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ASYMPTOTIC APPROXIMATION OF SOLUTION TO QUASILINEAR ELLIPTIC BOUNDARY-VALUE PROBLEM IN A TWO-LEVEL THICK JUNCTION OF TYPE 3:2:2

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We consider quasilinear elliptic boundary-value problem in a two-level thick junction Ω_ε of type $3:2:2$, which is the union of a cylinder Ω_0 and a large number of ε-periodically situated thin discs with varying thickness. Different Robin boundary conditions with perturbed parameters are given on the surfaces of the thin discs. The leading terms of the asymptotic expansion are constructed and the corresponding estimate in Sobolev space is obtained.

Introduction

A thick junction of type $m : k : d$ is a union of some domain, which is called the junction's body, and a large number of ε-periodically alternating thin domains, which are attached to some manifold (the joint zone) on the boundary of the junction's body. The small parameter ε characterizes distance between neighboring thin domains and their thickness. The type $m : k : d$ of a thick junction refers, respectively, to the limiting dimensions (as $\varepsilon \to 0$) of the junction's body, the joint zone and each of the attached thin domains.

The subject of the investigation of boundary-value problems in thick junctions is the asymptotic behavior of solutions to such problems as $\varepsilon \to 0$, i.e. when the number of the attached thin domains infinitely increases and their thickness tends to zero.

The first researches in this direction were carried out in [9, 10, 14], where the convergence theorems for Green function of the Neumann problem for the Helmholz equation in the junction's body were proved. In these papers either the assumption about the convergence of certain components of the boundary-value problem was made, or explicit representations of

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Figure 1: Heat radiator that has form of a thick junction of type $3:2:2$.

certain quantities were used, which was possible under certain configurations of the junction's body (the half-space). In [21]–[17], [23] thick junctions were classified, asymptotic methods for the investigation of main boundary-value problems of mathematical physics in thick junctions of different types were developed, the convergence theorems were proved, the first terms of asymptotic expansions were constructed, and the corresponding estimates were proved. It was shown that qualitative properties of solutions essentially depend on the junction's type and the conditions given on the boundaries of the attached thin domains (see also [2, 1, 18]).

As an extension of the investigation, in papers [5, 7, 22] thick junctions of more complicated geometric structure were considered, namely multi-level thick junctions. A multi-level thick junction is a thick junction, in which thin domains are divided into finitely many levels depending on their geometric structure and boundary conditions imposed on their surfaces. Besides, thin domains from each level ε-periodically alternate along the joint zone. In these papers linear boundary-value problems in thick junctions of types $2:1:1$ and $3:2:1$ were considered. Moreover, there a new qualitative difference in the asymptotic behavior of solutions to boundary-value problems in multi-level thick junctions was noticed, namely the "multi-phase" effect in the domain that is filled up simultaneously by the thin domains from different levels.

The successful applying in nanotechnology and microelectronics of constructions, which have form of thick junctions (see Fig. 1 and [11]–[13]), has lead to effective studying of boundary-value problems in thick junctions of various types and more complicated structure (see also [2]–[4], [16, 18]).

In the present paper we consider quasilinear parabolic boundary-value problem in a two-level thick junction of type $3:2:2$, which consists of a cylinder Ω_0 and a large number of thin annular discs with varying thickness, which are ε-periodically attached to Ω_0. Different nonhomogeneous Robin boundary conditions are given on the surfaces of the thin discs from various levels. The leading terms of the asymptotic expansion for a solution to this problem are constructed and the asymptotic estimate in Sobolev space is proved.

The outline of the paper is as follows. In Section 1 thick junction Ω_ε is described and quasilinear elliptic boundary-value problem in this thick junction is stated. In Section 2 outer and inner asymptotic expansions for the solution u_ε are constructed and homogenized boundary-value problem is obtained. In Section 3 approximation function R_ε for solution

u_ε is constructed and asymptotic estimate is proved. In Section 4 the obtained results are discussed.

1 Statement of Problem

Let $0 < d_0 < d_2 \le d_1$ and $0 < b_2 < b_1 < 1$; $h_i : [d_0,\, d_i] \to (0,\, 1)$, $i = 1, 2$, are piecewise smooth functions. Suppose that functions h_i satisfy the following conditions:

$$0 < b_i - \frac{h_i(s)}{2},\ b_i + \frac{h_i(s)}{2} < 1 \quad \forall s \in [d_0,\, d_i],\ i = 1,\, 2, \quad b_2 + \frac{h_2(s)}{2} < b_1 - \frac{h_1(s)}{2} \quad \forall s \in [d_0,\, d_2].$$

These inequalities imply that for all $s \in [d_0,\, d_i]$ the intervals

$$I_i(s) := \left(b_i - \frac{h_i(s)}{2},\, b_i + \frac{h_i(s)}{2}\right), \quad i = 1,\, 2,$$

belong to interval $(0,\, 1)$, don't have common points and don't adjoin.

We additionally assume that functions $h_1,\, h_2$ are constant in some neighborhood of d_0, i.e. there exists $\delta > 0$ such that $h_i(s) = h_i(d_0)$ for all $s \in [d_0,\, d_0 + \delta]$, $i = 1, 2$.

Consider a model thick junction Ω_ε of type $3 : 2 : 2$ (see Fig. 2) that consists of cylinder

$$\Omega_0 = \{x = (x_1,\, x_2,\, x_3) \in \mathbb{R}^3 :\ 0 < x_2 < l,\ r := \sqrt{x_1^2 + x_3^2} < d_0\}$$

and $2N$ thin annular discs

$$G_\varepsilon^{(1)}(j) = \{x \in \mathbb{R}^3 :\ |x_2 - \varepsilon(j + b_1)| < \frac{\varepsilon h_1(r)}{2},\ d_0 \le r < d_1\},$$

$$G_\varepsilon^{(2)}(j) = \{x \in \mathbb{R}^3 :\ |x_2 - \varepsilon(j + b_2)| < \frac{\varepsilon h_2(r)}{2},\ d_0 \le r < d_2\},$$

where $j = \overline{0,\, N-1}$, $\varepsilon = l/N$, i.e.

$$\Omega_\varepsilon = \Omega_0 \cup G_\varepsilon, \quad G_\varepsilon = G_\varepsilon^{(1)} \cup G_\varepsilon^{(2)}, \quad G_\varepsilon^{(1)} = \cup_{j=0}^{N-1} G_\varepsilon^{(1)}(j), \quad G_\varepsilon^{(2)} = \cup_{j=0}^{N-1} G_\varepsilon^{(2)}(j).$$

Here N is a large integer. Therefore, ε is a small parameter, which characterizes distance between neighboring thin discs and their thickness.

Denote by $S_\varepsilon^{(1)}$ and $S_\varepsilon^{(2)}$ the union of the lateral surfaces of the thin discs from the first and the second level, respectively, and by $S^\pm$ the bases of cylinder Ω_0, i.e.

$$S_\varepsilon^{(i)} := \{x \in \partial G_\varepsilon^{(i)} :\ |x_2 - \varepsilon(j + b_i)| = \varepsilon h_i(r)/2,\ j = \overline{0, N-1},\ r \in (d_0,\, d_i)\}, \quad i = 1,\, 2,$$

$$S^- = \{x \in \partial\Omega_0 :\ x_2 = 0\}, \quad S^+ = \{x \in \partial\Omega_0 :\ x_2 = l\}, \quad S^\pm = S^+ \cup S^-.$$

Also we introduce the following notations:

$$\overline{\Omega}_i = \overline{\Omega}_0 \cup \overline{D}_i, \quad D_i = \{x \in \mathbb{R}^3 :\ 0 < x_2 < l,\ d_0 < r < d_i\},\ i = 1,\, 2,$$

$$Q_0^{(i)} = \{x \in \partial\Omega_i :\ r = d_i\},\ i = \overline{0, 2}, \quad Q_\varepsilon^{(i)} = \{x \in \partial G_\varepsilon^{(i)} :\ r = d_i\},\ i = 1,\, 2,$$

$$\Upsilon_\varepsilon^{(i)} = S_\varepsilon^{(i)} \cup Q_\varepsilon^{(i)}, \quad \Theta_\varepsilon^{(i)} = G_\varepsilon^{(i)} \cap \partial\Omega_0,\ i = 1,\, 2, \quad \Theta_\varepsilon = \Theta_\varepsilon^{(1)} \cup \Theta_\varepsilon^{(2)}, \quad Q_\varepsilon^{(0)} = Q_0^{(0)} \setminus \Theta_\varepsilon.$$

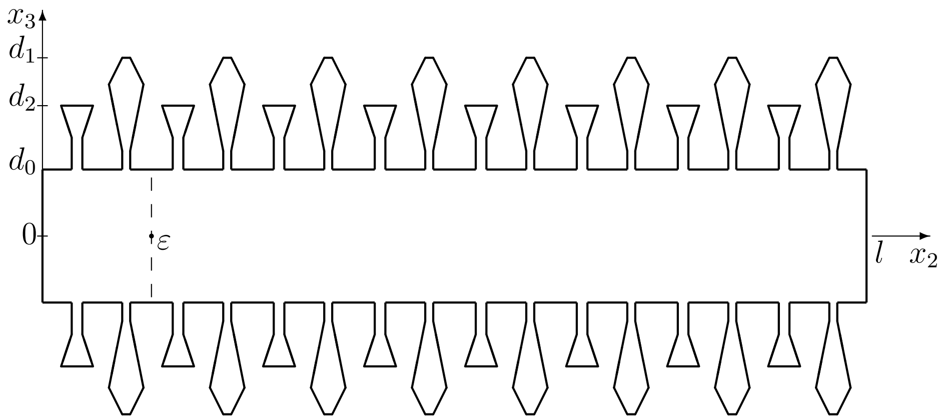

Figure 2: The cross-section of thick junction Ω_ε of type $3:2:2$ ($N=8$).

