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Classification of the extreme points of $\mathcal{L}_s({}^2l_\infty^3)$ by computation

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Let $l_\infty^3 = \mathbb{R}^3$ be endowed with the supremum norm. In [Comment. Math. 2017, **57** (1), 1–7], S.G. Kim classified the extreme points of the unit ball of $\mathcal{L}_s({}^2l_\infty^3)$ only using Mathematica 8, where $\mathcal{L}_s({}^2l_\infty^3)$ is the space of symmetric bilinear forms on l_∞^3. It seems to be interesting and meaningful to classify the extreme points of the unit ball of $\mathcal{L}_s({}^2l_\infty^3)$ without using Mathematica 8. The aim of this paper is to make such classification by mathematical calculations.

Key words and phrases: extreme point.

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1 Introduction

Throughout the paper, we let $n, m \in \mathbb{N}, n, m \geq 2$. We write B_E for the closed unit ball of a real Banach space E. The dual space of E is denoted by E^*. An element $x \in B_E$ is called an *extreme point* of B_E if $y, z \in B_E$ with $x = \frac{1}{2}(y+z)$ implies $x = y = z$. An element $x \in B_E$ is called an *exposed point* of B_E if there is $f \in E^*$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. It is easy to see that every exposed point of B_E is an extreme point. An element $x \in B_E$ is called a *smooth point* of B_E if there is unique $f \in E^*$ so that $f(x) = 1 = \|f\|$. We denote by $\operatorname{ext} B_E$, $\exp B_E$ and $\operatorname{sm} B_E$ the set of extreme points, the set of exposed points and the set of smooth points of B_E, respectively. A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form T on the product $E \times \cdots \times E$ such that $P(x) = T(x, \cdots, x)$ for every $x \in E$. We denote by $\mathcal{P}({}^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. We denote by $\mathcal{L}({}^nE)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{\|x_k\|=1} |T(x_1, \cdots, x_n)|$. Let $\mathcal{L}_s({}^nE)$ denote the closed subspace of all continuous symmetric n-linear forms on E. Notice that $\mathcal{L}({}^nE)$ is identified with the dual of n-fold projective tensor product $\hat{\bigotimes}_{\pi,n}E$. With this identification, the action of a continuous n-linear form T as a bounded linear functional on $\hat{\bigotimes}_{\pi,n}E$ is given by

$$\left\langle \sum_{i=1}^{k} x^{(1),i} \otimes \cdots \otimes x^{(n),i},\ T \right\rangle = \sum_{i=1}^{k} T\left(x^{(1),i}, \cdots, x^{(n),i}\right).$$

Notice also that $\mathcal{L}_s({}^nE)$ is identified with the dual of n-fold symmetric projective tensor product $\hat{\bigotimes}_{s,\pi,n}E$. With this identification, the action of a continuous symmetric n-linear form T as

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a bounded linear functional on $\hat{\bigotimes}_{s,\pi,n}E$ is given by

$$\Big\langle \sum_{i=1}^{k} \frac{1}{n!}\Big(\sum_{\sigma} x^{\sigma(1),i}\otimes\cdots\otimes x^{\sigma(n),i}\Big),\ T\Big\rangle = \sum_{i=1}^{k} T\Big(x^{(1),i},\cdots,x^{(n),i}\Big),$$

where σ goes over all permutations on $\{1,\ldots,n\}$.

Since the geometries of the unit balls of $\mathcal{P}(^nE)$ and $\mathcal{L}_s(^nE)$ are closely related with the geometry of the unit ball of E, it is interesting and significant to investigate the geometries of $\mathcal{P}(^nE)$ and $\mathcal{L}_s(^nE)$. For more details about applications and significance of the theory of polynomials and (symmetric) multilinear mappings on Banach spaces, we refer the reader to [8].

Let us introduce the history of classification problems of the extreme points, the exposed points and the smooth points of the unit ball of continuous n-homogeneous polynomials on a Banach space.

We let $l_p^n = \mathbb{R}^n$ for every $1 \le p \le \infty$ equipped with the l_p-norm. Y.S. Choi et al. [3–5] initiated and classified $\operatorname{ext} B_{\mathcal{P}(^2l_p^2)}$ for $p = 1,2$. Y.S. Choi and S.G. Kim [6] classified $\operatorname{sm} B_{\mathcal{P}(^2l_1)}$. B.C. Grecu [12] classified $\operatorname{ext} B_{\mathcal{P}(^2l_p^2)}$ for $1 < p < 2$ or $2 < p < \infty$. In the paper [45], S.G. Kim et al. showed that if E is a separable real Hilbert space with $\dim(E) \ge 2$, then, $\operatorname{ext} B_{\mathcal{P}(^2E)} = \exp B_{\mathcal{P}(^2E)}$. In [16], S.G. Kim classified $\exp B_{\mathcal{P}(^2l_p^2)}$ for $1 \le p \le \infty$, and in [18,20], he characterized $\operatorname{ext} B_{\mathcal{P}(^2d_*(1,w)^2)}$ and $\operatorname{sm} B_{\mathcal{P}(^2d_*(1,w)^2)}$, where $d_*(1,w)^2 = \mathbb{R}^2$ with the octagonal norm $\|(x,y)\|_w = \max\Big\{|x|,|y|,\frac{|x|+|y|}{1+w}\Big\}$ for $0 < w < 1$. In [25], S.G. Kim classified $\exp B_{\mathcal{P}(^2d_*(1,w)^2)}$ and showed that $\exp B_{\mathcal{P}(^2d_*(1,w)^2)} \ne \operatorname{ext} B_{\mathcal{P}(^2d_*(1,w)^2)}$. In [30,33,44], he classified $\operatorname{ext} B_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})}$, $\exp B_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})}$ and $\operatorname{sm} B_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})}$, where $\mathbb{R}^2_{h(\frac{1}{2})} = \mathbb{R}^2$ with the hexagonal norm $\|(x,y)\|_{h(\frac{1}{2})} = \max\Big\{|y|,|x|+\frac{1}{2}|y|\Big\}$.

Parallel to the classification problems of $\operatorname{ext} B_{\mathcal{P}(^nE)}$, $\exp B_{\mathcal{P}(^nE)}$ and $\operatorname{sm} B_{\mathcal{P}(^nE)}$, it seems to be natural and interesting to study the classification problems of the extreme points, the exposed points and the smooth points of the unit ball of continuous (symmetric) multilinear forms on a Banach space since (symmetric) multilinear forms on a Banach space is closely related with homogeneous polynomials in their definitions.

In [17,19,21,22,24,28,29,32,34,36,37,39], S.G. Kim classified $\operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^2)}$, $\operatorname{ext} B_{\mathcal{L}_s(^2d_*(1,w)^2)}$, $\operatorname{ext} B_{\mathcal{L}(^2d_*(1,w)^2)}$, $\exp B_{\mathcal{L}_s(^2d_*(1,w)^2)}$, $\operatorname{ext} B_{\mathcal{L}(^2d_*(1,w)^2)}$, $\operatorname{ext} B_{\mathcal{L}(^2l_\infty^n)}$, $\operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^n)}$, $\operatorname{ext} B_{\mathcal{L}(^nl_\infty^2)}$, $\operatorname{sm} B_{\mathcal{L}_s(^nl_\infty^2)}$ for every $n \ge 2$ and studied $\operatorname{ext} B_{\mathcal{L}(^2l_\infty)}$. He showed that $\exp B_{\mathcal{L}_s(^2l_\infty^n)} = \operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^n)}$, $\exp B_{\mathcal{L}_s(^nl_\infty^2)} = \operatorname{ext} B_{\mathcal{L}_s(^nl_\infty^2)}$, $\exp B_{\mathcal{L}(^2l_\infty^n)} = \operatorname{ext} B_{\mathcal{L}(^2l_\infty^n)}$, $\exp B_{\mathcal{L}(^nl_\infty^2)} = \operatorname{ext} B_{\mathcal{L}(^nl_\infty^2)}$, $|\operatorname{ext} B_{\mathcal{L}(^nl_\infty^2)}| = 2^{(2^n)}$, $|\operatorname{ext} B_{\mathcal{L}_s(^nl_\infty^2)}| = 2^{n+1}$ for every $n \ge 2$. In [2], M. Cavalcante et al. characterized $\operatorname{ext} B_{\mathcal{L}(^nl_\infty^m)}$. In [40], S.G. Kim classified $\operatorname{sm} B_{\mathcal{L}(^nl_\infty^m)}$ and $\operatorname{sm} B_{\mathcal{L}_s(^nl_\infty^m)}$ for every $n,m \ge 2$. In [38], S.G. Kim characterized $\operatorname{ext} B_{\mathcal{L}_s(^nl_\infty^m)}$, $\operatorname{ext} B_{\mathcal{L}(^nl_\infty^m)}$ and showed that $\exp B_{\mathcal{L}_s(^ml_\infty^m)} = \operatorname{ext} B_{\mathcal{L}_s(^nl_\infty^m)}$, $\exp B_{\mathcal{L}(^ml_\infty^m)} = \operatorname{ext} B_{\mathcal{L}(^nl_\infty^m)}$ for every $n,m \ge 2$. In [41], S.G. Kim classified extreme points and exposed points of the unit ball of the space of bilinear symmetric forms on the real Banach space of bilinear symmetric forms on l_∞^2. It is shown that for this case, the set of extreme points is equal to the set of exposed points. In [42], S.G. Kim characterized $\operatorname{ext} B_{\mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})}$ and $\operatorname{ext} B_{\mathcal{L}_s(^n\mathbb{R}^m_{\|\cdot\|})}$, where $\mathbb{R}^m_{\|\cdot\|}$ is $\mathbb{R}^m$ with a norm $\|\cdot\|$ such that $|\operatorname{ext} B_{\mathbb{R}^m_{\|\cdot\|}}| = 2m$ for $m \ge 2$. In [43], S.G. Kim characterized $\operatorname{ext} B_{\mathcal{L}(^nl_1)}$ for $n \ge 2$.

