ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2022, **14** (2), 564–581
doi:10.15330/cmp.14.2.564-581

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2022, Т.14, №2, С.564–581

# Linear Diophantine fuzzy subsets of polygroups

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Linear Diophantine fuzzy sets were recently introduced as a generalized form of fuzzy sets. The aim of this paper is to shed the light on the relationship between algebraic hyperstructures and linear Diophantine fuzzy sets through polygroups. More precisely, we introduce the concepts of linear Diophantine fuzzy subpolygroups of a polygroup, linear Diophantine fuzzy normal subpolygroups of a polygroup, and linear Diophantine anti-fuzzy subpolygroups of a polygroup. Furthermore, we study some of their properties and characterize them in relation to level and ceiling sets.

*Key words and phrases:* polygroup, linear Diophantine fuzzy set, linear Diophantine fuzzy subpolygroup, linear Diophantine fuzzy normal subpolygroup, level set, linear Diophantine anti-fuzzy subpolygroup, ceiling set.

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## Introduction

In classical set theory, the membership of an element in a set is considered according to the condition whether it belongs to the set or it does not belong to it. Such a definition fails to deal with many real life problems. Fuzzy set theory was introduced in 1965 by L.A. Zadeh [23], where the membership of an element in a fuzzy set is a real number in the unit interval $[0, 1]$. So that in a fuzzy set, the sum of degree of membership of an element with its degree of non-membership is equal to one. In 1986, K. Atanassov [6] generalized fuzzy sets by introducing intuitionistic fuzzy sets (IFS). Where in an IFS, the sum of degree of membership of an element with its degree of non-membership is less than or equal to one. Another generalization of fuzzy sets was introduced in 1987, when R.R. Yager [20] introduced fuzzy multisets, gave examples on them, and investigated their properties. An element of a fuzzy multiset can occur more than once with possibly the same or different membership values. In 2002, D. Ramot et al. introduced complex fuzzy sets and logic as a generalization of fuzzy sets and logic and presented some examples and results on them. For more details, see [17,18]. For other different generalizations of fuzzy sets, we refer to [8,21,22]. In 2019, M. Riaz and M. Hashmi [19] found that fuzzy sets and intuitionistic fuzzy sets have their own limitations related to the functions of membership and non-membership. To eliminate such limitations and by using reference parameters, they introduced a new generalization of fuzzy sets and called it *linear Diophantine fuzzy sets* (LDFS). Their proposed model enhances the existing methodologies and the decision

УДК 512.53
2020 *Mathematics Subject Classification:* 20N20.

maker can freely choose the grades without any limitation. More classification of the problem can be done by changing the physical sense of reference parameters. The latter in LDFS play an important role in the study of some problems whereas the previous generalizations of fuzzy sets cannot deal with parameterizations. S. Ayub et al. [7] established a robust fusion of binary relations and LDFSs and introduced the concept of linear Diophantine fuzzy relation (LDF-relation) by making the use of reference parameters corresponding to the membership and non-membership fuzzy relations.

A hypergroup represents a natural generalization of a group and it was introduced by F. Marty [16] in 1934 at the eighth Congress of Scandinavian Mathematicians. This was the introducing of a new branch of Mathematics known by "Algebraic Hypersructures". Since then, many researchers worked on this field from both directions: theory and applications. For details about algebraic hyperstructures, we refer to the related books in [10–12]. A special class of hypergroups, known as polygroups (or quasi-canonical hypergroups), was introduced and studied by S.D. Comer [9] in 1984. For details about polygroups, we refer to book [11] of B. Davvaz.

An interesting field of research, known as "Fuzzy algebraic hyperstructures", was introduced as a combination of fuzzy sets and algebraic hyperstructures. Many algebraists worked on this field by introducing new related concepts, studying their various properties, and applying them in other fields. For more details about this field of research, we refer to the book written by B. Davvaz and I. Cristea [12] in 2015 and for some interesting applications, we refer to [1–3]. This field had enlarged to include the new generalization of fuzzy sets. For example, M. Al-Tahan et al. studied complex fuzzy subpolygroups of polygroups [4] and fuzzy multi-polygroups [5].

In 2020, H. Kamaci [15] combined the notion of linear Diophantine fuzzy sets and algebraic structures and studied finite linear Diophantine fuzzy subsets of some algebraic structures (groups, rings, and fields). Inspired by the work in [15] related to linear Diophantine fuzzy subgroups of groups and by the work done by M. Al-Tahan et al. [4, 5] on the connection of some generalizations of fuzzy sets with polygroups, our paper discusses linear Diophantine fuzzy subsets of polygroups and it is organized as follows. After an Introduction, Section 2 presents a background about linear Diophantine fuzzy sets and polygroups that are needed throughout the paper. Then Section 3 and Section 4 define linear Diophantine fuzzy subpolygroups of polygroups and linear Diophantine fuzzy normal subpolygroups of polygroups respectively, present illustrative examples, and study their properties. Finally, Section 5 defines linear Diophantine anti-fuzzy (normal) subpolygroups of polygroups and studies their properties.

## 1 Preliminaries

In this section, we present some basic results and examples related to linear Diophantine fuzzy sets and polygroup theory that are needed throughout the paper. For more details about these topics, we refer to [9–12, 19].

### 1.1 Linear Diophantine Fuzzy sets

A fuzzy set is a generalization of a crisp set. A linear Diophantine fuzzy set (LDFS) is a new generalization of fuzzy set found by M. Riaz and M. Hashmi [19]. The proposed model of LDFS is more efficient and flexible rather than other generalizations of fuzzy set due to the use of reference parameters.

**Definition 1** ([23]). *Let $X$ be a universal set and $\mu : X \to [0,1]$. Then $A = \{(x, \mu(x)) : x \in X\}$ is a fuzzy set with membership function "$\mu$".*

**Definition 2** ([19]). *Let $X$ be a universal set. Then a linear Diophantine fuzzy set (LDFS) $D$ on the universal set $X$ is described in the following form*

$$D = \{(x, < U(x), V(x) >, < \alpha(x), \beta(x) >) : x \in X\},$$

*where $U(X), V(x) \in [0,1]$ are degrees of membership and non-membership respectively and $\alpha(x), \beta(x) \in [0,1]$ are reference parameters. The degrees satisfy $0 \leq \alpha(x) + \beta(x) \leq 1$ and $0 \leq \alpha(x)U(x) + \beta(x)V(x) \leq 1$ for all $x \in X$.*

**Example 1.** *In a high school, a representative is to be selected based on a certain criteria. The committee needs to identify the most suitable candidate among all students who applied for this post. The selection criteria looks at the performance, grades, knowledge, extracurricular activities the applicant is involved in, and other skills each applicant has. Assume that it is desired to determine the best-qualified candidate who meets the specified selection criteria that also has a high IQ level. Let $X = \{Ziad, Leyan, Tarek, Talia\}$ be the set of candidates selected for the interview. For the LDFS's construction, the reference parameters are considered as $\alpha$ = "high IQ level" and $\beta$ = "low IQ level". The result is the following LDFS*

$$D = \big\{(Ziad, < 0.73, 0.2 >, < 0.82, 0.1 >), (Leyan, < 0.65, 0.23 >, < 0.7, 0.15 >), (Tarek, < 0.6, 0.25 >, < 0.66, 0.2 >), (Talia, < 0.59, 0.3 >, < 0.6, 0.25 >)\big\}.$$

*$D(Ziad) = (< 0.73, 0.2 >, < 0.82, 0.1 >)$ implies that for Ziad, the degrees of membership and non-membership with respect to the selection criteria are 0.73 and 0.2 respectively, and the degrees of reference parameters: high IQ level and low IQ level are 0.82 and 0.1.*

*The committee may change the physical meaning of reference parameters to presentable/non-presentable, easy to adapt/not easy to adapt, etc and gets another LDFS. Here the reference parameters play an important role. They represent some specific property about candidates like whether they have high IQ level or not, are presentable or not, easy to adapt or not.*

**Example 2.** *Let $X = \{1, 2, 3, 4\}$ be a universal set and define $D$ on $X$ as follows*

$$D(1) = (< 0.1, 0.3 >, < 0.4, 0.5 >), \quad D(2) = (< 0.2, 0.4 >, < 0.34, 0.05 >),$$
$$D(3) = (< 0.8, 0.4 >, < 0.1, 0.5 >), \quad D(4) = (< 0.2, 0.1 >, < 0.3, 0.7 >).$$

