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# ON INVERSE TOPOLOGY PROBLEM FOR LAPLACE OPERATORS ON GRAPHS

Laplacian operators on finite compact metric graphs are considered under the assumption that matching conditions at graph vertices are of $\delta$ type. Under one additional assumption, the inverse topology problem is treated. Using the apparatus of boundary triples, we generalize and extend existing results on necessary conditions of isospectrality of two Laplacians defined on different graphs. A result is also given covering the case of Schrödinger operators.

*Key words and phrases:* quantum graphs, Schrödinger operator, Laplace operator, inverse spectral problem, boundary triples, isospectral graphs.

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## INTRODUCTION

In the present paper we focus our attention on the so-called quantum graph, i.e., a metric graph $\Gamma$ with an associated second-order differential operator acting in Hilbert space $L^2(\Gamma)$ of square summable functions with an additional assumption that functions belonging to the domain of the operator are coupled by certain matching conditions at graph vertices. Recently these operators have attracted a considerable interest of both physicists and mathematicians due to a number of important physical applications. Extensive literature on the subject is surveyed in, e.g., [4, 19].

The present paper is devoted to the study of the following inverse spectral problem for Laplace and Schrödinger operators on finite compact metric graphs: given spectral data (i.e., the spectrum of the operator), edge potentials and matching conditions, to reconstruct the underlying metric graph.

There exists an extensive literature devoted to the named problem. To name just a few, we would like to mention pioneering works [13,16,24] and later contributions [14,20,21]. Different approaches to the same problem were developed, e.g., in [2,3,23].

In our papers [6–9] we suggested an approach to inverse spectral problems on graphs based on the theory of boundary triples, leading to the asymptotic analysis of Weyl-Titchmarsh M-function of the graph. In the cited papers this argument was successfully applied to the study of a different (although related) inverse spectral problem on graphs. This approach will be also used throughout the present paper.

Here we consider the case of a general connected compact finite metric graph under the only *additional assumption* that (cf. [1]) *it does not contain: (i) loops; (ii) cycles with all edges having pairwise rationally dependent edge lengths.* This restriction is equivalent to the fact that the

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minimal operator *naturally associated* with the graph is simple, i.e., has no reducing self-adjoint "parts". We also assume that each graph vertex is allowed to have matching of $\delta$ type only (see Section 1 for definitions). The named class proves to be physically viable [10, 11].

The general case of arbitrary graphs with vertices of both $\delta$ and $\delta'$ types will be treated in a separate publication.

## 1 PRELIMINARIES

### 1.1 Definition of the Laplace operator on a quantum graph

We call $\Gamma = \Gamma(\mathbf{E}_\Gamma, \sigma)$ a finite compact metric graph, if it is a collection of a finite non-empty set $\mathbf{E}_\Gamma$ of compact intervals $e_j = [x_{2j-1}, x_{2j}]$, $j = 1, 2, \ldots, n$, called *edges*, and of a partition $\sigma$ of the set of endpoints $\{x_k\}_{k=1}^{2n}$ into $N$ classes, $\mathbf{V}_\Gamma = \bigcup_{m=1}^{N} V_m$. The equivalence classes $V_m$, $m = 1, 2, \ldots, N$ will be called *vertices* and the number of elements belonging to the set $V_m$ will be called the *valence* (or, alternatively, *degree*) of the vertex $V_m$ (denoted $\deg V_m \equiv \gamma_m$).

Whenever we need to consider a different graph $\tilde{\Gamma}$ of the same class alongside the graph $\Gamma$, we will use the same notation for all objects pertaining to it, having decorated each symbol ($n$, $N$, $\gamma_m$, etc.) with a tilde.

With a finite compact metric graph $\Gamma$ we associate Hilbert spaces $L_2(\Gamma) = \oplus_{j=1}^{n} L_2(e_j)$ and $W_2^2(\Gamma) = \oplus_{j=1}^{n} W_2^2(e_j)$. These spaces obviously do not feel the graph connectivity, being the same for each graph with the same number of edges of same lengths.