In thick junction Ω_ε we consider the quasilinear elliptic boundary-value problem

$$\begin{cases} -\Delta u_\varepsilon + \vartheta_0(u_\varepsilon) = f_\varepsilon & \text{in } \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon + \varepsilon\vartheta_1(u_\varepsilon) = \varepsilon^\beta g_\varepsilon & \text{on } S_\varepsilon^{(1)}, \\ \partial_\nu u_\varepsilon + \vartheta_1(u_\varepsilon) = 0 & \text{on } Q_\varepsilon^{(1)}, \\ \partial_\nu u_\varepsilon + \varepsilon^\alpha\vartheta_2(u_\varepsilon) = \varepsilon^\beta g_\varepsilon & \text{on } \Upsilon_\varepsilon^{(2)}, \\ \partial_\nu u_\varepsilon = 0 & \text{on } Q_\varepsilon^{(0)}, \\ \partial_{x_2}^p u_\varepsilon|_{S^-} = \partial_{x_2}^p u_\varepsilon|_{S^+}, & p=0,\,1, \\ [u_\varepsilon]|_{r=d_0} = [\partial_r u_\varepsilon]|_{r=d_0} = 0 & \text{on } \Theta_\varepsilon. \end{cases} \tag{1}$$

Here $\partial_\nu = \partial/\partial\nu$ is the outward normal derivative; $\alpha,\ \beta \geq 1$ are parameters; the square brackets denote the jump of the enclosed quantities. For the right-hand sides of problem (1) we assume that $f_\varepsilon \in L^2(\Omega_\varepsilon)$, $g_\varepsilon \in H^1(D_1)$ and

$$\exists C_0 > 0 \quad \exists \varepsilon_0 > 0 \quad \forall \varepsilon \in (0,\, \varepsilon_0): \quad \|g_\varepsilon\|_{L^2(D_1)} + \|\partial_{x_2} g_\varepsilon\|_{L^2(D_1)} \leq C_0.$$

Functions ϑ_i are Lipschitz-continuous (which is equal to $\vartheta_i \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$) and

$$\exists c_1,\, c_2 > 0: \quad c_1 \leq \vartheta_i'(s) \leq c_2 \quad \text{for a.e. } s \in \mathbb{R},\ i = \overline{0,\, 2}. \tag{2}$$

Consider spaces $H_\varepsilon = \{\varphi \in H^1(\Omega_\varepsilon): \ \varphi|_{S^-} = \varphi|_{S^+}\}$.

A function $u_\varepsilon \in H_\varepsilon$ is a weak solution to problem (1) if for any function $\varphi \in H_\varepsilon$ the following integral identity holds:

$$\int_{\Omega_\varepsilon} (\nabla u_\varepsilon \cdot \nabla\varphi + \vartheta_0(u_\varepsilon)\,\varphi)\,dx + \varepsilon\int_{S_\varepsilon^{(1)}} \vartheta_1(u_\varepsilon)\,\varphi\,d\sigma_x + \int_{Q_\varepsilon^{(1)}} \vartheta_1(u_\varepsilon)\,\varphi\,d\sigma_x$$
$$+\,\varepsilon^\alpha \int_{\Upsilon_\varepsilon^{(2)}} \vartheta_2(u_\varepsilon)\,\varphi\,d\sigma_x = \int_{\Omega_\varepsilon} f_\varepsilon\,\varphi\,dx + \varepsilon^\beta \int_{S_\varepsilon^{(1)}\cup\Upsilon_\varepsilon^{(2)}} g_\varepsilon\,\varphi\,d\sigma_x. \tag{3}$$

By the same arguments as in [16] we can prove that for any fixed $\varepsilon > 0$ there exists a unique weak solution to problem (1).

The aim is to study the asymptotic behavior of the solution to problem (1) as $\varepsilon \to 0$, i.e. when the number of the attached thin discs infinitely increases and their thickness tends to zero.

2 Formal Asymptotic Expansions for the Solution

Only in this Section for formal calculations we assume that functions f_ε, g_ε do not depend on ε, i.e. $f_\varepsilon = f_0$ in Ω_1 and $g_\varepsilon = g_0$ in D_1, and they are smooth in $\overline{\Omega}_1$ and $\overline{D}_1$, respectively.

2.1 Outer Expansions

We seek the leading terms of the asymptotic expansion for solution u_ε, restricted to Ω_0, in the form

$$u_\varepsilon(x) \approx u_0^+(x) + \sum_{k\geq 1} \varepsilon^k u_k^+(x), \quad x \in \Omega_0, \tag{4}$$

and, restricted to the thin discs $G_\varepsilon^{(i)}(j)$, $j = \overline{0, N-1}$, in the form

$$u_\varepsilon(x) \approx u_0^{i,-}(x) + \sum_{k\geq 1} \varepsilon^k u_k^{i,-}(x,\, \xi_2 - j), \quad x \in G_\varepsilon^{(i)}(j), \quad i = 1,\, 2, \tag{5}$$

where $\xi_2 = x_2/\varepsilon$.

Expansions (4) and (5) are usually called *outer expansions*.

With the help of Taylor's formula we get

$$\vartheta_0(u_\varepsilon(x)) = \vartheta_0(u_0^+(x)) + \vartheta_0'(\,\cdot\,) \sum_{k\geq 1} \varepsilon^k u_k^+(x), \quad x \in \Omega_0. \tag{6}$$

Plugging the series (4) into the first equation of problem (1) and the boundary conditions on $S^\pm$, using (6) and collecting coefficients of the same powers of ε, we get the following relations for function u_0^+:

$$\begin{cases} -\Delta u_0^+ + \vartheta_0(u_0^+) &=& f_0 & \text{in } \Omega_0, \\ \partial_{x_2}^p u_0^+|_{S^-} &=& \partial_{x_2}^p u_0^+|_{S^+}, & p = 0,\, 1. \end{cases}$$

Now let us find the limit relations in domains D_i, $i = 1,\, 2$, which are filled up by the thin discs from i-th level as ε tends to zero. Assuming for a moment that functions $u_k^{i,-}$ are smooth, we write their Taylor series with respect to x_2 at the point $\varepsilon(j + b_i)$ and pass to the "rapid" variable $\xi_2 = x_2/\varepsilon$. Then (5) takes the form

$$u_\varepsilon(x) \approx u_0^{i,-}(x_1,\, \varepsilon(j + b_i),\, x_3) + \sum_{k\geq 1} \varepsilon^k V_k^{i,j}(\tilde{x},\, \xi_2), \quad x \in G_\varepsilon^{(i)}(j), \tag{7}$$

where $\tilde{x} := (x_1,\, x_3)$, and

$$V_k^{i,j}(\tilde{x},\, \xi_2) = \sum_{m=0}^{k-1} \frac{(\xi_2 - j - b_i)^m}{m!} \frac{\partial^m u_{k-m}^{i,-}}{\partial x_2^m}(x_1,\, \varepsilon(j + b_i),\, x_3,\, \xi_2 - j) + \frac{(\xi_2 - j - b_i)^k}{k!} \frac{\partial^k u_0^{i,-}}{\partial x_2^k}(x_1,\, \varepsilon(j + b_i),\, x_3). \tag{8}$$

Further we will indicate arguments of functions only if their absence may cause confusion.

The outward unit normal to the lateral surfaces of the thin discs except a set of zero measure is as follows:

$$\nu_\varepsilon(x) = \frac{1}{\sqrt{1+4^{-1}\varepsilon^2|h_i'(r)|^2}}\left(-\frac{\varepsilon h_i'(r)x_1}{2r},\, \pm 1,\, -\frac{\varepsilon h_i'(r)x_3}{2r}\right), \quad x \in S_\varepsilon^{(i)},\ i=1,\,2, \tag{9}$$

where "$+$" and "$-$" refer, respectively, to the left and the right parts of the lateral surface of each thin disc. Obviously, $(1+\varepsilon^2 4^{-1}|h_i'(r)|^2)^{-\frac{1}{2}} = 1+\mathcal{O}(\varepsilon^2),\ \varepsilon \to 0.$

Again with the help of Taylor's formula we obtain

$$\vartheta_0(u_\varepsilon(x)) = \vartheta_0(u_0^{i,-}(x_1,\,\varepsilon(j+b_i),\,x_3)) + \vartheta_0'(\,\cdot\,)\sum_{k\geq 1}\varepsilon^k V_k^{i,j}(\tilde{x},\,\xi_2,\,t),\ x\in G_\varepsilon^{(i)}. \tag{10}$$

Let us put (7) into (1) instead of u_ε. Taking into account (9), (10) and that the Laplace operator in the variables $(\tilde{x},\,\xi_2)$ has the form $\Delta_x = \Delta_{\tilde{x}} + \varepsilon^{-2}\frac{\partial^2}{\partial\xi_2^2}$ and collecting coefficients of the same powers of ε, we arrive at one-dimensional boundary-value problems with respect to ξ_2 for functions $V_k^{i,j}$.

Problems for $V_1^{i,j}$ read

$$\left\{\begin{array}{rcl} \partial^2_{\xi_2\xi_2}V_1^{i,j} & = & 0,\quad \xi_2\in I_{h_i(r)}(j) := (-\frac{h_i(r)}{2}+j+b_i,\ \frac{h_i(r)}{2}+j+b_i),\\ \partial_{\xi_2}V_1^{i,j} & = & 0,\quad \xi_2 = \pm\frac{h_i(r)}{2}+j+b_i,\end{array}\right. \qquad (i=1,\,2) \tag{11}$$

where $\partial_{\xi_2} = \frac{\partial}{\partial\xi_2},\ \partial^2_{\xi_2\xi_2} = \frac{\partial^2}{\partial\xi_2^2}$. Here the variables $\tilde{x}$ are regarded as parameters.

