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ON SOME OF CONVERGENCE DOMAINS OF MULTIDIMENSIONAL S-FRACTIONS WITH INDEPENDENT VARIABLES

The convergence of multidimensional S-fractions with independent variables is investigated using the multidimensional generalization of the classical Worpitzky's criterion of convergence, the criterions of convergence of the branched continued fractions with independent variables, whose partial quotients are of the form $\frac{q_{i(k)}^{i_k} q_{i(k-1)}^{i_k-1}(1-q_{i(k-1)})z_{i(k)}}{1}$, and the convergence continuation theorem to extend the convergence, already known for a small domain (open connected set), to a larger domain. It is shown that the union of the intersections of the parabolic and circular domains is the domain of convergence of the multidimensional S-fraction with independent variables, and that the union of parabolic domains is the domain of convergence of the branched continued fraction with independent variables, reciprocal to it.

Key words and phrases: multidimensional S-fraction with independent variables, convergence.

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1 INTRODUCTION

It is well known (see, for example [2, 7]) that the branched continued fractions with independent variables are an efficient tool for the approximation of analytic multivariable functions, which are represented by multiple power series. One of the important problem for these branched continued fractions is to establish the widest domains (open connected sets) of their convergence. Convergence domains have been given in [1,2,8,11] for multidimensional regular C-fractions with independent variables, in [4] for multidimensional regular S-fractions with independent variables, in [9] for multidimensional g-fractions with independent variables, in [6] for multidimensional associated fractions with independent variables and in [6, 10] for multidimensional J-fractions with independent variables.

Let N be a fixed natural number and

$$\mathcal{I}_k = \{i(k) : \ i(k) = (i_1, i_2, \ldots, i_k), \ 1 \le i_p \le i_{p-1}, \ 1 \le p \le k, \ i_0 = N\}, \ k \ge 1,$$

be the sets of multiindices. In addition, let $i(0) = 0$ and $\mathcal{I}_0 = \{0\}$.

We investigate here the convergence of multidimensional S-fraction with independent variables

$$1 + \sum_{i_1=1}^{N} \frac{c_{i(1)} z_{i_1}}{1} \underset{+}{} \sum_{i_2=1}^{i_1} \frac{c_{i(2)} z_{i_2}}{1} \underset{+}{} \sum_{i_3=1}^{i_2} \frac{c_{i(3)} z_{i_3}}{1} \underset{+}{} \cdots, \tag{1}$$

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where the $c_{i(k)} > 0$ for all $i(k) \in \mathcal{I}_k$, $k \geq 1$, $\mathbf{z} = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N$, and reciprocal to it

$$\frac{1}{1} + \sum_{i_1=1}^{N} \frac{c_{i(1)} z_{i_1}}{1} + \sum_{i_2=1}^{i_1} \frac{c_{i(2)} z_{i_2}}{1} + \sum_{i_3=1}^{i_2} \frac{c_{i(3)} z_{i_3}}{1} + \cdots. \tag{2}$$

2 CONVERGENCE

We introduce the following notation $Q_{i(n)}^{(n)}(\mathbf{z}) \equiv 1$, $i(n) \in \mathcal{I}_n$, $n \geq 1$, and

$$Q_{i(k)}^{(n)}(\mathbf{z}) = 1 + \sum_{i_{k+1}=1}^{i_k} \frac{c_{i(k+1)} z_{i_{k+1}}}{1} + \sum_{i_{k+2}=1}^{i_{k+1}} \frac{c_{i(k+2)} z_{i_{k+2}}}{1} + \cdots + \sum_{i_n=1}^{i_{n-1}} \frac{c_{i(n)} z_{i_n}}{1},$$

where $i(k) \in \mathcal{I}_k,\ 1 \leq k \leq n-1,\ n \geq 2$. It is clear that the following recurrence relations hold

$$Q_{i(k)}^{(n)}(\mathbf{z}) = 1 + \sum_{i_{k+1}=1}^{i_k} \frac{c_{i(k+1)} z_{i_{k+1}}}{Q_{i(k+1)}^{(n)}(\mathbf{z})} \text{ for all } i(k) \in \mathcal{I}_k,\ 1 \leq k \leq n-1,\ n \geq 2. \tag{3}$$

Let $f_n(\mathbf{z}) = 1 + \sum_{i_1=1}^{N} (c_{i(1)} z_{i_1} / Q_{i(1)}^{(n)}(\mathbf{z}))$ be the nth approximant of (1), $n \geq 1$.

