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Generalized fractional inequalities of the Hermite-Hadamard type via new Katugampola generalized fractional integrals

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A new generalization of the Katugampola generalized fractional integrals in terms of the Mittag-Leffler functions is proposed. Consequently, new generalizations of the Hermite-Hadamard inequalities by this newly proposed fractional integral operator, for a positive convex stochastic process, are established. Other known results are easily deduced as particular cases of these inequalities. The obtained results also hold for any convex function.

Key words and phrases: Hermite-Hadamard inequalities, Mittag-Leffler function, generalized Katugampola fractional integral, convex and positive stochastic process.

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Introduction

The Mittag-Leffler function is considered as the mother of all other functions because of its applications in fractional calculus. In this paper, we generalize the Hermite-Hadamard (HH) inequalities for new Katugampola generalized fractional integrals via Mittag-Leffler functions for convex-positive stochastic processes.

Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to be convex if for all $x, y \in I$ and $\lambda \in [0, 1]$, the following inequality holds:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

For this class of functions, the following theorem is known.

Theorem 1 (Hermite-Hadamard inequality)**.** *Let $f : I \to \mathbb{R}$ be a convex function, and $a, b \in I$ with $a < b$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

There are many extensions by means of convexity, analogues of inequality (1) (with the Riemann-Liouville, Hadamard, Erdélyi-Kober, Katugampola, Weyl and Liouville fractional integrals) that abound in the literature. Also, inequalities akin to the above double inequality have been established for different classes of functions. See the following papers [4, 6, 7, 9, 10, 12–14] and their references for more.

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Not long ago, U.N. Katugampola [5] unified the aforementioned six integral operators as follows.

Suppose $X_c^p(a,b)$, $c \in \mathbb{R}$, denote the set of complex valued Lebesgue measurable functions f on $[a,b]$ with the norm

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty,$$

and

$$\|f\|_{X_c^\infty} = \sup_{x\in(a,b)} \text{ess}\, |t^c f(t)|.$$

Let $f \in X_c^p(a,b)$, $\alpha > 0$ and $\beta, \rho, \eta, \kappa \in \mathbb{R}$. Then the left (and respectively the right) fractional integral of f is given by

$$\left({}^\rho\mathcal{I}^{\alpha,\beta}_{a^+,\eta,\kappa} f\right)(t) = \frac{\rho^{1-\beta} t^\kappa}{\Gamma(\alpha)} \int_a^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho(\eta+1)-1} f(s) ds, \quad 0 \le a < t < b \le \infty,$$

and

$$\left({}^\rho\mathcal{I}^{\alpha,\beta}_{b^-,\eta,\kappa} f\right)(t) = \frac{\rho^{1-\beta} t^{\rho\eta}}{\Gamma(\alpha)} \int_t^b (s^\rho - t^\rho)^{\alpha-1} s^{\kappa+\rho-1} f(s) ds, \quad 0 \le a < t < b \le \infty.$$

For $\eta = 0$, $\beta = \alpha$ and $\kappa = 0$, one obtains, from the above defined operators, the so-called Katugampola integrals.

On the other hand, in 1980, K. Nikodem [8] introduced the notion of convex stochastic processes (with $(\Omega, \mathcal{F}, P)$ a probability space) and proposed the following definition. A stochastic process $X : I \times \Omega \to \mathbb{R}$ is said to be convex if

$$X(\lambda a + (1-\lambda)b, .) \le \lambda X(a, .) + (1-\lambda) X(b, .)$$

holds almost everywhere for all $a, b \in I$ and $\lambda \in [0,1]$. If we put $\lambda = \frac{1}{2}$ in the above inequality, then the process X is Jensen-convex or $\frac{1}{2}$-convex. A stochastic process X is termed concave if $-X$ is convex. For a stochastic process, we have the following result.

Theorem 2 ([6]). *Let $X : I \times \Omega \to \mathbb{R}$ be a convex and mean square continuous process in the interval $I \times \Omega$. Then*

$$X\left(\frac{a+b}{2}, .\right) \le \frac{1}{b-a} \int_a^b X(t, .) dt \le \frac{X(a, .) + X(b, .)}{2} \tag{2}$$

holds almost everywhere.

Recently, B. Ahmad et al. [1] gave the definition of the new fractional integrals.

Definition 1 ([1]). *Let $u \in L_1(a,b)$. The fractional integrals $\mathcal{I}_a^\alpha$ and $\mathcal{I}_b^\alpha$ of order $\alpha \in (0,1)$ are defined by*

$$\mathcal{I}_a^\alpha u(x) = \frac{1}{\alpha} \int_a^x \exp\left(-\frac{1-\alpha}{\alpha}(x-s)\right) u(s) ds, \quad x > a, \tag{3}$$

and

$$\mathcal{I}_b^\alpha u(x) = \frac{1}{\alpha} \int_x^b \exp\left(-\frac{1-\alpha}{\alpha}(s-x)\right) u(s) ds, \quad x < b, \tag{4}$$

respectively.

Theorem 3 ([1]). *Let $u : [a,b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $u \in L_1(a,b)$. If u is a convex function on $[a,b]$, then the following inequalities for fractional integrals* (3) *and* (4) *hold:*

$$u\left(\frac{a+b}{2}\right) \le \frac{1-\alpha}{2\left(1-\exp\left(-\frac{1-\alpha}{\alpha}(b-a)\right)\right)}\left[\mathcal{I}_a^\alpha u(b) + \mathcal{I}_b^\alpha u(a)\right] \le \frac{u(a)+u(b)}{2}. \tag{5}$$

In this article, we propose the extension and generalization of the Katugampola generalized integrals for Mittag-Leffler functions as follows.