*Then $D$ is an LDFS on $X$.*

**Remark 1.** *A fuzzy set $A$ on a universal set $X$ with a membership function $\mu$ is a special case of linear Diophantine fuzzy set. This is easily seen as*

$$A = \{(x, < \mu(x), 0 >, < 1, 0 >) : x \in X\}$$

*is an LDFS on $X$.*

**Definition 3** ([19]). *Let $X$ be a universal set and $D_1, D_2$ be LDFSs on $X$. Then*

(1) *the complement of $D_1$, denoted by $D_1^c$, is defined as*

$$D_1^c = \{(x, < V_1(x), U_1(x) >, < \beta_1(x), \alpha_1(x) >) : x \in X\},$$

(2) *$D_1$ is subset of $D_2$, denoted by $D_1 \subseteq D_2$, if $D_1(x) \leq D_2(x)$ for all $x \in X$, i.e. $U_1(x) \leq U_2(x)$, $V_1(x) \geq V_2(x)$, $\alpha_1(x) \leq \alpha_2(x)$ and $\beta_1(x) \geq \beta_2(x)$ for all $x \in X$,*

(3) *$D_1 = D_2$ if $D_1 \subseteq D_2$ and $D_2 \subseteq D_1$,*

(4) *the intersection of $D_1$ and $D_2$, denoted by $D_1 \cap D_2$, is defined as*

$$\{(x, < U_1(x) \wedge U_2(x), V_1(x) \vee V_2(x) >, < \alpha_1(x) \wedge \alpha_2(x), \beta_1(x) \vee \beta_2(x) >) : x \in X\},$$

(5) *the union of $D_1$ and $D_2$, denoted by $D_1 \cup D_2$, is defined as*

$$\{(x, < U_1(x) \vee U_2(x), V_1(x) \wedge V_2(x) >, < \alpha_1(x) \vee \alpha_2(x), \beta_1(x) \wedge \beta_2(x) >) : x \in X\}.$$

**Example 3.** *Let $X = \{1, 2, 3, 4\}$ be a universal set and $D$ be the LDFS on $X$ defined in Example 2. Then the LDFS $D^c$ on $X$ is defined as follows*

$$D^c(1) = (< 0.3, 0.1 >, < 0.5, 0.4 >), \quad D^c(2) = (< 0.4, 0.2 >, < 0.05, 0.34 >),$$
$$D^c(3) = (< 0.4, 0.8 >, < 0.5, 0.1 >), \quad D^c(4) = (< 0.1, 0.2 >, < 0.7, 0.3 >).$$

**Proposition 1** ([19]). *Let $X$ be a universal set and $D_1, D_2$ be LDFSs on $X$. Then*

(1) $(D_1^c)^c = D_1$,

(2) *if $D_1 \subseteq D_2$, then $D_2^c \subseteq D_1^c$,*

(3) $(D_1 \cap D_2)^c = D_1^c \cup D_2^c$,

(4) $(D_1 \cup D_2)^c = D_1^c \cap D_2^c$.

**Definition 4.** *Let $X_1, X_2$ be universal sets and $D_1, D_2$ be LDFSs on $X_1, X_2$ respectively. Then the Cartesian product $D_1 \times D_2$ of $D_1$ and $D_2$ is defined as*

$$\{((x, y), < U_1(x) \wedge U_2(y), V_1(x) \vee V_2(y) >, < \alpha_1(x) \wedge \alpha_2(y), \beta_1(x) \vee \beta_2(y) >) : (x, y) \in X \times Y\}.$$

**Proposition 2** ([19]). *Let $X_1, X_2$ be universal sets and $D_1, D_2$ be LDFSs on $X_1, X_2$ respectively. Then the Cartesian product $D_1 \times D_2$ of $D_1$ and $D_2$ is an LDFS.*

**Example 4.** *Let $X_1 = \{0, 1\}$ and $X_2 = \{p, q\}$ be two universal sets and define the LDFSs $D_1, D_2$ on $X_1, X_2$ respectively as follows*

$$D_1 = \{(0, < 0.3, 0.1 >, < 0.5, 0.4 >), (1, < 0.4, 0.1 >, < 0.3, 0.6 >)\},$$
$$D_2 = \{(p, < 0.2, 0.04 >, < 0.4, 0.6 >), (q, < 0.3, 0.2 >, < 0.4, 0.4 >)\}.$$

*Then the LDFS $D = D_1 \times D_2$ on $X_1 \times X_2$ is defined as follows:*

$$D((0, p)) = (< 0.2, 0.1 >, < 0.4, 0.6 >), \quad D((0, q)) = (< 0.3, 0.2 >, < 0.4, 0.4 >),$$
$$D((1, p)) = (< 0.2, 0.1 >, < 0.3, 0.6 >), \quad D((1, q)) = (< 0.3, 0.2 >, < 0.3, 0.6 >).$$

### 1.2 Polygroups

Let $P$ be a non-empty set and $\mathcal{P}^*(P)$ be the family of all non-empty subsets of $P$. Then a mapping $\circ : P \times P \to \mathcal{P}^*(P)$ is called a *binary hyperoperation* on $P$. The couple $(P, \circ)$ is called a *hypergroupoid*.

In the above definition, if $X$ and $Y$ are two non-empty subsets of $P$ and $p \in P$, then we define

$$X \circ Y = \bigcup_{\substack{a \in X \\ b \in Y}} a \circ b, \quad p \circ X = \{p\} \circ X \quad \text{and} \quad X \circ p = X \circ \{p\}.$$

**Definition 5** ([9]). *A polygroup is a system $\langle P, \circ, e, ^{-1} \rangle$, where $e \in P$, $^{-1} : P \to P$ is a unitary operation on $P$, "$\circ$" maps $P \times P$ into $\mathcal{P}^*(P)$, and the following axioms hold for all $x, y, z \in P$:*

1) $(x \circ y) \circ z = x \circ (y \circ z)$,
2) $e \circ x = x \circ e = \{x\}$,
3) $x \in y \circ z$ *implies* $y \in x \circ z^{-1}$ *and* $z \in y^{-1} \circ x$.

The following properties follow easily from the axioms of Definition 5:

$$e \in x \circ x^{-1} \cap x^{-1} \circ x, \quad e^{-1} = e, \ (x^{-1})^{-1} = x \quad \text{and} \quad (x \circ y)^{-1} = y^{-1} \circ x^{-1}.$$

A polygroup $\langle P, \circ, e, ^{-1} \rangle$ is said to be commutative if $x \circ y = y \circ x$ for all $x, y \in P$. For simplicity, we write $x$ instead of $\{x\}$ for all $x$ in the polygroup $\langle P, \circ, e, ^{-1} \rangle$.

**Definition 6** ([11]). *Let $\langle M, \circ_1, e_1, ^{-1} \rangle, \langle N, \circ_2, e_2, ^{-1} \rangle$ be two polygroups. Then the productional polygroup $\langle M \times N, \circ, (e_1, e_2), ^{-1} \rangle$ is defined as follows*

$$(x_1, y_1) \circ (x_2, y_2) = \{(x_3, y_3) : x_3 \in x_1 \circ_1 x_2, y_3 \in y_1 \circ_2 y_2\}.$$

In the above definition, $(M \times N, \circ)$ with identity "$(e_1, e_2)$" and unitary operation "$^{-1}$" is a polygroup.