We shall prove the following result.

Theorem 1. *A multidimensional S-fraction with independent variables* (1)*, where the* $c_{i(k)}$*,* $i(k) \in \mathcal{I}_k$*,* $k \geq 2$*, satisfy the inequalities*

$$c_{i(k)} \leq q_{i(k)}^{i_k} q_{i(k-1)}^{i_k - 1} (1 - q_{i(k-1)}) \text{ for all } i(k) \in \mathcal{I}_k,\ k \geq 2, \tag{4}$$

where $\{q_{i(k)}\}_{i(k) \in \mathcal{I}_k,\, k \in \mathbb{N}}$ *is a sequence of real numbers such that*

$$0 < q_{i(k)} < 1 \text{ for all } i(k) \in \mathcal{I}_k,\ k \geq 1, \tag{5}$$

converges to a function holomorphic in the domain

$$P_M = \bigcup_{\alpha \in (-\pi/2, \pi/2)} \left\{ \mathbf{z} \in \mathbb{C}^N :\ |z_k| - \mathrm{Re}(z_k e^{-2i\alpha}) < 2\cos^2(\alpha),\ |z_k| < M,\ 1 \leq k \leq N \right\} \tag{6}$$

for every constant $M > 0$*. The convergence is uniform on every compact subset of* P_M*.*

Proof. Let α be an arbitrary number from the interval $(-\pi/2, \pi/2)$, n be an arbitrary natural number, and let $\mathbf{z}$ be an arbitrary fixed point from domain (6). By induction on k for each multiindex $i(k) \in \mathcal{I}_k$ we show that the following inequalities are valid

$$\mathrm{Re}(Q_{i(k)}^{(n)}(\mathbf{z}) e^{-i\alpha}) > q_{i(k)}^{i_k} \cos(\alpha) > 0, \tag{7}$$

where $1 \leq k \leq n$.

It is clear that for $k = n$, $i(n) \in \mathcal{I}_n$, relations (7) hold. By induction hypothesis that (7) hold for $k = r+1$, $r \leq n-1$, $i(r+1) \in \mathcal{I}_{r+1}$, we prove (7) for $k = r$ and for each $i(r) \in \mathcal{I}_r$. Indeed, use of relations (3) for the arbitrary multiindex $i(r) \in \mathcal{I}_r$ lead to

$$Q_{i(r)}^{(n)}(\mathbf{z})e^{-i\alpha} = e^{-i\alpha} + \sum_{i_{r+1}=1}^{i_r} \frac{c_{i(r+1)} z_{i_{r+1}} e^{-2i\alpha}}{Q_{i(r+1)}^{(n)}(\mathbf{z})e^{-i\alpha}}.$$

In the proof of lemma 4.41 [12] it is shown that if $x \geq c > 0$ and $v^2 \leq 4u + 4$,

$$\min_{-\infty<y<+\infty} \operatorname{Re}\left(\frac{u+iv}{x+iy}\right) = -\frac{\sqrt{u^2+v^2}-u}{2x}. \tag{8}$$

We set

$$u = \operatorname{Re}(z_{i_{r+1}} e^{-2i\alpha}),\ v = \operatorname{Im}(z_{i_{r+1}} e^{-2i\alpha}),\ x = \operatorname{Re}(Q_{i(r+1)}^{(n)}(\mathbf{z})e^{-i\alpha}),\ y = \operatorname{Re}(Q_{i(r+1)}^{(n)}(\mathbf{z})e^{-i\alpha}).$$

Then from (6) it is easy to show that $v^2 \leq 4u + 4$ for each index i_{r+1}, $1 \leq i_{r+1} \leq i_r$.

