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Composition of entire and analytic functions in the unit ball

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In this paper, we investigate a composition of entire function of several complex variables and analytic function in the unit ball. We modified early known results with conditions providing equivalence of boundedness of L-index in a direction for such a composition and boundedness of l-index of initial function of one variable, where the continuous function $L: \mathbb{B}^n \to \mathbb{R}_+$ is constructed by the continuous function $l: \mathbb{C}^m \to \mathbb{R}_+$. Taking into account new ideas from recent results on composition of entire functions, we remove a condition that a directional derivative of the inner function Φ in the composition does not equal to zero. Instead of the condition we construct a greater function $L(z)$ for which $F(z) = f(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}})$ has bounded L-index in a direction, where $f\colon \mathbb{C}^m \to \mathbb{C}$ is an entire function of bounded l-index in the direction $(1, \ldots, 1)$, $\Phi\colon \mathbb{B}^n \to \mathbb{C}$ is an analytic function in the unit ball.

We weaken the condition $|\partial_{\mathbf{b}}^k \Phi(z)| \le K |\partial_{\mathbf{b}} \Phi(z)|^k$ for all $z \in \mathbb{B}^n$, where $K \ge 1$ is a constant, $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ is a given direction and

$$\partial_{\mathbf{b}} F(z) := \sum_{j=1}^{n} \frac{\partial F(z)}{\partial z_j} b_j,\ \partial_{\mathbf{b}}^k F(z) := \partial_{\mathbf{b}} \big(\partial_{\mathbf{b}}^{k-1} F(z)\big).$$

It is replaced by the condition $|\partial_{\mathbf{b}}^k \Phi(z)| \le K (l(\Phi(z)))^{1/(N_{\mathbf{1}}(f,l)+1)} |\partial_{\mathbf{b}} \Phi(z)|^k$, where $N_{\mathbf{1}}(f,l)$ is the l-index of the function f in the direction $\mathbf{1} = (1, \ldots, 1)$. The described result is an improvement of previous one. It is also a new result for the one-dimensional case $n = 1$, $m = 1$, i.e. for an analytic function Φ in the unit disc and for an entire function $f : \mathbb{C} \to \mathbb{C}$ of bounded l-index.

Key words and phrases: analytic function, unit disc, entire function, bounded L-index in direction, composite function, bounded l-index.

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Introduction

In this paper, the main objects of investigation are functions analytic in the unit ball. Recently in [1] we obtained a modification of some results on composition of entire functions and its boundedness of L-index in direction, where $L : \mathbb{C}^n \to \mathbb{R}_+$ is a continuous function. The first result of such type in [4] contains a condition that a directional derivative of inner entire function $\Phi : \mathbb{C}^n \to \mathbb{C}$ in a composition does not equal to zero. We completely removed this condition in [1] and rewrite proof with some greater function L than in [4]. Also there was

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obtained a weakening of the condition $|\partial^k_{\mathbf{b}}\Phi(z)| \leq K|\partial_{\mathbf{b}}\Phi(z)|^k$ for all $z \in \mathbb{C}^n$. It was replaced by $|\partial^k_{\mathbf{b}}\Phi(z)| \leq K(l(\Phi(z)))^{1/(N_{\mathbf{1}}(f,l)+1)}|\partial_{\mathbf{b}}\Phi(z)|^k$, where $N_{\mathbf{1}}(f,l)$ is the l-index in the direction $(1,\ldots,1)$ of the function f (see details in Theorem 1 from [1]).

In the present paper, we will consider a similar problem for functions analytic in the unit ball. The paper is an addendum to [5]. There were obtained sufficient conditions of boundedness of L-index in direction for a composition of entire function of several complex variables and analytic function in the unit ball (see below Theorem 3). By analogy to [1] we will modify this result.

Note that the notion of entire function $f\colon \mathbb{C} \to \mathbb{C}$ of bounded index ($l \equiv 1$) was firstly appeared in papers of J. Mac-Donnell and B. Lepson [13, 14] by the following property of the central index of entire function. Let $f(z) = \sum\limits_{n=0}^{+\infty} f_n(a)(z-a)^n$, where $f_n(a) = \frac{f^{(n)}(a)}{n!}$. Let $\mu_f(r,a) = \max\{|f_n(a)|r^n\colon n \geq 0\}$ be the maximal term of power series at the point $z = a$, and $\nu_f(r,a) = \max\{n\colon |f_n(a)|r^n = \mu_f(r,a)\}$ be its central index, $r > 0$. An entire function is a function of bounded index, if $\sup\{\nu_f(1,a)\colon a \in \mathbb{C}\}$ is finite. This simple definition has led us to appear sufficiently meaningful theory of functions having bounded index. In this theory, there was obtained some description of properties of analytic solutions of differential equations. If we choose $r = 1/l(a)$ instead of $r = 1$ in the definition, then we obtained a concept of entire function of bounded l-index with a positive continuous fixed function $l\colon \mathbb{C} \to (0,+\infty)$.

