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ON ORDERS OF TWO TRANSFORMATION SEMIGROUPS OF THE BOOLEAN

We consider the semigroup $\mathcal{O}(\mathcal{B}_n)$ of all order–preserving transformations $\varphi : \mathcal{B}_n \to \mathcal{B}_n$ of ordered by inclusion boolean $\mathcal{B}_n$ of n-element set (i.e. such transformations that $A \subseteq B$ implies $\varphi(A) \subseteq \varphi(B)$) and its subsemigroup $\mathcal{C}(\mathcal{B}_n)$ of those transformations for which $\varphi(A) \subseteq A$ for all $A \in \mathcal{B}_n$. Orders of these semigroups are calculated.

Key words and phrases: semigroup, order-preserving transformation, order-decreasing transformation, monotone boolean functions.

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INTRODUCTION

For every partial order $\leq$ on a set M, two types of transformations of this set arise in a natural way. A transformation $\varphi : M \to M$ is called *order–preserving*, if for every $a, b \in M$, $a \leq b$ implies $\varphi(a) \leq \varphi(b)$, and *order–decreasing*, if for arbitrary $a \in M$ the inequality $\varphi(a) \leq a$ holds. Both order–preserving and order–decreasing transformations form a semigroup with respect to the composition of transformations. These semigroups are denoted by $\mathcal{O}(M, \leq)$ and $\mathcal{F}(M, \leq)$ correspondingly (or just $\mathcal{O}(M)$ and $\mathcal{F}(M)$, if it is clear what the partial order is).

It is clear that studying of semigroups $\mathcal{O}(M)$ and $\mathcal{F}(M)$ began from the simplest case, when M is a finite chain. The semigroup $\mathcal{O}_n$ of order–preserving transformations of n-element chain appeared first in paper [1] and corresponding semigroup $\mathcal{F}_n$ in the book [2].

Studying of these semigroups have been done intensively during last twenty years (see last chapter of [3] and references). For other partial orders such semigroups have been studied relatively small (see [4]).

We consider the boolean $\mathcal{B}_n$ — the set of all subsets of a n-element set $N = \{1, 2, \ldots, n\}$ naturally ordered by inclusion. In [5] it is proved that order of a semigroup $\mathcal{F}(\mathcal{B}_n)$ is equal to $2^{n \cdot 2^{n-1}}$. We calculate orders of semigroups $\mathcal{O}(\mathcal{B}_n)$ and $\mathcal{C}(\mathcal{B}_n) = \mathcal{O}(\mathcal{B}_n) \cap \mathcal{F}(\mathcal{B}_n)$.

ORDERS OF SEMIGROUPS $\mathcal{O}(\mathcal{B}_n)$ AND $\mathcal{C}(\mathcal{B}_n)$

For every subset $A \subseteq N$ we can build a vector

$$\boldsymbol{x}_A = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \{0,1\}^n, \qquad \text{where} \qquad \alpha_k = 1 \Leftrightarrow k \in A\,,$$

such that inclusion relation $\subseteq$ on $\mathcal{B}_n$ induces partial order $\preceq$ on the set $B_n = \{0,1\}^n$ of boolean vectors of length n. Recall that boolean function $f : B_n \to \{0,1\}$ is called *monotone*, if for every $\boldsymbol{x}, \boldsymbol{y} \in B_n$, $\boldsymbol{x} \preceq \boldsymbol{y}$ implies $f(\boldsymbol{x}) \preceq f(\boldsymbol{y})$. Denote by M_n the set of all monotone boolean functions $f : B_n \to \{0,1\}$.

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Theorem 1. $|\mathcal{O}(\mathcal{B}_n)| = |M_n|^n$.

Proof. Let $\varphi \in \mathcal{O}(\mathcal{B}_n)$. For every k, $1 \le k \le n$, consider a boolean function $\varphi_k : B_n \to \{0,1\}$ defined by the rule
$$\varphi_k(\boldsymbol{x}_A) = 1 \quad \text{if and only if} \quad k \in \varphi(A)\,.$$
It is clear that for every k function φ_k is monotone.

Consider the map
$$\mathcal{O}(\mathcal{B}_n) \longrightarrow M_n \times M_n \times \cdots \times M_n, \qquad \varphi \longmapsto (\varphi_1, \varphi_2, \ldots, \varphi_n)\,.$$
It is obvious that this map is injective. It suffices to show that it is surjective, i.e. every collection $(\psi_1, \psi_2, \ldots, \psi_n)$ of monotone boolean functions corresponds to some transformation φ in $\mathcal{O}(\mathcal{B}_n)$. It is not hard to build such transformation φ. Really, for every subset $A \subseteq N$ define
$$\varphi(A) = \{k \mid \psi_k(\boldsymbol{x}_A) = 1\}\,,$$
then, the transformation $\varphi : \mathcal{B}_n \to \mathcal{B}_n$ will be order–preserving and for every k we will have $\varphi_k = \psi_k$. □

Theorem 2. $|\mathcal{C}(\mathcal{B}_n)| = |M_{n-1}|^n$.

Proof. Since $\mathcal{C}(\mathcal{B}_n) \subseteq \mathcal{O}(\mathcal{B}_n)$, similarly, as in the proof of previous theorem, for every transformation $\varphi \in \mathcal{C}(\mathcal{B}_n)$ we can construct a collection $(\varphi_1, \varphi_2, \ldots, \varphi_n)$ of monotone boolean functions. Note that for order–decreasing transformation φ, from $k \notin A$ it follows that $k \notin \varphi(A)$. Thus, if $k \notin A$, then $\varphi_k(\boldsymbol{x}_A) = 0$.