It follows from (11) that $V_1^{i,j}$ do not depend on ξ_2. Therefore, $V_1^{i,j}$ are equal to some functions $\varphi_1^{(i)}(x_1,\,\varepsilon(j+b_i),\,x_3),\ x\in G_\varepsilon^{(i)}(j)$, which will be defined later. Then, due to (8) we have

$$\begin{aligned} u_1^{i,-}(x_1,\,\varepsilon(j+b_i),\,x_3,\,\xi_2-j) &= \varphi_1^{(i)}(x_1,\,\varepsilon(j+b_i),\,x_3)\\ &\quad -(\xi_2-j-b_i)\partial_{x_2}u_0^{i,-}(x_1,\,\varepsilon(j+b_i),\,x_3),\quad x\in G_\varepsilon^{(i)}(j). \end{aligned} \tag{12}$$

Boundary-value problems for $V_2^{1,j}$ and $V_2^{2,j}$ have the view

$$\left\{\begin{array}{ll} -\partial^2_{\xi_2\xi_2}V_2^{1,j} = (\Delta_{\tilde{x}}u_0^{1,-} - \vartheta_0(u_0^{1,-}) + f_0)|_{x_2=\varepsilon(j+b_1)}, & \xi_2\in I_{h_1(r)}(j),\\ \pm\partial_{\xi_2}V_2^{1,j} = (2^{-1}\nabla_{\tilde{x}}h_1\cdot\nabla_{\tilde{x}}u_0^{1,-} - \vartheta_1(u_0^{1,-}) + \delta_{\beta,1}g_0)|_{x_2=\varepsilon(j+b_1)}, & \xi_2 = \pm\frac{h_1(r)}{2}+j+b_1,\end{array}\right. \tag{13}$$

and

$$\left\{\begin{array}{ll} -\partial^2_{\xi_2\xi_2}V_2^{2,j} = (\Delta_{\tilde{x}}u_0^{2,-} - \vartheta_0(u_0^{2,-}) + f_0)|_{x_2=\varepsilon(j+b_2)}, & \xi_2\in I_{h_2(r)}(j),\\ \pm\partial_{\xi_2}V_2^{2,j} = (2^{-1}\nabla_{\tilde{x}}h_2\cdot\nabla_{\tilde{x}}u_0^{2,-} - \delta_{\alpha,1}\vartheta_2(u_0^{2,-}) + \delta_{\beta,1}g_0)|_{x_2=\varepsilon(j+b_2)}, & \xi_2 = \pm\frac{h_2(r)}{2}+j+b_2,\end{array}\right. \tag{14}$$

respectively, where $\delta_{\alpha,1},\ \delta_{\beta,1}$ are Kronecker's symbols.

The solvability conditions for problems (13) and (14) read

$$\begin{aligned} -\operatorname{div}_{\tilde{x}}(h_1\nabla_{\tilde{x}}u_0^{1,-}) + h_1\vartheta_0(u_0^{1,-}) + 2\vartheta_1(u_0^{1,-}) &= h_1 f_0 + 2\delta_{\beta,1}g_0,\\ x_2 &= \varepsilon(j+b_1),\ r\in(d_0,\,d_1), \end{aligned} \tag{15}$$

$$-\operatorname{div}_{\tilde{x}}(h_2\nabla_{\tilde{x}}u_0^{2,-}) + h_2\vartheta_0(u_0^{2,-}) + 2\delta_{\alpha,1}\vartheta_2(u_0^{2,-}) = h_2 f_0 + 2\delta_{\beta,1}g_0,$$
$$x_2 = \varepsilon(j+b_2),\ r\in(d_0,\, d_2), \quad (16)$$

respectively.

Putting (7) into the Robin boundary conditions on $Q_\varepsilon^{(i)}$, we get

$$\partial_r u_0^{1,-} + \vartheta_1(u_0^{1,-}) = 0, \quad x\in Q_\varepsilon^{(1)},\ x_2=\varepsilon(j+b_1), \quad (17)$$
$$\partial_r u_0^{2,-} = 0, \quad x\in Q_\varepsilon^{(2)},\ x_2=\varepsilon(j+b_2). \quad (18)$$

In order to find conditions in joint zone $Q_0^{(0)}$ we use the method of matched asymptotic expansions for outer expansions (4), (7) and an inner expansion which will be constructed in the next subsection.

2.2 Inner Expansion

In a neighborhood of joint zone $Q_0^{(0)}$ we introduce the "rapid" coordinates $\xi = (\xi_1,\, \xi_2)$, where $\xi_1 = -(r-d_0)/\varepsilon$ and $\xi_2 = x_2/\varepsilon$. Here $(r,\, x_2,\, \theta)\in\mathbb{R}^3$ are cylindric coordinates: $r=\sqrt{x_1^2+x_3^2}$, $\tan(\theta)=x_3/x_1$. The Laplace operator in the coordinates $(\xi_1,\, \xi_2,\, \theta)$ has the form

$$\Delta_x = \varepsilon^{-2}\Delta_\xi - \varepsilon^{-1}\frac{1}{d_0-\varepsilon\xi_1}\frac{\partial}{\partial\xi_1} + \frac{1}{(d_0-\varepsilon\xi_1)^2}\frac{\partial^2}{\partial\theta^2}. \quad (19)$$

We seek the leading terms of the inner expansion in a neighborhood of $Q_0^{(0)}$ in the form

$$u_\varepsilon(x) \approx u_0^+(x)|_{r=d_0} + \varepsilon\big(Z_1(\xi)\partial_{x_2}u_0^+(x)|_{r=d_0}$$
$$- \big(\eta(x_2)\Xi_1(\xi) + (1-\eta(x_2))\Xi_2(\xi)\big)\partial_r u_0^+(x)|_{r=d_0}\big) + \ldots, \quad (20)$$

where $Z_1,\ \Xi_1,\ \Xi_2$ are some functions, which are 1-periodic with respect to ξ_2 and defined in the union $\Pi := \Pi^+\cup\Pi_1^-\cup\Pi_2^-$ of semiinfinite strips

$$\Pi^+ = \{\xi\in\mathbb{R}^2:\ \xi_1>0,\ \xi_2\in(0,\,1)\}, \qquad \Pi_i^- = \{\xi\in\mathbb{R}^2:\ \xi_1\le 0,\ \xi_2\in I_i(d_0)\}, \quad i=1,\,2,$$

(see definition of $I_i(d_0)$ in Section 1), η is some function, which will be defined from matching conditions.

Putting (20) into the differential equation of problem (1) with regard to (19) and into the corresponding boundary conditions and collecting coefficients of the same powers of ε, we get the junction-layer problems for functions $Z_1,\ \Xi_1,\ \Xi_2$. Functions Ξ_1 and Ξ_2 are solutions to the following homogeneous problem:

$$\begin{cases} -\Delta_\xi\Xi = 0 & \text{in } \Pi, \\ \partial_{\xi_2}\Xi = 0 & \text{on } (\partial\Pi_1^-\cup\partial\Pi_2^-)\cap\{\xi\in\mathbb{R}^2:\ \xi_1<0\}, \\ \partial_{\xi_1}\Xi = 0 & \text{on } \partial\Pi\cap\{\xi\in\mathbb{R}^2:\ \xi_1=0\}, \\ \partial_{\xi_2}^p\Xi|_{\xi_2=0} = \partial_{\xi_2}^p\Xi|_{\xi_2=1}, & p=0,\,1,\ \xi_1>0. \end{cases} \quad (21)$$

Main asymptotic relations for functions $\Xi_1,\ \Xi_2$ can be obtained from general results on the asymptotic behavior of solutions to elliptic problems in domains with different exits to infinity (see, for instance, [25]). However, for domain Π, we can define more exactly the asymptotic relations for junction-layer solutions $\Xi_1,\ \Xi_2$ in the same way as in papers [20, 15].

Proposition 2.1. *There exist two solutions* $\Xi_1,\ \Xi_2 \in H^1_{\sharp,\mathrm{loc}}(\Pi)$ *to problem (21), which have the following differentiable asymptotics:*

$$\Xi_1 = \begin{cases} \xi_1 + \mathcal{O}(\exp(-2\pi\xi_1)), & \xi_1 \to +\infty,\ \xi \in \Pi^+, \\ \alpha_1^{(1)} + \mathcal{O}(\exp(\pi h_1^{-1}(d_0)\xi_1)), & \xi_1 \to -\infty,\ \xi \in \Pi_1^-, \\ h_2^{-1}(d_0)\xi_1 + \alpha_1^{(2)} + \mathcal{O}(\exp(\pi h_2^{-1}(d_0)\xi_1)), & \xi_1 \to -\infty,\ \xi \in \Pi_2^-, \end{cases} \tag{22}$$

$$\Xi_2 = \begin{cases} \xi_1 + \mathcal{O}(\exp(-2\pi\xi_1)), & \xi_1 \to +\infty,\ \xi \in \Pi^+, \\ h_1^{-1}(d_0)\xi_1 + \alpha_2^{(1)} + \mathcal{O}(\exp(\pi h_1^{-1}(d_0)\xi_1)), & \xi_1 \to -\infty,\ \xi \in \Pi_1^-, \\ \alpha_2^{(2)} + \mathcal{O}(\exp(\pi h_2^{-1}(d_0)\xi_1)), & \xi_1 \to -\infty,\ \xi \in \Pi_2^-. \end{cases} \tag{23}$$

Here $H^1_{\sharp,\mathrm{loc}}(\Pi) = \{u : \Pi \to \mathbb{R} :\ \ u(\xi_1, 0) = u(\xi_1, 1)$ *for any* $\xi_1 > 0,\ u \in H^1(\Pi_R)$ *for any* $R > 0\}$, $\Pi_R = \{\xi \in \Pi :\ -R < \xi_1 < R\}$; $\alpha_1^{(i)},\ \alpha_2^{(i)},\ i = 1, 2,$ *are some fixed constants.*

Any other solution to problem (21), which has a polynomial growth at infinity, can be represented as a linear combination $c_0 + c_1\Xi_1 + c_2\Xi_2$.

