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Bernstein-Jackson-type inequalities with exact constants in Orlicz spaces

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We establish the Bernstein and Jackson type inequalities with exact constants for estimations of best approximations by exponential type functions in Orlicz spaces $L_M(\mathbb{R}^n)$. For this purpose, we use a special scale of approximation spaces $\mathcal{B}^s_\tau(M)$ that are interpolation spaces between the subspace $\mathscr{E}_M$ of exponential type functions and the space $L_M(\mathbb{R}^n)$. These approximation spaces are defined using a functional $E\,(t, f)$ that plays a similar role as the module of smoothness. The constants in obtained inequalities are expressed using a normalization factor $N_{\vartheta,q}$ that is determined by the parameters τ and s of the approximation space $\mathcal{B}^s_\tau(M)$.

Key words and phrases: Bernstein and Jackson inequalities, best approximation, Orlicz space.

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1 Introduction

The classical Bernstein inequalities characterize the smoothness properties of a function depending on the speed of convergence to zero of its approximation by polynomials or entire exponential type functions. The Jackson inequalities indicate that smoothness of the function implies a quick decreasing to zero of its error of approximation by corresponding approximating aggregates (see, e.g., [2, Section 7], [15]). Investigation of the relation between the smoothness properties of functions and the possible orders of their approximations were carried out by many authors on various classes of functions [5, 6]. These results are extended to approximations in a Banach space, in particular, to spectral approximations for a linear closed unbounded operator [7, 10, 12]. In this case, special approximation scales of analytic vectors of finite exponential types and abstract Besov-type approximation spaces are considered.

It is important to determine exact estimates of the constants in the Bernstein and Jackson type inequalities, which allow us to estimate the best approximation errors by analytic vectors of an operator in a Banach space [8, 9].

In addition to the classical direction of approximation theory it should be mentioned the studies on direct and inverse approximation theorems in Orlicz function spaces [3, 11].

The purpose of this paper is to prove the Bernstein and Jackson type inequalities by explicitly calculated constants on approximation scales of exponential type functions in the Orlicz space. The constants in such inequalities are expressed using the special normalization factor that is determined by parameters of the approximation spaces.

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2 Preliminaries

Let us recall some concepts of the theory of Orlicz spaces (see, e.g., [13]). Let $M(t)$ be an arbitrary Young function, i.e. $M(t)$ is a non-decreasing convex function defined for $t \geq 0$ such that $M(0) = 0$, $M(t) \geq 0$ and $M(t) \not\equiv 0$. The function

$$\overline{M}(s) = \max_{t \geq 0}(ts - M(t))$$

is a conjugate function to M. Further, let $L_M(\mathbb{R}^n)$ be a space of measurable functions $f(x)$ with the Luxemburg norm

$$\|f\|_{(M)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} M(|f(x)|/\lambda)dx \leq 1\right\} < \infty. \tag{1}$$

Then $L_M(\mathbb{R}^n)$ is a Banach space and is called the Orlicz space. The norm (1) is equivalent to the Orlicz norm

$$\|f\|_M = \sup_{\rho(g,\overline{M}) \leq 1} \left|\int_{\mathbb{R}^n} f(x)g(x)dx\right|,$$

where $\rho(g, \overline{M}) = \int_{\mathbb{R}^n} \overline{M}(|g(x)|)dx < \infty$, so that

$$\|f\|_{(M)} \leq \|f\|_M \leq 2\|f\|_{(M)}.$$

Note that $\|f\|_{(M)} = \|f\|_{L_q(\mathbb{R}^n)}$ in the case $M(t) = t^q$ for $1 \leq q < \infty$, and $\|f\|_{(M)} = \|f\|_{L_\infty(\mathbb{R}^n)}$ when $M(t) = 0$ for $0 \leq t \leq 1$ and $M(t) = \infty$ for $t > 1$.