For a fixed k the set $\{A \in \mathcal{B}_n \mid k \in A\}$ is naturally identified with the boolean $\mathcal{B}(N \setminus \{k\})$ (which is isomorphic to the boolean $\mathcal{B}_{n-1}$) as a poset. After that, restriction $\widetilde{\varphi}_k$ of the function φ_k to the set of boolean vectors $\{\boldsymbol{x}_A \mid k \in A\}$ we can consider as a boolean function of $n-1$ arguments
$$\widetilde{\varphi}_k(\alpha_1, \ldots, \alpha_{n-1}) = \varphi_k(\alpha_1, \ldots, \alpha_{k-1}, 1, \alpha_k, \ldots, \alpha_{n-1}). \tag{1}$$
Since for $k \notin A$ we have $\varphi_k(\boldsymbol{x}_A) = 0$, then the function $\widetilde{\varphi}_k$ is still monotone, i.e. $\widetilde{\varphi}_k \in M_{n-1}$. This gives us the map
$$\mathcal{C}(\mathcal{B}_n) \longrightarrow M_{n-1} \times M_{n-1} \times \cdots \times M_{n-1}, \qquad \varphi \longmapsto (\widetilde{\varphi}_1, \widetilde{\varphi}_2, \ldots, \widetilde{\varphi}_n)\,, \tag{2}$$
which, obviously, will be injective.

Show that this map is surjective, i.e. every collection $(\psi_1, \psi_2, \ldots, \psi_n)$ of monotone functions from M_{n-1} corresponds to some transformation φ in $\mathcal{C}(\mathcal{B}_n)$. For arbitrary k, $1 \le k \le n$, and a boolean vector $\boldsymbol{x} = (\alpha_1, \ldots, \alpha_n)$ denote $\boldsymbol{x}^{(k)} = (\alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_n)$. For every subset $A \subseteq N$ define
$$\varphi(A) = \{k \in A \mid \psi_k(\boldsymbol{x}_A^{(k)}) = 1\}\,. \tag{3}$$
The fact that the transformation $\varphi : \mathcal{B}_n \to \mathcal{B}_n$ is order-decreasing follows from (3). Now, let $A \supset B$. If $k \in \varphi(B)$, then $\psi_k(\boldsymbol{x}_B^{(k)}) = 1$. However, $\boldsymbol{x}_A \succeq \boldsymbol{x}_B$, and since ψ_k is monotone, it follows that $\psi_k(\boldsymbol{x}_A^{(k)}) = 1$. Hence, $k \in \varphi(A)$ and $\varphi(A) \supset \varphi(B)$. Therefore, transformation φ is order–preserving too and $\varphi \in \mathcal{C}(\mathcal{B}_n)$. Finally, from equalities (3) and (1) it follows that for every boolean vector $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$
$$\widetilde{\varphi}_k(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) = 1 \quad \Longleftrightarrow \quad \psi_k(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) = 1\,,$$
that is, $\widetilde{\varphi}_k = \psi_k$ for all k. Thus, a transformation φ is a pre-image of a collection $(\psi_1, \psi_2, \ldots, \psi_n)$ with respect to a map (2).

Therefore, the map (2) is a bijection and $|\mathcal{C}(\mathcal{B}_n)| = |M_{n-1}|^n$. □

From theorems 1 and 2 the next corollary follows.

Corollary. $|\mathcal{O}(\mathcal{B}_n)|^{n+1} = |\mathcal{C}(\mathcal{B}_{n+1})|^n$.

Note that the numbers $|M_n|$ are called *Dedekind numbers*. They arise in many problems of algebra and combinatorics. In particular, the number $|M_n|$ is equal to the order of the free distributive lattice of rank n completed with zero and unit. The problem of the computation of these numbers is very difficult. For the moment there are neither known formulas nor even algorithms, that are more effective than exhaustive search of monotone functions. One can find estimates of different kinds (lower, upper, asymptotic) for these numbers in a serious survey [6].

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Розглядаються напівгрупа $\mathcal{O}(\mathcal{B}_n)$ всіх монотонних перетворень $\varphi : \mathcal{B}_n \to \mathcal{B}_n$ впорядкованого за включенням булеану $\mathcal{B}_n$ n-елементної множини (тобто тих перетворень, для яких із $A \subseteq B$ випливає $\varphi(A) \subseteq \varphi(B)$) та її піднапівгрупа $\mathcal{C}(\mathcal{B}_n)$ тих перетворень, для яких $\varphi(A) \subseteq A$ для всіх $A \in \mathcal{B}_n$. Підраховано порядки цих напівгруп.

Ключові слова і фрази: напівгрупа, монотонне перетворення, стискуюче перетворення, монотонні булеві функції.

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Рассматриваются полугруппа $\mathcal{O}(\mathcal{B}_n)$ всех изотонных преобразований $\varphi : \mathcal{B}_n \to \mathcal{B}_n$ упорядоченного отношением включения булеана $\mathcal{B}_n$ n-элементного множества (то есть тех преобразований, для которых из $A \subseteq B$ следует $\varphi(A) \subseteq \varphi(B)$) и ее подполугруппа $\mathcal{C}(\mathcal{B}_n)$ тех преобразований, для которых $\varphi(A) \subseteq A$ для всех $A \in \mathcal{B}_n$. Подсчитаны порядки этих полугрупп.

Ключевые слова и фразы: полугруппа, изотонное преобразование, преобразование с убывающим порядком, монотонные булевы функции.