Function Z_1 is a solution to the following problem:

$$\begin{cases} -\Delta_\xi Z = 0 & \text{in } \Pi, \\ \partial_{\xi_2} Z = -1 & \text{on } (\partial\Pi_1^- \cup \partial\Pi_2^-) \cap \{\xi \in \mathbb{R}^2 :\ \xi_1 < 0\}, \\ \partial_{\xi_1} Z = 0 & \text{on } \partial\Pi \cap \{\xi \in \mathbb{R}^2 :\ \xi_1 = 0\}, \\ \partial^p_{\xi_2} Z|_{\xi_2=0} = \partial^p_{\xi_2} Z|_{\xi_2=1}, & p = 0, 1,\ \xi_1 > 0. \end{cases} \tag{24}$$

Similarly to [20, 15, 24] it is easy to verify that there exists a unique solution $Z_1 \in H^1_{\sharp,\mathrm{loc}}(\Pi)$ with the following asymptotics:

$$Z = \begin{cases} \mathcal{O}(\exp(-2\pi\xi_1)), & \xi_1 \to +\infty,\ \xi \in \Pi^+, \\ -\xi_2 + b_1 + \alpha_3^{(1)} + \mathcal{O}(\exp(\pi h_1^{-1}(d_0)\xi_1)), & \xi_1 \to -\infty,\ \xi \in \Pi_1^-, \\ -\xi_2 + b_2 + \alpha_3^{(2)} + \mathcal{O}(\exp(\pi h_2^{-1}(d_0)\xi_1)), & \xi_1 \to -\infty,\ \xi \in \Pi_2^-. \end{cases} \tag{25}$$

Now let us verify matching conditions for outer expansions (4), (5) and inner expansion (20), namely, the leading terms of the asymptotics of the outer expansions as $\xi_1 \to \pm 0$ must coincide with the leading terms of the asymptotics of the inner expansion as $\xi_1 \to \pm\infty$. Near the point $(x_1, \varepsilon(j + b_i), x_3) \in Q_0^{(0)}$ function u_0^+ has the following asymptotics:

$$u_0^+(x) \approx u_0^+(x_1, \varepsilon(j + b_i), x_3)|_{r=d_0} + \varepsilon(\xi_2 - j - b_i)\partial_{x_2} u_0^+(x_1, \varepsilon(j + b_i), x_3)|_{r=d_0} \\ - \varepsilon\xi_1 \partial_r u_0^+(x_1, \varepsilon(j + b_i), x_3)|_{r=d_0} + \dots \quad \text{as } \xi_1 \to 0+,\ (x) \in \Omega_0.$$

Taking into account the asymptotics of Z_1, Ξ_1 and Ξ_2 as $\xi_1 \to +\infty$, we see that the matching conditions are satisfied for expansions (4) and (20).

The asymptotics of (5) in the neighborhood of $(x_1, \varepsilon(j + b_i), x_3) \in Q_0^{(0)}$ are equal to

$$u_0^{i,-}(x_1, \varepsilon(j + b_i), x_3)|_{r=d_0} + \varepsilon\big(\varphi_1^{(i)}(x_1, \varepsilon(j + b_i), x_3)|_{r=d_0} \\ - \xi_1 \partial_r u_0^{i,-}(x_1, \varepsilon(j + b_i), x_3)|_{r=d_0}\big) + \dots \quad \text{as } \xi_1 \to 0-,\ x \in G_\varepsilon^{(i)}(j),\ i = 1, 2. \tag{26}$$

It follows from (22), (23) and (25) that the first terms of the asymptotics of (20) in the neighborhood of $(x_1,\, \varepsilon(j+b_i),\, x_3) \in Q_0^{(0)}$ are

$$\begin{aligned} & u_0^+(x_1,\, \varepsilon(j+b_1),\, x_3)|_{r=d_0} + \varepsilon\big(\alpha_3^{(1)} \partial_{x_2} u_0^+(x_1,\, \varepsilon(j+b_1),\, x_3)|_{r=d_0} \\ & \quad - \big(\alpha_1^{(1)} \eta(\varepsilon(j+b_1)) + (h_1^{-1}(d_0)\xi_1 + \alpha_2^{(1)})(1-\eta(\varepsilon(j+b_1)))\big)\partial_r u_0^+(x_1,\, \varepsilon(j+b_1),\, x_3)|_{r=d_0}\big) \\ & \qquad\qquad \text{as } \xi_1 \to -\infty,\ x \in G_\varepsilon^{(1)}(j), \end{aligned} \tag{27}$$

and

$$\begin{aligned} & u_0^+(x_1,\, \varepsilon(j+b_2),\, x_3)|_{r=d_0} + \varepsilon\,\big(\alpha_3^{(2)} \partial_{x_2} u_0^+(x_1,\, \varepsilon(j+b_2),\, x_3)|_{r=d_0} \\ & \quad - \big((h_2^{-1}(d_0)\xi_1 + \alpha_1^{(2)})\eta(\varepsilon(j+b_2)) + \alpha_2^{(2)}(1-\eta(\varepsilon(j+b_2)))\big)\partial_r u_0^+(x_1,\, \varepsilon(j+b_2),\, x_3)|_{r=d_0}\big) \\ & \qquad\qquad \text{as } \xi_1 \to -\infty,\ x \in G_\varepsilon^{(2)}(j). \end{aligned} \tag{28}$$

Comparing the first terms of (26), (27) and (28), we get

$$u_0^+(x) = u_0^{i,-}(x), \quad x \in Q_0^{(0)},\ x_2 = \varepsilon(j+b_i),\ i = 1,\, 2. \tag{29}$$

Comparing the second terms of (26), (27) and (28), we find that

$$\varphi_1^{(i)}(x) = \alpha_3^{(i)} \partial_{x_2} u_0^+(x), \quad x \in Q_0^{(0)},\ x_2 = \varepsilon(j+b_i),\ i = 1,\, 2, \tag{30}$$

and

$$\begin{array}{rcl} (1-\eta)h_1^{-1}(d_0)\partial_r u_0^+(x) & = & \partial_r u_0^{1,-}(x),\ \ x \in Q_0^{(0)},\ x_2 = \varepsilon(j+b_1) \\ \eta h_2^{-1}(d_0)\partial_r u_0^+(x) & = & \partial_r u_0^{2,-}(x),\ \ x \in Q_0^{(0)},\ x_2 = \varepsilon(j+b_2). \end{array} \tag{31}$$

Since the points $\{\varepsilon(j+b_i):\ j = \overline{0, N-1}\}$, $i = 1,\, 2$, make up the ε-net of the segment $[0,\, l]$, we can extend equalities (12), (15), (16) in domains D_i, equalities (17), (18) in $Q_0^{(1)}$ and $Q_0^{(2)}$, respectively, and equalities (29), (30) and (31) in $Q_0^{(0)}$. As a result, from equalities (31) we derive the relation

$$\eta(x_2) = \frac{h_2(d_0)\partial_r u_0^{2,-}|_{r=d_0}}{h_1(d_0)\partial_r u_0^{1,-}|_{r=d_0} + h_2(d_0)\partial_r u_0^{2,-}|_{r=d_0}}, \quad x_2 \in (0,\, l),$$

and obtain

$$\partial_r u_0^+ = h_1(d_0)\partial_r u_0^{1,-} + h_2(d_0)\partial_r u_0^{2,-}, \quad x \in Q_0^{(0)}.$$

By virtue of (29) and (30) we can define $\varphi_1^{(i)}$ as follows:

$$\varphi_1^{(i)}(x) = \alpha_3^{(i)} \partial_{x_2} u_0^{i,-}(x), \quad x \in D_i,\ i = 1,\, 2.$$

2.3 The Homogenized Problem

With the help of the first terms u_0^+, $u_0^{1,-}$ and $u_0^{2,-}$ of asymptotic expansions (4) and (5) we define multi-sheeted function

$$\mathbf{U}_0(x) = \begin{cases} u_0^+(x), & x \in \Omega_0, \\ u_0^{1,-}(x), & x \in D_1, \\ u_0^{2,-}(x), & x \in D_2, \end{cases}$$

or in a short form $\mathbf{U}_0 = (u_0^+,\, u_0^{1,-},\, u_0^{2,-})$. It follows from the foregoing that the components of function $\mathbf{U}_0$ must satisfy the relations

$$
\left\{
\begin{array}{rcll}
-\Delta u_0^+ + \vartheta_0(u_0^+) & = & f_0 & \text{in } \Omega_0, \\
\partial_{x_2}^p u_0^+|_{S^-} & = & \partial_{x_2}^p u_0^+|_{S^+}, & p = 0,\, 1, \\
-\mathrm{div}_{\tilde{x}}(h_1(r)\nabla_{\tilde{x}} u_0^{1,-}) + h_1(r)\vartheta_0(u_0^{1,-}) & & & \\
+2\vartheta_1(u_0^{1,-}) & = & h_1(r) f_0 + 2\delta_{\beta,1} g_0 & \text{in } D_1, \\
\partial_\nu u_0^{1,-} + \vartheta_1(u_0^{1,-}) & = & 0 & \text{on } Q_0^{(1)}, \\
-\mathrm{div}_{\tilde{x}}\big(h_2(r)\nabla_{\tilde{x}} u_0^{2,-}\big) + h_2(r)\vartheta_0(u_0^{2,-}) & & & \\
+2\delta_{\alpha,1}\vartheta_2(u_0^{2,-}) & = & h_2(r) f_0 + 2\delta_{\beta,1} g_0 & \text{in } D_2, \\
\partial_\nu u_0^{2,-} & = & 0 & \text{on } Q_0^{(2)}, \\
u_0^+|_{Q_0^{(0)}} & = & u_0^{1,-}|_{Q_0^{(0)}} = u_0^{2,-}|_{Q_0^{(0)}}, & \\
\partial_r u_0^+ & = & h_1(d_0)\partial_r u_0^{1,-} + h_2(d_0)\partial_r u_0^{2,-} & \text{on } Q_0^{(0)}.
\end{array}
\right.
\tag{32}
$$

These relations form *the homogenized problem* for problem (1).