On the other hand, the properties of the maximal term and the central index of the power series underlie in the construction of the Wiman-Valiron theory, which, in particular, has also effective applications to the analytical theory of differential equations. At the same time, the results of both theories on the solutions of differential equations naturally complement each other. However, in the case of entire or analytical functions of several complex variables we do not fully observe this fact. In view of simplicity of the concept of bounded index the approaches and methods of the theory of functions of bounded l-index successfully can be completely transferred to many classes of analytic functions of several variables. But we can not claim similar facts about the Wiman-Valiron method. At most, in some cases of spaces of analytic functions of several variables it is possible to deduce only results concerning, for example, such an internal question in this theory, as proof of analogues of Wiman's inequality [9–12, 16]. And even in the case of analytic in the unit disc functions, the question of a complete analogue of the Wiman-Valiron theory is still not fully understood. To be fair, the main results of Wiman-Valiron theory can be applied to every entire function, in contrast to the fact that not every entire function has a function l, for which this entire function has a bounded l-index.

1 Main notations, definitions and results

Let $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$ be a unit ball, $L : \mathbb{B}^n \to \mathbb{R}_+$ be a continuous function, $\mathbf{b} = (b_1,\ldots,b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a fixed direction, where $\mathbf{0} = (0,\ldots,0)$, $\mathbf{1} = (1,\ldots,1)$. For $z \in \mathbb{B}^n$ we denote $D_z = \{t \in \mathbb{C} : |t| \leq \frac{1-|z|}{|\mathbf{b}|}\}$,

$$\lambda_{\mathbf{b}}(\eta) = \sup_{z\in\mathbb{B}^n}\sup_{t_1,t_2\in D_z}\left\{\frac{L(z+t_1\mathbf{b})}{L(z+t_2\mathbf{b})}\colon |t_1-t_2| \leq \frac{\eta}{\min\{L(z+t_1\mathbf{b}),L(z+t_2\mathbf{b})\}}\right\}.$$

The notation $Q_{\mathbf{b}}(\mathbb{B}^n)$ stands for a class of positive continuous functions $L : \mathbb{B}^n \to \mathbb{R}_+$, satisfying

$$\lambda_{\mathbf{b}}(\eta) < +\infty \quad \forall \eta \in [0, \beta], \tag{1}$$

and

$$L(z) > \frac{\beta|\mathbf{b}|}{1-|z|},$$

where $\beta > 1$ is some constant.

Similarly, $Q^n_{\mathbf{b}}$ stands for a class of positive continuous functions $L : \mathbb{C}^n \to \mathbb{R}_+$, satisfying (1) for all $\eta \in [0; +\infty)$ with

$$\lambda_{\mathbf{b}}(\eta) = \sup_{z\in\mathbb{C}^n} \sup_{t_1,t_2\in\mathbb{C}} \left\{ \frac{L(z+t_1\mathbf{b})}{L(z+t_2\mathbf{b})} : |t_1 - t_2| \le \frac{\eta}{\min\{L(z+t_1\mathbf{b}), L(z+t_2\mathbf{b})\}} \right\}.$$

Let $L\colon \mathbb{B}^n \to \mathbb{R}_+$ be any fixed continuous function. Analytic function $F : \mathbb{B}^n \to \mathbb{C}$ is called a function of *bounded L-index in a direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ (see [2]), if there exists $m_0 \in \mathbb{Z}_+$ such that for every $m \in \mathbb{Z}_+$ and for each $z \in \mathbb{B}^n$

$$\frac{|\partial^m_{\mathbf{b}} F(z)|}{m! L^m(z)} \le \max_{0\le k\le m_0} \frac{|\partial^k_{\mathbf{b}} F(z)|}{k! L^k(z)}, \tag{2}$$

where $\partial^0_{\mathbf{b}} F(z) = F(z), \partial_{\mathbf{b}} F(z) = \sum\limits_{j=1}^{n} \frac{\partial F(z)}{\partial z_j} b_j$, $\partial^k_{\mathbf{b}} F(z) = \partial_{\mathbf{b}} \left(\partial^{k-1}_{\mathbf{b}} F(z)\right)$, $k \ge 2$. The least such integer $m_0 = m_0(\mathbf{b})$ is called the L-index in the direction $\mathbf{b}$ of the analytic function F and is denoted by $N_{\mathbf{b}}(F, L) = m_0$.