Now, using (4)–(8) and induction hypothesis, we obtain that

$$\begin{aligned}\operatorname{Re}(Q_{i(r)}^{(n)}(\mathbf{z})e^{-i\alpha}) &\geq \cos(\alpha) - \sum_{i_{r+1}=1}^{i_r} \frac{q_{i(r+1)}^{i_{r+1}} q_{i(r)}^{i_{r+1}-1}(1-q_{i(r)})(|z_{i_{r+1}}| - \operatorname{Re}(z_{i_{r+1}} e^{-2i\alpha}))}{2\operatorname{Re}(Q_{i(r+1)}^{(n)}(\mathbf{z})e^{-i\alpha})} \\ &> \cos(\alpha) - \sum_{i_{r+1}=1}^{i_r} q_{i(r)}^{i_{r+1}-1}(1-q_{i(r)})\cos(\alpha) = q_{i(r)}^{i_r}\cos(\alpha) > 0.\end{aligned}$$

It follows from (7) that $Q_{i(k)}^{(n)}(\mathbf{z}) \neq 0$ for all $i(k) \in \mathcal{I}_k$, $1 \leq k \leq n$, $n \geq 1$, and for all $\mathbf{z}$ from domain (6). Thus, the approximants $f_n(\mathbf{z})$, $n \geq 1$, of (1) form a sequence of functions holomorphic in P_M.

Again, let α be an arbitrary number from the interval $(-\pi/2, \pi/2)$. And, let

$$P_{\alpha,\sigma,M} = \left\{\mathbf{z} \in \mathbb{C}^N :\ |z_k| - \operatorname{Re}(z_k e^{-2i\alpha}) < 2\sigma\cos^2(\alpha),\ |z_k| < \sigma M,\ 1 \leq k \leq N\right\}, \tag{9}$$

where $0 < \sigma < 1$. We set $c = \max\{c_1, c_2, \ldots, c_N\}$.

Using (7), for the arbitrary $\mathbf{z} \in P_{\alpha,\sigma,M}$, $P_{\alpha,\sigma,M} \subset P_M$, we obtain for $n \geq 1$

$$|f_n(\mathbf{z})| \leq 1 + \sum_{i_1=1}^{N} \frac{c_{i(1)}|z_{i_1}|}{\operatorname{Re}(Q_{i(1)}^{(n)}(\mathbf{z})e^{-i\alpha})} < 1 + \sum_{i_1=1}^{N} \frac{c\sigma M}{q_{i(1)}^{i_1}\cos(\alpha)} = C(P_{\alpha,\sigma,M}),$$

where the constant $C(P_{\alpha,\sigma,M})$ depends only on the domain (9), i.e. the sequence $\{f_n(\mathbf{z})\}$ is uniformly bounded in $P_{\alpha,\sigma,M}$.

Let K be an arbitrary compact subset of P_M. Let us cover K with domains of form (9). From this cover we choose the finite subcover $P_{\alpha_1,\sigma_1,M}, P_{\alpha_2,\sigma_2,M}, \ldots, P_{\alpha_r,\sigma_r,M}$. We set

$$C(K) = \max\left\{C(P_{\alpha_1,\sigma_1,M}), C(P_{\alpha_2,\sigma_2,M}), \ldots, C(P_{\alpha_r,\sigma_r,M})\right\}.$$

Then for arbitrary $\mathbf{z} \in K$ we obtain $|f_n(\mathbf{z})| \leq C(K)$, for $n \geq 1$, i.e. the sequence $\{f_n(\mathbf{z})\}$ is uniformly bounded on each compact subset of the domain (6).