Definition 2. *Let $f \in X_c^p(a,b)$, $\alpha > 0$ and $\beta, \rho, \eta, \kappa \in \mathbb{R}$. Then the left (and respectively the right) fractional integral of f is given by*

$$\left({}^\rho I_{a^+,\eta,\kappa}^{\alpha,\beta} f\right)(t) = \frac{\rho^{1-\beta} t^\kappa}{\Gamma(\alpha)} \int_a^t E_{\rho,\rho}\left[-\frac{1-\alpha}{\alpha}(t^\rho - s^\rho)\right] s^{\rho(\eta+1)-1} f(s)ds, \quad 0 \le a < t < b \le \infty,$$

and

$$\left({}^\rho I_{b^-,\eta,\kappa}^{\alpha,\beta} f\right)(t) = \frac{\rho^{1-\beta} t^{\rho\eta}}{\Gamma(\alpha)} \int_t^b E_{\rho,\rho}\left[-\frac{1-\alpha}{\alpha}(s^\rho - t^\rho)\right] s^{\kappa+\rho-1} f(s)ds, \quad 0 \le a < t < b \le \infty.$$

The function $E_{\alpha,\beta}(.)$ is the Mittag-Leffler function given by

$$E_{\alpha,\beta}(.) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta)}, \quad \alpha,\ \beta > 0,$$

with $E_{\alpha,1}(z) = E_\alpha(z)$, $E_1(z) = e^z$ and $E_{1,2} = \frac{e^z-1}{z}$. It satisfies the following integral relations.

Lemma 1 ([2,3]). *Let $\alpha, \beta \in \mathbb{C}$ and $\mathcal{R}(\alpha) > 0$, $\mathcal{R}(\beta) > 0$, then*

$$\int_0^z t^{\beta-1} E_{\alpha,\beta}(xt^\alpha)dt = z^\beta E_{\alpha,\beta+1}(xz^\alpha). \tag{6}$$

Lemma 2 ([2]). *For $\beta + r > 0$ and $r \in \mathbb{R}$, we have*

$$\int_0^1 t^{\beta+r} E_{\alpha,\beta+r}(t^\alpha)dt = E_{\alpha,\beta+r+1}(1) - E_{\alpha,\beta+r+2}(1).$$

Lemma 3 ([11]). *For $\mathcal{R}(\alpha) > 0$, $\mathcal{R}(\beta) > 0$, $\mathcal{R}(\rho) > 0$, $\mathcal{R}(\sigma) > 0$, $\gamma > 0$, we have*

$$\int_0^1 z^{\rho-1}(1-z)^{\sigma-1} E_{\alpha,\beta}(xz^\gamma)dz = \Gamma(\sigma)\,{}_2\psi_2\left[\begin{matrix}(\rho,\gamma),(1,1) \\ \\ (\beta,\alpha),(\sigma+\rho,\gamma)\end{matrix}\middle|\, x\right], \tag{7}$$

where ${}_2\psi_2$ is the generalized Wright function and $\alpha, \beta, \rho, \sigma \in \mathbb{C}$.

Special case of equation (7) *for $\rho = \beta$, $\gamma = \alpha$ is*

$$\int_0^1 z^{\beta-1}(1-z)^{\sigma-1} E_{\alpha,\beta}(xz^\alpha)dz = \Gamma(\sigma) E_{\alpha,\sigma+\beta}(x), \tag{8}$$

where $\alpha > 0$; β, $\sigma \in \mathbb{C}$, $\mathcal{R}(\beta) > 0$, $\mathcal{R}(\sigma) > 0$.

The particular case of equation (8) *when $\sigma = 1$ is*

$$\int_0^1 z^{\beta-1} E_{\alpha,\beta}(xz^\alpha)dz = E_{\alpha,1+\beta}(x). \tag{9}$$

Therefore, the goal of this article is to give a broader generalization of inequality (5) by means of this new generalized fractional integral operators within the framework of the convex stochastic processes. From this, inequalities involving the Riemann-Liouville, Hadamard, Erdélyi-Kober, Katugampola, Weyl and Liouville fractional integrals are deduced as particular cases.

This article is arranged as follows. Section 1 contains the statements and proofs of the main results, which are the generalization of the Hermite-Hadamard inequalities in view of the new fractional integral. A brief summary of the results is given in Section 2.

1 Main results

Here, we define $\lambda = \frac{1-\alpha}{\alpha}(b^{\tilde{\rho}} - a^{\tilde{\rho}})$ with $\tilde{\rho} := \rho(\eta+1)$. We begin by presenting a generalization of Theorem 3 in view of convex stochastic processes.

1.1 Generalized fractional Hermite-Hadamard inequality

Theorem 4. *Let $\alpha > 0$, $\rho > 0$, $\beta > 0$ and $\eta > 0$. Let $X : [a^{\rho(\eta+1)}, b^{\rho(\eta+1)}] \times \Omega \to \mathbb{R}$ be a positive stochastic process with $0 \le a < b$ and $X(t,.) \in X_c^p(a^{\rho(\eta+1)}, b^{\rho(\eta+1)})$. If $X(t,.)$ is convex, then the following inequality holds almost everywhere*

$$X\left(\frac{a^{\rho(\eta+1)} + b^{\rho(\eta+1)}}{2}, .\right) \le \frac{\Gamma(\alpha)}{2\rho^{1-\beta}(b^{\rho(\eta+1)} - a^{\rho(\eta+1)})} \frac{1}{E_{\rho(\eta+1),\rho(\eta+1)+1}(-\lambda)}$$
$$\times \left[\frac{1}{a^{k\rho(\eta+1)}} {}^{\rho}I^{\alpha,\beta}_{b^{\rho(\eta+1)^-},\eta,\kappa} X(a^{\rho(\eta+1)}, .) + \frac{1}{b^{\rho^2\eta(\eta+1)}} {}^{\rho}I^{\alpha,\beta}_{a^{\rho(\eta+1)^+},\eta,\rho\eta} X(b^{\rho(\eta+1)}, .)\right]$$
$$\le \frac{X(a^{\rho(\eta+1)}, .) + X(b^{\rho(\eta+1)}, .)}{2}.$$