We introduce space $\mathcal{V}_0 := L^2(\Omega_0) \times L^2(D_1) \times L^2(D_2)$ of multi-sheeted functions with the scalar product

$$
(\mathbf{u}, \mathbf{v})_{\mathcal{V}_0} = \int_{\Omega_0} u_0 v_0\, dx + \sum_{i=1}^{2} \int_{D_i} u_i v_i\, dx,
$$

where

$$
\mathbf{u}(x) = \left\{ \begin{array}{ll} u_0(x), & x \in \Omega_0, \\ u_1(x), & x \in D_1, \\ u_2(x), & x \in D_2, \end{array} \right. \qquad \text{and} \qquad \mathbf{v}(x) = \left\{ \begin{array}{ll} v_0(x), & x \in \Omega_0, \\ v_1(x), & x \in D_1, \\ v_2(x), & x \in D_2, \end{array} \right.
$$

or in a short form $\mathbf{u} = (u_0,\, u_1,\, u_2)$ and $\mathbf{v} = (v_0,\, v_1,\, v_2)$, belong to $\mathcal{V}_0$. Also we introduce anisotropic Sobolev space of multi-sheeted functions

$$
\begin{aligned}
\mathcal{H}_0 := \{\mathbf{u} = (u_0,\, u_1,\, u_2) \in \mathcal{V}_0 :\ & u_0 \in H^1(\Omega_0),\ u_0|_{S^-} = u_0|_{S^+}; \\
& \exists\, \partial_{x_j} u_i \in L^2(D_i),\ j = 1,\, 3,\ i = 1,\, 2;\ u_0|_{Q_0^{(0)}} = u_1|_{Q_0^{(0)}} = u_2|_{Q_0^{(0)}}\}
\end{aligned}
$$

with the inner product

$$
(\mathbf{u}, \mathbf{v})_{\mathcal{H}_0} = \int_{\Omega_0} (\nabla u_0 \cdot \nabla v_0 + u_0 v_0)\, dx + \sum_{i=1}^{2} \int_{D_i} (\nabla_{\tilde{x}} u_i \cdot \nabla_{\tilde{x}} v_i + u_i v_i)\, dx.
$$

It is obvious that $\mathcal{H}_0$ is continuously embedded in $\mathcal{V}_0$.

A function $\mathbf{U}_0 = (u_0^+,\, u_0^{1,-},\, u_0^{1,-}) \in \mathcal{H}_0$ is a weak solution to problem (32) if for any function $\varphi = (\varphi_0,\, \varphi_1,\, \varphi_2) \in \mathcal{H}_0$ the integral identity

$$
\int_{\Omega_0} (\nabla u_0^+ \cdot \nabla \varphi_0 + \vartheta_0(u_0^+)\, \varphi_0)\, dx + \sum_{i=1}^{2} \int_{D_i} h_i(r)(\nabla_{\tilde{x}} u_0^{i,-} \cdot \nabla_{\tilde{x}} \varphi_i + \vartheta_0(u_0^{i,-})\, \varphi_i)\, dx
$$

$$+2\int_{D_1}\vartheta_1(u_0^{1,-})\,\varphi_1\,dx + h_1(d_1)\int_{Q_0^{(1)}}\vartheta_1(u_0^{1,-})\,\varphi_1\,d\sigma_x + 2\delta_{\alpha,1}\int_{D_2}\vartheta_2(u_0^{2,-})\,\varphi_2\,dx$$

$$= \int_{\Omega_0} f_0\varphi_0\,dx + \sum_{i=1}^{2}\int_{D_i}(h_i f_0 + 2\delta_{\beta,1}g_0)\varphi_i\,dx$$

holds.

Using the properties of functions ϑ_i similarly as in [16] we can prove that there exists a unique weak solution to problem (32).

3 Approximation and Asymptotic Estimates

Let $\mathbf{U}_0 = (u_0^+,\, u_0^{1,-},\, u_0^{2,-})$ be the unique weak solution to problem (32). With the help of $\mathbf{U}_0$ and solutions $Z_1,\ \Xi_1,\ \Xi_2$ of junction-layer problems (21) and (24) we construct the main terms of expansions (4), (5) and (20). Consider smooth cut-off function $\chi_0(r)$, which is equal to 1 as $|r-d_0|<\delta_0/2$ and 0 as $|r-d_0|>\delta_0$, where $\delta_0\in(0,\,\delta)$ is some fixed number. Matching the outer expansions with the inner expansion with the help of χ_0, we define approximation function R_ε:

$$R_\varepsilon(x) := R_\varepsilon^+(x) = u_0^+(x) + \varepsilon\,\chi_0(r)\mathcal{N}^+(\xi,\,x_2,\,\theta), \quad x\in\Omega_0, \tag{33}$$

$$R_\varepsilon(x) := R_\varepsilon^{i,-}(x) = u_0^{i,-}(x) + \varepsilon\left(\tilde{Y}_i\left(\frac{x_2}{\varepsilon}\right)\partial_{x_2}u_0^{i,-}(x) + \chi_0(r)\mathcal{N}^{i,-}(\xi,\,x_2,\,\theta)\right),$$
$$x\in G_\varepsilon^{(i)}(j), \quad i=1,\,2. \tag{34}$$

Here

$$\mathcal{N}^+(\xi,\,x_2,\,\theta) = Z_1(\xi)\partial_{x_2}u_0^+|_{r=d_0} + \big(\xi_1 - \eta(x_2)\Xi_1(\xi) - (1-\eta(x_2))\Xi_2(\xi)\big)\partial_r u_0^+|_{r=d_0},$$

$$\mathcal{N}^{i,-}(\xi,\,x_2,\,\theta) = (Z_1(\xi) - \tilde{Y}_i(\xi_2))\partial_{x_2}u_0^+|_{r=d_0}$$
$$+ \big(\mathcal{Y}_i(\xi_1,\,x_2) - \eta(x_2)\Xi_1(\xi) - (1-\eta(x_2))\Xi_2(\xi)\big)\partial_r u_0^+|_{r=d_0},$$

where $\tilde{Y}_i(s) := -s + [s] + b_i + \alpha_3^{(i)}$, $[s]$ is the integer part of $s\in\mathbb{R}$, $i=1,\,2$, and

$$\begin{array}{lcl} \mathcal{Y}_1(\xi_1,\,x_2) & := & h_1^{-1}(d_0)\xi_1(1-\eta(x_2)), \\ \mathcal{Y}_2(\xi_1,\,x_2) & := & h_2^{-1}(d_0)\xi_1\eta(x_2), \end{array} \qquad \xi_1\le 0,\ x_2\in(0,\,l).$$

Obviously, $R_\varepsilon\in H_\varepsilon$.

Theorem 1. *Let $f_0\in H^3(\Omega_1)$, $\partial_{x_2}^p f_0|_{S^-} = \partial_{x_2}^p f_0|_{S^+}$, $p=0,\,1$, $g_0\in H^1(D_1)$.*

Then for any $\mu>0$ there exist positive constants ε_0, c_0 such that for all $\varepsilon\in(0,\,\varepsilon_0)$ the difference between solution u_ε to problem (1) and approximation function R_ε defined by (33) and (34), where $\mathbf{U}_0 = (u_0^+,\, u_0^{1,-},\, u_0^{2,-})$ is a weak solution to problem (32), satisfies the inequality

$$\|u_\varepsilon - R_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le c_0\left(\|f_\varepsilon - f_0\|_{L^2(\Omega_\varepsilon)} + \varepsilon^{1-\mu} + \varepsilon^{\delta_{\alpha,1}(2-\alpha)+\alpha-1} + \varepsilon^{\beta-1}\|g_\varepsilon - g_0\|_{L^2(D_1)}^{\delta_{\beta,1}}\right). \tag{35}$$

Proof. **Discrepancies in domain Ω_0.** It follows from the first two relations in (32) and from the theorem's assumptions that $\partial^2_{x_2x_2}u_0^+|_{S^-} = \partial^2_{x_2x_2}u_0^+|_{S^+}$. Then, according to the properties of Z_1, Ξ_1, Ξ_2 and u_0^+, function R_ε^+ satisfies the boundary conditions of problem (1) on $\partial\Omega_\varepsilon \cap \partial\Omega_0$.

Problems (21) and (24) imply

$$\Delta_\xi \mathcal{N}^+ = 0, \quad \Delta_\xi \mathcal{N}^{i,-} = 0 \qquad \xi \in \Pi,\ x_2 \in (0, l),\ \theta \in [0, 2\pi],\ i = 1, 2. \tag{36}$$

Let us consider the obvious equality

$$\Delta_{\tilde{x}}(\chi_0(r)\mathcal{N}) = \mathrm{div}_{\tilde{x}}(\mathcal{N}\nabla_{\tilde{x}}\chi_0(r)) + \nabla_{\tilde{x}}\chi_0(r)\cdot\nabla_{\tilde{x}}\mathcal{N} + \chi_0(r)\Delta_{\tilde{x}}\mathcal{N}, \qquad \mathcal{N} = \mathcal{N}(\xi, x_2, \theta). \tag{37}$$

Using (19), (32), (36) and (37), we get

$$\begin{aligned} -\Delta R_\varepsilon^+(x) - f_\varepsilon(x) &= f_0(x) - f_\varepsilon(x) - \vartheta_0(u_0^+(x)) + \chi_0(r)(r^{-1}\partial_{\xi_1}\mathcal{N}^+(\xi, x_2, \theta) \\ &-2\partial^2_{\xi_2x_2}\mathcal{N}^+(\xi, x_2, \theta)) - \varepsilon\,\mathrm{div}_{\tilde{x}}(\mathcal{N}^+|_{\xi_1=-(r-d_0)/\varepsilon}\nabla_{\tilde{x}}\chi_0(r)) + \chi_0'(r)\partial_{\xi_1}\mathcal{N}^+(\xi, x_2, \theta) \\ &-\varepsilon\chi_0(r)\partial^2_{x_2x_2}\mathcal{N}^+(\xi, x_2, \theta) - \varepsilon\, r^{-2}\chi_0(r)\,\partial^2_{\theta\theta}\mathcal{N}^+(\xi, x_2, \theta), \quad x \in \Omega_0. \end{aligned} \tag{38}$$

