{d_{\mu(G)}(u_i) + d_{\mu(G)}(u_j)} = \sum_{v_iv_j \notin E(G)} \frac{(d_G(v_i)+1)(d_G(v_j)+1)}{d_G(v_i) + d_G(v_j) + 2}, \text{ by Lemma 2.}
$$

By Lemma 1, we obtain

$$
\begin{aligned}
C_2' &\leq \frac{1}{4} \sum_{v_iv_j \notin E(G)} (d_G(v_i)+1)(d_G(v_j)+1)\Big(\frac{1}{d_G(v_i) + d_G(v_j)} + \frac{1}{2}\Big) \\
&= \frac{1}{4} \sum_{v_iv_j \notin E(G)} \Big(\frac{d_G(v_i)d_G(v_j)}{d_G(v_i) + d_G(v_j)} + \frac{d_G(v_i)d_G(v_j)}{2} + \frac{d_G(v_i) + d_G(v_j)}{2} + \frac{1}{d_G(v_i) + d_G(v_j)} + \frac{3}{2}\Big) \\
&= \frac{1}{4}\Big(\overline{ISI}(G) + \frac{\overline{M_2}(G)}{2} + \frac{\overline{M_1}(G)}{2} + \frac{\overline{H}(G)}{2} + \frac{3}{2}\Big(\frac{n(n-1)}{2} - m\Big)\Big).
\end{aligned}
$$

Case 2.2: $u_iu_j \notin E(\mu(G))$ and $v_iv_j \in E(G)$.

$$
C_2'' = \sum_{u_iu_j \notin E(\mu(G))} \frac{d_{\mu(G)}(u_i)d_{\mu(G)}(u_j)}{d_{\mu(G)}(u_i) + d_{\mu(G)}(u_j)} = \sum_{v_iv_j \in E(G)} \frac{(d_G(v_i)+1)(d_G(v_j)+1)}{d_G(v_i) + d_G(v_j) + 2}, \text{ by Lemma 2.}
$$

Apply Lemma 1, we have

$$
\begin{aligned}
C_2'' &\leq \frac{1}{4} \sum_{v_iv_j \in E(G)} \Big(\frac{d_G(v_i)d_G(v_j)}{d_G(v_i) + d_G(v_j)} + \frac{d_G(v_i)d_G(v_j)}{2} + \frac{d_G(v_i) + d_G(v_j)}{2} + \frac{1}{d_G(v_i) + d_G(v_j)} + \frac{3}{2}\Big) \\
&= \frac{1}{4}\Big(ISI(G) + \frac{M_2(G)}{2} + \frac{M_1(G)}{2} + \frac{H(G)}{2} + \frac{3m}{2}\Big).
\end{aligned}
$$

If $u_iu_j \notin E(\mu(G))$, then there are m edges $v_iv_j \in E(G)$ and $\frac{n(n-1)}{2} - m$ nonadjacent vertex pairs $\{v_i, v_j\}$ in G as well as $\mu(G)$. By Cases 2.1 and 2.2, we have the contribution of nonadjacent vertex pair of case 2 is given by

$$
\begin{aligned}
C_2 &= \Big(\frac{n(n-1)}{2} - m\Big)C_2' + mC_2'' \\
&= \frac{1}{4}\Big(\frac{n(n-1)}{2} - m\Big)\Big(\overline{ISI}(G) + \frac{\overline{M_2}(G)}{2} + \frac{\overline{M_1}(G)}{2} + \frac{\overline{H}(G)}{2} + \frac{3n(n-1)}{4} - \frac{3m}{2}\Big) \\
&\quad + \frac{m}{4}\Big(ISI(G) + \frac{M_2(G)}{2} + \frac{M_1(G)}{2} + \frac{H(G)}{2} + \frac{3m}{2}\Big).
\end{aligned}
$$

Case 3: The nonadjacent vertex pairs $\{u_i, v_i\}$ in $\mu(G)$ for each $i = 1, 2, \ldots, n$.

$$\begin{aligned} C_3 &= \sum_{i=1}^{n} \frac{d_{\mu(G)}(u_i)d_{\mu(G)}(v_i)}{d_{\mu(G)}(u_i) + d_{\mu(G)}(v_i)} = \sum_{i=1}^{n} \frac{2(d_G(v_i)+1)d_G(v_i)}{3d_G(v_i)+1}, \qquad \text{by Lemma 2} \\ &\le \frac{1}{4}\sum_{i=1}^{n}\Big(2d_G^2(v_i) + 2d_G(v_i)\Big)\Big(\frac{1}{3d_G(v_i)} + 1\Big), \qquad \text{by Lemma 1} \\ &= \frac{1}{4}\Big(2M_1(G) + \frac{16m}{3} + \frac{2n}{3}\Big). \end{aligned}$$