Denote $D^\alpha f = \dfrac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$ and $\alpha! = \alpha_1! \cdot \ldots \cdot \alpha_n!$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$. In what follows, $\alpha \geq 0$ means $\alpha_1 \geq 0, \ldots, \alpha_n \geq 0$.

A function $f \in L_M(\mathbb{R}^n)$ has an exponential type $\nu > 0$ if the series

$$\tilde{f}(\zeta) := \sum_{\alpha \geq 0} \frac{\zeta^{|\alpha|}}{\alpha!} \|D^\alpha f\|_{(M)}$$

is such that $\nu = \limsup_{|\alpha| \to \infty} \|D^\alpha f\|_{(M)}^{1/|\alpha|} < \infty$. For the behavior of the sequence $\|D^\alpha f\|_{(M)}$, $\alpha \geq 0$, we refer the reader to [1, Theorem 2].

3 Subspaces of exponential type functions in the Orlicz space

In this section, we describe the subspaces of exponential type functions in the Orlicz space $L_M(\mathbb{R}^n)$. For any $\nu > 0$ we define the subspace

$$\mathscr{E}_M^\nu = \left\{f(x) \in L_M(\mathbb{R}^n) : \|f\|_{\mathscr{E}_M^\nu} < \infty\right\}$$

endowed with the norm $\|f\|_{\mathscr{E}_M^\nu} = \sum_{\alpha \geq 0} \|D^\alpha f\|_{(M)}/\nu^{|\alpha|}$.

Proposition 1. (a) *For any $\mu > \nu$ the following contractive inclusions hold*

$$\mathscr{E}_M^\nu \hookrightarrow \mathscr{E}_M^\mu \hookrightarrow L_M(\mathbb{R}^n).$$

(b) *The space $\mathscr{E}_M^\nu$ is complete.*

Proof. (a) The inequalities $\|f\|_{\mathscr{E}_M^\mu} \le \|f\|_{\mathscr{E}_M^\nu}$ and $\|f\|_{(M)} \le \|f\|_{\mathscr{E}_M^\mu}$ yield the contractive inclusions $\mathscr{E}_M^\nu \hookrightarrow \mathscr{E}_M^\mu$ and $\mathscr{E}_M^\mu \hookrightarrow L_M(\mathbb{R}^n)$, respectively.

(b) Let (f_n) be a fundamental sequence in $\mathscr{E}_M^\nu$. Thus, for any $\varepsilon > 0$ there exists n_ε such that $\|f_n - f_m\|_{\mathscr{E}_M^\nu} < \varepsilon$ for $n, m > n_\varepsilon$. Then the sequences (f_n) and $\left(\|D^\alpha f_n\|_{(M)}/\nu^{|\alpha|}\right)$ for all $\alpha \ge 0$ are fundamental in $L_M(\mathbb{R}^n)$. It follows $\|f_n\|_{\mathscr{E}_M^\nu} \le \|f_n - f_{n_\varepsilon}\|_{\mathscr{E}_M^\nu} + \|f_{n_\varepsilon}\|_{\mathscr{E}_M^\nu} \le \varepsilon + \|f_{n_\varepsilon}\|_{\mathscr{E}_M^\nu}$ for any $n \ge n_\varepsilon$.

There exist $f, g_\alpha \in L_M(\mathbb{R}^n)$ such that $f_n \to f$ and $\|D^\alpha f_n\|_{(M)}/\nu^{|\alpha|} \to g_\alpha$ in $L_M(\mathbb{R}^n)$ for $\alpha \ge 0$. Thus, the equality $g_\alpha = \|D^\alpha f\|_{(M)}/\nu^{|\alpha|}$ with $f \in L_M(\mathbb{R}^n)$ holds. Hence, we obtain $\|D^\alpha f_n\|_{(M)}/\nu^{|\alpha|} \to \|D^\alpha f\|_{(M)}/\nu^{|\alpha|}$ for all $\alpha \ge 0$. Taking the limit in $L_M(\mathbb{R}^n)$ as $n \to \infty$, we find $\|f\|_{\mathscr{E}_M^\nu} \le \|f_{n_\varepsilon}\|_{\mathscr{E}_M^\nu} + \varepsilon$, that is $f \in \mathscr{E}_M^\nu$, since $f_{n_\varepsilon} \in \mathscr{E}_M^\nu$. Thus, $\mathscr{E}_M^\nu$ is complete. □