We multiply (38) by a test function $\psi \in H_\varepsilon$, integrate by parts in Ω_0 and take into account the boundary conditions, satisfied by R_ε^+. This yields

$$\int_{\Omega_0}(\nabla R_\varepsilon^+\cdot\nabla\psi + \vartheta_0(R_\varepsilon^+)\psi)\,dx - \int_{\Theta_\varepsilon}\partial_r R_\varepsilon^+\psi\,d\sigma_x - \int_{\Omega_0} f_\varepsilon\psi\,dx = I_0^+(\varepsilon, \psi) + \ldots + I_4^+(\varepsilon, \psi), \tag{39}$$

where

$$\begin{aligned} I_0^+(\varepsilon, \psi) &:= \int_{\Omega_0}(f_0 - f_\varepsilon)\psi\,dx, \\ I_1^+(\varepsilon, \psi) &:= \int_{\Omega_0}(\vartheta_0(R_\varepsilon^+) - \vartheta_0(u_0^+))\psi\,dx, \\ I_2^+(\varepsilon, \psi) &:= \int_{\Omega_0}\chi_0(r^{-1}\partial_{\xi_1}\mathcal{N}^+ - \partial^2_{x_2\xi_2}\mathcal{N}^+)\psi\,dx, \\ I_3^+(\varepsilon, \psi) &:= \varepsilon\int_{\Omega_0}\mathcal{N}^+\nabla_{\tilde{x}}\chi_0\cdot\nabla_{\tilde{x}}\psi\,dx + \int_{\Omega_0}\chi_0'\partial_{\xi_1}\mathcal{N}^+\psi\,dx, \\ I_4^+(\varepsilon, \psi) &:= \varepsilon\int_{\Omega_0}\chi_0\partial_{x_2}\mathcal{N}^+\partial_{x_2}\psi\,dx + \varepsilon\int_{\Omega_0}r^{-2}\chi_0\partial_\theta\mathcal{N}^+\partial_\theta\psi\,dx. \end{aligned}$$

Discrepancies in the thin discs. One can readily check that

$$\begin{aligned} \partial_r R_\varepsilon^{1,-} &= -\vartheta_1(u_0^{1,-}) - \varepsilon\tilde{Y}_1\left(\frac{x_2}{\varepsilon}\right)\partial_{x_2}\vartheta_1(u_0^{1,-}), & x \in Q_\varepsilon^{(1)}, \\ \partial_r R_\varepsilon^{2,-} &= 0, & x \in Q_\varepsilon^{(2)}, \end{aligned} \tag{40}$$

$$\partial_r R_\varepsilon^{i,-} = \varepsilon\tilde{Y}_i\left(\frac{x_2}{\varepsilon}\right)\partial^2_{rx_2}u_0^{i,-} + \partial_r R_\varepsilon^+, \quad x \in \Theta_\varepsilon^{(i)},\ i = 1, 2. \tag{41}$$

Taking into account (9) and that functions h_i are constant in a neighborhood of d_0, we derive that

$$\begin{aligned} \partial_\nu R_\varepsilon^{i,-} = \frac{\varepsilon}{\sqrt{1 + 4^{-1}\varepsilon^2|h_i'(r)|^2}}&\left(\pm\tilde{Y}_i\left(\frac{x_2}{\varepsilon}\right)\partial^2_{x_2x_2}u_0^{i,-} \pm \chi_0\frac{\partial}{\partial x_2}(\mathcal{N}^{i,-}|_{\xi_2=x_2/\varepsilon})\right. \\ &\left.-\frac{1}{2}\nabla_{\tilde{x}}h_i\cdot\nabla_{\tilde{x}}(u_0^{i,-} + \varepsilon\tilde{Y}_i\left(\frac{x_2}{\varepsilon}\right)\partial_{x_2}u_0^{i,-})\right), \quad x \in S_\varepsilon^{(i)},\ i = 1, 2, \end{aligned} \tag{42}$$

where "+" and "−" refer to the left and the right parts of the lateral surfaces of the thin discs, respectively.

Relations (19), (32), (36) and (37) yield

$$
\begin{aligned}
-\Delta R_\varepsilon^{i,-}(x) - f_\varepsilon(x) = f_0(x) - f_\varepsilon(x) - \vartheta_0(u_0^{i,-})\\
+\chi_0(r)(r^{-1}\partial_{\xi_1}\mathcal{N}^{i,-}(\xi,\, x_2,\, \theta) - 2\partial^2_{\xi_2 x_2}\mathcal{N}^{i,-}(\xi,\, x_2,\, \theta)) - \varepsilon \mathrm{div}_{\tilde{x}}(\mathcal{N}^{i,-}|_{\xi_1=-(r-d_0)/\varepsilon}\nabla_{\tilde{x}}\chi_0(r))\\
+\chi_0'(r)\partial_{\xi_1}\mathcal{N}^{i,-}(\xi,\, x_2,\, \theta) - \varepsilon\chi_0(r)\partial^2_{x_2x_2}\mathcal{N}^{i,-}(\xi,\, x_2,\, \theta) - \varepsilon\chi_0(r)r^{-2}\partial^2_{\theta\theta}\mathcal{N}^{i,-}(\xi,\, x_2,\, \theta)\\
+\nabla_{\tilde{x}}(\ln h_i(r))\cdot\nabla_{\tilde{x}}u_0^{i,-} - \varepsilon\mathrm{div}\big(\tilde{Y}_i\left(\frac{x_2}{\varepsilon}\right)\nabla(\partial_{x_2}u_0^{i,-})\big)\\
-2(1-\delta_{i,2}(1-\delta_{\alpha,1}))h_i^{-1}(r)\vartheta_i(u_0^{i,-}) + 2\delta_{\beta,1}h_i^{-1}(r)g_0(x), \quad x\in G_\varepsilon^{(i)}.
\end{aligned}
\tag{43}
$$

Consider the integral identity

$$
\int_{S_\varepsilon^{(i)}} \frac{\varepsilon h_i(r)}{2\sqrt{1+4^{-1}\varepsilon^2|h_i'(r)|^2}}\,\varphi\, d\sigma_x = \int_{G_\varepsilon^{(i)}}\varphi\, dx - \varepsilon\int_{G_\varepsilon^{(i)}} Y_i\left(\frac{x_2}{\varepsilon}\right)\partial_{x_2}\varphi\, dx, \qquad i=1,\,2, \tag{44}
$$

where $Y_i(s) = -s + [s] + b_i$ and $[s]$ is the integer part of s, $\varphi\in H^1(G_\varepsilon^{(i)})$ is an arbitrary function. We multiply (43) by a test function $\psi\in H_\varepsilon$ and integrate by parts in $G_\varepsilon^{(i)}$, using (44) and taking into account relations (40), (41), (42). This yields

$$
\begin{aligned}
\int_{G_\varepsilon^{(1)}}(\nabla R_\varepsilon^{1,-}\cdot\nabla\psi + \vartheta_0(R_\varepsilon^{1,-})\psi)\,dx + \varepsilon\int_{S_\varepsilon^{(1)}}\vartheta_1(R_\varepsilon^{1,-})\psi\, d\sigma_x + \int_{Q_\varepsilon^{(1)}}\vartheta_1(R_\varepsilon^{1,-})\psi\, d\sigma_x\\
+\int_{\Theta_\varepsilon^{(1)}}\partial_r R_\varepsilon^{+}\psi\, d\sigma_x - \int_{G_\varepsilon^{(1)}} f_\varepsilon\psi\, dx - \varepsilon^\beta\int_{S_\varepsilon^{(1)}} g_\varepsilon\psi\, d\sigma_x = I_0^{1,-}(\varepsilon,\,\psi)+\ldots+I_7^{1,-}(\varepsilon,\,\psi)
\end{aligned}
\tag{45}
$$

and

$$
\begin{aligned}
\int_{G_\varepsilon^{(2)}}(\nabla R_\varepsilon^{2,-}\cdot\nabla\psi + \vartheta_0(R_\varepsilon^{1,-})\psi)\,dx + \varepsilon^\alpha\int_{\Upsilon_\varepsilon^{(2)}}\vartheta_2(R_\varepsilon^{2,-})\psi\, d\sigma_x + \int_{\Theta_\varepsilon^{(2)}}\partial_r R_\varepsilon^{+}\psi\, d\sigma_x\\
-\int_{G_\varepsilon^{(2)}} f_\varepsilon\psi\, dx - \varepsilon^\beta\int_{\Upsilon_\varepsilon^{(2)}} g_\varepsilon\psi\, d\sigma_x = I_0^{2,-}(\varepsilon,\,\psi)+\ldots+I_7^{2,-}(\varepsilon,\,\psi)
\end{aligned}
\tag{46}
$$

for all $\psi\in H_\varepsilon$, where

$$
\begin{aligned}
I_0^{i,-}(\varepsilon,\,\psi) &:= \int_{G_\varepsilon^{(i)}}(f_0 - f_\varepsilon)\psi\, dx,\\
I_1^{i,-}(\varepsilon,\,\psi) &:= \int_{G_\varepsilon^{(i)}}(\vartheta_0(R_\varepsilon^{i,-}) - \vartheta_0(u_0^{i,-}))\psi\, dx,\\
I_2^{i,-}(\varepsilon,\,\psi) &:= \int_{G_\varepsilon^{(i)}}\chi_0(r^{-1}\partial_{\xi_1}\mathcal{N}^{i,-} - \partial^2_{x_2\xi_2}\mathcal{N}^{i,-})\psi\, dx,\\
I_3^{i,-}(\varepsilon,\,\psi) &:= \varepsilon\int_{G_\varepsilon^{(i)}}\mathcal{N}^{i,-}\nabla_{\tilde{x}}\chi_0\cdot\nabla_{\tilde{x}}\psi\, dx + \int_{G_\varepsilon^{(i)}}\chi_0'\partial_{\xi_1}\mathcal{N}^{i,-}\psi\, dx,\\
I_4^{i,-}(\varepsilon,\,\psi) &:= \varepsilon\int_{G_\varepsilon^{(i)}}\chi_0\partial_{x_2}\mathcal{N}^{i,-}\partial_{x_2}\psi\, dx + \varepsilon\int_{G_\varepsilon^{(i)}} r^{-2}\chi_0\partial_\theta\mathcal{N}^{i,-}\partial_\theta\psi\, dx,\\
I_5^{i,-}(\varepsilon,\,\psi) &:= \varepsilon\int_{G_\varepsilon^{(i)}} Y_i\left(\frac{x_2}{\varepsilon}\right)\partial_{x_2}(\psi\nabla_{\tilde{x}}u_0^{i,-}\cdot\nabla_{\tilde{x}}\ln h_i)\,dx\\
&\quad +\varepsilon\int_{G_\varepsilon^{(i)}}\tilde{Y}_i\left(\frac{x_2}{\varepsilon}\right)\nabla(\partial_{x_2}u_0^{i,-})\cdot\nabla\psi\, dx, \qquad i=1,\,2,
\end{aligned}
$$