Proof. For $\tilde{\rho} := \rho(\eta+1)$, $t \in [a,b]$ and $u, v \in [a,b]$, we define $u^{\tilde{\rho}}$ and $v^{\tilde{\rho}}$ as follows:

$$u^{\tilde{\rho}} = t^{\tilde{\rho}} a^{\tilde{\rho}} + (1 - t^{\tilde{\rho}}) b^{\tilde{\rho}}, \quad v^{\tilde{\rho}} = (1 - t^{\tilde{\rho}}) a^{\tilde{\rho}} + t^{\tilde{\rho}} b^{\tilde{\rho}}.$$

Then, $u^{\tilde{\rho}} + v^{\tilde{\rho}} = a^{\tilde{\rho}} + b^{\tilde{\rho}}$. Since X is a convex stochastic process, we have

$$X\left(\frac{u^{\tilde{\rho}} + v^{\tilde{\rho}}}{2}, .\right) \le \frac{X(u^{\tilde{\rho}}, .) + X(v^{\tilde{\rho}}, .)}{2},$$

and obtain

$$2X\left(\frac{a^{\tilde{\rho}} + b^{\tilde{\rho}}}{2}, .\right) \le X(t^{\tilde{\rho}} a^{\tilde{\rho}} + (1 - t^{\tilde{\rho}}) b^{\tilde{\rho}}, .) + X((1 - t^{\tilde{\rho}}) a^{\tilde{\rho}} + t^{\tilde{\rho}} b^{\tilde{\rho}}, .).$$

Now, multiplying both sides of the above inequality by $t^{\tilde{\rho}-1} E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}})$, $\tilde{\rho} > 0$ and integrating over t in the interval $[0,1]$ we obtain

$$2X\left(\frac{a^{\tilde{\rho}} + b^{\tilde{\rho}}}{2}, .\right) \int_0^1 t^{\tilde{\rho}-1} E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}})\, dt \le \int_0^1 t^{\tilde{\rho}-1} E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}}) X(t^{\tilde{\rho}} a^{\tilde{\rho}} + (1 - t^{\tilde{\rho}}) b^{\tilde{\rho}}, .)\, dt$$
$$+ \int_0^1 t^{\tilde{\rho}-1} E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}}) X((1 - t^{\tilde{\rho}}) a^{\tilde{\rho}} + t^{\tilde{\rho}} b^{\tilde{\rho}}, .)\, dt.$$

Thus, by equation (9), we have

$$2E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda) X\left(\frac{a^{\tilde{\rho}} + b^{\tilde{\rho}}}{2}, .\right) \le \int_0^1 t^{\tilde{\rho}-1} E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}}) X(t^{\tilde{\rho}} a^{\tilde{\rho}} + (1 - t^{\tilde{\rho}}) b^{\tilde{\rho}}, .)\, dt$$
$$+ \int_0^1 t^{\tilde{\rho}-1} E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}}) X((1 - t^{\tilde{\rho}}) a^{\tilde{\rho}} + t^{\tilde{\rho}} b^{\tilde{\rho}}, .)\, dt. \tag{10}$$

Now, we compute the right hand side of the inequality using the definition of the generalized integrals. From the definition of $u^{\tilde{\rho}}$ above, applying the following $t^{\tilde{\rho}} = \frac{b^{\tilde{\rho}} - u^{\tilde{\rho}}}{b^{\tilde{\rho}} - a^{\tilde{\rho}}}$ and

$\frac{u^{\tilde{\rho}-1}}{b^{\tilde{\rho}}-a^{\tilde{\rho}}}du = -t^{\tilde{\rho}-1}dt$ on the first integral, we get

$$\begin{aligned}\int_0^1 t^{\tilde{\rho}-1}E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}})X(t^{\tilde{\rho}}a^{\tilde{\rho}}+(1-t^{\tilde{\rho}})b^{\tilde{\rho}},.)dt &= -\int_b^a E_{\tilde{\rho},\tilde{\rho}}\left[-\lambda\left(\frac{b^{\tilde{\rho}}-u^{\tilde{\rho}}}{b^{\tilde{\rho}}-a^{\tilde{\rho}}}\right)\right]X(u^{\tilde{\rho}},.)\frac{u^{\tilde{\rho}-1}}{b^{\tilde{\rho}}-a^{\tilde{\rho}}}\,du \\ &= \frac{1}{\left(b^{\tilde{\rho}}-a^{\tilde{\rho}}\right)}\int_a^b u^{\tilde{\rho}-1}E_{\tilde{\rho},\tilde{\rho}}\left[-\lambda\left(\frac{b^{\tilde{\rho}}-u^{\tilde{\rho}}}{b^{\tilde{\rho}}-a^{\tilde{\rho}}}\right)\right]X(u^{\tilde{\rho}},.)\,du \\ &= \frac{\Gamma(\alpha)}{\rho^{1-\beta}(b^{\tilde{\rho}})^{\rho\eta}}\frac{1}{\left(b^{\tilde{\rho}}-a^{\tilde{\rho}}\right)}{}^{\rho}I^{\alpha,\beta}_{a^{\tilde{\rho}+},\eta,\rho\eta}X(b^{\tilde{\rho}},.).\end{aligned}$$

On the second integral, following similar steps, we also obtain

$$\int_0^1 t^{\tilde{\rho}-1}E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}})X((1-t^{\tilde{\rho}})a^{\tilde{\rho}}+t^{\tilde{\rho}}b^{\tilde{\rho}},.)\,dt = \frac{\Gamma(\alpha)}{\rho^{1-\beta}(a^{\tilde{\rho}})^k}\frac{1}{\left(b^{\tilde{\rho}}-a^{\tilde{\rho}}\right)}{}^{\rho}I^{\alpha,\beta}_{b^{\tilde{\rho}-},\eta,\kappa}X(a^{\tilde{\rho}},.).$$