We consider the subspace $\mathscr{E}_M = \bigcup_{\nu>0} \mathscr{E}_M^\nu$ in $L_M(\mathbb{R}^n)$ endowed with the quasinorm

$$\|f\|_{\mathscr{E}_M} = \|f\|_{(M)} + \inf\{\nu > 0 \colon f \in \mathscr{E}_M^\nu\}. \tag{2}$$

Let us clarify that for any $f \in \mathscr{E}_M^\nu$, $g \in \mathscr{E}_M^\mu$,

$$\|f + g\|_{\mathscr{E}_M} \le \|f\|_{\mathscr{E}_M} + \|g\|_{\mathscr{E}_M},$$

since $\inf\left\{\nu + \mu > 0 \colon f + g \in \mathscr{E}_M^{\nu+\mu}\right\} \le \inf\left\{\nu + \mu > 0 \colon f \in \mathscr{E}_M^\nu,\, g \in \mathscr{E}_M^\mu\right\}$.

Proposition 2. *The quasinormed space $(\mathscr{E}_M, \|\cdot\|_{\mathscr{E}_M})$ is complete.*

Proof. We reason similarly to [8, Theorem 1(a)]. Let (f_n) be a fundamental sequence in $\mathscr{E}_M$. Then there exists $\nu > 0$ such that $\|f_n\|_{\mathscr{E}_M} < \nu$ for $n \in \mathbb{N}$. It follows that $(f_n) \subset \mathscr{E}_M^\nu$, since $\inf\{\mu \colon (f_n) \subset \mathscr{E}_M\} < \nu$. Consider the corresponding sequence of functions $\left(\tilde{f}_n(\zeta)\right)$ in variable $\zeta \in \mathbb{R}$ of an exponential type ν. The following sequence

$$\{[0,\infty) \ni \zeta \mapsto (f_n - f_m)\tilde{}(\zeta)\exp(-\zeta\nu) \colon n, m \in \mathbb{N}\}$$

is bounded. Thus, according to the Bernstein compactness theorem [15, Theorem 3.3.6], for any $\varepsilon > 0$ there exist $n_\varepsilon \in \mathbb{N}$ and $B_\varepsilon > 0$ such that

$$\sup_{|\zeta| \le B_\varepsilon} (f_{n_i} - f_{m_i})\tilde{}(\zeta)\exp(-\zeta\nu) < \varepsilon \quad \text{for all} \quad n_i, m_i \ge n_\varepsilon.$$

Therefore, for all $n_i, m_i \ge n_\varepsilon$,

$$\|f_{n_i} - f_{m_i}\|_{\mathscr{E}_M^{2\nu}} \le 2\nu \left(\int_0^{B_\varepsilon} + \int_{B_\varepsilon}^\infty\right) (f_{n_i} - f_{m_i})\tilde{}(\zeta)\exp(-2\zeta\nu)\, d\zeta < 4\varepsilon.$$

Hence, (f_{n_i}) is fundamental sequence in $\mathscr{E}_M^{2\nu}$ and there exists $f_0 \in \mathscr{E}_M^{2\nu}$ such that $f_{n_i} \to f_0$ as $i \to \infty$, $f_n \to f_0$ in $\mathscr{E}_M^{2\nu}$. So, $\mathscr{E}_M$ is complete. □