We refer the reader to [1,6,7,9–11,13–15,23,26,27,31,35–38,46–53] and references therein for some recent work about extremal properties of homogeneous polynomials and multilinear forms on Banach spaces. For the applications of extreme point theory in optimization and optimal control theory, we refer the reader to [54,55].

The aim of this paper is to classify the extreme points of the unit ball of $\mathcal{L}_s(^2l_\infty^3)$ by mathematical calculations.

2 Results

In [28], S.G. Kim classified $\text{ext}\, B_{\mathcal{L}_s(^2l_\infty^3)}$ only using Mathematica 8. Using the classification it was also shown that every extreme point of the unit ball of $\mathcal{L}_s(^2l_\infty^3)$ is exposed. It seems to be interesting and meaningful to classify $\text{ext}\, B_{\mathcal{L}_s(^2l_\infty^3)}$ without using Mathematica 8. We will classify $\text{ext}\, B_{\mathcal{L}_s(^2l_\infty^3)}$ by mathematical calculations.

Let $l_\infty^3 = \mathbb{R}^3$ with the supremum norm. If $T \in \mathcal{L}_s(^2l_\infty^3)$ and $(x_1, y_1, z_1),\ (x_2, y_2, z_2) \in l_\infty^3$, then

$$T((x_1,y_1,z_1),(x_2,y_2,z_2)) = ax_1x_2 + by_1y_2 + cz_1z_2 + d_{12}(x_1y_2 + x_2y_1) + d_{13}(x_1z_2 + x_2z_1) + d_{23}(y_1z_2 + y_2z_1)$$

for some $a, b, c, d_{12}, d_{13}, d_{23} \in \mathbb{R}$. For simplicity, we denote $T = (a, b, c, 2d_{12}, 2d_{13}, 2d_{23})$.

Theorem 1 ([28]). *Let $T = (a, b, c, 2d_{12}, 2d_{13}, 2d_{23}) \in \mathcal{L}_s(^2l_\infty^3)$. Then,*

$$\|T\| = \max\{2|d_{12}| + |a+b-c|, 2|d_{13}| + |a-b+c|, 2|d_{23}| + |-a+b+c|, 2|d_{12}+d_{13}| + |a+b+c+2d_{23}|, 2|d_{12}-d_{13}| + |a+b+c-2d_{23}|\}.$$

Note that if $\|T\| = 1$, then $|a| \le 1, |b| \le 1, |c| \le 1$ and $2|d_{ij}| \le 1$ for $1 \le i < j \le 3$. For $T = (a, b, c, 2d_{12}, 2d_{13}, 2d_{23}) \in \mathcal{L}_s(^2l_\infty^3)$, we let

$$\begin{aligned} T_1 &:= (a, b, c, 2d_{12}, -2d_{13}, -2d_{23}), & S_1 &:= (b, a, c, 2d_{12}, 2d_{23}, 2d_{13}), \\ T_2 &:= (a, b, c, -2d_{12}, -2d_{13}, 2d_{23}), & S_2 &:= (c, a, b, 2d_{13}, 2d_{23}, 2d_{12}), \\ T_3 &:= (a, c, b, 2d_{13}, 2d_{12}, 2d_{23}), & S_3 &:= (c, b, a, 2d_{23}, 2d_{13}, 2d_{12}). \end{aligned}$$

Then, $T_k, S_k \in \mathcal{L}_s(^2l_\infty^3)$ with $\|T_k\| = \|S_k\| = \|T\|$ for $k = 1, 2, 3$.

Notice that if $T = (a, b, c, 2d_{12}, 2d_{13}, 2d_{23}) \in \mathcal{L}_s(^2l_\infty^3)$, we may assume that $a \ge |b| \ge |c|$ and

$$d_{12}d_{13} \ge 0. \tag{1}$$

Theorem 2. *Let $T = (a, b, c, 2d_{12}, 2d_{13}, 2d_{23}) \in \mathcal{L}_s(^2l_\infty^3)$. Then, the following are equivalent:*

(a) $T \in \text{ext}\, B_{\mathcal{L}_s(^2l_\infty^3)}$;

(b) $T_k, S_k \in \text{ext}\, B_{\mathcal{L}_s(^2l_\infty^3)}$ *for some* $k = 1, 2, 3$.

Proof. It is obvious. □

Theorem 3. *Let $x, a_j, b_j, c, d \in \mathbb{R}$ for $1 \le j \le 6$ and $T(x) = (a_1x + b_1, \ldots, a_6x + b_6) \in \mathcal{L}_s(^2l_\infty^3)$. Suppose that $\|T(x)\| \le 1$ for all $c < x < d$. If $x_0 \in \mathbb{R}$ be such that $c < x_0 < d$ and $\|T(x_0)\| = 1$, then $T(x_0) \notin \text{ext}\, B_{\mathcal{L}_s(^2l_\infty^3)}$.*

Proof. Let $\delta > 0$ be such that $c < x_0 - \delta < x_0 + \delta < d$. Define $T_j \in \mathcal{L}_s(^2l_\infty^3)$ by

$$T_1 = (a_1(x_0+\delta) + b_1, \ldots, a_6(x_0+\delta) + b_6) \ \text{ and } \ T_2 = (a_1(x_0-\delta) + b_1, \ldots, a_6(x_0-\delta) + b_6).$$

By the hypothesis, $\|T_i\| \le 1$ for $i = 1, 2$. Since $T_i \ne T(x_0)$ for $i = 1, 2$ and $T(x_0) = \frac{1}{2}(T_1 + T_2)$, $T(x_0) \notin \text{ext}\, B_{\mathcal{L}_s(^2l_\infty^3)}$. □

Lemma 1. *Let $x, y, z \in \mathbb{R}$. Then,*
(*a*) $1 = |x + y| = |x - y|$ *if and only if* $(|x| = 1, y = 0)$ *or* $(x = 0, |y| = 1)$*;*
(*b*) *if* $|1 + z| \leq 1,\ |1 - z| \leq 1$, *then* $z = 0$.

Proof. (a) *Necessity*. Suppose the contrary. Then, $x \neq 0$ and $y \neq 0$. Without loss of generality we may assume that $0 < y \leq x < 1$. It follows that $1 = x + y = x - y$, which shows that $x = 1$, a contradiction. Sufficiency is obvious.

(b) Since $2 = |(1+z) + (1-z)| \leq |1+z| + |1-z| \leq 2$ and $|1 \pm z| \leq 1$, $|1-z| = |1+z| = 1$. By (a), $z = 0$. □

Lemma 2. *Let $a, b \in \mathbb{R}$ be such that $|a| + |b| = 1$. Then, the following are equivalent:*
(i) $(|a| = 1, b = 0)$ *or* $(a = 0, |b| = 1)$;
(ii) *if $\epsilon, \delta \in \mathbb{R}$ satisfies $|a + \epsilon| + |b + \delta| \leq 1$ and $|a - \epsilon| + |b - \delta| \leq 1$, then $\epsilon = \delta = 0$.*

Proof. By symmetry, we may assume that $|a| \geq |b|$.

$(i) \Rightarrow (ii)$ Suppose that $|a| = 1, b = 0$ and let $\epsilon, \delta \in \mathbb{R}$ be such that $|a + \epsilon| + |b + \delta| \leq 1$ and $|a - \epsilon| + |b - \delta| \leq 1$. Then, $|a + \epsilon| + |\delta| \leq 1$ and $|a - \epsilon| + |\delta| \leq 1$, which shows that $1 \geq |a| + |\epsilon| + |\delta| = 1 + |\epsilon| + |\delta|$. Therefore, $\epsilon = \delta = 0$.