$$I_6^{1,-}(\varepsilon,\,\psi) := -\varepsilon\int_{S_\varepsilon^{(1)}} \frac{\vartheta_1(u_0^{1,-})\psi}{\sqrt{1+4^{-1}\varepsilon^2|h_1'(r)|^2}}\,d\sigma_x - 2\varepsilon\int_{G_\varepsilon^{(1)}} Y_1\left(\frac{x_2}{\varepsilon}\right) h_1^{-1}\partial_{x_2}(\vartheta_1(u_0^{1,-})\psi)\,dx$$

$$+\varepsilon\int_{S_\varepsilon^{(1)}} \vartheta_1(R_\varepsilon^{1,-})\psi\,d\sigma_x + \int_{Q_\varepsilon^{(1)}} (\vartheta_1(R_\varepsilon^{1,-}) - \vartheta_1(u_0^{1,-}) - \varepsilon\tilde{Y}_1\left(\frac{x_2}{\varepsilon}\right)\partial_{x_2}\vartheta_1(u_0^{1,-}))\psi\,d\sigma_x,$$

$$I_6^{2,-}(\varepsilon,\,\psi) := -\varepsilon\delta_{\alpha,1}\int_{S_\varepsilon^{(2)}} \frac{\vartheta_2(u_0^{2,-})\psi}{\sqrt{1+4^{-1}\varepsilon^2|h_2'(r)|^2}}\,d\sigma_x$$

$$-2\varepsilon\delta_{\alpha,1}\int_{G_\varepsilon^{(2)}} Y_2\left(\frac{x_2}{\varepsilon}\right) h_2^{-1}\partial_{x_2}(\vartheta_2(u_0^{2,-})\psi)\,dx + \varepsilon^\alpha\int_{\Upsilon_\varepsilon^{(2)}} \vartheta_2(R_\varepsilon^{2,-})\psi\,d\sigma_x,$$

$$I_7^{1,-}(\varepsilon,\,\psi) := \varepsilon\delta_{\beta,1}\int_{S_\varepsilon^{(1)}} \frac{g_0\psi}{\sqrt{1+4^{-1}\varepsilon^2|h_1'(r)|^2}}\,d\sigma_x$$

$$+2\varepsilon\delta_{\beta,1}\int_{G_\varepsilon^{(1)}} Y_1\left(\frac{x_2}{\varepsilon}\right) h_1^{-1}\partial_{x_2}(g_0\psi)\,dx - \varepsilon^\beta\int_{S_\varepsilon^{(1)}} g_\varepsilon\psi\,d\sigma_x,$$

$$I_7^{2,-}(\varepsilon,\,\psi) := \varepsilon\delta_{\beta,1}\int_{S_\varepsilon^{(2)}} \frac{g_0\psi}{\sqrt{1+4^{-1}\varepsilon^2|h_2'(r)|^2}}\,d\sigma_x$$

$$+2\varepsilon\delta_{\beta,1}\int_{G_\varepsilon^{(2)}} Y_2\left(\frac{x_2}{\varepsilon}\right) h_2^{-1}\partial_{x_2}(g_0\psi)\,dx - \varepsilon^\beta\int_{\Upsilon_\varepsilon^{(2)}} g_\varepsilon\psi\,d\sigma_x.$$

Asymptotic estimates. After summing (39), (45) and (46) we see that function R_ε defined by (33) and (34) satisfies the integral identity

$$\int_{\Omega_\varepsilon} (\nabla R_\varepsilon\cdot\nabla\psi + \vartheta_0(R_\varepsilon)\psi)\,dx + \varepsilon\int_{S_\varepsilon^{(1)}} \vartheta_1(R_\varepsilon)\psi\,d\sigma_x + \int_{Q_\varepsilon^{(1)}} \vartheta_1(R_\varepsilon)\psi\,d\sigma_x$$

$$+\,\varepsilon^\alpha\int_{\Upsilon_\varepsilon^{(2)}} \vartheta_2(R_\varepsilon)\psi\,d\sigma_x - \int_{\Omega_\varepsilon} f_\varepsilon\psi\,dx - \varepsilon^\beta\int_{S_\varepsilon^{(1)}\cup\Upsilon_\varepsilon^{(2)}} g_\varepsilon\psi\,d\sigma_x = F_\varepsilon(\psi) \quad (47)$$

for any $\psi\in H_\varepsilon$, where $F_\varepsilon(\psi) := I_0^\pm + \ldots + I_4^\pm + I_5^- + I_6^- + I_7^-$, $I_k^- := I_k^{1,-} + I_k^{2,-}$, $k=\overline{0,7}$, $I_m^\pm := I_m^+ + I_m^-$, $m=\overline{0,4}$.

It follows from (3) and (47) that

$$\int_{\Omega_\varepsilon} (\nabla(R_\varepsilon - u_\varepsilon)\cdot\nabla\psi + (\vartheta_0(R_\varepsilon) - \vartheta_0(u_\varepsilon))\psi)\,dx + \varepsilon\int_{S_\varepsilon^{(1)}} (\vartheta_1(R_\varepsilon) - \vartheta_1(u_\varepsilon))\psi\,d\sigma_x$$

$$+\int_{Q_\varepsilon^{(1)}} (\vartheta_1(R_\varepsilon) - \vartheta_1(u_\varepsilon))\psi\,d\sigma_x + \varepsilon^\alpha\int_{\Upsilon_\varepsilon^{(2)}} (\vartheta_2(R_\varepsilon) - \vartheta_2(u_\varepsilon))\psi\,d\sigma_x = F_\varepsilon(\psi) \quad (48)$$

for all $\psi\in H_\varepsilon$.

Now we are going to estimate $F_\varepsilon(\psi)$.

With the help of Cauchy-Schwartz-Bunyakovskii inequality we obtain

$$|I_0^\pm(\varepsilon,\,\psi)| \le \|f_\varepsilon - f_0\|_{L^2(\Omega_\varepsilon)}\,\|\psi\|_{H^1(\Omega_\varepsilon)}.$$

Remark 3.1. *Here and further all constants c_i, C_i in asymptotic estimates are independent of ε.*

With the help of (2), Taylor's formula and Cauchy-Schwartz-Bunyakovskii inequality we derive that

$$|I_1^+(\varepsilon,\,\psi)| = \varepsilon \left| \int_{\Omega_0} \vartheta_0'(\,\cdot\,)\chi_0 \mathcal{N}^+ \psi \, dx \right| \le \varepsilon c_0 \|\psi\|_{H^1(\Omega_\varepsilon)}.$$

Similarly we estimate I_1^-. Thus, $|I_1^\pm(\varepsilon,\,\psi)| \le \varepsilon C_1 \|\psi\|_{H^1(\Omega_\varepsilon)}$.

Since functions $\partial_{\xi_1}\mathcal{N}^+,\ \partial^2_{x_2\xi_2}\mathcal{N}^+,\ \partial_{\xi_1}\mathcal{N}^{i,-},\ \partial^2_{x_2\xi_2}\mathcal{N}^{i,-}$ exponentially decrease as $|\xi_1| \to \infty$ (see (22), (23) and (25)), then from Lemma 3.1 in [6] we derive that

$$\forall \mu > 0 \quad \exists C_2 > 0 \quad \exists \varepsilon_0 > 0 \quad \forall \varepsilon \in (0,\,\varepsilon_0): \quad |I_2^\pm(\varepsilon,\,\psi)| \le \varepsilon^{1-\mu} C_2 \|\psi\|_{H^1(\Omega_\varepsilon)}.$$

The integrals in $I_3^\pm(\varepsilon,\,\psi)$ are in fact over

$$\operatorname{supp}(\chi_0'(r)) \cap \Omega_\varepsilon = \{x \in \Omega_\varepsilon : \ \ \delta_0/2 < |r - d_0| < \delta_0\},$$

where, according to (21) and (24), functions $\mathcal{N}^+,\ \partial_{\xi_1}\mathcal{N}^+,\ \partial_{\xi_1}\mathcal{N}^{i,-}$ are exponentially small, and function $\mathcal{N}^{i,-}$ can be estimated by some constant c_1. Thus,

$$|I_3^\pm(\varepsilon,\,\psi)| \le \varepsilon C_3 \|\psi\|_{H^1(\Omega_\varepsilon)}.$$

The integrals in $I_4^\pm$ are over $\{x \in \mathbb{R}^3 : \ \ |r - d_0| < \delta_0\}$ and they can be estimated, extracting if necessary the exponentially decreasing part in the corresponding integrand and then using Cauchy-Schwartz-Bunyakovskii inequality. Consider, for example, the integral

$$\begin{aligned}
\left| \int_{G_\varepsilon^{(1)}} \chi_0 \partial_{x_2}\mathcal{N}^{1,-} \partial_{x_2}\psi \, dx \right| &= \Bigg| \int_{G_\varepsilon^{(1)}} \chi_0 \Big((Z_1 - \tilde{Y}_1) \partial^2_{x_2x_2} u_0^+|_{r=d_0} \\
&\quad - (h_1^{-1}(d_0)\xi_1 + \Xi_1 - \Xi_2)\eta' \partial_r u_0^+|_{r=d_0} \\
&\quad + (h_1^{-1}(d_0)\xi_1(1-\eta) - \eta\Xi_1 - (1-\eta)\Xi_2)\partial^2_{x_2 r} u_0^+|_{r=d_0} \Big) \partial_{x_2}\psi \, dx \Bigg| \\
&\le c_2 \|\psi\|_{H^1(\Omega_\varepsilon)} \Bigg(\sqrt{\int_0^T \int_{G_\varepsilon^{(1)}} \chi_0 |Z_1 - \tilde{Y}_1|^2 \, dx dt} \\
&\quad + \|\alpha_1^{(1)}\eta + \alpha_2^{(1)}(1-\eta) + (\alpha_1^{(1)} - \alpha_2^{(1)})\eta'\|_{L^2(G_\varepsilon^{(1)})} \\
&\quad + \sqrt{\int_{G_\varepsilon^{(1)}} \chi_0 \big| h_1^{-1}(d_0)\xi_1 + (\Xi_1 - \alpha_1^{(1)}) - (\Xi_2 - \alpha_2^{(1)}) \big|^2 dx} \\
&\quad + \sqrt{\int_{G_\varepsilon^{(1)}} \chi_0 \big| \eta(\Xi_1 - \alpha_1^{(1)}) + (1-\eta)(\Xi_2 - h_1^{-1}(d_0)\xi_1 - \alpha_2^{(1)}) \big|^2 dx} \Bigg) \\
&\le c_3 \, \|\psi\|_{H^1(\Omega_\varepsilon)} \Bigg(\sqrt{2\pi l d_0 \varepsilon} \|Z_1 - \tilde{Y}_1\|_{L^2(\Pi_1^-)} + \sqrt{|G_\varepsilon^{(1)}|} \\
&\quad + \sqrt{2\pi l d_0 \varepsilon} \, \|h_1^{-1}(d_0)\xi_1 + (\Xi_1 - \alpha_1^{(1)}) - (\Xi_2 - \alpha_2^{(1)})\|_{L^2(\Pi_1^-)} \\
&\quad + \sqrt{2\pi l d_0 \varepsilon} \, \|\eta(\Xi_1 - \alpha_1^{(1)}) + (1-\eta)(\Xi_2 - h_1^{-1}(d_0)\xi_1 - \alpha_2^{(1)})\|_{L^2(\Pi_1^-)} \Bigg),
\end{aligned}$$