For a smooth enough function $f \in L_2(\Gamma)$, we will use throughout the following definition of the normal derivative on a finite compact metric graph

$$\partial_n f(x_j) = \begin{cases} f'(x_j), & \text{if } x_j \text{ is the left endpoint of the edge,} \\ -f'(x_j), & \text{if } x_j \text{ is the right endpoint of the edge.} \end{cases}$$

If $f \in \oplus_{j=1}^{n} W_2^2(e_j)$ and $\alpha_m$ is a complex number (referred to below as a coupling constant), the condition of continuity of the function $f$ through the vertex $V_m$ (i.e., $f(x_j) = f(x_k)$ if $x_j, x_k \in V_m$) together with the condition

$$\sum_{x_j \in V_m} \partial_n f(x_j) = \alpha_m f(V_m)$$

is called $\delta$-type matching at the vertex $V_m$.

Note that the $\delta$-type matching condition in a particular case when $\alpha_m = 0$ reduces to the so-called standard, or Kirchhoff, matching condition at the vertex $V_m$.

The graph Laplacian $A_{\vec{\alpha}}$ on a graph $\Gamma$ with $\delta$-type matching conditions is the operator of negative second derivative in the Hilbert space $L_2(\Gamma)$ on the domain of functions belonging to the Sobolev space $\oplus_{j=1}^{n} W_2^2(e_j)$ and satisfying $\delta$-type matching conditions at every vertex $V_m$, $m = 1, 2, \ldots, N$. The corresponding Schrödinger operator on the same graph is defined likewise on the same domain in the case of summable edge potentials.

Provided that all coupling constants $\alpha_m$, $m = 1 \ldots N$, are real, it is easy to ascertain that the operator $A_{\vec{\alpha}}$ is a proper self-adjoint extension of a closed symmetric operator $A_{\min}$ in Hilbert space $L_2(\Gamma)$ [10, 15].

Clearly, the self-adjoint operator thus defined on a finite compact metric graph has purely discrete spectrum that accumulates to $+\infty$.

Note that w.l.o.g. each edge $e_j$ of the graph $\Gamma$ can be considered to be an interval $[0, l_j]$, where $l_j = x_{2j} - x_{2j-1}$, $j = 1, \ldots, n$ is the length of the corresponding edge. Throughout the present paper we will therefore only consider this situation.

### 1.2 Boundary triples and the Weyl-Titchmarsh matrix M-function

The analysis presented in the present paper is essentially based on the theory of boundary triples [5,12,17,18] applied to the class of operators introduced above. Two fundamental concepts of this theory are those of a boundary triple and of the Titchmarsh-Weyl generalized matrix-function. Assume that $A_{\min}$ is a symmetric densely defined operator in Hilbert space $H$, and that its deficiency indices are equal. Put $A_{\max} := A^*_{\min}$.

The property of the Weyl-Titchmarsh $M$-function that makes it the tool of choice for the analysis of isospectral Laplacians on graphs can be formulated in the following way: provided that $A_B$ is an almost solvable extension of a simple[1] symmetric operator $A_{\min}$ parameterized by a (self-adjoint) matrix $B$, $\lambda_0 \in \rho(A_B)$ if and only if $(B - M(\lambda))^{-1}$ admits analytic continuation into the point $\lambda_0$.

In [8], we have obtained the following

**Proposition 1** ( [8]). *Let $\Gamma$ be a finite compact metric graph having no loops and with coupling of $\delta$ type at all vertices. There exists a closed densely defined symmetric operator $A_{\min}$ and a boundary triple such that the operator $A_{\vec{\alpha}}$ is an almost solvable extension of $A_{\min}$, for which the parameterizing matrix $B$ is nothing but* $\operatorname{diag}\{\alpha_1, \dots, \alpha_N\}$*, whereas the generalized Weyl-Titchmarsh $M$-function is a $N \times N$ matrix with matrix elements given by the following formula*

$$m_{jk}(\lambda) = \begin{cases} -\mu \sum_{e_t \in E_k} \cot \mu l_t, & j = k, \\ \mu \sum_{e_t \in C_{kj}} \frac{1}{\sin \mu l_t}, & j \neq k,\ V_j \text{ is adjacent to } V_k, \\ 0, & j \neq k,\ V_j \text{ is a vertex not adjacent to } V_k. \end{cases}$$