We use a real K-method interpolation (see, e.g., [2,16]). Let $0 < \vartheta < 1$ and $1 \le q \le \infty$. For a compatible couple $(\mathscr{E}_M, L_M(\mathbb{R}^n))$, we define the interpolation space

$$(\mathscr{E}_M, L_M(\mathbb{R}^n))_{\vartheta,q} = \left\{f \in \mathscr{E}_M + L_M(\mathbb{R}^n) \colon \|f\|_{(\mathscr{E}_M, L_M(\mathbb{R}^n))_{\vartheta,q}} < \infty\right\},$$

$$\|f\|_{(\mathscr{E}_M, L_M(\mathbb{R}^n))_{\vartheta,q}} = \begin{cases} \left(\int_0^\infty \left[t^{-\vartheta}K(t,f)\right]^q \dfrac{dt}{t}\right)^{1/q} & \text{if } q < \infty, \\ \sup\limits_{0<t<\infty} t^{-\vartheta}K(t,f) & \text{if } q = \infty, \end{cases}$$

where $K(t,f) = K(t,f;\mathscr{E}_M, L_M(\mathbb{R}^n)) = \inf_{f=f_0+f_1} \left(\|f_0\|^2_{\mathscr{E}_M} + t^2 \|f_1\|^2_{(M)} \right)^{1/2}$, $t > 0$.

For $\tau = \vartheta q$ and $s+1 = 1/\vartheta$ we assign the subspace

$$\mathcal{B}^s_\tau(M) = \left\{ f \in \mathscr{E}_M + L_M(\mathbb{R}^n) \colon \|f\|_{\mathcal{B}^s_\tau(M)} < \infty \right\},$$

$$\|f\|_{\mathcal{B}^s_\tau(M)} = \begin{cases} \left(\int_0^\infty \left[t^s E\left(t,f;\mathscr{E}_M, L_M(\mathbb{R}^n)\right) \right]^\tau \dfrac{dt}{t} \right)^{1/\tau} & \text{if } \tau < \infty, \\ \sup\limits_{0<t<\infty} t^s E\left(t,f;\mathscr{E}_M, L_M(\mathbb{R}^n)\right) & \text{if } \tau = \infty, \end{cases}$$

where $E\left(t,f\right) = E\left(t,f;\mathscr{E}_M, L_M(\mathbb{R}^n)\right) = \inf\left\{ \|f - f_0\|_{(M)} : f_0 \in \mathscr{E}_M, \|f_0\|_{\mathscr{E}_M} \le t \right\}$ and $t > 0$.

With regarding to the quasinorm $\|\cdot\|_{\mathcal{B}^s_\tau(M)}$, we note the following. If $\|f\|_{\mathcal{B}^s_\tau(M)} = 0$ then $E(t,f) = 0$ for all t. There exist $f_n \in \mathscr{E}_M$ $(n \in \mathbb{N})$ such that $\|f - f_n\|_{(M)} \to 0$ and $\|f_n\|_{\mathscr{E}_M} \to 0$ as $n \to \infty$. Hence, $f_n \to 0$ in $\mathscr{E}_M$ and $f_n \to f$ in $L_M(\mathbb{R}^n)$. Since $\mathscr{E}_M$ is the subspace of $L_M(\mathbb{R}^n)$, so $f_n \to 0$ in $L_M(\mathbb{R}^n)$ and $f = 0$. For $0 < \varepsilon < 1$ we have

$$\|f+g\|_{\mathcal{B}^s_\tau(M)} \le \max\left(1, 2^{-1/\tau'}\right) \left[\left(\int_0^\infty \left[t^s E(\varepsilon t, f) \right]^\tau \frac{dt}{t} \right)^{1/\tau} + \left(\int_0^\infty \left[t^s E((1-\varepsilon)t, g) \right]^\tau \frac{dt}{t} \right)^{1/\tau} \right].$$

Setting $\kappa = 2\max\left(1, 2^{-1/\tau'}\right)$, we get

$$\|f+g\|_{\mathcal{B}^s_\tau(M)} \le \kappa \left(\varepsilon^{-s} \|f\|_{\mathcal{B}^s_\tau(M)} + (1-\varepsilon)^{-s} \|g\|_{\mathcal{B}^s_\tau(M)} \right).$$