Let $L\colon \mathbb{C}^n \to \mathbb{R}_+$ be any fixed continuous function. An entire function $F(z)$, $z \in \mathbb{C}^n$, is called *a function of bounded L-index in a direction* $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ (see [4]), if there exists $m_0 \in \mathbb{Z}_+$ such that for every $m \in \mathbb{Z}_+$ and every $z \in \mathbb{C}^n$ inequality (2) holds. The least such integer m_0 is called the L-index in the direction $\mathbf{b}$ and is denoted by $N_{\mathbf{b}}(F, L)$. In the case $n = 1$ and $\mathbf{b} = 1$ we obtain the definition of entire function of one variable of bounded l-index (see [6,7,15]) and $N(F, L) := N_1(F, L)$; in the case $n = 1$, $\mathbf{b} = 1$ and $L(z) \equiv 1$ it is reduced to the definition of function of bounded index, supposed by B. Lepson [13].

To prove main theorem we need auxiliary propositions. They are analogs of Hayman's Theorem for entire functions and analytic functions in the unit ball. It was firstly proved by W. Hayman [8] for entire functions of one variable having bounded index.

Theorem 1 ([3]). *Let $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ and $L \in Q^n_{\mathbf{b}}$. An entire function $F(z)$ has bounded L-index in the direction $\mathbf{b}$ if and only if there exist numbers $p \in \mathbb{Z}_+$, $R > 0$ and $C > 0$ such that for every $z \in \mathbb{C}^n$, $|z| \ge R$,*

$$\frac{|\partial^{p+1}_{\mathbf{b}} F(z)|}{L^{p+1}(z)} \le C \max\left\{ \frac{|\partial^k_{\mathbf{b}} F(z)|}{L^k(z)} : 0 \le k \le p \right\}. \tag{3}$$

Theorem 2 ([2]). *Let $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ and $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$. An analytic function $F : \mathbb{B}^n \to \mathbb{C}$ is of bounded L-index in the direction $\mathbf{b}$ if and only if there exist $p \in \mathbb{Z}_+$ and $C > 0$ such that for every $z \in \mathbb{B}^n$ inequality* (3) *holds.*

Note that the parameter $p = N_{\mathbf{b}}(F, L)$ is required in Theorem 1 and Theorem 2 (see their proofs in [2,3]).

There was obtained the following result on boundedness of L-index in direction for a composition of functions analytic in the unit ball.

Let $\Phi : \mathbb{B}^n \to \mathbb{C}$ be an analytic function, satisfying

$$|\partial_{\mathbf{b}}^j \Phi(z)| \le K|\partial_{\mathbf{b}}\Phi(z)|^j, \quad K \equiv \text{const} > 0, \tag{4}$$

for all $z \in \mathbb{B}^n$ and for all $j \le p$, where p is some positive integer number.

Theorem 3 ([5]). *Let $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, $f : \mathbb{C}^m \to \mathbb{C}$ be an entire function, $\Phi : \mathbb{B}^n \to \mathbb{C}$ be an analytic function such that $\partial_{\mathbf{b}}\Phi(z) \neq 0$ for all $z \in \mathbb{B}^n$. Suppose that $l \in Q_{\mathbf{1}}^m$, $l(w) \ge 1$, $w \in \mathbb{C}^m$, $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, $L(z) = \big|\partial_{\mathbf{b}}\Phi(z)\big| l(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}})$.*

If the entire function f has bounded l-index in the direction $\mathbf{1}$ and the function Φ satisfies (4) *with $p = N_{\mathbf{1}}(f, l)$ then the analytic function $F(z) = f(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}})$ has bounded L-index in the direction $\mathbf{b}$.*

And if the analytic function $F(z) = f(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}})$ has bounded L-index in the direction $\mathbf{b}$ and the function Φ satisfies (4) *with $p = N_{\mathbf{b}}(F, L)$ then the entire function f has bounded l-index in the direction $\mathbf{1}$.*

As in [1] for entire functions, we state that in Theorem 3 the conditions $\partial_{\mathbf{b}}\Phi(z) \neq 0$ and $|\partial_{\mathbf{b}}^j \Phi(z)| \le K|\partial_{\mathbf{b}}\Phi(z)|^j$ for every $j \in \{1, \ldots, p\}$ are generated by method of proof. In fact, we can remove it and prove more general proposition with some greater function L.

Our main result is the following.