Case 4: The nonadjacent vertex pairs $\{u_i, v_j\}$ in $\mu(G)$.

$$C_4 = \sum_{u_iv_j \notin E(\mu(G))} \frac{d_{\mu(G)}(u_i)d_{\mu(G)}(v_j)}{d_{\mu(G)}(u_i) + d_{\mu(G)}(v_j)} = \sum_{v_iv_j \notin E(G)} \frac{2(d_G(v_i)+1)d_G(v_j)}{d_G(v_i) + 2d_G(v_j) + 1}, \qquad \text{by Lemma 2.}$$

For any vertex $v_i \in V(G)$, we have $\delta(G) \le d_G(v_i) \le \Delta(G)$. Thus

$$C_4 \le \Big(\frac{n(n-1)}{2} - m\Big)\frac{2\Delta(G)(\Delta(G)+1)}{3\delta(G)+1}.$$

Case 5: The nonadjacent vertex pairs $\{w, v_i\}$ in $\mu(G)$ for each $i = 1, 2, \ldots, n$.

$$\begin{aligned} C_5 &= \sum_{v_iw \notin E(\mu(G))} \frac{d_{\mu(G)}(v_i)d_{\mu(G)}(w)}{d_{\mu(G)}(v_i) + d_{\mu(G)}(w)} = \sum_{v_i \in V(G)} \frac{2(n+1)d_G(v_i)}{2d_G(v_i) + (n+1)}, \qquad \text{by Lemma 2} \\ &\le \frac{1}{4}\sum_{v_i \in V(G)} 2(n+1)d_G^2(v_i)\Big(\frac{1}{2d_G(v_i)} + \frac{1}{n+1}\Big), \qquad \text{by Lemma 1} \\ &= \frac{1}{4}\Big(n(n+1) + 4m\Big). \end{aligned}$$

From the above five cases of nonadjacent vertex pairs, we can obtain the desired results. This completes the proof. □

1.2 Double graph

Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. The vertices of the double graph G^* are given by the two sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Thus for each vertex $v_i \in V(G)$, there are two vertices x_i and y_i in $V(G^*)$. The *double graph* G^* includes the initial edge set of each copies of G, and for any edge $v_iv_j \in E(G)$, two more edges x_iy_j and x_jy_i are added. For a given vertex v in G, let $D_G(v) = \sum_{uv \notin E(G)} \frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)}$. Now we find the exact value of the inverse sum indeg index and its coindex for double graph of a given graph.

Theorem 3. *The inverse sum indeg index of the double graph G^* of a graph G is given by $ISI(G^*) = 8\,ISI(G)$.*

Proof. From the definition of double graph it is clear that $d_{G^*}(x_i) = d_{G^*}(y_i) = 2d_G(v_i)$, where $v_i \in V(G)$ and $x_i, y_i \in V(G^*)$ are corresponding clone vertices of v_i.

Thus from the definition of ISI, we have

$$\begin{aligned} ISI(G^*) &= \sum_{uv\in E(G^*)} \frac{d_{G^*}(u)d_{G^*}(v)}{d_{G^*}(u)+d_{G^*}(v)} = \sum_{x_ix_j\in E(G^*)} \frac{d_{G^*}(x_i)d_{G^*}(x_j)}{d_{G^*}(x_i)+d_{G^*}(x_j)} \\ &+ \sum_{y_iy_j\in E(G^*)} \frac{d_{G^*}(y_i)d_{G^*}(y_j)}{d_{G^*}(y_i)+d_{G^*}(y_j)} + \sum_{x_iy_j\in E(G^*)} \frac{d_{G^*}(x_i)d_{G^*}(y_j)}{d_{G^*}(x_i)+d_{G^*}(y_j)} \\ &+ \sum_{x_jy_i\in E(G^*)} \frac{d_{G^*}(x_j)d_{G^*}(y_i)}{d_{G^*}(x_j)+d_{G^*}(y_i)} = 4\sum_{v_iv_j\in E(G)} \frac{4\,d_G(v_i)d_G(v_j)}{2d_G(v_i)+2d_G(v_j)} = 8\,ISI(G). \end{aligned}$$

□

Theorem 4. *Let G be a connected graph with n vertices and m edges. Then $\overline{ISI}(G^*) = 8\overline{ISI}(G)+2m$.*

Proof. Let $V(G)=\{v_1,v_2,\dots,v_n\}$. Suppose that x_i and y_i are the corresponding clone vertices, in G^*, of v_i for each $i\in\{1,2,\dots,n\}$. For any given vertex v_i in G and its clone vertices x_i and y_i, $d_{G^*}(x_i)=d_{G^*}(y_i)=2d_G(v_i)$ by the definition of double graph.