**Example 5.** *Let $P = \{e, m, n\}$ and $(P, \circ)$ be defined by Table 1.*

| $\circ$ | $e$ | $m$ | $n$ |
|---|---|---|---|
| $e$ | $e$ | $m$ | $n$ |
| $m$ | $m$ | $\{e, n\}$ | $\{m, n\}$ |
| $n$ | $n$ | $\{m, n\}$ | $\{e, m\}$ |

**Table 1.** The polygroup $\langle P, \circ, e^{-1} \rangle$

*Then $\langle P, \circ, e, ^{-1} \rangle$ is a commutative polygroup.*

**Example 6** ([14]). *Let $P_1 = \{e, a, b, c\}$ and $(P_1, \cdot)$ be defined by Table 2. Then $\langle P_1, \cdot, e, ^{-1} \rangle$ is a non-commutative polygroup.*

**Example 7** ([14]). *Let $P_2 = \{e, a, b, c, d, f, g\}$ and $(P_2, \cdot)$ be defined by Table 3. Then $\langle P_2, \cdot, e, ^{-1} \rangle$ is a non-commutative polygroup.*

**Remark 2.** *Every group is a polygroup.*

| $\cdot$ | $e$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $a$ | $P_1$ | $c$ |
| $b$ | $b$ | $\{e,a,b\}$ | $b$ | $\{b,c\}$ |
| $c$ | $c$ | $\{a,c\}$ | $c$ | $P_1$ |

**Table 2.** The polygroup $\langle P_1,\cdot,e,^{-1}\rangle$

| $\cdot$ | $e$ | $a$ | $b$ | $c$ | $d$ | $f$ | $g$ |
|---|---|---|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ | $d$ | $f$ | $g$ |
| $a$ | $a$ | $e$ | $b$ | $c$ | $d$ | $f$ | $g$ |
| $b$ | $b$ | $b$ | $\{e,a\}$ | $g$ | $f$ | $d$ | $c$ |
| $c$ | $c$ | $c$ | $f$ | $\{e,a\}$ | $g$ | $b$ | $d$ |
| $d$ | $d$ | $d$ | $g$ | $f$ | $\{e,a\}$ | $c$ | $b$ |
| $f$ | $f$ | $f$ | $c$ | $d$ | $b$ | $g$ | $\{e,a\}$ |
| $g$ | $g$ | $g$ | $d$ | $b$ | $c$ | $\{e,a\}$ | $f$ |

**Table 3.** The polygroup $\langle P_2,\cdot,e,^{-1}\rangle$

Let $\langle P,\circ,e,^{-1}\rangle$ be a polygroup and $K\subseteq P$. Then $S$ is a subpolygroup of $P$ if for all $x,y\in S$ we have that $x\circ y\subseteq S$ and $x^{-1}\in S$.

Let $\langle P,\circ,e,^{-1}\rangle$ be a polygroup and $S$ be a subpolygroup of $P$. Then $S$ is a normal subpolygroup of $P$ if for all $p\in P$ we have that $p\circ S\circ p^{-1}\subseteq S$.

**Example 8.** *Let $\langle P_2,\cdot,e,^{-1}\rangle$ be the polygroup in Example 7. Then $\{e,a,f,g\}$ is a normal subpolygroup of $P_2$.*

In [14], M. Jafarpour et al. described a method to get a polygroup from a group. Let $(G,\cdot)$ be a group, $a\notin G$, and $P_G=G\cup\{a\}$. Define "$\circ$" on $P_G$ as follows:

(1) $a\circ a=e$;

(2) $e\circ x=x\circ e=x$ for all $x\in P_G$;

(3) $a\circ x=x\circ a=x$ for all $x\in P_G-\{e,a\}$;

(4) $x\circ y=x\cdot y$ for all $x,y\in G$ and $y\neq x^{-1}$;

(5) $x\circ x^{-1}=\{e,a\}$ for all $x\in P_G-\{e,a\}$.

**Proposition 3** ([14]). *If $(G,\cdot)$ is a group, then $\langle P_G,\circ,e,^{-1}\rangle$ is a polygroup.*

**Theorem 1.** *Let $(G,\cdot)$ be a group and $N\neq\varnothing\subseteq P_G$. Then $N$ is a subpolygroup of $P_G$ if and only if $N=\{e\}$ or $N=P_S$ for some subgroup $S$ of $G$.*

*Proof.* Let $S$ be a subgroup of $G$. Having $e\in S$ implies that $e\in P_S$ and hence $P_S\neq\varnothing$. Let $x\in P_S$. Then

$$x^{-1}=\begin{cases}a, & \text{if } x=a,\\ x^{-1}, & \text{if } x\in S\end{cases}\in P_S.$$

For $x, y \in P_S$, we have

$$x \circ y = \begin{cases} xy, & \text{if } x, y \in S \text{ and } y \neq x^{-1}, \\ x, & \text{if } y = a \text{ and } x \in S, \\ y, & \text{if } x = a \text{ and } y \in S, \\ e, & \text{if } x = y = a, \\ \{e, a\}, & \text{if } x, y \in S \text{ and } y = x^{-1} \end{cases} \subseteq P_S.$$

Thus, $P_S$ is a subpolygroup of $P_G$.

Conversely, let $N \neq \{e\}$ be a subpolygroup of $P_G$. Then there exists $x \neq e \in N$. Since $N$ is a subpolygroup of $P_G$, it follows that $x^{-1} \in N$ and hence, $x \circ x^{-1} = \{e, a\} \subseteq N$. Having $a \in N$ asserts that we can write $N = S \cup \{a\}$ (with $a \notin S$). We need to show that $S$ is a subgroup of $G$. Let $x \in S$. Then $x \neq a \in N$ and hence, $x^{-1} \neq a \in N$. Thus, $x^{-1} \in S$. Let $x, y \in S$. Then $x \circ y = \begin{cases} xy \neq a, & \text{if } y \neq x^{-1}, \\ \{e, a\}, & \text{otherwise} \end{cases} \subseteq N$. Thus, $xy \in S$. □

**Corollary 1.** *Let $(\mathbb{Z}, +)$ be the group of integers under standard addition of integers and $S$ be a subpolygroup of the polygroup $P_{\mathbb{Z}}$. Then $S = \{0\}$ or $S = P_{n\mathbb{Z}}$ for some integer $n$.*

**Theorem 2.** *Let $(G, \cdot)$ be a non-trivial group and $N \neq \varnothing \subseteq P_G$. Then $N$ is a normal subpolygroup of $P_G$ if and only if $N = P_S$ for some normal subgroup $S$ of $G$.*

*Proof.* Let $S$ be a normal subgroup of $G$. Theorem 1 asserts that $P_S$ is a subpolygroup of $P_G$. We need to show that $x \circ P_S \circ x^{-1} \subseteq P_S$. If $x \in P_S$, we are done. If $x \notin P_S$, then $x \neq a$ and $x \notin S$. We get that for all $z \in P_S$,

$$x \circ z \circ x^{-1} = \begin{cases} \{e, a\}, & \text{if } z \in \{e, a\}, \\ xzx^{-1}, & \text{otherwise} \end{cases} \subseteq P_S.$$

Conversely, let $N$ be a normal subpolygroup of $P_G$. Theorem 1 asserts that $N = \{e\}$ or there exists a subgroup $S$ of $G$ such that $N = P_S$. Since there exists $x \neq e \in G$, it follows that $x \circ e \circ x^{-1} = \{e, a\} \nsubseteq \{e\}$ and hence $\{e\}$ is not a normal subpolygroup of $P_G$. We need to show that $xSx^{-1} \subseteq S$ for all $x \in G$. Having $x \circ (S \cup \{a\}) \circ x^{-1} \subseteq S \cup \{a\}$ for all $x \in G \cup \{a\}$ implies that $xSx^{-1} \subseteq S$ for all $x \in G$. □

## 2 Linear Diophantine fuzzy subpolygroups

In this section and inspired by linear Diophantine fuzzy subgroups of a group in [15], we define linear Diophantine fuzzy subpolygroups of a polygroup and study its properties under various operations of LDFSs. Moreover, we find a relation between LDF-subpolygroups of a polygroup and its level sets.

Since every group is a polygroup, it follows that the results of this section generalizes the results in [15] about LDF-subgroups of a group.

Let $X$ be a universal set and $D$ be an LDFS on $X$ given as follows

$$D = \{(x, < U(x), V(x) >, < \alpha(x), \beta(x) >) : x \in X\},$$

where $U(X), V(x) \in [0, 1]$ are degrees of membership and non-membership respectively and $\alpha(x), \beta(x) \in [0, 1]$ are reference parameters. The degrees satisfy $0 \leq \alpha(x) + \beta(x) \leq 1$ and $0 \leq \alpha(x)U(x) + \beta(x)V(x) \leq 1$ for all $x \in X$.

For $x, y \in X$ we have

*(1)* $D(x) \wedge D(y) = (< u, v >, < \alpha, \beta >)$, where $u = U(x) \wedge U(y)$, $v = V(x) \vee V(y)$, $\alpha = \alpha(x) \wedge \alpha(y)$, $v = \beta(x) \vee \beta(y)$;

*(2)* $D(x) \vee D(y) = (< u, v >, < \alpha, \beta >)$, where $u = U(x) \vee U(y)$, $v = V(x) \wedge V(y)$, $\alpha = \alpha(x) \vee \alpha(y)$, $v = \beta(x) \wedge \beta(y)$.

**Definition 7** ([15]). *Let $(G, \cdot)$ be a group and $D$ be an LDFS of $P$. Then $D$ is a linear Diophantine fuzzy subgroup (LDF-subgroup) of $G$ if the following conditions hold for all $x, y \in G$:*

*(1)* $D(x \cdot y) \geq D(x) \wedge D(y)$;

*(2)* $D(x^{-1}) \geq D(x)$.