Theorem 4. *Let $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, $l \in Q_{\mathbf{1}}^m$, $l(w) \ge 1$, $w \in \mathbb{C}^m$, $f : \mathbb{C}^m \to \mathbb{C}$ be an entire function of bounded l-index in the direction $\mathbf{1}$, $\Phi : \mathbb{B}^n \to \mathbb{C}$ be an analytic function.*

Suppose that $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$ with

$$L(z) = \max\{1, |\partial_{\mathbf{b}}\Phi(z)|\}\, l(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}}), \tag{5}$$

and for all $z \in \mathbb{B}^n$ and $k \in \{1, 2, \ldots, N_{\mathbf{1}}(f, l) + 1\}$ one has

$$|\partial_{\mathbf{b}}^k \Phi(z)| \le K(l(\Phi(z)))^{1/(N_{\mathbf{1}}(f,l)+1)}|\partial_{\mathbf{b}}\Phi(z)|^k, \tag{6}$$

where $K \ge 1$ is a constant. Then the analytic function $F(z) = f(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}})$ has bounded L-index in the direction $\mathbf{b}$.

Theorem 4 is new even in one-dimensional case, i.e. for analytic in the unit disc functions. Note that in [5] Theorem 4 is formulated without condition (6). There is written "The proof of this theorem is similar to proof of Theorem 3 and also use analogs of Hayman's Theorem for entire functions of bounded L-index in direction". But careful analysis of the proof of Theorem 4 shows that the condition is important. And now we do not know whether is it possible to remove this condition (6). But condition (6) is significantly weaker than condition (4) by some arbitrariness in choice of the function l. This arbitrariness is generated the fact if a function has bounded l-index then the function has bounded l_1-index for any function l_1 such that $l_1(z) \ge l(z)$ for all $z \in \mathbb{C}$.

2 Proof of Theorem 4

Proof of Theorem 4. Our proof combines ideas from proofs of corresponding theorems in [5] and [1]. Denote $\nabla f = \partial_{\mathbf{1}} f = \sum_{j=1}^{m} \frac{\partial f}{\partial z_j}$, $\nabla^k f \equiv \partial_{\mathbf{1}}^k f$ for $k \geq 2$. In the proof of Theorem 3 in [5] there was established the following formula by method of mathematical induction

$$\partial_{\mathbf{b}}^k F(z) = \nabla^k f(\Phi(z), \ldots, \Phi(z)) \left(\partial_{\mathbf{b}} \Phi(z)\right)^k + \sum_{j=1}^{k-1} \nabla^j f(\Phi(z), \ldots, \Phi(z)) Q_{j,k}(z), \qquad (7)$$

where

$$Q_{j,k}(z) = \sum_{\substack{n_1+2n_2+\ldots+kn_k=k \\ 0 \leq n_1 \leq j-1}} c_{j,k,n_1,\ldots,n_k} \left(\partial_{\mathbf{b}} \Phi(z)\right)^{n_1} \left(\partial_{\mathbf{b}}^2 \Phi(z)\right)^{n_2} \ldots \left(\partial_{\mathbf{b}}^k \Phi(z)\right)^{n_k},$$

and $c_{j,k,n_1,\ldots,n_k}$ are non-negative integer numbers.

There was also deduced that

$$\nabla^k f(\Phi(z), \ldots, \Phi(z)) = \frac{\partial_{\mathbf{b}}^k F(z)}{\left(\partial_{\mathbf{b}} \Phi(z)\right)^k} + \frac{1}{\left(\partial_{\mathbf{b}} \Phi(z)\right)^{2k}} \sum_{j=1}^{k-1} \partial_{\mathbf{b}}^j F(z) \left(\partial_{\mathbf{b}} \Phi(z)\right)^j Q_{j,k}^*(z), \qquad (8)$$

where

$$Q_{j,k}^*(z) = \sum_{m_1+2m_2+\ldots+km_k=2(k-j)} b_{j,k,m_1,\ldots,m_k} \left(\partial_{\mathbf{b}} \Phi(z)\right)^{m_1} \left(\partial_{\mathbf{b}}^2 \Phi(z)\right)^{m_2} \ldots \left(\partial_{\mathbf{b}}^k \Phi(z)\right)^{m_k},$$

and $b_{j,k,m_1,\ldots,m_k}$ are some integer coefficients.