Substituting the integrals into (10), we obtain

$$\begin{aligned}&2E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda)X\left(\frac{a^{\tilde{\rho}}+b^{\tilde{\rho}}}{2},.\right) \\ &\qquad \le \frac{\Gamma(\alpha)}{\rho^{1-\beta}}\frac{1}{\left(b^{\tilde{\rho}}-a^{\tilde{\rho}}\right)}\left(\frac{1}{(b^{\tilde{\rho}})^{\rho\eta}}{}^{\rho}I^{\alpha,\beta}_{a^{\tilde{\rho}+},\eta,\rho\eta}X(b^{\tilde{\rho}},.)+\frac{1}{(a^{\tilde{\rho}})^k}{}^{\rho}I^{\alpha,\beta}_{b^{\tilde{\rho}-},\eta,\kappa}X(a^{\tilde{\rho}},.)\right).\end{aligned} \tag{11}$$

To obtain the other part of the inequality, we use the convex property of the process X as follows

$$\begin{aligned}X(t^{\tilde{\rho}}a^{\tilde{\rho}}+(1-t^{\tilde{\rho}})b^{\tilde{\rho}},.) &\le t^{\tilde{\rho}}X(a^{\tilde{\rho}},.)+(1-t^{\tilde{\rho}})X(b^{\tilde{\rho}},.), \\ X((1-t^{\tilde{\rho}})a^{\tilde{\rho}}+t^{\tilde{\rho}}b^{\tilde{\rho}},.) &\le (1-t^{\tilde{\rho}})X(a^{\tilde{\rho}},.)+t^{\tilde{\rho}}X(b^{\tilde{\rho}},.).\end{aligned}$$

Adding the two inequalities, we obtain

$$X(t^{\tilde{\rho}}a^{\tilde{\rho}}+(1-t^{\tilde{\rho}})b^{\tilde{\rho}},.)+X((1-t^{\tilde{\rho}})a^{\tilde{\rho}}+t^{\tilde{\rho}}b^{\tilde{\rho}},.) \le X(a^{\tilde{\rho}},.)+X(b^{\tilde{\rho}},.).$$

Multiplying through by $t^{\tilde{\rho}-1}E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}})$, $\tilde{\rho}>0$, and integrating the resulting inequality over t, in the interval $[0,1]$, we obtain

$$\begin{aligned}&\frac{\Gamma(\alpha)}{\rho^{1-\beta}}\frac{1}{\left(b^{\tilde{\rho}}-a^{\tilde{\rho}}\right)}\left(\frac{1}{(b^{\tilde{\rho}})^{\rho\eta}}{}^{\rho}I^{\alpha,\beta}_{a^{\tilde{\rho}+},\eta,\rho\eta}X(b^{\tilde{\rho}},.)+\frac{1}{(a^{\tilde{\rho}})^k}{}^{\rho}I^{\alpha,\beta}_{b^{\tilde{\rho}-},\eta,\kappa}X(a^{\tilde{\rho}},.)\right) \\ &\qquad\qquad \le \left(X(a^{\tilde{\rho}},.)+X(b^{\tilde{\rho}},.)\right)E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda).\end{aligned} \tag{12}$$

Thus, combining (11) and (12), we get

$$\begin{aligned}2E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda)X\left(\frac{a^{\tilde{\rho}}+b^{\tilde{\rho}}}{2},.\right) &\le \frac{\Gamma(\alpha)}{\rho^{1-\beta}}\frac{1}{\left(b^{\tilde{\rho}}-a^{\tilde{\rho}}\right)}\left(\frac{1}{(b^{\tilde{\rho}})^{\rho\eta}}{}^{\rho}I^{\alpha,\beta}_{a^{\tilde{\rho}+},\eta,\rho\eta}X(b^{\tilde{\rho}},.)+\frac{1}{(a^{\tilde{\rho}})^k}{}^{\rho}I^{\alpha,\beta}_{b^{\tilde{\rho}-},\eta,\kappa}X(a^{\tilde{\rho}},.)\right) \\ &\le \left(X(a^{\tilde{\rho}},.)+X(b^{\tilde{\rho}},.)\right)E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda).\end{aligned} \tag{13}$$

Finally, divide all sides of (13) by $2E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda)$ and the intended result follows. □

In what follows, we deduce the generalized Hermite-Hadamard inequality for the generalized Katugampola fractional integrals of any convex function.

Corollary 1. *Let* $f \in X_c^p(a^{\rho(\eta+1)}, b^{\rho(\eta+1)})$. *If* $f : [a^{\rho(\eta+1)}, b^{\rho(\eta+1)}] \to \mathbb{R}$ *is a convex function with* $0 \le a < b, \alpha > 0$, $\rho > 0$, $\beta > 0$ *and* $\eta > 0$*, then*

$$\begin{aligned} f\left(\frac{a^{\rho(\eta+1)} + b^{\rho(\eta+1)}}{2}\right) &\le \frac{\Gamma(\alpha)}{2\rho^{1-\beta}(b^{\rho(\eta+1)} - a^{\rho(\eta+1)})} \frac{1}{E_{\rho(\eta+1),\rho(\eta+1)+1}(-\lambda)} \\ &\times \left[\frac{1}{a^{k\rho(\eta+1)}} {}^{\rho}I^{\alpha,\beta}_{b^{\rho(\eta+1)^-},\eta,\kappa} f(a^{\rho(\eta+1)}) + \frac{1}{b^{\rho^2\eta(\eta+1)}} {}^{\rho}I^{\alpha,\beta}_{a^{\rho(\eta+1)^+},\eta,\rho\eta} f(b^{\rho(\eta+1)})\right] \\ &\le \frac{f(a^{\rho(\eta+1)}) + f(b^{\rho(\eta+1)})}{2}. \end{aligned}$$