$(ii) \Rightarrow (i)$ Suppose the contrary. Then $0 < |b| \leq |a| < 1$. Let $t > 0$ be such that $t|a| < |b|$. Let $\epsilon := t|a|\mathrm{sign}(a)$ and $\delta := -t|a|\mathrm{sign}(b)$. Notice that $\epsilon \neq 0$ and $\delta \neq 0$. It follows that

$$|a + \epsilon| + |b + \delta| = (|a| + t|a|) + (|b| - t|a|) = |a| + |b| = 1$$

and

$$|a - \epsilon| + |b - \delta| = (|a| - t|a|) + (|b| + t|a|) = |a| + |b| = 1.$$

This is a contradiction. □

Proposition 1. *Let $T = (a, b, c, 2d_{12}, 2d_{13}, 2d_{23}) \in \mathcal{L}_s(^2 l_\infty^3)$ be such that $\|T\| = 1$. If $|a| = 1$ or $|b| = 1$ or $|c| = 1$, then $T = \pm(1, 0, 0, 0, 0, 0)$ or $\pm(0, 1, 0, 0, 0, 0)$ or $\pm(0, 0, 1, 0, 0, 0)$, respectively.*

Proof. By (1) we may assume that $a \geq |b| \geq |c|$. Hence, $|a| = 1$. Without loss of generality we may assume that $a = 1$. By Theorem 1, we have

$$1 \geq |a + b - c| = |1 + b - c|, \qquad 1 \geq |a - b + c| = |1 - b + c|.$$

By Lemma 1 (b), $b = c$. By Theorem 1, we have

$$1 \geq 2|d_{12}| + |a + b - c| = 1 + 2|d_{12}|, \qquad 1 \geq 2|d_{13}| + |a - b + c| = 1 + 2|d_{13}|,$$

which shows that $d_{12} = d_{13} = 0$. By Theorem 1, we have

$$\begin{aligned} 1 &\geq 2|d_{12} + d_{13}| + |a + b + c + 2d_{23}| = |1 + 2b + 2d_{23}|, \\ 1 &\geq 2|d_{12} - d_{13}| + |a + b + c - 2d_{23}| = |1 + 2b - 2d_{23}|, \end{aligned} \tag{2}$$

which implies that $|1 + 2b| \leq 1$. By Theorem 1, $|1 - 2b| \leq 2|d_{23}| + |1 - 2b| \leq 1$. By Lemma 1 (b), $b = c = 0$. By (2), $d_{23} = 0$. Therefore, $T = \pm(1, 0, 0, 0, 0, 0)$. We complete the proof. □

Proposition 2. *Let $T = (a, b, c, 2d_{12}, 2d_{13}, 2d_{23}) \in \mathcal{L}_s(^2 l_\infty^3)$ be such that $\|T\| = 1$ and $d_{12} = d_{13} = d_{23} = 0$. If at least three among $|a + b + c|, |a + b - c|, |a - b + c|, |-a + b + c|$ equal to 1, then $T \in \{\pm(1, 0, 0, 0, 0, 0), \pm(0, 1, 0, 0, 0, 0), \pm(0, 0, 1, 0, 0, 0)\}$.*

Proof. By (1) we may assume that $a \geq |b| \geq |c|$. By symmetry it suffices to show the theorem when $|a+b-c| = |a-b+c| = |-a+b+c| = 1$. By Theorem 1, $|a+b+c| \leq 1$ since $\|T\| = 1$. Since $|a \pm (b-c)| = 1$, by Lemma 1, $|a| = 1, b = c$. By Proposition 1, $T = \pm(1,0,0,0,0,0)$. □

Proposition 3. *Let* $T = (a,b,c,2d_{12},2d_{13},2d_{23}) \in \mathcal{L}_s(^2l^3_\infty)$ *be such that* $\|T\| = 1$ *and* $d_{12} = d_{13} = d_{23} = 0$. *Then,* $T \in \operatorname{ext} B_{\mathcal{L}_s(^2l^3_\infty)}$ *if and only if*

$$T \in \{\pm(1,0,0,0,0,0), \pm(0,1,0,0,0,0), \pm(0,0,1,0,0,0)\}.$$

Proof. Necessity. By Theorem 1,

$$|a+b+c| \leq 1, |a+b-c| \leq 1, |a-b+c| \leq 1, |-a+b+c| \leq 1.$$

Claim. At least three among $|a+b+c|, |a+b-c|, |a-b+c|, |-a+b+c|$ equal to 1.

Otherwise. We have two cases.

Case 1. $|a+b+c| = 1, |a+b-c| = 1, |a-b+c| < 1, |-a+b+c| < 1$.

By Lemma 1, $1 = |a+b| = a+b, c = 0$. Obviously, $|a-b| < 1$ and $T = (a, 1-a, 0,0,0,0)$ for $0 < a < 1$. By Theorem 3, T is not extreme. This is a contradiction.

Case 2. $|-a+b+c| = 1, |a-b+c| = 1, |a+b+c| < 1, |a+b-c| < 1$.

By analogous arguments as Case 1, $T = (a, a-1, 0,0,0,0)$ for $0 < a < 1$. By Theorem 3, T is not extreme. This is a contradiction.

Therefore, the claim holds. By Proposition 2, necessity follows. Sufficiency is obvious. □

Proposition 4. *Let* $T = (a,b,c,2d_{12},2d_{13},2d_{23}) \in \mathcal{L}_s(^2l^3_\infty)$ *be such that* $\|T\| = 1$ *and at least one among* $2|d_{12}|, 2|d_{13}|, 2|d_{23}|$ *equals to 1. Then,* $T \in \operatorname{ext} B_{\mathcal{L}_s(^2l^3_\infty)}$ *if and only if*

$$\begin{aligned} T \in \Big\{ &\pm\Big(\frac{1}{2}, -\frac{1}{2}, 0, 1, 0, 0\Big), \pm\Big(\frac{1}{2}, -\frac{1}{2}, 0, -1, 0, 0\Big), \pm\Big(\frac{1}{2}, 0, -\frac{1}{2}, 0, 1, 0\Big), \\ &\pm\Big(\frac{1}{2}, 0, -\frac{1}{2}, 0, -1, 0\Big), \pm\Big(0, \frac{1}{2}, -\frac{1}{2}, 0, 0, 1\Big), \pm\Big(0, \frac{1}{2}, -\frac{1}{2}, 0, 0, -1\Big) \Big\}. \end{aligned}$$

Proof. By (1) we may assume that $a \geq |b| \geq |c|$.

Necessity. By Theorem 2, it suffices to show the assertion for the case $2|d_{12}| = 1$. By Theorem 1, $0 = a+b-c = a+b+c = d_{13} = d_{23}$, which shows that $0 = c = a+b$, $-\frac{1}{2} \leq a \leq \frac{1}{2}$ and $T = (a, -a, 0, \pm 1, 0, 0)$ for $-\frac{1}{2} \leq a \leq \frac{1}{2}$. Since T is extreme, by Theorem 3, $a = \pm\frac{1}{2}$ and hence, $T = \pm(\frac{1}{2}, -\frac{1}{2}, 0, \pm 1, 0, 0)$.

Sufficiency. By Theorem 2, it suffices to show that $T = (\frac{1}{2}, -\frac{1}{2}, 0, 1, 0, 0) \in \operatorname{ext} B_{\mathcal{L}_s(^2l^3_\infty)}$.

Let $R_j \in \mathcal{L}_s(^2l^3_\infty)$ be such that

$$R_1 = \Big(\frac{1}{2} + \epsilon_1, -\frac{1}{2} + \epsilon_2, \epsilon_3, 1 + \delta_{12}, \delta_{13}, \delta_{23}\Big)$$

and

$$R_2 = \Big(\frac{1}{2} - \epsilon_1, -\frac{1}{2} - \epsilon_2, -\epsilon_3, 1 - \delta_{12}, -\delta_{13}, -\delta_{23}\Big)$$

with $\|R_1\| = \|R_2\| = 1$ for some $\epsilon_i, \delta_{ij} \in \mathbb{R}$ for $i,j = 1,2,3$ with $i < j$. By Theorem 1, it follows that

$$0 = \delta_{12} = \delta_{13} = \delta_{23}, \quad 0 = \epsilon_1 + \epsilon_2 + \epsilon_3, \quad 0 = -\epsilon_1 + \epsilon_2 + \epsilon_3, \quad 0 = \epsilon_1 - \epsilon_2 + \epsilon_3,$$

which show that $\epsilon_i = \delta_{ij} = 0$ for $i,j = 1,2,3$ with $i < j$. Hence, $R_1 = R_2 = T$ and hence $T \in \operatorname{ext} B_{\mathcal{L}_s(^2l^3_\infty)}$. □

Proposition 5. *Let $T = (a, b, c, 2d_{12}, 2d_{13}, 2d_{23}) \in \mathcal{L}_s({}^2 l_\infty^3)$ be such that $\|T\| = 1$ and $2|d_{12}| = 2|d_{13}| = 2|d_{23}| = \frac{1}{2}$. Then $T \in \text{ext}\, B_{\mathcal{L}_s({}^2 l_\infty^3)}$ if and only if*

$$\begin{aligned} T \in \Big\{ & \pm\Big(\frac{1}{2}, 0, 0, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\Big), \pm\Big(\frac{1}{2}, 0, 0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\Big), \pm\Big(\frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\Big), \\ & \pm\Big(0, \frac{1}{2}, 0, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\Big), \pm\Big(0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\Big), \pm\Big(0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\Big), \\ & \pm\Big(0, 0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\Big), \pm\Big(0, 0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\Big), \pm\Big(0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\Big), \\ & \pm\Big(-\frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\Big), \pm\Big(0, -\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\Big), \pm\Big(0, 0, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\Big)\Big\}. \end{aligned}$$

Proof. By (1) we may assume that $a \geq |b| \geq |c|$ and $d_{12} d_{13} \geq 0$.