Denote $L_0(z) = l(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}}) |\partial_{\mathbf{b}} \Phi(z)|$. Taking into account (7) and (5), for $k = p+1$ we have

$$\begin{aligned} \frac{|\partial_{\mathbf{b}}^{p+1} F(z)|}{L_0^{p+1}(z)} &\leq \frac{|\nabla^{p+1} f(\Phi(z), \ldots, \Phi(z))|}{L_0^{p+1}(z)} |\partial_{\mathbf{b}} \Phi(z)|^{p+1} \\ &\quad + \sum_{j=1}^{p} \frac{|\nabla^j f(\Phi(z), \ldots, \Phi(z))| |Q_{j,p+1}(z)|}{L_0^{p+1}(z)} \\ &\leq \frac{|\nabla^{p+1} f(\Phi(z), \ldots, \Phi(z))| \, |\partial_{\mathbf{b}} \Phi(z)|^{p+1}}{l^{p+1}(\Phi(z), \ldots, \Phi(z)) \, |\partial_{\mathbf{b}} \Phi(z)|^{p+1}} \\ &\quad + \sum_{j=1}^{p} \frac{|\nabla^j f(\Phi(z), \ldots, \Phi(z))|}{l^j(\Phi(z), \ldots, \Phi(z))} \cdot \frac{|Q_{j,p+1}(z)| l^j(\Phi(z), \ldots, \Phi(z))}{|\partial_{\mathbf{b}} \Phi(z)|^{p+1} \, l^{p+1}(\Phi(z), \ldots, \Phi(z))}. \end{aligned} \qquad (9)$$

Let f be an entire function of bounded l-index in the direction $\mathbf{1}$. By Theorem 1 inequality (3) holds for $n = m$, $F = f$, $L = l$, $\mathbf{b} = \mathbf{1}$ and $p = N_{\mathbf{1}}(f, l)$, namely

$$\frac{|\nabla^{p+1} f(\tau)|}{l^{p+1}(\tau)} \le C \max\left\{\frac{|\nabla^k f(\tau)|}{l^k(\tau)} \colon\ 0 \le k \le p\right\} \quad \forall \tau \in \mathbb{C}^m.$$

Applying to (9) this inequality with $\tau = (\Phi(z), \ldots, \Phi(z))$, we obtain

$$\begin{aligned}\frac{|\partial_{\mathbf{b}}^{p+1} F(z)|}{L_0^{p+1}(z)} &\le \max\left\{\frac{|\nabla^k f(\Phi(z), \ldots, \Phi(z))|}{l^k(\Phi(z), \ldots, \Phi(z))} \colon\ 0 \le k \le p\right\} \\ &\quad\times\left(C + \sum_{j=1}^{p} \frac{|Q_{j,p+1}(z)| l^{j-p-1}(\Phi(z), \ldots, \Phi(z))}{|\partial_{\mathbf{b}} \Phi(z)|^{p+1}}\right) \\ &\le \max\left\{\frac{|\nabla^k f(\Phi(z), \ldots, \Phi(z))|}{l^k(\Phi(z), \ldots, \Phi(z))} \colon\ 0 \le k \le p\right\} \\ &\quad\times\left(C + \sum_{j=1}^{p} \sum_{\substack{\sum_{k=1}^{p+1} k n_k = p+1 \\ 0 \le n_1 \le j-1}} c_{j,p+1,n_1,\ldots,n_{p+1}} \frac{|(\partial_{\mathbf{b}} \Phi(z))^{n_1} (\partial_{\mathbf{b}}^2 \Phi(z))^{n_2} \ldots \left(\partial_{\mathbf{b}}^{p+1} \Phi(z)\right)^{n_{p+1}}|}{l^{p+1-j}(\Phi(z), \ldots, \Phi(z))\, |\partial_{\mathbf{b}} \Phi(z)|^{p+1}}\right).\end{aligned} \tag{10}$$

In view of condition (6) inequality (10) yields

$$\begin{aligned}\frac{|\partial_{\mathbf{b}}^{p+1} F(z)|}{L_0^{p+1}(z)} &\le \max\left\{\frac{|\nabla^k f(\Phi(z))|}{l^k(\Phi(z), \ldots, \Phi(z))} \colon 0 \le k \le p\right\} \\ &\quad\times\left(C + \sum_{j=1}^{p} \sum_{\substack{\sum_{k=1}^{p+1} k n_k = p+1 \\ 0 \le n_1 \le j-1}} \frac{c_{j,p+1,n_1,\ldots,n_{p+1}} K^{p+1} l(\Phi(z), \ldots, \Phi(z)) |\partial_{\mathbf{b}} \Phi(z)|^{p+1}}{l^{p+1-j}(\Phi(z), \ldots, \Phi(z))\, |\partial_{\mathbf{b}} \Phi(z)|^{p+1}}\right) \\ &\le \max\left\{\frac{|\nabla^k f(\Phi(z), \ldots, \Phi(z))|}{l^k(\Phi(z), \ldots, \Phi(z))} \colon 0 \le k \le p\right\} \\ &\quad\times\left(C + \sum_{j=1}^{p} \sum_{\substack{\sum_{k=1}^{p+1} k n_k = p+1 \\ 0 \le n_1 \le j-1}} \frac{c_{j,p+1,n_1,\ldots,n_{p+1}} K^{p+1}}{l^{p-j}(\Phi(z, \ldots, \Phi(z)))}\right).\end{aligned} \tag{11}$$