With an appropriate choice of ε, we have

$$\|f+g\|_{\mathcal{B}^s_\tau(M)} \le \kappa \left(\|f\|^{1/s}_{\mathcal{B}^s_\tau(M)} + \|g\|^{1/s}_{\mathcal{B}^s_\tau(M)} \right)^s.$$

Proposition 3. *The following isomorphism with equivalent quasinorms*

$$\mathcal{B}^s_\tau(M) = \left(\mathscr{E}_M, L_M(\mathbb{R}^n)\right)^{1/\vartheta}_{\vartheta,q} \tag{3}$$

holds, where $\left(\mathscr{E}_M, L_M(\mathbb{R}^n)\right)^{1/\vartheta}_{\vartheta,q}$ *is endowed with the quasinorm* $\|f\|^{1/\vartheta}_{(\mathscr{E}_M, L_M(\mathbb{R}^n))_{\vartheta,q}}$.

Proof. Let us reason as in [9, Theorem 2]. We use the following normalization factor

$$N_{\vartheta,q} := \begin{cases} \left(\int_0^\infty t^{q(1-\vartheta)-1} (1+t^2)^{-q/2} dt \right)^{-1/q} & \text{if } q < \infty, \\ \vartheta^{-\vartheta/2} (1-\vartheta)^{-(1-\vartheta)/2} & \text{if } q = \infty, \end{cases}$$

which is characterized in [4]. For $1 \le q < \infty$ and $f \in \mathscr{E}_M$, we obtain

$$\|f\|^q_{(\mathscr{E}_M, L_M(\mathbb{R}^n))_{\vartheta,q}} \le \|f\|^q_{\mathscr{E}_M} \|f\|^q_{(M)} \int_0^\infty \frac{t^{(1-\vartheta)q}}{(\|f\|^2_{\mathscr{E}_M} + t^2 \|f\|^2_{(M)})^{q/2}} \frac{dt}{t} = N^{-q}_{\vartheta,q} \|f\|^{q(1-\vartheta)}_{\mathscr{E}_M} \|f\|^{q\vartheta}_{(M)}.$$

In the case $q = \infty$, we obtain

$$\|f\|_{(\mathscr{E}_M, L_M(\mathbb{R}^n))_{\vartheta,\infty}} \le \|f\|^{1-\vartheta}_{\mathscr{E}_M} \|f\|^\vartheta_{(M)} \sup_{t>0} \frac{t^{1-\vartheta}}{\sqrt{1+t^2}} = N^{-1}_{\vartheta,\infty} \|f\|^{1-\vartheta}_{\mathscr{E}_M} \|f\|^\vartheta_{(M)}.$$

From [16, Remark 3.1] it follows

$$K_\infty(t,f) \le K(t,f) \le \sqrt{2}K_\infty(t,f), \tag{4}$$

where $K_\infty(t,f) = \inf\limits_{f=f_0+f_1} \max\Big(\|f_0\|_{\mathscr{E}_M}, t\|f_1\|_{(M)}\Big)$. Further,

$$\min(1,t)/\sqrt{2} \le t/\sqrt{1+t^2} \le \min(1,t). \tag{5}$$

Taking into account (4) and (5), the inequalities (6) from [8] can be written as

$$\|f\|^q_{(\mathscr{E}_M, L_M(\mathbb{R}^n))_{\vartheta,q}} \le 2^{q/2}(\vartheta q^2)^{-1}\|f\|^{\vartheta q}_{\mathcal{B}^s_\tau(M)} \le 2^{q/2}\|f\|^q_{(\mathscr{E}_M, L_M(\mathbb{R}^n))_{\vartheta,q}} \quad \text{with} \quad \tau = \vartheta q.$$