Necessity. By Theorem 2, it suffices to show the assertion for the case $2d_{12} = 2d_{13} = \frac{1}{2}$. By Theorem 1, we have

$$a + b + c = 2d_{23}, \quad |a + b - c| \leq \frac{1}{2}, \quad |a - b + c| \leq \frac{1}{2}, \quad |-a + b + c| \leq \frac{1}{2}, \quad |a + b + c + 2d_{23}| \leq 1.$$

Hence, $|a + b + c + 2d_{23}| = 1$.

Claim. At least two among $|a + b - c|, |a - b + c|, |-a + b + c|$ equal to $\frac{1}{2}$.

Otherwise. Then,

$$\begin{aligned} & \Big(|a + b - c| = \frac{1}{2}, |a - b + c| < \frac{1}{2}, |-a + b + c| < \frac{1}{2}\Big), \\ & \Big(|a + b - c| < \frac{1}{2}, |a - b + c| = \frac{1}{2}, |-a + b + c| < \frac{1}{2}\Big), \\ & \Big(|a + b - c| < \frac{1}{2}, |a - b + c| < \frac{1}{2}, |-a + b + c| = \frac{1}{2}\Big) \text{ or} \\ & \Big(|a + b - c| < \frac{1}{2}, |a - b + c| < \frac{1}{2}, |-a + b + c| < \frac{1}{2}\Big). \end{aligned}$$

Suppose that $|a + b - c| = \frac{1}{2}$, $|a - b + c| < \frac{1}{2}$ and $|-a + b + c| < \frac{1}{2}$. Let $R_j \in \mathcal{L}_s({}^2 l_\infty^3)$ be such that

$$R_1 = \Big(a, b + \frac{1}{N}, c + \frac{1}{N}, 2d_{12}, 2d_{13}, 2d_{23}\Big) \quad \text{and} \quad R_2 = \Big(a, b - \frac{1}{N}, c - \frac{1}{N}, 2d_{12}, 2d_{13}, 2d_{23}\Big),$$

where $|-a + b + c| + \frac{2}{N} < \frac{1}{2}$. Then, $\|R_1\| = \|R_2\| = 1$ and $T = \frac{1}{2}(R_1 + R_2)$ and, hence, T is not extreme. This is a contradiction. Similarly, if T satisfies the other cases, we may reach to a contradiction. Therefore, we have shown the claim.

Hence, $(|a + b - c| = |a - b + c| = \frac{1}{2})$, $(|a + b - c| = |-a + b + c| = \frac{1}{2})$ or $(|a - b + c| = |-a + b + c| = \frac{1}{2})$.

By symmetry, we may assume that $|a + b - c| = |a - b + c| = \frac{1}{2}$. Since $|2a \pm 2(b - c)| = 1$, by Lemma 1, $|a| = \frac{1}{2}, b = c$. Since $|2a - 4b| \leq 1$, $|2a + 4b| = 1$, by Lemma 1, $b = c = 0$, $a = -2d_{23} = \frac{1}{2}$ and $T = \Big(\frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\Big)$.

Sufficiency. By Theorem 2, it suffices to show that $T = \Big(\frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\Big) \in \text{ext}\, B_{\mathcal{L}_s({}^2 l_\infty^3)}$.

Let $R_j \in \mathcal{L}_s({}^2 l_\infty^3)$ be such that

$$R_1 = \Big(\frac{1}{2} + \epsilon_1, \epsilon_2, \epsilon_3, \frac{1}{2} + \delta_{12}, \frac{1}{2} + \delta_{13}, -\frac{1}{2} + \delta_{23}\Big)$$

and

$$R_2 = \Big(\frac{1}{2} - \epsilon_1, -\epsilon_2, -\epsilon_3, \frac{1}{2} - \delta_{12}, \frac{1}{2} - \delta_{13}, -\frac{1}{2} - \delta_{23}\Big)$$

with $\|R_1\| = \|R_2\| = 1$ for some $\epsilon_i, \delta_{ij} \in \mathbb{R}$, $i, j = 1, 2, 3$ with $i < j$.

By Theorem 1, it follows that

$$\begin{aligned}
0 &= \delta_{12} - \delta_{13} = \delta_{12} + \delta_{13}, \\
0 &= \epsilon_1 + \epsilon_2 + \epsilon_3 - \delta_{23}, \\
0 &= \epsilon_1 + \epsilon_2 + \epsilon_3 + \delta_{23}, \\
0 &= -\epsilon_1 + \epsilon_2 + \epsilon_3, \\
0 &= \epsilon_1 - \epsilon_2 + \epsilon_3,
\end{aligned}$$

which show that $\epsilon_i = \delta_{ij} = 0$ for $i, j = 1, 2, 3$ with $i < j$. Hence, $R_1 = R_2 = T$ and, hence, $T \in \operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^3)}$. $\square$

We are in position to show the main result.

Theorem 4.

$$\begin{aligned}
\operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^3)} = \Big\{ &\pm (1,0,0,0,0,0), \pm(0,1,0,0,0,0), \pm(0,0,1,0,0,0), \\
&\pm \Big(\frac{1}{2}, -\frac{1}{2}, 0, 1, 0, 0\Big), \pm\Big(\frac{1}{2}, -\frac{1}{2}, 0, -1, 0, 0\Big), \pm\Big(\frac{1}{2}, 0, -\frac{1}{2}, 0, 1, 0\Big), \\
&\pm \Big(\frac{1}{2}, 0, -\frac{1}{2}, 0, -1, 0\Big), \pm\Big(0, \frac{1}{2}, -\frac{1}{2}, 0, 0, 1\Big), \pm\Big(0, \frac{1}{2}, -\frac{1}{2}, 0, 0, -1\Big), \\
&\pm \Big(\frac{1}{2}, 0, 0, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\Big), \pm\Big(\frac{1}{2}, 0, 0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\Big), \pm\Big(\frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\Big), \\
&\pm \Big(0, \frac{1}{2}, 0, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\Big), \pm\Big(0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\Big), \pm\Big(0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\Big), \\
&\pm \Big(0, 0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\Big), \pm\Big(0, 0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\Big), \pm\Big(0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\Big), \\
&\pm \Big(-\frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\Big), \pm\Big(0, -\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\Big), \pm\Big(0, 0, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\Big)\Big\}.
\end{aligned}$$

Proof. By Theorem 2 and Propositions 1–5, the 42 bilinear forms in the list of Theorem 4 are extreme.

Let $T = (a, b, c, 2d_{12}, 2d_{13}, 2d_{23}) \in \operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^3)}$. By (1) we may assume that $a \geq |b| \geq |c|$ and $d_{12}d_{13} \geq 0$. We will show that T is contained in the list of Theorem 4.

Let $W_1 := 2d_{12}$, $Z_1 := a + b - c$, $W_2 := 2d_{13}$, $Z_2 := a - b + c$, $W_3 := 2d_{23}$, $Z_3 := -a + b + c$, $W_4 := 2(d_{12} + d_{13})$, $Z_4 := a + b + c + 2d_{23}$, $W_5 := 2(d_{12} - d_{13})$, $Z_5 := a + b + c - 2d_{23}$, $B_j := |W_j| + |Z_j|$ for $j = 1, \ldots, 5$.

Remark 1. *Notice that if* $S = (a', b', c', 2d'_{12}, 2d'_{13}, 2d'_{23}) \in \mathcal{L}_s(^2l_\infty^3)$ *is an element in the list of Theorem 4, then* $B'_j = 1$ *for all* $j = 1, \ldots, 5$*, where*

$$\begin{aligned}
B'_1 &:= |2d'_{12}| + |a' + b' - c'|, \\
B'_2 &:= |2d'_{13}| + |a' - b' + c'|, \\
B'_3 &:= |2d'_{23}| + |-a' + b' + c'|, \\
B'_4 &:= 2|d'_{12} + d'_{13}| + |a' + b' + c' + 2d'_{23}|, \\
B'_5 &:= 2|d'_{12} - d'_{13}| + |a' + b' + c' - 2d'_{23}|.
\end{aligned}$$

Claim. $B_j = 1$ for all $j = 1, \ldots, 5$.