We will use that $l(\Phi(z)) \ge 1$. Then from (11) it follows

$$\frac{|\partial_{\mathbf{b}}^{p+1} F(z)|}{L_0^{p+1}(z)} \le C_1 \max\left\{\frac{|\nabla^k f(\Phi(z), \ldots, \Phi(z))|}{l^k(\Phi(z), \ldots, \Phi(z))} \colon 0 \le k \le p\right\}, \tag{12}$$

where

$$C_1 = C + K^{p+1} \sum_{j=1}^{p} \sum_{\substack{\sum_{k=1}^{p+1} k n_k = p+1 \\ 0 \le n_1 \le j-1}} c_{j,p+1,n_1,\ldots,n_{p+1}}.$$

Applying equality (8), we can estimate the fraction $\frac{|\nabla^k f(\Phi(z),\ldots,\Phi(z))|}{l^k(\Phi(z),\ldots,\Phi(z))}$ as follows

$$\begin{aligned}
\frac{|\nabla^k f(\Phi(z),\ldots,\Phi(z))|}{l^k(\Phi(z),\ldots,\Phi(z))} &\leq \frac{|\partial^k_{\mathbf{b}} F(z)|}{l^k(\Phi(z),\ldots,\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^k} \\
&\quad + \sum_{j=1}^{k-1} \frac{|\partial^j_{\mathbf{b}} F(z)||Q^*_{j,k}(z)|}{l^k(\Phi(z),\ldots,\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{2k-j}} \\
&\leq \max_{1\leq j\leq k}\left\{\frac{|\partial^j_{\mathbf{b}}\Phi(z)|}{l^j(\Phi(z),\ldots,\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^j}\right\} \\
&\quad \times\left(1+\sum_{j=1}^{k-1}\frac{|Q^*_{j,k}(z)|}{l^{k-j}(\Phi(z),\ldots,\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{2(k-j)}}\right) \\
&\leq \max\left\{\frac{|\partial^j_{\mathbf{b}}\Phi(z)|}{l^j(\Phi(z),\ldots,\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^j} : 1\leq j\leq k\right\} \\
&\quad \times\Bigg(1+\sum_{j=1}^{k-1}\sum_{\sum_{i=1}^k im_i=2(k-j)} |b_{j,k,m_1,\ldots,m_k}| \\
&\qquad\qquad \times \frac{|(\partial_{\mathbf{b}}\Phi(z))^{m_1}(\partial^2_{\mathbf{b}}\Phi(z))^{m_2}\ldots(\partial^k_{\mathbf{b}}\Phi(z))^{m_k}|}{l^{k-j}(\Phi(z),\ldots,\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{2(k-j)}}\Bigg).
\end{aligned} \tag{13}$$

From inequalities (6) and $l(w)\geq 1$ it follows that $|\partial^s_{\mathbf{b}}\Phi(z)|\leq Kl^{s/2}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^s$, because $s/2\geq 1/(N_{\mathbf{1}}(f,l)+1)$ for $s\in\{1,2,\ldots,N_{\mathbf{1}}(f,l)+1\}$ and $N_{\mathbf{1}}(f,l)\geq 1$. Applying this inequality to (13), we deduce

$$\begin{aligned}
\frac{|\nabla^k f(\Phi(z),\ldots,\Phi(z))|}{l^k(\Phi(z))} &\leq \max\left\{\frac{|\partial^j_{\mathbf{b}} F(z)|}{l^j(\Phi(z),\ldots,\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^j} : 1\leq j\leq k\right\} \\
&\quad \times\Bigg(1+\sum_{j=1}^{k-1}\sum_{\sum_{i=1}^k im_i=2(k-j)} |b_{j,k,m_1,\ldots,m_k}|K^{m_1+m_2+\ldots+m_k} \\
&\quad \times\frac{(l(\Phi(z),\ldots,\Phi(z)))^{(m_1+2m_2+\ldots+km_k)/2}|\partial_{\mathbf{b}}\Phi(z)|^{m_1+2m_2+\ldots+km_k}}{l^{k-j}(\Phi(z),\ldots,\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{2(k-j)}}\Bigg) \\
&\leq C_2\max\left\{\frac{|\partial^j_{\mathbf{b}}\Phi(z)|}{l^j(\Phi(z),\ldots,\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^j} : 1\leq j\leq k\right\},
\end{aligned}$$

where

$$C = 1+\sum_{j=1}^{k-1}\sum_{\sum_{i=1}^k im_i=2(k-j)} |b_{j,k,m_1,\ldots,m_k}|K^{m_1+m_2+\ldots+m_k}.$$