For $v_i,v_j\in V(G)$, if $v_iv_j\notin E(G)$, then $x_ix_j\notin E(G)$,$y_iy_j\notin E(G)$,$x_iy_j\notin E(G)$and $y_ix_j\notin E(G)$.

Hence we only consider total contribution of the following three types of nonadjacent vertex pairs to calculate $\overline{ISI}(G)$.

Case 1: The nonadjacent vertex pairs $\{x_i,x_j\}$ and $\{y_i,y_j\}$, where $v_iv_j\notin E(G)$.

$$\begin{aligned} \sum_{y_iy_j\notin E(G^*)} \frac{d_{G^*}(y_i)d_{G^*}(y_j)}{d_{G^*}(y_i)+d_{G^*}(y_j)} &= \sum_{x_ix_j\notin E(G^*)} \frac{d_{G^*}(x_i)d_{G^*}(x_j)}{d_{G^*}(x_i)+d_{G^*}(x_j)} = \sum_{v_iv_j\notin E(G)} \frac{4d_G(v_i)d_Gv_j)}{2d_G(v_i)+2d_Gv_j)} \\ &= 2\overline{ISI}(G). \end{aligned}$$

Case 2: The nonadjacent vertex pairs $\{x_i,y_i\}$ for each $i\in\{1,2,\dots,n\}$.

$$\sum_{i=1}^{n} \frac{d_{G^*}(x_i)d_{G^*}(y_i)}{d_{G^*}(x_i)+d_{G^*}(y_i)} = \sum_{i=1}^{n} \frac{4d_G(v_i)d_G(v_i)}{2d_G(v_i)+2d_G(v_i)} = \sum_{i=1}^{n} d_G(v_i) = 2m.$$

Case 3: The nonadjacent vertex pairs $\{x_i,y_j\}$ and $\{y_i,x_j\}$, where $v_iv_j\notin E(G)$.

For each x_i, there exist $n-1-d_G(v_i)$ vertices in the set $\{y_1,y_2,\dots,y_n\}$, among which every vertex together with x_i compose a nonadjacent vertex pairs of G^*. The total contribution of these $n-1-d_G(v_i)$ nonadjacent vertex pairs to calculate $\overline{ISI}(G^*)$ is

$$\sum_{x_iy_j\notin E(G^*)} \frac{d_{G^*}(x_i)d_{G^*}(y_j)}{d_{G^*}(x_i)+d_{G^*}(y_j)} = \sum_{v_iv_j\notin E(G^*)} \frac{4d_G(v_i)d_G(v_j)}{2d_G(v_i)+2d_G(v_j)} = 2D_G(v_i).$$

Hence

$$\sum_{i\neq j,\ x_iy_j\notin E(G^*)} \frac{d_{G^*}(x_i)d_{G^*}(y_j)}{d_{G^*}(x_i)+d_{G^*}(y_j)} = \sum_{i=1}^{n} 2D_G(v_i) = 4\overline{ISI}(G).$$

Hence

$$\begin{aligned} \overline{ISI}(G^*) &= \sum_{x_ix_j\notin E(G^*)} \frac{d_{G^*}(x_i)d_{G^*}(x_j)}{d_{G^*}(x_i)+d_{G^*}(x_j)} + \sum_{y_iy_j\notin E(G^*)} \frac{d_{G^*}(y_i)d_{G^*}(y_j)}{d_{G^*}(y_i)+d_{G^*}(y_j)} + \sum_{i=1}^{n} \frac{d_{G^*}(x_i)d_{G^*}(y_i)}{d_{G^*}(x_i)+d_{G^*}(y_i)} \\ &+ \sum_{i\neq j,\ x_iy_j\notin E(G^*)} \frac{d_{G^*}(x_i)d_{G^*}(y_j)}{d_{G^*}(x_i)+d_{G^*}(y_j)} = 8\overline{ISI}(G)+2m. \end{aligned}$$

□

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Оборотно підсумовуючий індег коіндекс $\overline{ISI}(G)$ простого зв'язного графу G визначено як сума доданків $\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)}$ по всіх ребрах uv, які не лежать у G, де $d_G(u)$ позначає степінь вершини u в G. У статті встановлено верхні обмеження на оборотно підсумовуючий індег коіндекс графу добутку вершин корони та графу Мицелскіана. Крім того отримано точне значення оборотного підсумовуючого індег індексу і коіндексу для подвійного графу.

Ключові слова і фрази: оборотно підсумовуючий індег індекс, граф вершин корони, граф Мицелскіана, подвійний граф.