Otherwise. Suppose that only two among $B_1, \ldots, B_5$ equal to 1. We will reach to a contradiction. Ten cases may occur:

$$\begin{aligned}
&(B_1 = B_2 = 1, B_3 < 1, B_4 < 1, B_5 < 1), \quad (B_1 = B_3 = 1, B_2 < 1, B_4 < 1, B_5 < 1),\\
&(B_1 = B_4 = 1, B_2 < 1, B_3 < 1, B_5 < 1), \quad (B_1 = B_5 = 1, B_2 < 1, B_3 < 1, B_4 < 1),\\
&(B_2 = B_3 = 1, B_1 < 1, B_4 < 1, B_5 < 1), \quad (B_2 = B_4 = 1, B_1 < 1, B_3 < 1, B_5 < 1),\\
&(B_2 = B_5 = 1, B_1 < 1, B_3 < 1, B_3 < 1), \quad (B_3 = B_4 = 1, B_1 < 1, B_2 < 1, B_5 < 1),\\
&(B_3 = B_5 = 1, B_1 < 1, B_2 < 1, B_4 < 1) \quad \text{or} \quad (B_4 = B_5 = 1, B_1 < 1, B_2 < 1, B_3 < 1).
\end{aligned}$$

By symmetry it is enough to consider the five cases:

$$\begin{aligned}
&(B_1 = B_2 = 1, B_3 < 1, B_4 < 1, B_5 < 1), \quad (B_1 = B_3 = 1, B_2 < 1, B_4 < 1, B_5 < 1),\\
&(B_1 = B_4 = 1, B_2 < 1, B_3 < 1, B_5 < 1), \quad (B_3 = B_4 = 1, B_1 < 1, B_2 < 1, B_5 < 1) \quad \text{or}\\
&(B_4 = B_5 = 1, B_1 < 1, B_2 < 1, B_3 < 1).
\end{aligned}$$

Suppose that $B_1 = B_2$ and the others are less than 1. Let $N \in \mathbb{N}$ such that $B_j + \frac{1}{N} < 1$ for $j = 3, 4, 5$. Let $T_j \in \mathcal{L}_s({}^2 l_\infty^3)$ be such that

$$T_1 = \Big(a, b, c, 2d_{12}, 2d_{13}, 2d_{23} + \frac{1}{N}\Big) \quad \text{and} \quad T_2 = \Big(a, b, c, 2d_{12}, 2d_{13}, 2d_{23} - \frac{1}{N}\Big).$$

By Theorem 1, $T_1 \neq T_2, \|T_j\| = 1$ and $T = \frac{1}{2}(T_1 + T_2)$. Hence T is not extreme. This is a contradiction.

Suppose that $B_1 = B_3$ and the others are less than 1. Let $N \in \mathbb{N}$ such that $B_j + \frac{2}{N} < 1$ for $j = 2, 4, 5$. Let $T_j \in \mathcal{L}_s({}^2 l_\infty^3)$ be such that

$$T_1 = \Big(a + \frac{1}{N}, b, c + \frac{1}{N}, 2d_{12}, 2d_{13}, 2d_{23}\Big) \quad \text{and} \quad T_2 = \Big(a - \frac{1}{N}, b, c - \frac{1}{N}, 2d_{12}, 2d_{13}, 2d_{23}\Big).$$

By Theorem 1, $T_1 \neq T_2, \|T_j\| = 1$ and $T = \frac{1}{2}(T_1 + T_2)$. Hence T is not extreme. This is a contradiction.

Suppose that $B_1 = B_4$ and the others are less than 1. Let $N \in \mathbb{N}$ such that $B_j + \frac{2}{N} < 1$ for $j = 2, 3, 5$. Let $T_j \in \mathcal{L}_s({}^2 l_\infty^3)$ be such that

$$T_1 = \Big(a + \frac{1}{N}, b, c + \frac{1}{N}, 2d_{12}, 2d_{13}, 2d_{23} - \frac{1}{N}\Big) \text{ and } T_2 = \Big(a - \frac{1}{N}, b, c - \frac{1}{N}, 2d_{12}, 2d_{13}, 2d_{23} + \frac{1}{N}\Big).$$

By Theorem 1, $T_1 \neq T_2, \|T_j\| = 1$ and $T = \frac{1}{2}(T_1 + T_2)$. Hence T is not extreme. This is a contradiction.

Suppose that $B_3 = B_4$ and the others are less than 1. Let $N \in \mathbb{N}$ such that $B_j + \frac{3}{N} < 1$ for $j = 1, 2, 5$. Let $T_j \in \mathcal{L}_s({}^2 l_\infty^3)$ be such that

$$T_1 = \Big(a, b + \frac{1}{N}, c - \frac{1}{N}, 2d_{12} + \frac{1}{N}, 2d_{13} - \frac{1}{N}, 2d_{23}\Big)$$

and

$$T_2 = \Big(a, b - \frac{1}{N}, c + \frac{1}{N}, 2d_{12} - \frac{1}{N}, 2d_{13} - \frac{1}{N}, 2d_{23}\Big).$$

By Theorem 1, $T_1 \neq T_2, \|T_j\| = 1$ and $T = \frac{1}{2}(T_1 + T_2)$. Hence T is not extreme. This is a contradiction.

Suppose that $B_4 = B_5$ and the others are less than 1. Let $N \in \mathbb{N}$ such that $B_j + \frac{2}{N} < 1$ for $j = 1, 2, 3$. Let $T_j \in \mathcal{L}_s(^2l_\infty^3)$ be such that

$$T_1 = \Big(a + \frac{1}{N}, b, c - \frac{1}{N}, 2d_{12}, 2d_{13}, 2d_{23}\Big) \quad \text{and} \quad T_2 = \Big(a + \frac{1}{N}, b, c - \frac{1}{N}, 2d_{12}, 2d_{13}, 2d_{23}\Big).$$

By Theorem 1, $T_1 \neq T_2, \|T_j\| = 1$ and $T = \frac{1}{2}(T_1 + T_2)$. Hence T is not extreme. This is a contradiction.

Suppose that only three among $B_1, \ldots, B_5$ equal to 1. We will reach to a contradiction. Ten cases may occur:

$$\begin{aligned}
&(B_1 = B_2 = B_3 = 1, B_4 < 1, B_5 < 1), \quad (B_1 = B_2 = B_4 = 1, B_3 < 1, B_5 < 1),\\
&(B_1 = B_2 = B_5 = 1, B_3 < 1, B_4 < 1), \quad (B_1 = B_3 = B_4 = 1, B_2 < 1, B_5 < 1),\\
&(B_1 = B_3 = B_5 = 1, B_2 < 1, B_4 < 1), \quad (B_2 = B_3 = B_4 = 1, B_1 < 1, B_5 < 1),\\
&(B_2 = B_3 = B_5 = 1, B_1 < 1, B_4 < 1), \quad (B_1 = B_4 = B_5 = 1, B_2 < 1, B_3 < 1),\\
&(B_2 = B_4 = B_5 = 1, B_1 < 1, B_3 < 1) \quad \text{or} \quad (B_3 = B_4 = B_5 = 1, B_1 < 1, B_2 < 1).
\end{aligned}$$

By symmetry it is enough to consider five cases:

$$\begin{aligned}
&(B_1 = B_2 = B_3 = 1, B_4 < 1, B_5 < 1), \quad (B_1 = B_2 = B_4 = 1, B_3 < 1, B_5 < 1),\\
&(B_1 = B_3 = B_4 = 1, B_2 < 1, B_5 < 1), \quad (B_1 = B_4 = B_5 = 1, B_2 < 1, B_4 < 1) \quad \text{or}\\
&(B_3 = B_4 = B_5 = 1, B_2 < 1, B_3 < 1).
\end{aligned}$$

Suppose that $B_1 = B_2 = B_3 = 1$ and $B_4 < 1, B_5 < 1$.

Since T is extreme, by Lemmas 1–2, we have

$$\begin{aligned}
&\Big[(2d_{12} = 0, |a + b - c| = 1) \text{ or } (2|d_{12}| = 1, |a + b - c| = 0)\Big],\\
&\Big[(2d_{13} = 0, |a - b + c| = 1) \text{ or } (2|d_{13}| = 1, |a - b + c| = 0)\Big] \quad \text{and}\\
&\Big[(2d_{23} = 0, |-a + b + c| = 1) \text{ or } (2|d_{23}| = 1, |-a + b + c| = 0)\Big].
\end{aligned}$$

If at least one among $2|d_{12}|, 2|d_{13}|, 2|d_{23}|$ equals to 1, then, by Proposition 4, T is contained in the list of Theorem 4. By Remark 1, $1 > B_4 = 1$, a contradiction. If $2d_{12} = 2d_{13} = 2d_{23} = 0$, then, by Proposition 3, T is contained in the list of Theorem 4. By Remark 1, $1 > B_4 = 1$, a contradiction.

Suppose that $B_1 = B_2 = B_4 = 1$ and $B_3 < 1, B_5 < 1$.

Since T is extreme, by Lemmas 1–2, we have

$$\begin{aligned}
&\Big[(2d_{12} = 0, |a + b - c| = 1) \text{ or } (2|d_{12}| = 1, |a + b - c| = 0)\Big],\\
&\Big[(2d_{13} = 0, |a - b + c| = 1) \text{ or } (2|d_{13}| = 1, |a - b + c| = 0)\Big] \quad \text{and}\\
&\Big[(2(d_{12} + d_{13}) = 0, |a + b + c + 2d_{23}| = 1) \text{ or } (2|d_{12} + d_{13}| = 1, |a + b + c + 2d_{23}| = 0)\Big].
\end{aligned}$$

We claim that $2|d_{12}| = 1$ or $2|d_{13}| = 1$. Otherwise. Then, $2d_{12} = 2d_{13} = 0$. Then,

$$|a + b - c| = |a - b + c| = 1.$$

By Lemma 1, $a = 1, |b-c| = 0$. By Proposition 1, $T = (1,0,0,0,0,0)$. Hence, $1 > B_3 = 1$, a contradiction.