Remark 1. *If we set* $\beta = \alpha, \kappa = \eta = 0$ *and* $\rho = 1$ *in Corollary 1, then we recover Theorem 3, since*

$$E_{1,2}(-\lambda) = \frac{\exp(-\lambda) - 1}{-\lambda} = \frac{1 - \exp(-\lambda)}{\lambda} = \frac{\alpha(1 - \exp(-\frac{1-\alpha}{\alpha}(b-a)))}{(1-\alpha)(b-a)}.$$

That is, we obtain

$$f\left(\frac{a+b}{2}\right) \le \frac{(1-\alpha)\Gamma(\alpha)}{2\alpha(1 - \exp(-\frac{1-\alpha}{\alpha}(b-a)))} \left[I^{\alpha,\alpha}_{b^-,0,0} f(a) + I^{\alpha,\alpha}_{a^+,0,0} f(b)\right] \le \frac{f(a) + f(b)}{2}. \tag{14}$$

Equation (14) *gives exactly the result (inequality) of Theorem 3 when* $\Gamma(\alpha) = \alpha$ *as given in Definition 1.*

Theorem 5. *Let* $\alpha > 0$, $\rho > 0$, $\beta > 0$ *and* $\eta > 0$. *Let* $X : [a^{\rho(\eta+1)}, b^{\rho(\eta+1)}] \times \Omega \to \mathbb{R}$ *be a convex, mean square differentiable stochastic process with* $0 \le a < b$ *and* $X(t,.) \in X_c^p(a^{\rho(\eta+1)}, b^{\rho(\eta+1)})$. *If* $X'(t,.)$ *is mean square differentiable, then the following inequality holds almost everywhere*

$$\begin{aligned} &\Bigg| \frac{X(a^{\tilde{\rho}},.)+X(b^{\tilde{\rho}},.)}{2} - \frac{\Gamma(\alpha)}{2\rho^{1-\beta}} \frac{1}{(b^{\tilde{\rho}} - a^{\tilde{\rho}})} \frac{1}{E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda)} \\ &\quad \times \left(\frac{1}{(b^{\tilde{\rho}})^{\rho\eta}} {}^{\rho}I^{\alpha,\beta}_{a^{\tilde{\rho}^+},\eta,\rho\eta} X(b^{\tilde{\rho}},.) + \frac{1}{(a^{\tilde{\rho}})^k} {}^{\rho}I^{\alpha,\beta}_{b^{\tilde{\rho}^-},\eta,\kappa} X(a^{\tilde{\rho}},.)\right) \Bigg| \\ &\quad \le \frac{3}{2}\tilde{\rho}(b^{\tilde{\rho}} - a^{\tilde{\rho}})^2 \sup_{c \in [a^{\tilde{\rho}}, b^{\tilde{\rho}}]} |X''(c,.)|. \end{aligned}$$

Proof. Following the proof of Theorem 4, we have

$$\begin{aligned} &\frac{\Gamma(\alpha)}{\rho^{1-\beta}} \frac{1}{(b^{\tilde{\rho}} - a^{\tilde{\rho}})} \left(\frac{1}{(b^{\tilde{\rho}})^{\rho\eta}} {}^{\rho}I^{\alpha,\beta}_{a^{\tilde{\rho}^+},\eta,\rho\eta} X(b^{\tilde{\rho}},.) + \frac{1}{(a^{\tilde{\rho}})^k} {}^{\rho}I^{\alpha,\beta}_{b^{\tilde{\rho}^-},\eta,\kappa} X(a^{\tilde{\rho}},.)\right) \\ &\quad = \int_0^1 t^{\tilde{\rho}-1} E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}}) X(t^{\tilde{\rho}} a^{\tilde{\rho}} + (1 - t^{\tilde{\rho}}) b^{\tilde{\rho}},.)\, dt \\ &\quad + \int_0^1 t^{\tilde{\rho}-1} E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}}) X((1 - t^{\tilde{\rho}}) a^{\tilde{\rho}} + t^{\tilde{\rho}} b^{\tilde{\rho}},.)\, dt. \end{aligned} \tag{15}$$

Integrating by parts with $u := X(t^{\tilde{\rho}}a^{\tilde{\rho}} + (1-t^{\tilde{\rho}})b^{\tilde{\rho}},.)$ and $dv := t^{\tilde{\rho}-1}E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}}) \iff v := \int t^{\tilde{\rho}-1}E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}})\,dt = t^{\tilde{\rho}}E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda t^{\tilde{\rho}})$ (using equation (6) in Lemma 1) we obtain

$$\begin{aligned}
\int_0^1 t^{\tilde{\rho}-1}E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}})X(t^{\tilde{\rho}}a^{\tilde{\rho}} + (1-t^{\tilde{\rho}})b^{\tilde{\rho}},.)\,dt &= t^{\tilde{\rho}}E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda t^{\tilde{\rho}})X(t^{\tilde{\rho}}a^{\tilde{\rho}} + (1-t^{\tilde{\rho}})b^{\tilde{\rho}},.)\Big|_0^1 \\
&\quad - \int_0^1 t^{\tilde{\rho}}E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}})X(t^{\tilde{\rho}}a^{\tilde{\rho}} + (1-t^{\tilde{\rho}})b^{\tilde{\rho}},.)\tilde{\rho}(a^{\tilde{\rho}} - b^{\tilde{\rho}})t^{\tilde{\rho}-1}\,dt \\
&= E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda)X(a^{\tilde{\rho}},.) - \tilde{\rho}(a^{\tilde{\rho}} - b^{\tilde{\rho}})\int_0^1 t^{2\tilde{\rho}-1}E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}})X'(t^{\tilde{\rho}}a^{\tilde{\rho}} + (1-t^{\tilde{\rho}})b^{\tilde{\rho}},.)\,dt \\
&= E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda)X(a^{\tilde{\rho}},.) + \tilde{\rho}(b^{\tilde{\rho}} - a^{\tilde{\rho}})\int_0^1 t^{2\tilde{\rho}-1}E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}})X'(t^{\tilde{\rho}}a^{\tilde{\rho}} + (1-t^{\tilde{\rho}})b^{\tilde{\rho}},.)\,dt.
\end{aligned} \tag{16}$$