**Definition 8.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D$ an LDFS of $P$. Then $D$ is a linear Diophantine fuzzy subpolygroup (LDF-subpolygroup) of $P$ if the following conditions hold for all $x, y \in P$:*

*(1)* $D(z) \geq D(x) \wedge D(y)$ *for all* $z \in x \circ y$;

*(2)* $D(x^{-1}) \geq D(x)$.

**Proposition 4.** *Let $\langle P, \cdot, e, ^{-1} \rangle$ be a polygroup and $D$ an LDF-subpolygroup of $P$. Then the following are true:*

*(1)* $D(e) \geq D(x)$ *for all* $x \in P$.

*(2)* $D(x^{-1}) = D(x)$ *for all* $x \in P$.

*(3)* $D(z) \geq D(x)$ *for all* $z \in x^k$ *and* $k \in \mathbb{Z}$.

*Proof.* The proof is straightforward. □

**Example 9.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be the polygroup defined in Example 5 and $D_1, D_2$ be the LDFSs on $P$ defined respectively as follow*

$$\{(e, < 0.6, 0.2 >, < 0.8, 0.1 >), (m, < 0.5, 0.3 >, < 0.5, 0.2 >), (n, < 0.5, 0.3 >, < 0.5, 0.2 >)\},$$
$$\{(e, < 0.4, 0.2 >, < 0.8, 0.1 >), (m, < 0.5, 0.3 >, < 0.5, 0.2 >), (n, < 0.5, 0.3 >, < 0.5, 0.2 >)\}.$$

*Then $D_1$ is an LDF-subpolygroup of $P$, whereas $D_2$ is not an LDF-subpolygroup of $P$.*

**Proposition 5.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D$ be the LDFS on $P$ defined as*

$$D(x) = D(y) = (< u, v >, < \alpha, \beta >)$$

*for all $x, y \in P$. Then $D$ is an LDF-subpolygroup of $P$. Here, $u, v, \alpha, \beta \in [0, 1]$, $\alpha + \beta \leq 1$ and $\alpha u + \beta v \leq 1$ .*

*Proof.* The proof is straightforward. □

Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D$ be the LDFS on $P$ defined as $D(x) = D(y)$ for all $x, y \in P$. Then $D$ is called the constant LDFS.

**Proposition 6.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D$ be the LDFS on $P$ defined as follows. For all $x, y \in P$*

$$D(x) = \begin{cases} t, & \text{if } x = e, \\ t', & \text{otherwise.} \end{cases}$$

*Here, $t = (< u, v >, < \alpha, \beta >)$, $t' = (< u', v' >, < \alpha', \beta' >)$, $t > t'$, $u, v, u', v', \alpha, \beta, \alpha', \beta'$ are real numbers between 0 and 1 (both inclusive), $\alpha + \beta, \alpha' + \beta' \leq 1$, $\alpha u + \beta v \leq 1$ and $\alpha' u' + \beta' v' \leq 1$. Then $D$ is an LDF-subpolygroup of $P$.*

*Proof.* Let $x, y \in P$ and $z \in x \circ y$. We consider the following two cases: $(x, y) = (e, e)$ and $(x, y) \neq (e, e)$. If $(x, y) = (e, e)$, then $z = e$ and $D(z) = t = D(x) \wedge D(y)$. If $(x, y) \neq (e, e)$, then

$$D(z) = \begin{cases} t \geq t' = D(x) \wedge D(y), & \text{if } z = e, \\ t' = D(x) \wedge D(y), & \text{otherwise.} \end{cases}$$

Moreover, $D(x^{-1}) = \begin{cases} t, & \text{if } x = e, \\ t', & \text{otherwise} \end{cases} = D(x)$. Therefore, $D$ is an LDF-subpolygroup of $P$. □

**Proposition 7.** *Let $(G, \cdot)$ be a group and $D$ be an LDF-subgroup of $G$. Then the LDFS $D'$ is an LDF-subpolygroup of $P_G$. Here, $D'$ is defined as follows*

$$D'(x) = \begin{cases} D(e), & \text{if } x \in \{e, a\}, \\ D(x), & \text{otherwise.} \end{cases}$$

*Proof.* Let $x, y \in P_G$ and $z \in x \circ y$. Then

$$D'(z) = \begin{cases} D(xy), & \text{if } x, y \in S \text{ and } y \neq x^{-1}, \\ D(x), & \text{if } y = a \text{ and } x \in G, \\ D(y), & \text{if } x = a \text{ and } y \in G, \\ D(e), & \text{if } x = y = a \text{ or } y = x^{-1}. \end{cases}$$

Since $D(xy) \geq D(x) \wedge D(y)$ for all $x, y \in G$ and $D'(a) = D'(e) = D(e)$, it follows that $D'(z) \geq D'(x) \wedge D'(y)$ for all $z \in x \circ y$. Moreover,

$$D'(x^{-1}) = \begin{cases} D(e), & \text{if } x \in \{e, a\}, \\ D(x^{-1}), & \text{otherwise} \end{cases} \geq D'(x).$$

Therefore, $D'$ is an LDF-subpolygroup of $P_G$. □

**Example 10.** *Let $(\mathbb{Z}, +)$ be the group of integers under standard addition of integers. One can easily see that the LDFS $D$ is an LDF-subgroup of $\mathbb{Z}$ where $D$ is defined as follows*

$$D(x) = \begin{cases} (< 0.8, 0.1 >, < 0.5, 0.4 >), & \text{if } x \in 2\mathbb{Z}, \\ (< 0.4, 0.6 >, < 0.1, 0.7 >), & \text{otherwise.} \end{cases}$$

*Proposition 7 asserts that the LDFS $D'$ is an LDF-subpolygroup of $P_{\mathbb{Z}}$. Here, $D'$ is defined as follows*

$$D'(x) = \begin{cases} (< 0.8, 0.1 >, < 0.5, 0.4 >), & \text{if } x \in 2\mathbb{Z} \cup \{a\}, \\ (< 0.4, 0.6 >, < 0.1, 0.7 >), & \text{otherwise.} \end{cases}$$

**Proposition 8.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D_1, D_2$ be LDF-subpolygroups of $P$. Then $D_1 \cap D_2$ is an LDF-subpolygroup of $P$.*

*Proof.* Let $x, y \in P$,

$$D_1 = \{(x, < U_1(x), V_1(x) >, < \alpha_1(x), \beta_1(x) >) : x \in P\},$$
$$D_2 = \{(x, < U_2(x), V_2(x) >, < \alpha_2(x), \beta_2(x) >) : x \in P\}$$

be LDF-subpolygroups of $P$ and

$$D = D_1 \cap D_2 = \{(x, < U(x), V(x) >, < \alpha(x), \beta(x) >) : x \in P\}.$$

For all $z \in x \circ y$, we have

$$U_1(z) \geq U_1(x) \wedge U_1(y), V_1(z) \leq V_1(x) \vee V_1(y), \alpha_1(z) \geq \alpha_1(x) \wedge \alpha_1(y), \beta_1(z) \leq \beta_1(x) \vee \beta_1(y),$$
$$U_2(z) \geq U_2(x) \wedge U_2(y), V_2(z) \leq V_2(x) \vee V_2(y), \alpha_2(z) \geq \alpha_2(x) \wedge \alpha_2(y), \beta_2(z) \leq \beta_2(x) \vee \beta_2(y).$$

We get

$$U(z) = U_1(z) \wedge U_2(z) \geq U_1(x) \wedge U_1(y) \wedge U_2(x) \wedge U_2(y) = U(x) \wedge U(y),$$
$$V(z) = V_1(z) \vee V_2(z) \leq V_1(x) \vee V_1(y) \vee V_2(x) \vee V_2(y) = V(x) \vee V(y),$$
$$\alpha(z) = \alpha_1(z) \wedge \alpha_2(z) \geq \alpha_1(x) \wedge \alpha_1(y) \wedge \alpha_2(x) \wedge \alpha_2(y) = \alpha(x) \wedge \alpha(y),$$
$$\beta(z) = \beta_1(z) \vee \beta_2(z) \leq \beta_1(x) \vee \beta_1(y) \vee \beta_2(x) \vee \beta_2(y) = \beta(x) \vee \beta(y).$$