Suppose that $B_1 = B_3 = B_4 = 1$ and $B_2 < 1, B_5 < 1$.

Since T is extreme, by Lemmas 1–2, we have

$$\Big[(2d_{12} = 0, |a+b-c| = 1) \text{ or } (2|d_{12}| = 1, |a+b-c| = 0)\Big],$$
$$\Big[(2d_{23} = 0, |-a+b+c| = 1) \text{ or } (2|d_{23}| = 1, |-a+b+c| = 0)\Big], \quad \text{and}$$
$$\Big[(2(d_{12}+d_{13}) = 0, |a+b+c+2d_{23}| = 1) \text{ or } (2|d_{12}+d_{13}| = 1, |a+b+c+2d_{23}| = 0)\Big].$$

If $2|d_{12}| = 1$ or $2|d_{23}| = 1$, by Proposition 4, T is contained in the list of Theorem 4. By Remark 1, $1 > B_2 = 1$, a contradiction.

Suppose that $2|d_{12}| = 2|d_{23}| = 0$. Then,

$$(2d_{13} = 0, |a+b+c| = 1) \text{ or } (2|d_{13}| = 1, |a+b+c| = 0).$$

By Propositions 3–4, T is contained in the list of Theorem 4. By Remark 1, $1 > B_2 = 1$, a contradiction.

Suppose that $B_1 = B_4 = B_5 = 1$ and $B_2 < 1, B_3 < 1$.

Since T is extreme, by Lemmas 1–2, we have

$$\Big[(2d_{12} = 0, |a+b-c| = 1) \text{ or } (2|d_{12}| = 1, |a+b-c| = 0)\Big],$$
$$\Big[(2(d_{12}+d_{13}) = 0, |a+b+c+2d_{23}| = 1) \text{ or } (2|d_{12}+d_{13}| = 1, |a+b+c+2d_{23}| = 0)\Big], \quad \text{and}$$
$$\Big[(2(d_{12}-d_{13}) = 0, |a+b+c-2d_{23}| = 1) \text{ or } (2|d_{12}-d_{13}| = 1, |a+b+c-2d_{23}| = 0)\Big].$$

If $2|d_{12}| = 1$, by Proposition 4, T is contained in the list of Theorem 4. By Remark 1, $1 > B_2 = 1$, a contradiction. Suppose that $2|d_{12}| = 0$. Then,

$$\Big[(2d_{13} = 0, |a+b+c+2d_{23}| = 1) \text{ or } (2|d_{13}| = 1, |a+b+c+2d_{23}| = 0)\Big] \quad \text{and}$$
$$\Big[(2d_{13} = 0, |a+b+c-2d_{23}| = 1) \text{ or } (2|d_{13}| = 1, |a+b+c-2d_{23}| = 0)\Big].$$

If $2|d_{13}| = 1$, by Proposition 4, T is contained in the list of Theorem 4. By Remark 1, $1 > B_2 = 1$, a contradiction. Suppose that $2|d_{13}| = 0$. Then, $|(a+b+c) \pm 2d_{23}| = 1$. By Lemma 1,

$$(a+b+c = 0, 2|d_{23}| = 1) \text{ or } (|a+b+c| = 1, 2|d_{23}| = 0).$$

If $2|d_{23}| = 1$, by Proposition 4, T is contained in the list of Theorem 4. By Remark 1, $1 > B_2 = 1$, a contradiction. If $2|d_{23}| = 0$, then $d_{12} = d_{13} = d_{23} = 0$. By Proposition 3, T is contained in the list of Theorem 4. By Remark 1, $1 > B_2 = 1$, a contradiction.

Suppose that $B_3 = B_4 = B_5 = 1$ and $B_1 < 1, B_2 < 1$.

Since T is extreme, by Lemmas 1–2, we have

$$\Big[(2d_{23} = 0, |-a+b+c| = 1) \text{ or } (2|d_{23}| = 1, |-a+b+c| = 0)\Big],$$
$$\Big[(2(d_{12}+d_{13}) = 0, |a+b+c+2d_{23}| = 1) \text{ or } (2|d_{12}+d_{13}| = 1, |a+b+c+2d_{23}| = 0)\Big] \quad \text{and}$$
$$\Big[(2(d_{12}-d_{13}) = 0, |a+b+c-2d_{23}| = 1) \text{ or } (2|d_{12}-d_{13}| = 1, |a+b+c-2d_{23}| = 0)\Big].$$

If $2|d_{23}| = 1$, by Proposition 4, T is contained in the list of Theorem 4. By Remark 1, $1 > B_1 = 1$, a contradiction.

Let $2|d_{23}| = 0$. Suppose that $2(d_{12} + d_{13}) = 0, |a + b + c + 2d_{23}| = 1$. If $2(d_{12} - d_{13}) = 0$, $|a + b + c - 2d_{23}| = 1$, then

$$d_{12} = d_{13} \quad \text{and} \quad [(a + b + c = 0, 2|d_{23}| = 1) \quad \text{or} \quad (|a + b + c| = 1, 2d_{23} = 0)].$$

By Propositions 3–4, T is contained in the list of Theorem 4. By Remark 1, $1 > B_1 = 1$, a contradiction.

Suppose that $2|d_{12} + d_{13}| = 1, |a + b + c + 2d_{23}| = 0$. Then, $|2d_{12} \pm 2d_{13}| = 1$. By Lemma 2,

$$(|2d_{12}| = 1, |2d_{13}| = 0) \quad \text{or} \quad (|2d_{12}| = 0, |2d_{13}| = 1).$$

By Proposition 4, T is contained in the list of Theorem 4. By Remark 1, $1 > B_1 = 1$, a contradiction.

Suppose that only four among $B_1, \ldots, B_5$ equal to 1. We will reach to a contradiction.

Five cases may occur:

$$\begin{aligned}
&(B_1 = B_2 = B_3 = B_4 = 1, B_5 < 1), \quad (B_1 = B_2 = B_3 = B_5 = 1, B_4 < 1),\\
&(B_1 = B_2 = B_4 = B_5 = 1, B_3 < 1), \quad (B_1 = B_3 = B_4 = B_5 = 1, B_2 < 1) \quad \text{or}\\
&(B_2 = B_3 = B_4 = B_5 = 1, B_1 < 1).
\end{aligned}$$

By symmetry it is enough to consider three cases:

$$\begin{aligned}
&(B_1 = B_2 = B_3 = B_4 = 1, B_5 < 1), \quad (B_1 = B_2 = B_4 = B_5 = 1, B_3 < 1) \quad \text{or}\\
&(B_2 = B_3 = B_4 = B_5 = 1, B_1 < 1).
\end{aligned}$$

Suppose that $B_1 = B_2 = B_3 = B_4 = 1, B_5 < 1$.

Since T is extreme, by Lemmas 1–2, we have

$$\begin{aligned}
&\Big\{\Big[(2d_{12} = 0, |a + b - c| = 1) \text{ or } (2|d_{12}| = 1, |a + b - c| = 0)\Big], \quad \text{and}\\
&\quad \Big[(2d_{13} = 0, |a - b + c| = 1) \text{ or } (2|d_{13}| = 1, |a - b + c| = 0)\Big]\Big\},\\
&\Big\{\Big[(2d_{12} = 0, |a + b - c| = 1) \text{ or } (2|d_{12}| = 1, |a + b - c| = 0)\Big] \quad \text{and}\\
&\quad \Big[(2d_{23} = 0, |-a + b + c| = 1) \text{ or } (2|d_{23}| = 1, |-a + b + c| = 0)\Big]\Big\},\\
&\Big\{\Big[(2d_{12} = 0, |a + b - c| = 1) \text{ or } (2|d_{12}| = 1, |a + b - c| = 0)\Big] \quad \text{and}\\
&\quad \Big[(2(d_{12} + d_{13}) = 0, |a + b + c + 2d_{23}| = 1) \text{ or } (2|d_{12} + d_{13}| = 1, |a + b + c + 2d_{23}| = 0)\Big]\Big\},\\
&\Big\{\Big[(2d_{13} = 0, |a - b + c| = 1) \text{ or } (2|d_{13}| = 1, |a - b + c| = 0)\Big] \quad \text{and}\\
&\quad \Big[(2d_{23} = 0, |-a + b + c| = 1) \text{ or } (2|d_{23}| = 1, |-a + b + c| = 0)\Big]\Big\},\\
&\Big\{\Big[(2d_{13} = 0, |a - b + c| = 1) \text{ or } (2|d_{13}| = 1, |a - b + c| = 0)\Big] \quad \text{and}\\
&\quad \Big[(2(d_{12} + d_{13}) = 0, |a + b + c + 2d_{23}| = 1) \text{ or } (2|d_{12} + d_{13}| = 1, |a + b + c + 2d_{23}| = 0)\Big]\Big\} \quad \text{or}\\
&\Big\{\Big[(2d_{23} = 0, |-a + b + c| = 1) \text{ or } (2|d_{23}| = 1, |-a + b + c| = 0)\Big] \quad \text{and}\\
&\quad \Big[(2(d_{12} + d_{13}) = 0, |a + b + c + 2d_{23}| = 1) \text{ or } (2|d_{12} + d_{13}| = 1, |a + b + c + 2d_{23}| = 0)\Big]\Big\}.
\end{aligned}$$