Similarly, the second integral in equation (15) becomes

$$\begin{aligned}
&\int_0^1 t^{\tilde{\rho}-1}E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}})X((1-t^{\tilde{\rho}})a^{\tilde{\rho}} + t^{\tilde{\rho}}b^{\tilde{\rho}},.)\,dt \\
&\quad = E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda)X(b^{\tilde{\rho}},.) - \tilde{\rho}(b^{\tilde{\rho}} - a^{\tilde{\rho}})\int_0^1 t^{2\tilde{\rho}-1}E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}})X'((1-t^{\tilde{\rho}})a^{\tilde{\rho}} + t^{\tilde{\rho}}b^{\tilde{\rho}},.)\,dt.
\end{aligned} \tag{17}$$

Now, combining equations (16) and (17) into (15) we have

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{\rho^{1-\beta}}\frac{1}{(b^{\tilde{\rho}} - a^{\tilde{\rho}})}\left(\frac{1}{(b^{\tilde{\rho}})^{\rho\eta}}{}^{\rho}I^{\alpha,\beta}_{a^{\tilde{\rho}+},\eta,\rho\eta}X(b^{\tilde{\rho}},.) + \frac{1}{(a^{\tilde{\rho}})^{k}}{}^{\rho}I^{\alpha,\beta}_{b^{\tilde{\rho}-},\eta,\kappa}X(a^{\tilde{\rho}},.)\right) \\
&\quad = E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda)\big(X(a^{\tilde{\rho}},.) + X(b^{\tilde{\rho}},.)\big) - \tilde{\rho}(b^{\tilde{\rho}} - a^{\tilde{\rho}}) \\
&\qquad \times \int_0^1 t^{2\tilde{\rho}-1}E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}})\Big[X'((1-t^{\tilde{\rho}})a^{\tilde{\rho}} + t^{\tilde{\rho}}b^{\tilde{\rho}},.) - X'(t^{\tilde{\rho}}a^{\tilde{\rho}} + (1-t^{\tilde{\rho}})b^{\tilde{\rho}},.)\Big]\,dt.
\end{aligned} \tag{18}$$

Rearranging equation (18) we obtain

$$\begin{aligned}
&E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda)\big(X(a^{\tilde{\rho}},.) + X(b^{\tilde{\rho}},.)\big) - \frac{\Gamma(\alpha)}{\rho^{1-\beta}}\frac{1}{(b^{\tilde{\rho}} - a^{\tilde{\rho}})} \\
&\qquad \times \left(\frac{1}{(b^{\tilde{\rho}})^{\rho\eta}}{}^{\rho}I^{\alpha,\beta}_{a^{\tilde{\rho}+},\eta,\rho\eta}X(b^{\tilde{\rho}},.) + \frac{1}{(a^{\tilde{\rho}})^{k}}{}^{\rho}I^{\alpha,\beta}_{b^{\tilde{\rho}-},\eta,\kappa}X(a^{\tilde{\rho}},.)\right) \\
&\quad = \tilde{\rho}(b^{\tilde{\rho}} - a^{\tilde{\rho}})\int_0^1 t^{2\tilde{\rho}-1}E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}}) \\
&\qquad \times \Big[X'((1-t^{\tilde{\rho}})a^{\tilde{\rho}} + t^{\tilde{\rho}}b^{\tilde{\rho}},.) - X'(t^{\tilde{\rho}}a^{\tilde{\rho}} + (1-t^{\tilde{\rho}})b^{\tilde{\rho}},.)\Big]\,dt.
\end{aligned} \tag{19}$$

Next, in equation (19), apply the Mean Value Theorem for derivative on X' for some $c \in [a^{\tilde{\rho}}, b^{\tilde{\rho}}]$, thus

$$\begin{aligned}
&E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda)\big(X(a^{\tilde{\rho}},.) + X(b^{\tilde{\rho}},.)\big) - \frac{\Gamma(\alpha)}{\rho^{1-\beta}}\frac{1}{(b^{\tilde{\rho}} - a^{\tilde{\rho}})} \\
&\qquad \times \left(\frac{1}{(b^{\tilde{\rho}})^{\rho\eta}}{}^{\rho}I^{\alpha,\beta}_{a^{\tilde{\rho}+},\eta,\rho\eta}X(b^{\tilde{\rho}},.) + \frac{1}{(a^{\tilde{\rho}})^{k}}{}^{\rho}I^{\alpha,\beta}_{b^{\tilde{\rho}-},\eta,\kappa}X(a^{\tilde{\rho}},.)\right) \\
&\quad = \tilde{\rho}(b^{\tilde{\rho}} - a^{\tilde{\rho}})^2\int_0^1 t^{2\tilde{\rho}-1}E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}})(1-2t^{\tilde{\rho}})X''(c,.)\,dt.
\end{aligned} \tag{20}$$

Divide both sides of equation (20) by $2E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda)$ and take the absolute values of both sides to guarantee

$$\begin{aligned}\Bigg| \frac{X(a^{\tilde{\rho}},.)+X(b^{\tilde{\rho}},.)}{2} &- \frac{\Gamma(\alpha)}{2\rho^{1-\beta}}\frac{1}{(b^{\tilde{\rho}}-a^{\tilde{\rho}})}\frac{1}{E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda)} \\ &\times\left(\frac{1}{(b^{\tilde{\rho}})^{\rho\eta}}{}^{\rho}I^{\alpha,\beta}_{a^{\tilde{\rho}+},\eta,\rho\eta}X(b^{\tilde{\rho}},.)+\frac{1}{(a^{\tilde{\rho}})^{k}}{}^{\rho}I^{\alpha,\beta}_{b^{\tilde{\rho}-},\eta,\kappa}X(a^{\tilde{\rho}},.)\right)\Bigg| \\ &\leq \tilde{\rho}(b^{\tilde{\rho}}-a^{\tilde{\rho}})^2\frac{1}{2E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda)}|X''(c,.)|\int_0^1 t^{2\tilde{\rho}-1}E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}})|1-2t^{\tilde{\rho}}|\,dt. \end{aligned} \tag{21}$$