*Here $\mu = \sqrt{\lambda}$ (the branch such that $\operatorname{Im} \mu \geq 0$), $l_t$ is the length of $e_t$, $E_k$ is the set of graph edges incident to the vertex $V_k$, $C_{kj}$ is the set of graph edges connecting vertices $V_k$ and $V_j$.*

The result of [1] further implies that under the additional assumption formulated in Introduction the minimal operator $A_{\min}$ is simple. This means that *each* eigenvalue of $A_{\vec{\alpha}}$ is a pole of the meromorphic matrix-function $(B - M(\lambda))^{-1}$ for $B = \operatorname{diag}\{\alpha_1, \dots, \alpha_N\}$ with multiplicity equal to the multiplicity of the eigenvalue.

In essence, we will build our analysis upon the foundation provided by Proposition 1 and the latter remark.

## 2 ISOSPECTRALITY OF GRAPH LAPLACIANS

In the present Section, we formulate the main results of the paper. We start with the following

**Theorem 1.** *Let $\Gamma$ and $\tilde{\Gamma}$ be two finite compact metric graphs subject to the assumption of Introduction with all vertices of $\delta$ type. Let $A_{\vec{\alpha}}$, $\tilde{A}_{\tilde{\vec{\alpha}}}$ be two graph Laplacians on $\Gamma$ and $\tilde{\Gamma}$, parameterized by coupling constants $\{\alpha_k\}$ and $\{\tilde{\alpha}_k\}$, respectively. If (point) spectra of the operators $A_{\vec{\alpha}}$ and $\tilde{A}_{\tilde{\vec{\alpha}}}$ coincide counting multiplicities, then (i) total lengths of $\Gamma$ and $\tilde{\Gamma}$ are equal, $\sum_i l_i = \sum_i \tilde{l}_i$; (ii) Euler characteristics[2] of $\Gamma$ and $\tilde{\Gamma}$ are equal, $\chi_\Gamma = \chi_{\tilde{\Gamma}}$; (iii) the set equality $\Sigma = \tilde{\Sigma}$ holds, where $\Sigma := \{\sigma_1, \dots, \sigma_m\}$ is the set of non-zero elements of the list $\{\alpha_i/\gamma_i\}_{i=1}^N$ and $\tilde{\Sigma}$ is defined analogously for the graph $\tilde{\Gamma}$.*

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[1] I.e., there exists no reducing subspace $H_0$ such that the restriction $A_{\min}|H_0$ is a selfadjoint operator in $H_0$.

[2] Recall that the Euler characteristic of a graph is the difference between the number of vertices and the number of edges.

**Remark 1.** *The implication (i) also follows from the Weyl-type asymptotics of (discrete) spectra which evidently holds for both Laplacians.*

*The implication (ii) is a generalization of [21,23] where this result was proved in the case of Kirchhoff matching conditions at all vertices to the general case of arbitrary $\delta$ coupling.*

*Note finally that the set equality of (iii) is only meaningful if at least some coupling constants of $A_{\vec\alpha}$ are non-zero (and hence the same number of coupling constants pertaining to $\tilde A_{\tilde{\vec\alpha}}$ is non-zero). Therefore, the case of Kirchhoff matching turns out to be the most complicated as (iii) then yields no information.*