Thus, the isomorphism (3) is valid. □

Next, we establish some interpolation properties of the spaces $\mathcal{B}^s_\tau(M)$. For $0 < s_0, s_1 < \infty$, $0 < \vartheta < 1$ and $0 < \tau, \tau_0, \tau_1 < \infty$, we consider the interpolation space

$$\big(B^{s_0}_{\tau_0}(M),\, B^{s_1}_{\tau_1}(M)\big)_{\vartheta,\tau} = \Big\{ f \in B^{s_0}_{\tau_0}(M) + B^{s_1}_{\tau_1}(M)\colon \|f\|_{(B^{s_0}_{\tau_0}(M), B^{s_1}_{\tau_1}(M))_{\vartheta,\tau}} < \infty \Big\}$$

endowed with the quasinorm

$$\|f\|_{(B^{s_0}_{\tau_0}(M), B^{s_1}_{\tau_1}(M))_{\vartheta,\tau}} = \left(\int_0^\infty \Big[t^{-\vartheta}K(t,f; B^{s_0}_{\tau_0}(M),\, B^{s_1}_{\tau_1}(M))\Big]^\tau \frac{dt}{t} \right)^{1/\tau},$$

where $K(t,f; B^{s_0}_{\tau_0}(M),\, B^{s_1}_{\tau_1}(M)) = \inf\limits_{f=f_0+f_1} \left(\|f_0\|^2_{B^{s_0}_{\tau_0}(M)} + t^2\|f_1\|^2_{B^{s_1}_{\tau_1}(M)} \right)^{1/2}$, $t > 0$.

Proposition 4. *If $s = (1-\vartheta)s_0 + \vartheta s_1$ with $s_0 \ne s_1$, then*

$$\big(B^{s_0}_{\tau_0}(M),\, B^{s_1}_{\tau_1}(M)\big)_{\vartheta,\tau} = B^s_\tau(M). \tag{6}$$

If $0 < \tau \le \varrho < \infty$, then the following continuous embedding holds

$$B^s_\tau(M) \hookrightarrow B^s_\varrho(M). \tag{7}$$

Proof. By the reiteration theorem [2, Theorem 3.11.5] for the indices $\vartheta = (1-\eta)\vartheta_0 + \eta\vartheta_1$ with $\vartheta_i = 1/(s_i+1)$, $i = 0,1$, $\vartheta = 1/(s+1)$, $\tau = g\vartheta$ and $0 < \eta < 1$, we obtain

$$\Big(\big[B^{s_0}_{\tau_0}(M)\big]^{\vartheta_0},\, \big[B^{s_1}_{\tau_1}(M)\big]^{\vartheta_1}\Big)_{\eta,g} = \big[B^s_\tau(M)\big]^\vartheta. \tag{8}$$

Applying the power theorem [2, Theorem 3.11.6], we obtain

$$\Big(\big[B^{s_0}_{\tau_0}(M)\big]^{\vartheta_0},\, \big[B^{s_1}_{\tau_1}(M)\big]^{\vartheta_1}\Big)_{\eta,g} = \big(B^{s_0}_{\tau_0}(M),\, B^{s_1}_{\tau_1}(M)\big)^\vartheta_{\varrho,\tau} \tag{9}$$

with $\varrho = \eta\vartheta_1/\vartheta$. The equalities (8) and (9) for $s = (1-\varrho)s_0 + \varrho s_1$ yield (6) with $\varrho = \vartheta$.

For any $f \in \big(B^{s_0}_{\tau_0}(M),\, B^{s_1}_{\tau_1}(M)\big)_{\vartheta,\tau}$ there exists $c > 0$ such that

$$\begin{aligned} \|f\|_{\left(B^{s_0}_{\tau_0}(M), B^{s_1}_{\tau_1}(M)\right)_{\vartheta,\varrho}} &\le \Big(\sup_{t>0} t^{-\vartheta}K(t,f; B^{s_0}_{\tau_0}(M), B^{s_1}_{\tau_1}(M)) \Big)^{(1-\tau/\varrho)} \\ &\quad \times \left(\int_0^\infty \Big[t^{-\vartheta}K(t,f; B^{s_0}_{\tau_0}(M), B^{s_1}_{\tau_1}(M))\Big]^\tau \frac{dt}{t} \right)^{1/\varrho} \\ &\le c\|f\|_{\left(B^{s_0}_{\tau_0}(M), B^{s_1}_{\tau_1}(M)\right)_{\vartheta,\tau}}. \end{aligned}$$