Then from inequality (12) we get

$$\begin{aligned}
\frac{|\partial^{p+1}_{\mathbf{b}}F(z)|}{L_0^{p+1}(z)} &\leq C_1\max\left\{\frac{|f^{(k)}(\Phi(z),\ldots,\Phi(z))|}{l^k(\Phi(z),\ldots,\Phi(z))} : 0\leq k\leq p\right\} \\
&\leq C_1C_2\max\left\{\frac{|\partial^j_{\mathbf{b}}F(z)|}{L_0^j(z)} : 0\leq j\leq p\right\},\quad p=N_{\mathbf{1}}(f,l).
\end{aligned} \tag{14}$$

The last inequality is proved for all z such that $\partial_{\mathbf{b}}\Phi(z) \neq 0$ and by the condtion $N_{\mathbf{1}}(f,l) \geq 1$. If $N_{\mathbf{1}}(f,l) = 0$ then $p = 0$ and from (12) we deduce

$$\frac{|\partial_{\mathbf{b}} F(z)|}{L_0(z)} \leq C_1 |f(\Phi(z),\ldots,\Phi(z))| = C_1|F(z)|.$$

Thus, (14) is proved for all $N_{\mathbf{1}}(f,l)$.

Remind that $L(z) = l(\Phi(z),\ldots,\Phi(z)) \max\{1, |\partial_{\mathbf{b}}\Phi(z)|\}$. Rewrite inequality (14) in the following form

$$\frac{\left|\partial_{\mathbf{b}}^{p+1} F(z)\right|}{L^{p+1}(z)} \cdot \frac{L^{p+1}(z)}{L_0^{p+1}(z)} \leq C_1 C_2 \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z)|}{L^k(z)} \frac{L^k(z)}{L_0^k(z)} \colon\ 0 \leq k \leq p \right\}.$$

Then

$$\begin{aligned}
\frac{\left|\partial_{\mathbf{b}}^{p+1} F(z)\right|}{L^{p+1}(z)} &\leq C_1 C_2 \frac{L_0^{p+1}(z)}{L^{p+1}(z)} \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z)|}{L^k(z)} \frac{L^k(z)}{L_0^k(z)} \colon\ 0 \leq k \leq p \right\} \\
&\leq C_1 C_2 \frac{L_0^{p+1}(z)}{L^{p+1}(z)} \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z)|}{L^k(z)} \colon\ 0 \leq k \leq p \right\} \max\left\{ \frac{L^k(z)}{L_0^k(z)} \colon\ 0 \leq k \leq p \right\} \qquad (15) \\
&= C_1 C_2 \frac{(L_0(z)/L(z))^{p+1}}{\min\limits_{0 \leq k \leq p} (L_0(z)/L(z))^k} \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z)|}{L^k(z)} \colon\ 0 \leq k \leq p \right\}.
\end{aligned}$$

Let $t_0 = t(z) = L_0(z)/L(z)$ and $k_0 \leq p$ $(k_0 \in \mathbb{Z}_+)$ be such that $(t_0)^{k_0} = \min_{0 \leq k \leq p} t_0^k$. One should observe that $t_0 \in (0,1]$ and $p + 1 - k_0 \geq 1$. Hence,

$$\frac{t_0^{p+1}}{t_0^{k_0}} = t_0^{p+1-k_0} \leq t_0 \leq 1.$$

Therefore,

$$\frac{(L_0(z)/L(z))^{p+1}}{\min_{0 \leq k \leq p} (L_0(z)/L(z))^k} = t_0^{p+1-k_0} \leq t_0 \leq 1.$$

Thus, from inequality (15) we get

$$\frac{\left|\partial_{\mathbf{b}}^{p+1} F(z)\right|}{L^{p+1}(z)} \leq C_1 C_2 \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z)|}{L^k(z)} \colon\ 0 \leq k \leq p \right\} \qquad (16)$$

for all z such that $\partial_{\mathbf{b}}\Phi(z) \neq 0$.