The latter implies that $D(z) \geq D(x) \wedge D(y)$ for all $z \in x \circ y$. Moreover, having

$$U_1(x^{-1}) \geq U_1(x),\ V_1(x^{-1}) \leq V_1(x), \alpha_1(x^{-1}) \geq \alpha_1(x), \beta_1(x^{-1}) \leq \beta_1(x),$$
$$U_2(x^{-1}) \geq U_2(x),\ V_2(x^{-1}) \leq V_2(x),\ \alpha_2(x^{-1}) \geq \alpha_2(x), \beta_2(x^{-1}) \leq \beta_2(x)$$

implies that $U(x^{-1}) = U_1(x^{-1}) \wedge U_2(x^{-1}) \geq U(x)$, $V(x^{-1}) = V_1(x^{-1}) \vee V_2(x^{-1}) \leq V(x)$, $\alpha(x^{-1}) = \alpha_1(x^{-1}) \wedge \alpha_2(x^{-1}) \geq \alpha(x)$, and $\beta(x^{-1}) = \beta_1(x^{-1}) \vee \beta_2(x^{-1}) \leq \beta(x)$. The latter implies that $D(x^{-1}) \geq D(x)$ for all $x \in P$. Therefore, $D$ is an LDF-supolygroup of $P$. ☐

**Remark 3.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D_1, D_2$ be LDF-subpolygroups of $P$. Then $D_1 \cup D_2$ is not necessary an LDF-subpolygroup of $P$.*

We illustrate Remark 3 by Example 11.

**Example 11.** *Let $(\mathbb{Z}, +)$ be the group of integers under standard addition and $D_1, D_2$ be the LDFS on the polygroup $P_{\mathbb{Z}}$ defined as follows*

$$D_1(x) = \begin{cases} (< 0.9, 0.1 >, < 0.5, 0.4 >), & \textit{if } x \in 2\mathbb{Z} \cup \{a\}, \\ (< 0.4, 0.6 >, < 0.1, 0.7 >), & \textit{otherwise,} \end{cases}$$

$$D_2(x) = \begin{cases} (< 0.7, 0.1 >, < 0.5, 0.4 >), & \textit{if } x \in 3\mathbb{Z} \cup \{a\}, \\ (< 0.3, 0.6 >, < 0.1, 0.7 >), & \textit{otherwise.} \end{cases}$$

*One can easily see that $D_1$ and $D_2$ are LDF-subpolygroups of $P_{\mathbb{Z}}$. Having*

$$(D_1 \cup D_2)(2) = (< 0.9, 0.1 >, < 0.5, 0.4 >),$$
$$(D_1 \cup D_2)(3) = (< 0.7, 0.1 >, < 0.5, 0.4 >),$$
$$(D_1 \cup D_2)(5) = (< 0.4, 0.6 >, < 0.1, 0.7 >)$$

*and $5 = 2 \circ 3$ implies that $(D_1 \cup D_2)(5) \ngeq (D_1 \cup D_2)(2) \wedge (D_1 \cup D_2)(3)$. Thus, $D_1 \cup D_2$ is not an LDF-subpolygroup of $P_{\mathbb{Z}}$.*

**Proposition 9.** *Let $\langle P_1, \circ_1, e_1, ^{-1} \rangle$, $\langle P_2, \circ_2, e_2, ^{-1} \rangle$ be polygroups and $D_1, D_2$ be LDF-subpolygroups of $P_1, P_2$ respectively. Then $D_1 \times D_2$ is an LDF-subpolygroup of the productional polygroup $P_1 \times P_2$.*

*Proof.* The proof is straightforward. □

**Definition 9.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D$ be an LDFS of $P$. If $u, v, \alpha, \beta \in [0,1]$ satisfying $0 \leq \alpha + \beta \leq 1$ and $0 \leq \alpha u + \beta v \leq 1$, then the level set of $P$ corresponding to $t = (< u, v >, < \alpha, \beta >)$ is defined as follows*

$$D_t = \{x \in P : D(x) \geq t\}.$$

**Example 12.** *Let $(\mathbb{Z}, +)$ be the group of integers under standard addition of integers and $D$ be the LDF-subpolygroup of $P_{\mathbb{Z}}$ defined as follows*

$$D(x) = \begin{cases} (< 0.8, 0.1 >, < 0.5, 0.4 >), & \text{if } x \in 2\mathbb{Z} \cup \{a\}, \\ (< 0.4, 0.6 >, < 0.1, 0.7 >), & \text{otherwise.} \end{cases}$$

*Then for $t_1 = (< 0.3, 0.7 >, < 0.05, 0.75 >)$, $t_2 = (< 0.7, 0.3 >, < 0.4, 0.65 >)$ and $t_3 = (< 0.9, 0.7 >, < 0.05, 0.75 >)$ the corresponding level sets are $P_{\mathbb{Z}}$, $P_{2\mathbb{Z}}$ and $\varnothing$ respectively.*

**Theorem 3.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D$ be an LDFS of $P$. Then $D$ is an LDF-subpolygroup of $P$ if and only if $D_t$ is either the empty set or a subpolygroup of $P$ for all $t = (< u, v >, < \alpha, \beta >)$. Here, $u, v, \alpha, \beta \in [0,1]$ satisfying $0 \leq \alpha + \beta \leq 1$ and $0 \leq \alpha u + \beta v \leq 1$.*

*Proof.* Let $D$ be an LDF-subpolygroup of $P$ and $D_t \neq \varnothing$. If $x, y \in D_t$, then $D(x), D(y) \geq t$. Since $D$ is an LDF-subpolygroup of $P$, it follows that $D(z) \geq D(x) \wedge D(y) \geq t$ all $z \in x \circ y$. The latter implies that $z \in D_t$. Moreover, $x^{-1} \in D_t$ as $D(x^{-1}) \geq D(x) \geq t$. Thus, $D_t$ is a subpolygroup of $P$.

Conversely, let $x, y \in P$ with $D(x) = t_1, D(y) = t_2$ and $t = t_1 \wedge t_2$. Then $x, y \in D_t$. Since $D_t$ is a subpolygroup of $P$, it follows that $z \in D_t$ for all $z \in x \circ y$. The latter implies that $D(z) \geq t = D(x) \wedge D(y)$. Moreover, having $x \in D_{t_1}$ and $D_{t_1}$ a subpolygroup of $P$ implies that $x^{-1} \in D_{t_1}$. The latter implies $D(x^{-1}) \geq t = D(x)$. Thus, $D$ is an LDF-subpolygroup of $P$. □

**Proposition 10.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup. Then every subpolygroup of $P$ is a level set of $P$ for some LDF-subpolygroup $D$ of $P$.*

*Proof.* Let $S$ be a subpolygroup of $P$ and define the LDFS $D$ on $P$ as follows

$$D(x) = \begin{cases} (< u, v >, < \alpha, \beta >), & \text{if } x \in S, \\ (< 0, 1 >, < 0, 1 >), & \text{otherwise.} \end{cases}$$

Here, $u, v, \alpha, \beta \in ]0, 1]$, $\alpha + \beta \leq 1$, and $\alpha u + \beta v \leq 1$. One can easily see that $D$ is an LDF-subpolygroup of $P$ and that $S = D_t$, where $t = (< u, v >, < \alpha, \beta >)$. □

## 3 Linear Diophantine fuzzy normal subpolygroups

In this section and inspired by linear Diophantine fuzzy normal subgroups of a group in [15], we define linear Diophantine fuzzy normal subpolygroups of a polygroup and study its properties under various operations of LDFSs. Moreover, we find a relation between LDF-normal subpolygroups of a polygroup and its level sets.

Since every group is a polygroup, it follows that the results of this section generalizes the results in [15] about LDF-normal subgroups of a group.