*Proof.* Let $\Pi(\lambda) = \prod_{e\in\mathbf{E}_\Gamma} \frac{\sin l_j\sqrt{\lambda}}{\sqrt{\lambda}}$ and let $\tilde\Pi(\lambda)$ be defined analogously for the graph $\tilde\Gamma$. Then the functions $\Pi(\lambda)\det(B - M(\lambda))$ and $\tilde\Pi(\lambda)\det(\tilde B - \tilde M(\lambda))$, where $M$ and $\tilde M$ are Weyl-Titchmarsh matrices of Proposition 1 pertaining to $\Gamma$ and $\tilde\Gamma$, respectively, are entire functions of exponential type of order not greater than $1/2$ (see [22, Chapter I]). Moreover, zeroes of these two functions are located precisely at the eigenvalues of the operators $A_{\vec\alpha}$, $\tilde A_{\tilde{\vec\alpha}}$, respectively (counting multiplicities). This follows from [5,12] using: (i) the fact that an M-matrix of Proposition 1 is a matrix-valued R-function with almost everywhere Hermitian boundary values on $\mathbb{R}$; (ii) the fact that the poles of an M-matrix of Proposition 1 are located at the eigenvalues of the Dirichlet decoupling of the graph $\Gamma$, counting multiplicities; (iii) the fact that within conditions of the Theorem, both $A_{\min}$ and $\tilde A_{\min}$ are simple.

Then the condition of isospectrality implies that the fraction $\frac{\Pi(\lambda)\det(B-M(\lambda))}{(\tilde\Pi(\lambda)\det(\tilde B-\tilde M(\lambda))}$ is [22] again an entire function of exponential type of order not greater than $1/2$. Applying the Hadamard theorem, one easily obtains

$$\frac{\Pi(\lambda)\det(B - M(\lambda))}{\tilde\Pi(\lambda)\det(\tilde B - \tilde M(\lambda))} = \exp(a) \tag{1}$$

for some finite constant $a$.

Consider asymptotic expansions of the functions $\det(B - M(\lambda))$ and $\det(\tilde B - \tilde M(\lambda))$ as $\lambda \to -\infty$ along the real line. Using the asymptotic expansion for $M(\lambda)$ following easily from Proposition 1 one has

$$\det(B - M(\lambda)) = \prod_{i=1}^{N}(\alpha_i + \gamma_i\tau) + o(\tau^{-M}); \quad \det(\tilde B - \tilde M(\lambda)) = \prod_{i=1}^{\tilde N}(\tilde\alpha_i + \tilde\gamma_i\tau) + o(\tau^{-M})$$

for any natural $M > 0$, where $\tau = -i\sqrt{\lambda} \to +\infty$. Using asymptotic expansions for $\Pi(\lambda)$ and $\tilde\Pi(\lambda)$ and (1), one immediately ascertains (i) and then (ii), which leads to

$$\frac{2^{\tilde n}}{2^n}\frac{\prod_{i=1}^{N}(\frac{\alpha_i}{\tau} + \gamma_i) + o(\tau^{-M})}{\prod_{i=1}^{\tilde N}(\frac{\tilde\alpha_i}{\tau} + \tilde\gamma_i) + o(\tau^{-M})} = \exp a, \tag{2}$$

wherefrom $\exp a = (2^{\tilde n}\prod_{i=1}^{N}\gamma_i)/(2^n\prod_{i=1}^{\tilde N}\tilde\gamma_i)$. One then divides both sides of (2) by $\exp a$. Taking the logarithm of the result, one arrives at

$$\sum_{i=1}^{N}\ln\Big(1 + \frac{\alpha_i}{\gamma_i}\frac{1}{\tau}\Big) - \sum_{i=1}^{\tilde N}\ln\Big(1 + \frac{\tilde\alpha_i}{\tilde\gamma_i}\frac{1}{\tau}\Big) + o(\tau^{-M}) = 0.$$

The Taylor expansion of logarithms yields that for any natural $M$

$$-\sum_{j=1}^{M}\frac{(-1)^j}{j\tau^j}\sum_{i=1}^{N}\Big(\frac{\alpha_i}{\gamma_i}\Big)^j + \sum_{j=1}^{M}\frac{(-1)^j}{j\tau^j}\sum_{i=1}^{\tilde N}\Big(\frac{\tilde\alpha_i}{\tilde\gamma_i}\Big)^j + o(\tau^{-M}) = 0.$$

Comparing coefficients at equal powers of $\tau$ now yields

$$\sum_{i=1}^{N} \frac{(-\alpha_i)^m}{\gamma_i^m} = \sum_{i=1}^{\tilde{N}} \frac{(-\tilde{\alpha}_i)^m}{\tilde{\gamma}_i^m}$$

for any natural $m$. Using the argument of [8, Lemma 5.1] now completes the proof. $\square$