where $|G_\varepsilon^{(1)}|$ is the measure of $G_\varepsilon^{(1)}$. Relations (22), (23) and (25) show that the norms in the right-hand side of the last inequality are bounded in ε. Similarly we can estimate the rest of the integrals in $I_4^\pm(\varepsilon,\,\psi)$. As a result, we obtain

$$|I_4^\pm(\varepsilon,\,\psi)| \le \varepsilon C_5 \|\psi\|_{H^1(\Omega_\varepsilon)}.$$

Remark 3.2. *Constants C_3 and C_4 depend on*

$$\sup_{x\in Q_0^{(0)},\ t\in(0,T)} \left| \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}} u_0^+(x) \right|, \quad |\alpha| = \alpha_1+\alpha_2+\alpha_3 \le 2,\ \alpha_k \ge 0,\ k=\overline{1,\,3}.$$

Extending homogenized problem (32) periodically in x_2 through the planes $\{x\in\mathbb{R}^3:\ x_2 = 0\}$ and $\{x\in\mathbb{R}^3:\ x_2=l\}$ and taking into account the assumptions for f_0 and g_0, by virtue of classical results on the smoothness of solutions to boundary-value problems we conclude that these quantities are bounded.

Since f_0 is smooth, then $\partial_{x_2} u_0^{i,-} \in H^1(D_i),\ i=1,\,2$. Consequently,

$$|I_5^-(\varepsilon,\,\psi)| \le \varepsilon c_4 \sum_{i=1}^{2} (\|u_0^{i,-}\|_{H^1(D_i)} + \|\partial_{x_2} u_0^{i,-}\|_{H^1(D_i)}) \|\psi\|_{H^1(\Omega_\varepsilon)} \le \varepsilon C_5 \|\psi\|_{H^1(\Omega_\varepsilon)}.$$

In order to estimate I_6^- we consider summand $I_6^{2,-}$ when $\alpha=1$. Obviously, the second integral in $I_6^{2,-}$ can be estimated by $\varepsilon c_5 \|\psi\|_{H^1(\Omega_\varepsilon)}$. Using Taylor's formula and obvious equality

$$1-\frac{1}{a} = \frac{a^2-1}{a^2+a} \qquad (a^2+a\neq 0)$$

we derive that the sum of the first and the third integrals in $I_6^{2,-}$ is equal to

$$4^{-1}\varepsilon^3 \int_{S_\varepsilon^{(2)}} \frac{|h_2'(r)|^2\,\vartheta_2(u_0^{2,-})\psi}{1+4^{-1}\varepsilon^2|h_2'(r)|^2+\sqrt{1+4^{-1}\varepsilon^2|h_2'(r)|^2}}\,d\sigma_x$$

$$\varepsilon^2 \int_{S_\varepsilon^{(2)}} \vartheta_2'(\,\cdot\,)(\tilde{Y}_2\left(\frac{x_2}{\varepsilon}\right)\partial_{x_2}u_0^{2,-} + \chi_0\mathcal{N}^{2,-})\psi\,dx + \varepsilon \int_{Q_\varepsilon^{(2)}} \vartheta_2(R_\varepsilon)\psi\,d\sigma_x$$

$$=: J_1(\varepsilon,\,\psi) + J_2(\varepsilon,\,\psi) + J_3(\varepsilon,\,\psi).$$

With the help of (2) and (44) we obtain $|J_1(\varepsilon,\,\psi)+J_2(\varepsilon,\,\psi)| \le \varepsilon c_6 \|\psi\|_{H^1(\Omega_\varepsilon)}$. Taking into account (2), properties of the trace operator and the fact that f_0 is smooth, we get $|J_3(\varepsilon,\,\psi)| \le \varepsilon c_7 \|\psi\|_{H^1(\Omega_\varepsilon)}$. Thus, in case $\alpha=1$ we have

$$|I_6^{2,-}(\varepsilon,\,\psi)| \le \varepsilon c_8 \|\psi\|_{H^1(\Omega_\varepsilon)}.$$

In case $\alpha>1$ with the help of (44) we obtain $|I_6^{2,-}(\varepsilon,\,\psi)| \le \varepsilon^{\alpha-1} c_9 \|\psi\|_{H^1(\Omega_\varepsilon)}$.

Similarly to $I_6^{2,-}(\varepsilon,\,\psi)$, we estimate $I_6^{1,-}(\varepsilon,\,\psi)$ and $I_7^-(\varepsilon,\,\psi)$. As a result, we get

$$|I_6^{1,-}(\varepsilon,\,\psi)| \le \varepsilon C_6 \|\psi\|_{H^1(\Omega_\varepsilon)}$$

and

$$|I_7^-(\varepsilon,\,\psi)| \le C_7 \begin{cases} (\varepsilon + \|g_0 - g_\varepsilon\|_{L^2(D_1)})\|\psi\|_{H^1(\Omega_\varepsilon)}, & \beta = 1, \\ \varepsilon^{\beta-1}\|\psi\|_{H^1(\Omega_\varepsilon)}, & \beta > 1. \end{cases}$$

Thus,

$$|F_\varepsilon(\psi)| \le C_8 \left(\|f_\varepsilon - f_0\|_{L^2(\Omega_\varepsilon)} + \varepsilon^{1-\mu} + \varepsilon^{\delta_{\alpha,1}(2-\alpha)+\alpha-1} + \varepsilon^{\beta-1}\|g_\varepsilon - g_0\|^{\delta_{\beta,1}}_{L^2(D_1)}\right)\|\psi\|_{H^1(\Omega_\varepsilon)}, \quad (49)$$

where $\mu > 0$ is an arbitrary number. Setting in (48) $\psi := R_\varepsilon - u_\varepsilon$ and using (49) and obvious inequality

$$(\vartheta_i(s_1) - \vartheta_i(s_2))(s_1 - s_2) \ge c_1(s_1 - s_2)^2 \qquad \forall s_1,\, s_2 \in \mathbb{R},\ i = \overline{0,2},$$

which follows from (2), we obtain estimate (35). □

4 Discussion of the Obtained Results

As we can see from the obtained results, the homogenized problem (32) for problem (1) is a nonstandard boundary-value problems for multi-sheeted function $\mathbf{U}_0$ in anisotropic Sobolev space $\mathcal{H}_0$ (see Section 2.3). This problem consists of three boundary-value problems (in domains Ω_0 and $D_i,\ i = 1,\, 2$), connected with each other by the conjugation conditions (on $Q_0^{(0)}$).

The nonhomogeneous Robin boundary conditions on the lateral surfaces of the thin discs in problem (1) are transformed as $\varepsilon \to 0$ into new summands in the differential equations in domains $D_i,\ i = 1,\, 2$, in problem (32). These summands show us the influence of the perturbed parameters α and β. If $\alpha > 1$, then summand $2\delta_{\alpha,1}\vartheta_1(u_0^{2,-})$ vanishes. From physical point of view this means that the outer heat conduction coefficient is too small, and we can neglect this heat exchange. If $\beta > 1$, then summands $2\delta_{\beta,1}g_0$ vanish, which means that the temperature of the environment is too small, and we can consider it equal to zero.

Also functions $h_i,\ i = 1,\, 2$, which describe the relative thickness of the thin discs from the i-th level, are transformed into the coefficients of the differential equations in domains D_i, respectively. The variable x_2 is involved as a parameter in the boundary-value problems in $D_i,\ i = 1,\, 2$, which shows us the influence of the type of thick junction Ω_ε on the asymptotic behavior of solution u_ε.

From results proved in the present paper it follows that for applied problems in thick junctions we can use the homogenized problem (32), which is simpler, instead of the initial problem (1) with sufficient plausibility.

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Садовий Д.Ю. *Асимптотична апроксимація розв'язку квазілінійної еліптичної крайової задачі в дворівневому густому з'єднанні типу 3:2:2* // Карпатські математичні публікації. — 2012. — Т.4, №2. — С. 297–315.

Розглядається квазілінійна еліптична крайова задача в дворівневому густому з'єднанні типу 3 : 2 : 2, яке є об'єднанням циліндру Ω_0 та великої кількості ε-періодично розташованих тонких дисків змінної товщини. На поверхнях тонких дисків з обох рівнів задані різні крайові умови третього роду зі збуреними параметрами. Будуються головні члени асимптотики та доводиться відповідна оцінка в просторі Соболєва.

Садовой Д.Ю. *Асимптотическая апроксимация решения квазилинейной эллиптической краевой задачи в двухуровневом густом соединении типа 3:2:2* // Карпатские математические публикации. — 2012. — Т.4, №2. — С. 297–315.

Рассматривается квазилинейная эллиптическая краевая задача в двухуровневом густом соединении Ω_ε типа 3 : 2 : 2, которое состоит из цилиндра Ω_0 и большого количества ε-периодически присоединённых тонких дисков переменной толщины. На поверхностях тонких дисков из обоих уровней задаются разные краевые условия третьего рода с возмущёнными коэффициентами. Строятся главные члены асимптотического разложения и показывается соответствующая оценка в пространстве Соболева.