**Definition 10.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D$ be an LDF-subpolygroup of $P$. Then $D$ is a linear Diophantine fuzzy normal subpolygroup (LDF-normal subpolygroup) of $P$ if for all $x, y \in P$, the following condition holds*

$$D(z) = D(z') \text{ for all } z \in x \circ y \text{ and } z' \in y \circ x.$$

**Example 13.** *Let $(\mathbb{Z}, +)$ be the group of integers under standard addition of integers. One can easily see that the LDFS $D$ is an LDF-normal subpolygroup of $P_{\mathbb{Z}}$. Here, $D$ is defined as follows*

$$D'(x) = \begin{cases} (<0.7, 0.1>, <0.6, 0.3>), & \text{if } x \in 3\mathbb{Z} \cup \{a\}, \\ (<0.45, 0.56>, <0.2, 0.5>), & \text{otherwise.} \end{cases}$$

**Proposition 11.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D$ be an LDF-normal subpolygroup of $P$. If $x, y \in P$ and $z, z' \in x \circ y$, then $D(z) = D(z')$.*

*Proof.* Let $D$ be an LDF-normal subpolygroup of $P$ and $x, y \in P$. Let $z, z' \in x \circ y$. Then $D(z) = D(x')$ and $D(z') = D(x')$ for all $x' \in y \circ x$. Thus, $D(z) = D(z')$. □

**Corollary 2.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D$ be an LDF-normal subpolygroup of $P$. If there exist $a, b \in P$ with $a \circ b = P$, then $D(x) = D(y)$ for all $x, y \in P$.*

*Proof.* The proof follows from Proposition 11. □

**Example 14.** *Let $\langle P_1, \cdot, e, ^{-1} \rangle$ be the polygroup defined in Example 6 and $D$ be any LDFS on $P_1$. Then $D$ is an LDF-normal subpolygroup of $P_1$ if and only if $D(e) = D(a) = D(b) = D(c)$. This is easily seen as $c \cdot c = P_1$.*

**Remark 4.** *In abelian groups, every LDF-subgroup is an LDF-normal subgroup. This may not hold for commutative polygroups.*

We illustrate Remark 4 by Example 15.

**Example 15.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be the commutative polygroup in Example 5 and*

$$D_1 = \{(e, <0.6, 0.2>, <0.8, 0.1>), (m, <0.5, 0.3>, <0.5, 0.2>), (n, <0.5, 0.3>, <0.5, 0.2>)\}.$$

*Then $D_1$ is an LDF-subpolygroup of $P$, but it is not an LDF-normal subpolygroup of $P$. This is clear as $n \circ n = \{e, m\}$ and $D_1(e) \neq D_1(m)$.*

**Proposition 12.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D$ be an LDF-subpolygroup of $P$. Then $D$ is an LDF-normal subpolygroup of $P$ if and only if $D(z) \geq D(y)$ for all $z \in x \circ y \circ x^{-1}$, where $x, y \in P$.*

*Proof.* Let $D$ be an LDF-normal subpolygroup of $P$, $x, y \in P$ and $z \in x \circ y \circ x^{-1}$. Then there exist $p \in y \circ x^{-1}$ such that $z \in x \circ p$. Having $p \in y \circ x^{-1}$ and $P$ a polygroup implies that $y \in p \circ x$. Having $z \in x \circ p$, $y \in p \circ x$ and $D$ an LDF-normal subpolygroup of $P$ implies that $D(z) = D(y)$.

Conversely, let $x, y \in P$, $z \in x \circ y$ and $z' \in y \circ x$. We get that $y \in z' \circ x^{-1}$ and hence $z \in x \circ y \subseteq x \circ z' \circ x^{-1}$. The latter implies that $D(z) \geq D(z')$. Similarly, we get $D(z') \geq D(z)$. □

**Theorem 4.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D$ be an LDFS of $P$. Then $D$ is an LDF-normal subpolygroup of $P$ if and only if $D_t$ is either the empty set or a normal subpolygroup of $P$ for all $t = (< u, v >, < \alpha, \beta >)$. Here, $u, v, \alpha, \beta \in [0, 1]$ satisfying $0 \leq \alpha + \beta \leq 1$ and $0 \leq \alpha u + \beta v \leq 1$.*

*Proof.* The proof follows from Theorem 3 and Proposition 12. □

**Example 16.** *Let $\langle P_2, \cdot, e, ^{-1} \rangle$ be the polygroup defined in Example 6 and $D$ be the LDFS on $P_2$ defined as follows*

$$D(x) = \begin{cases} (< 0.7, 0.1 >, < 0.5, 0.4 >), & \text{if } x \in \{e, a, f, g\}, \\ (< 0.4, 0.3 >, < 0.4, 0.6 >), & \text{otherwise.} \end{cases}$$

*Since for every $t = (< u, v >, < \alpha, \beta >)$ with $u, v, \alpha, \beta \in [0, 1]$ satisfying $0 \leq \alpha + \beta \leq 1$ and $0 \leq \alpha u + \beta v \leq 1$, $D_t \in \{\phi, \{e, a, f, g\}, P_2\}$, it follows that $D_t$ is either the empty set or a normal subpolygroup of $P_2$. The latter and Theorem 4 imply that $D$ is an LDF-normal subpolygroup of $P_2$.*

**Proposition 13.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup. Then every normal subpolygroup of $P$ is a level set of $P$ for some LDF-normal subpolygroup $D$ of $P$.*

*Proof.* Let $N$ be a normal subpolygroup of $P$ and define the LDFS $D$ on $P$ as follows

$$D(x) = \begin{cases} (< u, v >, < \alpha, \beta >), & \text{if } x \in N, \\ (< 0, 1 >, < 0, 1 >), & \text{otherwise.} \end{cases}$$

Here, $u, v, \alpha, \beta \in ]0, 1]$, $\alpha + \beta \leq 1$ and $\alpha u + \beta v \leq 1$. One can easily see that $D$ is an LDF-normal subpolygroup of $P$ and that $S = D_t$, where $t = (< u, v >, < \alpha, \beta >)$. □

**Proposition 14.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D_1, D_2$ be LDF-normal subpolygroups of $P$. Then $D_1 \cap D_2$ is an LDF-normal subpolygroup of $P$.*

*Proof.* The proof is similar to that of Proposition 8. □

**Remark 5.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D_1, D_2$ be LDF-normal subpolygroups of $P$. Then $D_1 \cup D_2$ is not necessary an LDF-normal subpolygroup of $P$.*

We illustrate Remark 5 by Example 17.

**Example 17.** *Let $(\mathbb{Z}, +)$ be the group of integers under addition and $D_1, D_2$ be the LDFS on $P_{\mathbb{Z}}$ defined in Example 11. One can easily see that $D_1$ and $D_2$ are LDF-normal subpolygroup of $P_{\mathbb{Z}}$, but $D_1 \cup D_2$ is not an LDF-normal subpolygroup of $P_{\mathbb{Z}}$.*

**Proposition 15.** *Let $\langle P_1, \circ_1, e_1, ^{-1} \rangle$, $\langle P_2, \circ_2, e_2, ^{-1} \rangle$ be polygroups and $D_1, D_2$ be LDF-normal subpolygroups of $P_1, P_2$ respectively. Then $D_1 \times D_2$ is an LDF-normal subpolygroup of the productional polygroup $P_1 \times P_2$.*

*Proof.* The proof is straightforward. □

## 4 Linear Diophantine anti-fuzzy (normal) subpolygroups

In this section, we define linear Diophantine anti-fuzzy (normal) subpolygroups of polygroups and study its properties under various operations of LDFSs. Moreover, we find a relation between LDAF-subpolygroups of a polygroup and its ceiling sets.

**Definition 11.** *Let $(G, \cdot)$ be a group and $D$ be an LDFS of $P$. Then $D$ is a linear Diophantine anti-fuzzy subgroup (LDAF-subgroup) of $G$ if the following conditions hold for all $x, y \in G$:*

*(1) $D(x \cdot y) \leq D(x) \vee D(y)$;*

*(2) $D(x^{-1}) \leq D(x)$.*

**Definition 12.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D$ be an LDFS of $P$. Then $D$ is a linear Diophantine anti-fuzzy subpolygroup (LDAF-subpolygroup) of $P$ if the following conditions hold for all $x, y \in P$:*

*(1) $D(z) \leq D(x) \vee D(y)$ for all $z \in x \circ y$;*

*(2) $D(x^{-1}) \leq D(x)$.*

Since every group is a polygroup, it follows that it suffices to elaborate the results of anti-fuzzy (normal) subpolygroups of a polygroup and they will be valid for anti-fuzzy (normal) subgroups of a group.