**Remark 2.** *Theorem 1 admits an extension to the case of graph Schrödinger operators. Indeed, assertions (i) and (ii) will be valid for a pair of Schrödinger operators on $\Gamma$ and $\tilde{\Gamma}$, respectively, if one requires that (a) all edge potentials have zero means, $\int_e q_e(x)dx = 0$ for any edge $e$, and (b) both minimal operators $A_{\min}$, $\tilde{A}_{\min}$ are simple. The proof follows the same argument as above, see [7] for necessary details.*

Our next result shows that even in the seemingly more complicated case of Kirchhoff matching one can in fact go one step further. The corresponding argument pertaining to the general situation of $\delta$ type matching as well as a detailed analysis of Schrödinger case to which the argument is also applicable will be scrutinized elsewhere. We have

**Theorem 2.** *Let $\Gamma$ and $\tilde{\Gamma}$ be two finite compact metric graphs subject to the assumption of Introduction with all vertices of $\delta$ type. Let $A_{\vec{0}}$, $\tilde{A}_{\vec{0}}$ be two graph Laplacians on $\Gamma$ and $\tilde{\Gamma}$ both with Kirchhoff matching conditions at all vertices. If (point) spectra of the operators $A_{\vec{0}}$ and $\tilde{A}_{\vec{0}}$ coincide counting multiplicities, then*

$$\frac{\prod_{i=1}^{n} l_i}{\prod_{i=1}^{N} \frac{\gamma_i}{2}} \sum_{T \in \mathcal{T}} w(T) = \frac{\prod_{i=1}^{\tilde{n}} \tilde{l}_i}{\prod_{i=1}^{\tilde{N}} \frac{\tilde{\gamma}_i}{2}} \sum_{T \in \tilde{\mathcal{T}}} w(T),$$

*where $\mathcal{T}$ and $\tilde{\mathcal{T}}$ are the sets of spanning trees for $\Gamma$ and $\tilde{\Gamma}$, respectively; the weight of the tree $w(T)$ is the product of inverse lengths over all edges forming the subgraph $T$, $w(T) = \prod_{e \in T} \frac{1}{l_e}$.*

*Proof.* Proceeding exactly as in the proof of Theorem 1 one gets the following identity

$$\frac{\Pi(\lambda) \det(-M(\lambda))}{\tilde{\Pi}(\lambda) \det(-\tilde{M}(\lambda))} = \frac{2^{\tilde{n}}}{2^n} \frac{\prod_{i=1}^{N} \gamma_i}{\prod_{i=1}^{\tilde{N}} \tilde{\gamma}_i}. \tag{3}$$

The asymptotic expansion of the latter as $\lambda \to 0$ proves to suffice our needs. Indeed, both $M(\lambda)$ and $\tilde{M}(\lambda)$ tend to negative weighted discrete Laplacians of $\Gamma$ and $\tilde{\Gamma}$, respectively (see [6] and Proposition 1), where the weight associated with any edge $e$ is nothing but its inverse length. Therefore, both M-matrices have exactly one eigenvalue, say, $\mu_1(\lambda)$ ($\tilde{\mu}_1(\lambda)$, resp.) zeroing out at $\lambda = 0$ (due to connectedness of both graphs, see [25]). Moreover, both $\mu_1$ and $\tilde{\mu}_1$ are analytic R-functions owing to the analytic properties of $M$ and $\tilde{M}$ and thus have simple zeroes at the named point. In fact, one can ascertain that $\mu_1'(0) = \sum_i l_i / N$ and the same holds true for $\tilde{\mu}_1'(0)$.