Given that $t\leq 1$, we estimate the integral in equation (21) as follows

$$\begin{aligned}\int_0^1 t^{2\tilde{\rho}-1}E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}})|1-2t^{\tilde{\rho}}|\,dt &\leq \int_0^1 t^{2\tilde{\rho}-1}E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}})[1+2t^{\tilde{\rho}}]\,dt \\ &= \int_0^1 t^{\tilde{\rho}}t^{\tilde{\rho}-1}E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}})[1+2t^{\tilde{\rho}}]\,dt \\ &\leq 3\int_0^1 t^{\tilde{\rho}-1}E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}})\,dt \\ &= 3E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda). \end{aligned}$$

Finally, equation (21) reads

$$\begin{aligned}\Bigg| \frac{X(a^{\tilde{\rho}},.)+X(b^{\tilde{\rho}},.)}{2} &- \frac{\Gamma(\alpha)}{2\rho^{1-\beta}}\frac{1}{(b^{\tilde{\rho}}-a^{\tilde{\rho}})}\frac{1}{E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda)} \\ &\times\left(\frac{1}{(b^{\tilde{\rho}})^{\rho\eta}}{}^{\rho}I^{\alpha,\beta}_{a^{\tilde{\rho}+},\eta,\rho\eta}X(b^{\tilde{\rho}},.)+\frac{1}{(a^{\tilde{\rho}})^{k}}{}^{\rho}I^{\alpha,\beta}_{b^{\tilde{\rho}-},\eta,\kappa}X(a^{\tilde{\rho}},.)\right)\Bigg| \\ &\leq 3\tilde{\rho}(b^{\tilde{\rho}}-a^{\tilde{\rho}})^2\frac{1}{2E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda)}E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda)\sup_{c\in[a^{\tilde{\rho}},b^{\tilde{\rho}}]}|X''(c,.)|, \end{aligned}$$

and the result follows. □

Theorem 6. *Let* $\alpha>0$, $\rho>0$, $\beta>0$ *and* $\eta>0$. *Let* $X:[a^{\rho(\eta+1)},b^{\rho(\eta+1)}]\times\Omega\to\mathbb{R}$ *be a convex, mean square differentiable stochastic process with* $0\leq a<b$ *and* $X(t,.)\in X^p_c(a^{\rho(\eta+1)},b^{\rho(\eta+1)})$. *If* $|X'(t,.)|$ *is convex, then the following inequality holds almost everywhere*

$$\begin{aligned}\Bigg| \frac{X(a^{\tilde{\rho}},.)+X(b^{\tilde{\rho}},.)}{2} &- \frac{\Gamma(\alpha)}{2\rho^{1-\beta}}\frac{1}{(b^{\tilde{\rho}}-a^{\tilde{\rho}})}\frac{1}{E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda)} \\ &\times\left(\frac{1}{(b^{\tilde{\rho}})^{\rho\eta}}{}^{\rho}I^{\alpha,\beta}_{a^{\tilde{\rho}+},\eta,\rho\eta}X(b^{\tilde{\rho}},.)+\frac{1}{(a^{\tilde{\rho}})^{k}}{}^{\rho}I^{\alpha,\beta}_{b^{\tilde{\rho}-},\eta,\kappa}X(a^{\tilde{\rho}},.)\right)\Bigg| \\ &\leq \tilde{\rho}(b^{\tilde{\rho}}-a^{\tilde{\rho}})\frac{|X'(a^{\tilde{\rho}},.)|+|X'(b^{\tilde{\rho}},.)|}{2}. \end{aligned}$$

Proof. Divide both sides of equation (19) by $2E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda)$ and take absolute values of both sides to give

$$\begin{aligned}
&\left| \frac{X(a^{\tilde{\rho}},.)+X(b^{\tilde{\rho}},.)}{2} - \frac{\Gamma(\alpha)}{2\rho^{1-\beta}} \frac{1}{(b^{\tilde{\rho}}-a^{\tilde{\rho}})} \frac{1}{E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda)} \right. \\
&\qquad \left. \times \left(\frac{1}{(b^{\tilde{\rho}})^{\rho\eta}} {}^{\rho}I^{\alpha,\beta}_{a^{\tilde{\rho}+},\eta,\rho\eta} X(b^{\tilde{\rho}},.) + \frac{1}{(a^{\tilde{\rho}})^{k}} {}^{\rho}I^{\alpha,\beta}_{b^{\tilde{\rho}-},\eta,\kappa} X(a^{\tilde{\rho}},.) \right) \right| \\
&= \tilde{\rho}(b^{\tilde{\rho}}-a^{\tilde{\rho}}) \frac{1}{2E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda)} \int_0^1 t^{2\tilde{\rho}-1} E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}}) \\
&\qquad \times \left| X'((1-t^{\tilde{\rho}})a^{\tilde{\rho}} + t^{\tilde{\rho}}b^{\tilde{\rho}},.) - X'(t^{\tilde{\rho}}a^{\tilde{\rho}} + (1-t^{\tilde{\rho}})b^{\tilde{\rho}},.) \right| dt \\
&\leq \tilde{\rho}(b^{\tilde{\rho}}-a^{\tilde{\rho}}) \frac{1}{2E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda)} \int_0^1 t^{2\tilde{\rho}-1} E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}}) \\
&\qquad \times \left| X'((1-t^{\tilde{\rho}})a^{\tilde{\rho}} + t^{\tilde{\rho}}b^{\tilde{\rho}},.) + X'(t^{\tilde{\rho}}a^{\tilde{\rho}} + (1-t^{\tilde{\rho}})b^{\tilde{\rho}},.) \right| dt.
\end{aligned} \tag{22}$$