By symmetry it is enough to consider four subcases:

$$\Big\{\Big[(2d_{12}=0,|a+b-c|=1) \text{ or } (2|d_{12}|=1,|a+b-c|=0)\Big] \quad \text{and}$$
$$\Big[(2d_{13}=0,|a-b+c|=1) \text{ or } (2|d_{13}|=1,|a-b+c|=0)\Big]\Big\},$$
$$\Big\{\Big[(2d_{12}=0,|a+b-c|=1) \text{ or } (2|d_{12}|=1,|a+b-c|=0)\Big] \quad \text{and}$$
$$\Big[(2d_{23}=0,|-a+b+c|=1) \text{ or } (2|d_{23}|=1,|-a+b+c|=0)\Big]\Big\},$$
$$\Big\{\Big[(2(d_{12}+d_{13})=0,|a+b+c+2d_{23}|=1) \text{ or } (2|d_{12}+d_{13}|=1,|a+b+c+2d_{23}|=0)\Big] \quad \text{and}$$
$$\Big[(2(d_{12}+d_{13})=0,|a+b+c+2d_{23}|=1) \text{ or } (2|d_{12}+d_{13}|=1,|a+b+c+2d_{23}|=0)\Big]\Big\} \quad \text{or}$$
$$\Big\{\Big[(2d_{23}=0,|-a+b+c|=1) \text{ or } (2|d_{23}|=1,|-a+b+c|=0)\Big] \quad \text{and}$$
$$\Big[(2(d_{12}+d_{13})=0,|a+b+c+2d_{23}|=1) \text{ or } (2|d_{12}+d_{13}|=1,|a+b+c+2d_{23}|=0)\Big]\Big\}.$$

Suppose that

$$\Big[(2d_{12}=0,|a+b-c|=1) \text{ or } (2|d_{12}|=1,|a+b-c|=0)\Big] \quad \text{and}$$
$$\Big[(2d_{13}=0,|a-b+c|=1) \text{ or } (2|d_{13}|=1,|a-b+c|=0)\Big].$$

If $2|d_{12}|=1$ or $2|d_{13}|=1$, by Proposition 4, T is contained in the list of Theorem 4. By Remark 1, $1>B_5=1$, a contradiction.

Let $d_{12}=d_{13}=0, |a+b-c|=|a-b+c|=1$. By Lemma 1, $a=1, b=c$. By Proposition 1, T is contained in the list of Theorem 4. By Remark 1, $1>B_5=1$, a contradiction.

Suppose that

$$\Big[(2d_{13}=0,|a-b+c|=1) \text{ or } (2|d_{13}|=1,|a-b+c|=0)\Big] \quad \text{and}$$
$$\Big[(2(d_{12}-d_{13})=0,|a+b+c-2d_{23}|=1) \text{ or } (2|d_{12}-d_{13}|=1,|a+b+c-2d_{23}|=0)\Big].$$

If $2|d_{13}|=1$, by Proposition 4, T is contained in the list of Theorem 4. By Remark 1, $1>B_5=1$, a contradiction.

Let $2d_{13}=d_{12}-d_{13}=0$. Then, $d_{12}=d_{13}=0, |a+b-c|=|a-b+c|=1$. By Lemma 1, $a=1, b=c$. By Proposition 1, T is contained in the list of Theorem 4. By Remark 1, $1>B_5=1$, a contradiction.

Suppose that

$$\Big[(2(d_{12}+d_{13})=0,|a+b+c+2d_{23}|=1) \text{ or } (2|d_{12}+d_{13}|=1,|a+b+c+2d_{23}|=0)\Big] \quad \text{and}$$
$$\Big[(2(d_{12}-d_{13})=0,|a+b+c-2d_{23}|=1) \text{ or } (2|d_{12}-d_{13}|=1,|a+b+c-2d_{23}|=0)\Big].$$

Let $2(d_{12}+d_{13})=2(d_{12}-d_{13})=0$. Since $|(a+b+c)\pm 2d_{23}|=1$, by Lemma 1,

$$(|a+b+c|=1, d_{23}=0) \text{ or } (|a+b+c|=0, 2|d_{23}|=1).$$

If $|a+b+c|=1, d_{23}=0$, then $d_{12}=d_{13}=d_{23}=0$. By Proposition 3, T is contained in the list of Theorem 4. By Remark 1, $1>B_5=1$, a contradiction.

If $|a+b+c| = 0, 2|d_{23}| = 1$, by Proposition 4, T is contained in the list of Theorem 4. By Remark 1, $1 > B_5 = 1$, a contradiction.

If $2|d_{12}+d_{13}| = 2|d_{12}-d_{13}| = 1$, by Lemma 1,

$$(|2d_{12}| = 1, d_{23} = 0) \text{ or } (d_{12} = 0, 2|d_{23}| = 1).$$

By Proposition 4, T is contained in the list of Theorem 4. By Remark 1, $1 > B_5 = 1$, a contradiction.

Let $(2(d_{12}+d_{13}) = 0, 2|d_{12}-d_{13}| = 1)$ or $(2(d_{12}+d_{13}) = 1, 2|d_{12}-d_{13}| = 0)$. Suppose that $(2(d_{12}+d_{13}) = 0, |a+b+c+2d_{23}| = 1)$ or $(2|d_{12}+d_{13}| = 0, |a+b+c+2d_{23}| = 1)$. Then, $4|d_{12}| = 4|d_{13}| = 4|d_{232}| = 1$. By Proposition 5, T is contained in the list of Theorem 4. By Remark 1, $1 > B_5 = 1$, a contradiction.

If $(2(d_{12}+d_{13}) = 1, |a+b+c+2d_{23}| = 0)$ or $(2|d_{12}+d_{13}| = 1, |a+b+c+2d_{23}| = 0)$, then $4|d_{12}| = 4|d_{13}| = 4|d_{232}| = 1$. By Proposition 5, T is contained in the list of Theorem 4. By Remark 1, $1 > B_5 = 1$, a contradiction.

Suppose that $B_1 = B_2 = B_4 = B_5 = 1, B_3 < 1$.

Suppose that

$$(2d_{12} = 0, |a+b-c| = 1) \quad \text{and} \quad (2d_{13} = 0, |a-b+c| = 1).$$

By Lemma 1, $a = 1, b = c$. By Proposition 1, T is contained in the list of Theorem 4. By Remark 1, $1 > B_3 = 1$, a contradiction.

Suppose that

$$(2d_{12} = 0, |a+b-c| = 1) \quad \text{and} \quad (2(d_{12}+d_{13}) = 0, |a+b+c+2d_{23}| = 1).$$

Then, $d_{12} = d_{13} = 0, a = 1, b = c, 2|d_{23}| = 1$. By Proposition 1, T is contained in the list of Theorem 4. By Remark 1, $1 > B_3 = 1$, a contradiction.

Suppose that

$$(2d_{13} = 0, |a-b+c| = 1) \quad \text{and} \quad (2(d_{12}-d_{13}) = 0, |a+b+c-2d_{23}| = 1).$$

Then, $d_{12} = d_{13} = 0, a = 1, b = c, 2|d_{23}| = 1$. By Proposition 1, T is contained in the list of Theorem 4. By Remark 1, $1 > B_3 = 1$, a contradiction.

Suppose that $B_2 = B_3 = B_4 = B_5 = 1, B_1 < 1$.

Suppose that

$$(2d_{13} = 0, |a-b+c| = 1) \quad \text{and} \quad (2d_{23} = 0, |-a+b+c| = 1).$$

Then, $d_{13} = d_{23}| = 0, |c| = 1, a = b$. By Proposition 1, T is contained in the list of Theorem 4. By Remark 1, $1 > B_1 = 1$, a contradiction.

Suppose that

$$(2d_{23} = 0, |-a+b+c| = 1) \quad \text{and} \quad (2(d_{12}+d_{13}) = 0, |a+b+c+2d_{23}| = 1).$$

Then, $d_{12} = d_{13} = d_{23} = 0$. By Proposition 3, T is contained in the list of Theorem 4. By Remark 1, $1 > B_1 = 1$, a contradiction.

Suppose that

$$(2d_{23} = 0, |-a+b+c| = 1) \quad \text{and} \quad (2|d_{12}+d_{13}| = 1, |a+b+c+2d_{23}| = 0).$$

Then, $d_{12} = d_{13} = 0, 2|d_{23}| = 1$. By Proposition 4, T is contained in the list of Theorem 4. By Remark 1, $1 > B_1 = 1$, a contradiction.