**Proposition 16.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D$ an LDF-subpolygroup of $P$. Then*

*(1) $D(e) \leq D(x)$ for all $x \in P$,*

*(2) $D(x^{-1}) = D(x)$ for all $x \in P$,*

*(3) $D(z) \leq D(x)$ for all $z \in x^k$ and $k \in \mathbb{Z}$.*

*Proof.* The proof is straightforward. □

**Example 18.** *Let $(\mathbb{Z}, +)$ be the group of integers under standard addition and $D$ be the LDFS on the polygroup $P_{\mathbb{Z}}$ defined as follows*

$$D(x) = \begin{cases} (<0.4, 0.6>, <0.1, 0.7>), & \text{if } x \in 5\mathbb{Z} \cup \{a\}, \\ (<0.9, 0.1>, <0.5, 0.4>), & \text{otherwise.} \end{cases}$$

*Then $D$ is an LDAF-subpolygroup of $P_{\mathbb{Z}}$.*

**Theorem 5.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D$ be an LDFS of $P$. Then $D$ is an LDAF-subpolygroup of $P$ if and only if $D^c$ is an LDF-subpolygroup of $P$.*

*Proof.* Let $D = \{(x, <U(x), V(x)>, <\alpha(x), \beta(x)>: x \in P\}$ be an LDAF-subpolygroup of $P$ and $x, y \in P$. Then for all $z \in x \circ y$

$$U(z) \leq U(x) \vee U(y), \quad V(z) \geq V(x) \wedge V(y), \quad \alpha(z) \leq \alpha(x) \vee \alpha(y), \quad \beta(z) \leq \beta(x) \wedge \beta(y).$$

The latter implies that $D^(c)(z) \geq D^c(x) \wedge D^c(y)$ for all $z \in x \circ y$. Moreover, having $U(x^{-1}) \leq U(x)$, $V(x^{-1}) \geq V(x)$, $\alpha(x^{-1}) \leq \alpha(x)$ and $\beta(x^{-1}) \geq \beta(x)$ implies $D^c(x^{-1}) \leq D^c(x)$. Thus, $D^c = \{(x, <V(x), U(x)>, <\beta(x), \alpha(x)>: x \in P\}$ is an LDF-subpolygroup of $P$. In a similar manner, we can prove that if $D^c$ is an LDF-subpolygroup of $P$ then $D$ is an LDAF-subpolygroup of $P$. □

**Corollary 3.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D$ be an LDFS of $P$. Then $D$ is an LDF-subpolygroup of $P$ if and only if $D^c$ is an LDAF-subpolygroup of $P$.*

*Proof.* The proof follows from Theorem 5. □

**Corollary 4.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D$ be an LDFS of $P$. Then $D$ and $D^c$ are LDF-subpolygroups of $P$ if and only if $D(x) = D(y)$ for all $x, y \in P$.*

**Corollary 5.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D$ an LDFS of $P$. Then $D$ and $D^c$ are LDAF-subpolygroups of $P$ if and only if $D(x) = D(y)$ for all $x, y \in P$.*

**Proposition 17.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D_1, D_2$ be LDAF-subpolygroups of $P$. Then $D_1 \cup D_2$ is an LDAF-subpolygroup of $P$.*

*Proof.* Let $D_1, D_2$ be LDAF-subpolygroups of $P$. Then, by means of Theorem 5, we get that $D_1^c, D_2^c$ are LDF-subpolygroups of $P$. Proposition 8 asserts that $D_1^c \cap D_2^c$ is an LDF-subpolygroup of $P$. Proposition 1 asserts that $(D_1 \cup D_2)^c = D_1^c \cap D_2^c$. The latter implies that $(D_1 \cup D_2)^c$ is and LDF-subpolygroup of $P$. Theorem 5 completes te proof. □

**Remark 6.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D_1, D_2$ be LDAF-subpolygroups of $P$. Then $D_1 \cap D_2$ is not necessary an LDAF-subpolygroup of $P$.*

We illustrate Remark 6 by Example 19.

**Example 19.** *Let $(\mathbb{Z}, +)$ be the group of integers under standard addition and $D_1, D_2$ be the LDFS on the polygroup $P_{\mathbb{Z}}$ defined as follows*

$$D_1(x) = \begin{cases} (< 0.4, 0.6 >, < 0.1, 0.7 >), & \text{if } x \in 2\mathbb{Z} \cup \{a\}, \\ (< 0.9, 0.1 >, < 0.5, 0.4 >), & \text{otherwise,} \end{cases}$$

$$D_2(x) = \begin{cases} (< 0.3, 0.6 >, < 0.1, 0.7 >), & \text{if } x \in 3\mathbb{Z} \cup \{a\}, \\ (< 0.7, 0.1 >, < 0.5, 0.4 >), & \text{otherwise.} \end{cases}$$

*One can easily see that $D_1$ and $D_2$ are LDF-subpolygroups of $P_{\mathbb{Z}}$. Having*

$$(D_1 \cap D_2)(2) = (< 0.4, 0.6 >, < 0.1, 0.7 >), \quad (D_1 \cap D_2)(3) = (< 0.3, 0.6 >, < 0.1, 0.7 >),$$
$$(D_1 \cap D_2)(5) = (< 0.7, 0.1 >, < 0.5, 0.4 >)$$

*and $5 = 2 \circ 3$ implies that $(D_1 \cap D_2)(5) \not\leq (D_1 \cap D_2)(2) \vee (D_1 \cap D_2)(3)$. Thus, $D_1 \cap D_2$ is not an LDAF-subpolygroup of $P_{\mathbb{Z}}$.*

**Definition 13.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D$ be an LDFS of $P$. If $u, v, \alpha, \beta \in [0, 1]$ satisfying $0 \leq \alpha + \beta \leq 1$ and $0 \leq \alpha u + \beta v \leq 1$, then the ceiling set of $P$ corresponding to $t = (< u, v >, < \alpha, \beta >)$ is given as follows*

$$D^t = \{x \in P : D(x) \leq t\}.$$

**Example 20.** *Let $(\mathbb{Z}, +)$ be the group of integers under standard addition and $D$ be the LDFS on the polygroup $P_{\mathbb{Z}}$ defined as follows*

$$D(x) = \begin{cases} (< 0.4, 0.6 >, < 0.1, 0.7 >), & \text{if } x \in 7\mathbb{Z} \cup \{a\}, \\ (< 0.9, 0.1 >, < 0.5, 0.4 >), & \text{otherwise.} \end{cases}$$

*The ceiling sets corresponding to $t_1 = (< 0.3, 0.6 >, < 0.1, 0.7 >)$, $t_2 = (< 0.5, 0.6 >, < 0.1, 0.7 >)$ and $t_3 = (< 0.9, 0.05 >, < 0.6, 0.1 >)$ are $\emptyset$, $P_{7\mathbb{Z}}$ and $P_{\mathbb{Z}}$ respectively.*

**Proposition 18.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup, $D$ be an LDFS of $P$, and $t = (< u, v >, < \alpha, \beta >)$, where $u, v, \alpha, \beta \in [0,1]$ satisfying $0 \leq \alpha + \beta \leq 1$ and $0 \leq \alpha u + \beta v \leq 1$. Then $D^t = D^c_t$.*

*Proof.* The proof is straightforward. □

**Theorem 6.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D$ be an LDFS of $P$. Then $D$ is an LDAF-subpolygroup of $P$ if and only if $D^t$ is either the empty set or a subpolygroup of $P$ for all $t = (< u, v >, < \alpha, \beta >)$. Here, $u, v, \alpha, \beta \in [0,1]$ satisfying $0 \leq \alpha + \beta \leq 1$ and $0 \leq \alpha u + \beta v \leq 1$.*

*Proof.* Let $D$ be an LDAF-subpolygroup of $P$. Then, by means of Theorem 5, $D^c$ is an LDF-subpolygroup of $P$. Theorem 3 asserts that $D^c_t$ is either the empty set or a subpolygroup of $P$. Proposition 18 implies that $D^t = D^c_t$ is either the empty set or a subpolygroup of $P$.

Conversely, let $D^t \neq \varnothing$ be a subpolygroup of $P$. Propsition 18 asserts that $D^c_t \neq \varnothing$ is a subpolygroup of $P$. Theorem 3 implies that $D^c$ is an LDF-subpolygroup of $P$ and hence, $D$ is an LDAF-subpolygroup of $P$ (by Theorem 5). □

**Proposition 19.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup. Then every subpolygroup of $P$ is a ceiling set of $P$ for some LDAF-subpolygroup $D$ of $P$.*

*Proof.* Let $S$ be a subpolygroup of $P$ and define the LDFS $D$ on $P$ as follows

$$D(x) = \begin{cases} (< 0, 1 >, < 0, 1 >), & \text{if } x \in S, \\ (< u, v >, < \alpha, \beta >), & \text{otherwise.} \end{cases}$$

Here, $u, v, \alpha, \beta \in [0, 1[$, $\alpha + \beta \leq 1$ and $\alpha u + \beta v \leq 1$. One can easily see that $D$ is an LDAF-subpolygroup of $P$ and that $S = D^t$, where $t = (< u, v >, < \alpha, \beta >)$. □

**Definition 14.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D$ an LDAF-subpolygroup of $P$. Then $D$ is a linear Diophantine anti-fuzzy normal subpolygroup (LDAF-normal subpolygroup) of $P$ if for all $x, y \in P$ the following condition holds*

$$D(z) = D(z') \text{ for all } z \in x \circ y \text{ and } z' \in y \circ x.$$

Next, we present some results of LDAF-normal subpolygroups of a polygroup and we omit their proofs because they are similar to that of LDAF-subpolygroups of a polygroup.