By triangle inequality and the convexity of $|X'(,.)|$, we have

$$\begin{aligned}
&|X'((1-t^{\tilde{\rho}})a^{\tilde{\rho}} + t^{\tilde{\rho}}b^{\tilde{\rho}},.) + X'(t^{\tilde{\rho}}a^{\tilde{\rho}}+(1-t^{\tilde{\rho}})b^{\tilde{\rho}},.)| \\
&\qquad \leq |X'((1-t^{\tilde{\rho}})a^{\tilde{\rho}} + t^{\tilde{\rho}}b^{\tilde{\rho}},.)| + |X'(t^{\tilde{\rho}}a^{\tilde{\rho}} + (1-t^{\tilde{\rho}})b^{\tilde{\rho}},.)| \\
&\qquad \leq |X'(a^{\tilde{\rho}},.)| + |X'(b^{\tilde{\rho}},.)|,
\end{aligned}$$

and equation (22) becomes

$$\begin{aligned}
&\left| \frac{X(a^{\tilde{\rho}},.)+X(b^{\tilde{\rho}},.)}{2} - \frac{\Gamma(\alpha)}{2\rho^{1-\beta}} \frac{1}{(b^{\tilde{\rho}}-a^{\tilde{\rho}})} \frac{1}{E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda)} \right. \\
&\qquad \left. \times \left(\frac{1}{(b^{\tilde{\rho}})^{\rho\eta}} {}^{\rho}I^{\alpha,\beta}_{a^{\tilde{\rho}+},\eta,\rho\eta} X(b^{\tilde{\rho}},.) + \frac{1}{(a^{\tilde{\rho}})^{k}} {}^{\rho}I^{\alpha,\beta}_{b^{\tilde{\rho}-},\eta,\kappa} X(a^{\tilde{\rho}},.) \right) \right| \\
&\qquad \leq \tilde{\rho}(b^{\tilde{\rho}}-a^{\tilde{\rho}}) \frac{1}{2E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda)} (|X'(a^{\tilde{\rho}},.)| + |X'(b^{\tilde{\rho}},.)|) \int_0^1 t^{2\tilde{\rho}-1} E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}})\, dt.
\end{aligned} \tag{23}$$

Since $t \leq 1$, we have that

$$\int_0^1 t^{2\tilde{\rho}-1} E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}})\, dt = \int_0^1 t^{\tilde{\rho}} t^{\tilde{\rho}-1} E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}})\, dt \leq \int_0^1 t^{\tilde{\rho}-1} E_{\tilde{\rho},\tilde{\rho}}(-\lambda t^{\tilde{\rho}})\, dt = E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda),$$

and therefore, equation (23) gives us

$$\begin{aligned}
&\left| \frac{X(a^{\tilde{\rho}},.)+X(b^{\tilde{\rho}},.)}{2} - \frac{\Gamma(\alpha)}{2\rho^{1-\beta}} \frac{1}{(b^{\tilde{\rho}}-a^{\tilde{\rho}})} \frac{1}{E_{\tilde{\rho},\tilde{\rho}+1}(-\lambda)} \left(\frac{1}{(b^{\tilde{\rho}})^{\rho\eta}} {}^{\rho}I^{\alpha,\beta}_{a^{\tilde{\rho}+},\eta,\rho\eta} X(b^{\tilde{\rho}},.) + \frac{1}{(a^{\tilde{\rho}})^{k}} {}^{\rho}I^{\alpha,\beta}_{b^{\tilde{\rho}-},\eta,\kappa} X(a^{\tilde{\rho}},.) \right) \right| \\
&\qquad \leq \tilde{\rho}(b^{\tilde{\rho}}-a^{\tilde{\rho}}) \frac{|X'(a^{\tilde{\rho}},.)| + |X'(b^{\tilde{\rho}},.)|}{2}.
\end{aligned}$$

□

Remark 2. *We can obtain the same result by using the convexity of X' instead, and by triangle inequality, thus*

$$|X'((1-t^{\tilde{\rho}})a^{\tilde{\rho}} + t^{\tilde{\rho}}b^{\tilde{\rho}},.) + X'(t^{\tilde{\rho}}a^{\tilde{\rho}} + (1-t^{\tilde{\rho}})b^{\tilde{\rho}},.)| \leq |X'(a^{\tilde{\rho}},.) + X'(b^{\tilde{\rho}},.)| \leq |X'(a^{\tilde{\rho}},.)| + |X'(b^{\tilde{\rho}},.)|.$$

2 Conclusion

New fractional inequalities of the Hermite-Hadamard type for positive-convex stochastic processes via new fractional integrals were established. The obtained results extend, generalize and unify some known results in the literature.

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Омаба М.Е. *Узагальнені дробові нерівності типу Ерміта-Адамара в термінах нових узагальнених дробових інтегралів Катугампола* // Карпатські матем. публ. — 2022. — Т.14, №2. — С. 475–484.

Запропоновано нове узагальнення для узагальнених дробових інтегралів Катугамполи в термінах функцій Міттаг-Леффлера. Встановлено нові узагальнення нерівностей Ерміта-Адамара за допомогою нещодавно запропонованого дробового інтегрального оператора для додатного опуклого випадкового процесу. Показано, що низку відомих результатів легко вивести як окремі випадки цих нерівностей. Отримані результати справедливі і для будь-якої опуклої функції.

Ключові слова і фрази: нерівності Ерміта-Адамара, функція Міттага-Леффлера, узагальнений дробовий інтеграл Катугамполи, опуклий і позитивний стохастичний процес.