Suppose that

$$\Big[(2(d_{12}+d_{13}) = 0, |a+b+c+2d_{23}| = 1) \text{ or } (2|d_{12}+d_{13}| = 1, |a+b+c+2d_{23}| = 0)\Big] \quad \text{and}$$
$$\Big[(2(d_{12}-d_{13}) = 0, |a+b+c-2d_{23}| = 1) \text{ or } (2|d_{12}-d_{13}| = 1, |a+b+c-2d_{23}| = 0)\Big].$$

If $(2(d_{12}+d_{13}) = 0, |a+b+c+2d_{23}| = 1)$ and $(2(d_{12}-d_{13}) = 0, |a+b+c-2d_{23}| = 1)$, then $2|d_{23}| = 1$. By Proposition 4, T is contained in the list of Theorem 4. By Remark 1, $1 > B_1 = 1$, a contradiction.

If $(2(d_{12}+d_{13}) = 0, |a+b+c+2d_{23}| = 1)$ and $(2|d_{12}-d_{13}| = 1, |a+b+c-2d_{23}| = 0)$, then $4|d_{12}| = 4|d_{13}| = 4|d_{23}| = 1$. By Proposition 5, T is contained in the list of Theorem 4. By Remark 1, $1 > B_1 = 1$, a contradiction.

If $(2(d_{12}+d_{13}) = 1, |a+b+c+2d_{23}| = 0)$ and $(2|d_{12}-d_{13}| = 0, |a+b+c-2d_{23}| = 1)$, then $4|d_{12}| = 4|d_{13}| = 4|d_{23}| = 1$. By Proposition 5, T is contained in the list of Theorem 4. By Remark 1, $1 > B_1 = 1$, a contradiction.

If $(2(d_{12}+d_{13}) = 1, |a+b+c+2d_{23}| = 0)$ and $(2|d_{12}-d_{13}| = 1, |a+b+c-2d_{23}| = 0)$, then $2|d_{12}| = 1$ or $2|d_{13}| = 1$. By Proposition 4, T is contained in the list of Theorem 4. By Remark 1, $1 > B_1 = 1$, a contradiction.

Therefore, $B_j = 1$ for all $j = 1, \ldots, 5$. Since T is extreme, by Lemma 2, there are $j_0 \in \{1, \ldots, 5\}$ such that

$$|W_{j_0} \pm Z_{j_0}| = 1.$$

By Theorem 2, it suffices to consider the cases $j_0 = 1, 3, 4$.

Case 1. $j_0 = 1$.

Then, $(2|d_{12}| = 1, a+b-c = 0)$ or $(2d_{12} = 0, |a-b+c| = 1)$. If $2|d_{12}| = 1$, $a+b-c = 0$, then $|-a+b+c| = |a-b+c| = 1, a+b+c = 0$. Hence, $T = \Big(\frac{1}{2}, -\frac{1}{2}, 0, \pm 1, 0, 0\Big)$. Suppose that $2d_{12} = 0, |a-b+c| = 1$. Since $|(a+b+c) \pm 2d_{23}| = 1 - 2|d_{13}|$, by Lemma 1, $(2|d_{23}| = 1 - 2|d_{13}|, a+b+c = 0)$ or $(2d_{23} = 0, |a+b+c| = 1 - 2|d_{13}|)$. Suppose that $2d_{23} = 0, |a+b+c| = 1 - 2|d_{13}|$. Since $|b \pm (a-c)| = 1$, by Lemma 1, $b = 0, a-c = |a-c| = 1$. Since $a \geq |b| \geq |c|$, $a = 1, b = c = d_{12} = d_{13} = d_{23} = 0$. Hence, $T = (1, 0, 0, 0, 0, 0)$. Suppose that $2|d_{23}| = 1 - 2|d_{13}|, a+b+c = 0$. Notice that $|a+b-c| = 1$. Without loss of generality, we may assume that $a+b-c = 1$. Hence, $c = -\frac{1}{2}$ and $b = \frac{1}{2} - a$ for $0 \leq a \leq 1$. Since $a \geq |b| \geq |c|$, $a = 1, b = -\frac{1}{2}$. Hence, $d_{12} = d_{13} = d_{23} = 0$. $1 = B_5 = 0$, a contradiction. Hence, the case $2|d_{23}| = 1 - 2|d_{13}|, a+b+c = 0$ can not happen.

Case 2. $j_0 = 3$.

Then, $(2|d_{23}| = 1, -a+b+c = 0)$ or $(2d_{23} = 0, |-a+b+c| = 1)$. If $2|d_{23}| = 1$, $T = (0, 0, 0, 0, 0, \pm 1)$. Then, $1 = B_1 = 0$, a contradiction. Therefore, the case $2|d_{23}| = 1$, $-a+b+c = 0$ can not happen. Suppose that $2d_{23} = 0, |-a+b+c| = 1$. Since $|2d_{12} \pm 2d_{13}| = 1 - |a+b+c|$, by Lemma 1, $(2|d_{12}| = 1 - |a+b+c|, 2d_{13} = 0)$ or $(2d_{12} = 0, 2|d_{13}| = 1 - |a+b+c|)$. Suppose that $2|d_{12}| = 1 - |a+b+c|, 2d_{13} = 0$. Since $|c \pm (a-b)| = 1$, by Lemma 1, $c = 0, a-b = |a-b| = 1$. Hence, $b = a-1$ and

$$T = (a, a-1, 0, \pm(1 - |1-2a|), 0, 0)$$

for $\frac{1}{2} \leq a \leq 1$. Since T is extreme, $T = (1,0,0,0,0,0)$ or $T = \left(\frac{1}{2}, -\frac{1}{2}, 0, \pm 1, 0, 0\right)$. Suppose that $2d_{12} = 0, 2|d_{13}| = 1 - |a+b+c|$. Since $|b \pm (a-c)| = 1$, by Lemma 1, $b = 0, a - c = |a-c| = 1$. Since $a \geq |b| \geq |c|$, $c = 0 = d_{13}$ and $T = (1,0,0,0,0,0)$.

Case 3. $j_0 = 4$.

Then, $(2(d_{12}+d_{13}) = 0, |a+b+c+2d_{23}| = 1)$ or $(2|d_{12}+d_{13}| = 1, a+b+c+2d_{23} = 0)$. Suppose that $2(d_{12}+d_{13}) = 0, |a+b+c+2d_{23}| = 1$. Then, $d_{13} = -d_{12} = 0$ because $d_{12}d_{13} \geq 0$. Hence, $|a+b+c| = |a-b+c| = |a+b-c| = 1$. By Proposition 2, $T = (1,0,0,0,0,0)$. Suppose that $2|d_{12}+d_{13}| = 1, a+b+c+2d_{23} = 0$. Without loss of generality, we may assume that $d_{12} \geq d_{13} \geq 0$. Notice that $2|d_{23}| = 2d_{13} = 1 - 2d_{12}$ and $|c + (a-b)| = |c - (a-b)|$. Let $l := |c + (a-b)|$. By Lemma 1, $(|c| = l, a - b = |a-b| = 0)$ or $(c = 0, a-b = |a-b| = l)$. Suppose that $|c| = l, a-b = |a-b| = 0$. Since $|2a \pm c| = 1 - 2d_{12}$, by Lemma 1, $2a = 1 - 2d_{12}, c = 0$. Hence,

$$T = \left(\frac{1-2d_{12}}{2}, \frac{1-2d_{12}}{2}, 0, 2d_{12}, 1-2d_{12}, \pm(1-2d_{12})\right)$$

for $0 \leq 2d_{12} \leq 1$. Since T is extreme, $T = (0,0,0,1,0,0)$ or $T = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, 1, \pm 1\right)$. This is a contradiction because $1 = B_2 = 0$ or $1 = B_4 = 3$ or $1 = B_5 = 3$. Therefore, the case $|c| = l, a-b = |a-b| = 0$ can not happen. Suppose that $c = 0, a-b = |a-b| = l$. Then, $a = \frac{1}{2}, b = \frac{1}{2} - 2d_{12}$ and $T = \left(\frac{1}{2}, \frac{1}{2} - 2d_{12}, 0, 1-2d_{12}, \pm(1-2d_{12})\right)$ for $\frac{1}{2} \leq 2d_{12} \leq 1$. Since T is extreme, $T = \left(\frac{1}{2}, -\frac{1}{2}, 0, 1, 0, 0\right)$ or $T = \left(\frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$.

Therefore, we complete the proof. □

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Нехай простір $l_\infty^3 = \mathbb{R}^3$ оснащено рівномірною нормою. У статті [Comment. Math. 2017, **57** (1), 1–7], С.Ґ. Кім класифікував екстремальні точки одиничної кулі простору $\mathcal{L}_s(^2l_\infty^3)$ використовуючи лише пакет Mathematica 8, де $\mathcal{L}_s(^2l_\infty^3)$ є простором симетричних білінійних форм на l_∞^3. Виглядає на те, що було б цікаво та важливо класифікувати екстремальні точки одиничної кулі простору $\mathcal{L}_s(^2l_\infty^3)$ без використання Mathematica 8. Метою цієї статті є зробити таку класифікацію за допомогою математичних обчислень.

Ключові слова і фрази: екстремальна точка.