**Theorem 7.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D$ an LDFS of $P$. Then $D$ is an LDAF-normal subpolygroup of $P$ if and only if $D^c$ is an LDF-normal subpolygroup of $P$.*

**Theorem 8.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D$ be an LDFS of $P$. Then $D$ is an LDAF-normal subpolygroup of $P$ if and only if $D^t$ is either the empty set or a normal subpolygroup of $P$ for all $t = (< u, v >, < \alpha, \beta >)$. Here, $u, v, \alpha, \beta \in [0,1]$ satisfying $0 \leq \alpha + \beta \leq 1$ and $0 \leq \alpha u + \beta v \leq 1$.*

**Proposition 20.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup. Then every normal subpolygroup of $P$ is a ceiling set of $P$ for some LDAF-normal subpolygroup $D$ of $P$.*

**Proposition 21.** *Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $D_1, D_2$ be LDAF-normal subpolygroups of $P$. Then $D_1 \cup D_2$ is an LDAF-normal subpolygroup of $P$.*

## 5 Conclusion

This paper introduced for the first time linear Diophantine (anti-) fuzzy algebraic hyperstructures. It studied some linear Diophantine fuzzy subsets of polygroups and investigated their properties. Since an LDFS is a fuzzy set, it follows that the results in this paper are a general form of the results related to (anti-) fuzzy (normal) subpolygroups of a polygroup. Moreover, it generalizes the work related to LDF-(normal) subgroups of a group. This is because every group is a polygroup.

For future research, it would be interesting to investigate the following.

1. Introduce new concepts related to linear Diophantine fuzzy subsets of other algebraic hyperstructures.

2. In the case of fuzzy algebraic hyperstructure, generalized fuzzy subpolygroups [13] were introduced and studied using the notions of "belongingness" and "quasi-coincidence". Is it possible to introduce a similar concept using linear Diophantine fuzzy sets?

3. Similar to complex fuzzy sets [17, 18], is it possible to introduce complex linear Diophantine fuzzy sets?

4. Similar to fuzzy multisets [20], is it possible to introduce linear Diophantine fuzzy multisets?

## References

[1] Al-Tahan M., Davvaz B. *A new relationship between intuitionistic fuzzy sets and genetics*. J. Classification 2019, **36**, 494–512. doi:10.1007/s00357-018-9276-8

[2] Al-Tahan M., Davvaz B. *Fuzzy subsets of the phenotypes of $F_2$-offspring*. Facta Univ. Ser. Math. Inform. 2019, **34** (4), 709–727. doi:10.22190/FUMI1904709A

[3] Al-Tahan M., Davvaz B. *The cyclic hypergroup associated with $S_n$, it's automorphism group and it's fuzzy grade*. Discrete Math. Algorithms Appl. 2018, **10** (5), 1850070. doi:10.1142/S1793830918500702

[4] Al-Tahan M., Davvaz B. *Complex fuzzy and generalized complex fuzzy subpolygroups of a polygroup*. Jordan J. Math. Stat. (JJMS) 2019, **12** (2), 151–173.

[5] Al- Tahan M., Hoskova-Mayerova S., Davvaz B. *Fuzzy multi-polygroups*. J. Intell Fuzzy Syst 2020, **38** (2), 2337–2345. doi:10.3233/JIFS-191262.

[6] Atanassov K. *Intuitionistic fuzzy sets*. Fuzzy Set Syst. 1986, **20** (1), 87–96. doi:10.1016/S0165-0114(86)80034-3

[7] Ayub S., Shabir M., Riaz M., Aslam M., Chinram R. *Linear Diophantine fuzzy relations and their algebraic properties with decision making*. Symmetry 2021, **13** (6), 945. doi:10.3390/sym13060945

[8] Chen J., Li S., Ma S., Wang X. *m-Polar fuzzy sets: an extension of bipolar fuzzy sets*. Sci. World J. 2014, **2014**, 8–11. doi:10.1155/2014/416530

[9] Comer S.D. *Polygroups derived from cogroups*. J. Algebra 1984, **89** (2), 397–405. doi:10.1016/0021-8693(84)90225-4

[10] Corsini P., Leoreanu V. Applications of hyperstructures theory. In: Szép J. (Ed.) Advances in Mathematics, 5. Springer, New York, 2003. doi:10.1007/978-1-4757-3714-1

[11] Davvaz B. Polygroup Theory and Related Systems. World Scientific Publishing, 2012. doi:10.1142/8593

[12] Davvaz B., Cristea I. Fuzzy Algebraic Hyperstructures In: Kacprzyk J. (Ed.) Studies in Fuzziness and Soft Computing, 321. Springer Cham, 2015.

[13] Davvaz B., Corsini P. *Generalized fuzzy polygroups*. Iran. J. Fuzzy Syst. 2006, **3** (1), 59–75. doi: 10.22111/ijfs.2006.438

[14] Jafarpour M., Aghabozorgt H., Davvaz B. *On nilpotent and solvable polygroups*. Bull. Iranian Math. Soc. 2013, **39** (3), 487–499.

[15] Kamaci H. *Linear diophantine fuzzy algebraic structures*. J. Ambient Intell. Human. Comput. 2021, **12**, 10353–10373. doi:10.1007/s12652-020-02826-x

[16] Marty F. Sur une generalization de la notion de group. 8*th* Congress Math. Scandenaves, 1934, 45–49.

[17] Ramot D., Milo R., Friedman M., Kandel A. *Complex fuzzy sets*. IEEE Transactions on Fuzzy Systems 2002, **10** (2), 171–186. doi:10.1109/91.995119

[18] Ramot D., Friedman M., Langholz G., Kandel A., Milo R. *On complex fuzzy sets*. IEEE Transactions on Fuzzy Systems 2003, **11** (4), 450–461. doi:10.1109/FUZZ.2001.1008861

[19] Riaz M., Hashmi M. *Linear Diophantine fuzzy set and its applications towards multi-attribute decision making problems*. J. Intell. Fuzzy Syst. 2019, **37** (4), 5417–5439. doi:10.3233/JIFS-190550

[20] Yager R.R. *On the theory of bags*. Int. J. Gen. Syst. 1986, **13** (1), 23–37. doi:10.1080/03081078608934952

[21] Yager R.R. *Pythagorean fuzzy subsets*. Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS) 2013, 57–61. doi:10.1109/IFSA-NAFIPS.2013.6608375

[22] Yager R.R. *Generalized orthopair fuzzy sets*. IEEE Transactions on Fuzzy Systems 2017, **25** (5), 1222–1230. doi:10.1109/TFUZZ.2016.2604005

[23] Zadeh L.A. *Fuzzy sets*. Information and Control 1965, **8** (3), 338–353. doi:10.1016/S0019-9958(65)90241-X

*Received 20.08.2021*
*Revised 08.11.2021*

Аль Таган М., Давваз Б., Парімала М., Аль-Касабех С. *Лінійні діофантові нечіткі підмножини полігруп* // Карпатські матем. публ. — 2022. — Т.14, №2. — С. 564–581.

Лінійні діофантові нечіткі множини нещодавно були представлені як узагальнена форма нечітких множин. Метою цієї статті є пролити світло на зв'язок між алгебраїчними гіперструктурами та лінійними діофантовими нечіткими множинами через полігрупи. Точніше, ми вводимо поняття лінійних діофантових нечітких підполігруп полігрупи, лінійних діофантових нечітких нормальних підполігруп полігрупи та лінійних діофантових анти-нечітких підполігруп полігрупи. Крім того, ми вивчаємо деякі з їхніх властивостей і характеризуємо їх у відношенні до множин рівня і стелі.

*Ключові слова і фрази:* полігрупа, лінійна діофантова нечітка множина, лінійна діофантова нечітка підполігрупа, лінійна діофантова нечітка нормальна підполігрупа, множина рівня, лінійна діофантова анти-нечітка підполігрупа, множина стелі.