If $\partial_{\mathbf{b}}\Phi(z) = 0$ then for any $k \in \{1,\ldots,N(f,l)+1\}$ inequality (6) implies $\partial_{\mathbf{b}}^k\Phi(z) = 0$. In view of (7) it means that $\partial_{\mathbf{b}}^k F(z) = 0$ for each $k \in \{1,\ldots,N(f,l)+1\}$. Thus, for the points z such that $\partial_{\mathbf{b}}\Phi(z) = 0$ inequality (16) is also satisfied.

Therefore, by Theorem 2 this inequality means that the function F has bounded L-index in the direction $\mathbf{b}$. $\square$

For $n = 1$ and $m = 1$ Theorem 4 yields the following corollary, where denoted $\mathbb{D} = \mathbb{B}^1$.

Corollary. *Let $l \in Q$, $l(w) \geq 1$, $w \in \mathbb{C}$, $f : \mathbb{C} \to \mathbb{C}$ be an entire function of bounded l-index, $\Phi : \mathbb{D} \to \mathbb{C}$ be an analytic function.*

Suppose that $L \in Q(\mathbb{D})$ with $L(z) = \max\{1, |\Phi'(z)|\}\, l(\Phi(z))$ and for all $z \in \mathbb{D}$ and $k \in \{1, 2, \ldots, N(f,l)+1\}$ one has

$$|\Phi^{(k)}(z)| \leq K(l(\Phi(z)))^{1/(N(f,l)+1)}|\Phi'(z)|^k,$$

where $K \geq 1$ is a constant.

Then the analytic function $F(z) = f(\Phi(z))$ has bounded L-index.

This corollary is new result for functions analytic in the unit disc.

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Бандура А.І., Скасків О.Б., Тимків І.Р. *Композиція цілої і аналітичної в одиничній кулі функцій* // Карпатські матем. публ. — 2022. — Т.14, №1. — С. 95–104.

У статті досліджується композиція цілої функції від багатьох комплексних змінних і аналітичної функції в одиничній кулі. У статті отримано певні нові версії встановлених раніше результатів, які містять умови, що забезпечують еквівалентність обмеженості L-індексу за напрямком такої композиції і обмеженість l-індексу початкової функції від однієї змінно, де $L : \mathbb{B}^n \to \mathbb{R}_+$ — неперервна функція, побудована за неперервною функцією $l : \mathbb{C}^m \to \mathbb{R}_+$. Застосовуючи деякі нові ідеї з недавніх результатів про композиції цілих функцій, ми знімаємо умову, що похідна за напрямком від внутрішньої функції Φ в композиції не дорівнює нулю. Власне, цієї умови позбуваємося, будуючи більшу функцію $L(z)$, для якої $F(z) = f(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ раз}})$ має обмежений L-індекс за напрямком, де $f\colon \mathbb{C}^m \to \mathbb{C}$ — ціла функція обмеженого l-індексу за напрямком $(1, \ldots, 1)$, $\Phi\colon \mathbb{B}^n \to \mathbb{C}$ — аналітична функція в одиничній кулі.

Ми послаблюємо умову $|\partial_{\mathbf{b}}^k \Phi(z)| \le K|\partial_{\mathbf{b}}\Phi(z)|^k$ для всіх $z \in \mathbb{B}^n$, де $K \ge 1$ — деяка стала, $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ — заданий напрямок, а

$$\partial_{\mathbf{b}} F(z) := \sum_{j=1}^{n} \frac{\partial F(z)}{\partial z_j} b_j,\ \partial_{\mathbf{b}}^k F(z) := \partial_{\mathbf{b}}\big(\partial_{\mathbf{b}}^{k-1} F(z)\big).$$

Вказану умову замінюємо на умову $|\partial_{\mathbf{b}}^k \Phi(z)| \le K(l(\Phi(z)))^{1/(N_{\mathbf{1}}(f,l)+1)}|\partial_{\mathbf{b}}\Phi(z)|^k$, де $N_{\mathbf{1}}(f,l)$ — l-індекс функції f за напрямком $\mathbf{1} = (1, \ldots, 1)$. Отриманий результат покращує попередній результат і є також новим в одновимірному випадку $n = 1$, $m = 1$, тобто, якщо Φ є аналітичною функцією в одиничному крузі та $f : \mathbb{C} \to \mathbb{C}$ — цілою функцією обмеженого l-індексу.

Ключові слова і фрази: аналітична функція, одиничний круг, ціла функція, обмежений L-індекс за напрямком, складена функція, обмежений l-індекс.