It follows that $\big(B^{s_0}_{\tau_0}(M),\, B^{s_1}_{\tau_1}(M)\big)_{\vartheta,\tau} \hookrightarrow \big(B^{s_0}_{\tau_0}(M),\, B^{s_1}_{\tau_1}(M)\big)_{\vartheta,\varrho}$. Now using (6), we at once obtain (7). □

4 Bernstein-Jackson-type inequalities

Let us establish the Bernstein and Jackson type inequalities that give an estimate of approximation errors by means of elements $\mathscr{E}_M$ in the Orlicz space $L_M(\mathbb{R}^n)$.

Theorem 1. *For each $f \in \mathscr{E}_M$ the following Bernstein-type inequality*

$$\|f\|_{\mathcal{B}^s_\tau(M)} \leq c_{s,\tau} \|f\|^s_{\mathscr{E}_M} \|f\|_{(M)} \tag{10}$$

holds with $c_{s,\tau} = (\tau^2(1+s))^{1/\tau} N^{-(1+s)}_{1/(1+s),\tau(1+s)}$ if $\tau < \infty$ and $c_{s,\infty} = N^{-(1+s)}_{1/(1+s),\infty}$.

For each $f \in \mathcal{B}^s_\tau(M)$ the following Jackson-type inequality

$$E(t,f) \leq t^{-s} c_{s,\tau} \|f\|_{\mathcal{B}^s_\tau(M)} \tag{11}$$

holds with $c_{s,\tau} = 2^{(1+s)/2}(\tau^2(1+s))^{-1/\tau} N^{(1+s)}_{1/(1+s),\tau(1+s)}$ if $\tau < \infty$ and $c_{s,\infty} = 1$.

Proof. It suffices to apply [8, Theorem 2] and Proposition 3. □

Corollary 1. *If $\tau = 2/(1+s)$, then the following inequalities hold:*

$$\|f\|_{\mathcal{B}^s_\tau(M)} \leq c_s \|f\|^s_{\mathscr{E}_M} \|f\|_{(M)}, \quad f \in \mathscr{E}_M, \tag{12}$$

with $c_s = [(1+s)(2\pi)^{-1} \sin(\pi/(1+s))]^{-(1+s)/2}$ and

$$E(t,f) \leq t^{-s} c_s \|f\|_{\mathcal{B}^s_\tau(M)}, \quad f \in \mathcal{B}^s_\tau(M), \tag{13}$$

with $c_s = [(1+s)\pi^{-1} \sin(\pi/(1+s))]^{(1+s)/2}$.

The inequalities (12) and (13) follow from the inequalities (10) and (11) correspondingly, taking into account that $N_{\vartheta,2} = ((2/\pi)\sin(\pi\vartheta))^{1/2}$ (see [14, Exercise B.5]) and $\vartheta = 1/(s+1)$.

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Встановлено нерівності типу Бернштейна і Джексона з точними константами для оцінок найкращих наближень функціями експоненціального типу в просторах Орліча $L_M(\mathbb{R}^n)$. Для цього ми використовуємо спеціальну шкалу апроксимаційних просторів $\mathcal{B}^s_\tau(M)$, які є інтерполяційними просторами між підпростором $\mathscr{E}_M$ функцій експоненціального типу і простором $L_M(\mathbb{R}^n)$. Ці апроксимаційні простори визначаються за допомогою функціоналу $E\,(t,f)$, який відіграє таку ж роль, як модуль гладкості. Константи в отриманих нерівностях виражено через коефіцієнт нормалізації $N_{\vartheta,q}$, який залежить від параметрів τ і s апроксимаційного простору $\mathcal{B}^s_\tau(M)$.

Ключові слова і фрази: нерівності Бернштейна і Джексона, найкраща апроксимація, простір Орліча.