Indeed, the kernel of $M(0)$ is generated by the vector $\vec{1} \equiv (1, 1, \dots, 1)^T$. An analytic expansion of the equality $M(\lambda)(\vec{1} + \lambda f_1 + O(\lambda^2)) = \lambda(\mu_1'(0) + O(\lambda))(\vec{1} + \lambda f_1 + O(\lambda^2))$ then yields, considering linear in $\lambda$ terms only

$$M_1 \vec{1} + M_0 f_1 = \mu_1'(0)\vec{1},$$

where $M(\lambda) = M_0 + \lambda M_1 + O(\lambda^2)$. The solvability condition of the latter is nothing but

$$\langle \mu_1'(0)\vec{1} - M_1\vec{1}, \vec{1} \rangle = 0,$$

from where the claim follows by the property $\langle M_1\vec{1}, \vec{1} \rangle = \sum_i l_i$, which in turn follows trivially from Proposition 1 by Taylor series expansion.

Using Theorem 1 yet again, one reduces (3) to

$$2^n \frac{\prod_{i=1}^n l_i}{N \prod_{i=1}^N \gamma_i} \prod_{k=2}^N \mu_k(0) = 2^{\tilde{n}} \frac{\prod_{i=1}^{\tilde{n}} \tilde{l}_i}{\tilde{N} \prod_{i=1}^{\tilde{N}} \tilde{\gamma}_i} \prod_{k=2}^{\tilde{N}} \tilde{\mu}_k(0),$$

where $\mu_2(0), \dots, \mu_N(0)$ and $\tilde{\mu}_2(0), \dots, \tilde{\mu}_{\tilde{N}}(0)$ are non-zero (positive) eigenvalues of weighted discrete Laplacians, associated with $\Gamma$ and $\tilde{\Gamma}$, respectively. Using the generalized Matrix-Tree Theorem [25], one finally has

$$\frac{1}{N} \prod_{k=2}^N \mu_k(0) = \sum_{T \in \mathcal{T}} w(T); \quad \frac{1}{\tilde{N}} \prod_{k=2}^{\tilde{N}} \tilde{\mu}_k(0) = \sum_{T \in \tilde{\mathcal{T}}} w(T).$$

Using the assertion (ii) of Theorem 1, one now easily completes the proof. $\square$

**Example 1.** *Assume that $\Gamma$ is a tree with Kirchhoff matching at all vertices. The assumption of Introduction is surely met. Then Theorem 1 (ii) yields that $\tilde{\Gamma}$ has to be a tree as well provided that the condition of isospectrality is satisfied, in line with results of [21, 23]. Theorem 2, however, leads to the following new strong additional condition, necessary for iospectrality*

$$\prod_{i=1}^N \frac{\gamma_i}{2} = \prod_{i=1}^{\tilde{N}} \frac{\tilde{\gamma}_i}{2}.$$

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Єршова Ю.Ю., Карпенко І.І., Кисельов О.В. *Про обернену задачу відновлення топології для операторів Лапласа на графах* // Карпатські матем. публ. — 2014. — Т.6, №2. — С. 230–236.

Розглядаються оператори Лапласа на скінченних компактних метричних графах у припущенні, що умови зв'язку в вершинах графа мають $\delta$-тип. При ще одному додатковому припущенні вивчається задача відновлення топології графа. З використанням апарата теорії граничних трійок узагальнені та доповнені результати, що вже існують, про необхідні умови ізоспектральності двох операторів Лапласа, котрі задані на різноманітних графах. Також наведений один окремий результат для оператора Шредінгера.

*Ключові слова і фрази:* квантові графи, оператор Шредінгера, оператор Лапласа, обернена спектральна задача, граничні трійки, ізоспектральні графи.

Ершова Ю.Ю., Карпенко И.И., Киселёв А.В. *Об обратной задаче восстановления топологии для операторов Лапласа на графах* // Карпатские матем. публ. — 2014. — Т.6, №2. — С. 230–236.

Рассматриваются операторы Лапласа на конечных компактных метрических графах в предположении, что условия связи в вершинах графа имеют $\delta$-тип. При одном дополнительном предположении изучается задача восстановления топологии графа. С использованием аппарата теории граничных троек обобщены и дополнены существующие результаты о необходимых условиях изоспектральности двух операторов Лапласа, заданных на различных графах. Также приведен один частный результат для оператора Шредингера.

*Ключевые слова и фразы:* квантовые графы, оператор Шредингера, оператор Лапласа, обратная спектральная задача, граничные тройки, изоспектральные графы.