Let $b = \min\{1, M, q_1/(2c), q_2/(2^2c), \ldots, q_N/(2^Nc)\}$ and let

$$\Delta_R = \left\{ \mathbf{z} \in \mathbb{R}^N : \ 0 < z_k < R < b, \ 1 \le k \le N \right\}.$$

Evidently $\Delta_R \subset P_M$ for each $0 < R < b$, in particular, say $\Delta_{b/2} \subset P_M$. Then for the arbitrary $\mathbf{z} \in \Delta_R$, $\Delta_R \subset P_M$, we obtain

$$|c_{i(1)} z_{i_1}| < bc \le 2^{-i_1} q_{i(1)}^{i_1} \text{ for all } i(1) \in \mathcal{I}_1,$$
$$|c_{i(k)} z_{i_k}| < q_{i(k)}^{i_k} q_{i(k-1)}^{i_k - 1} (1 - q_{i(k-1)}) \text{ for all } i(k) \in \mathcal{I}_k, \ k \ge 2.$$

It follows from theorem 1 [8], with $q_0 = 1/2$, that (1) converges in the domain Δ_R. Hence, by theorem 2.17 [3, p. 66] (see also theorem 24.2 [13, pp. 108–109]) the multidimensional S-fraction with independent variables (1) converges uniformly on compact subsets of P_M to a holomorphic function. □

The following two theorem can be proved in much the same way as theorem 1 using theorem 1 and 5 [5], respectively.

Theorem 2. *A multidimensional S-fraction with independent variables* (2)*, where the* $c_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \ge 1$, *satisfy the inequalities* $c_{i(k)} \le q_{i(k)}^{i_k} q_{i(k-1)}^{i_k - 1} (1 - q_{i(k-1)})$ *for all* $i(k) \in \mathcal{I}_k$, $k \ge 1$, *where* $\{q_{i(k)}\}_{i(k) \in \mathcal{I}_k,\, k \in \mathbb{N}_0}$ *is a sequence of real numbers such that* $0 < q_{i(k)} < 1$ *for all* $i(k) \in \mathcal{I}_k$, $k \ge 0$, *converges to a function holomorphic in the domain*

$$D = \bigcup_{\alpha \in (-\pi/2, \pi/2)} \left\{ \mathbf{z} \in \mathbb{C}^N : \ |z_k| - \mathrm{Re}(z_k e^{-2i\alpha}) < 2\cos^2(\alpha), \ 1 \le k \le N \right\}. \tag{10}$$

The convergence is uniform on every compact subset of D.

Theorem 3. *A multidimensional S-fraction with independent variables* (2)*, where the* $c_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \ge 1$, *satisfy the inequalities* $c_{i(1)} \le q_{i(1)}^{i_1 + 1}$ *for all* $i(1) \in \mathcal{I}_1$ *and* $c_{i(k)} \le q_{i(k)}^{i_k} q_{i(k-1)}^{i_k - 1} (1 - q_{i(k-1)})$ *for all* $i(k) \in \mathcal{I}_k$, $k \ge 2$, *where* $\{q_{i(k)}\}_{i(k) \in \mathcal{I}_k,\, k \in \mathbb{N}}$ *is a sequence of real numbers satisfying the inequalities* (5) *and* $\sum_{i_1=1}^{N} q_{i(1)} < 1$, *converges to a function holomorphic in the domain* (10). *The convergence is uniform on every compact subset of* D.

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Дмитришин Р.І. *Про деякі області збіжності багатовимірних S-дробів з нерівнозначними змінними* // Карпатські матем. публ. — 2019. — Т.11, №1. — С. 54–58.

Досліджується збіжність багатовимірних S-дробів з нерівнозначними змінними із використанням багатовимірного узагальнення класичної ознаки збіжності Ворпітського, ознак збіжності для гіллястих ланцюгових дробів з нерівнозначними змінними, частинні ланки яких мають вигляд $\frac{q_{i(k)}^{i_k} q_{i(k-1)}^{i_k-1}(1-q_{i(k-1)})z_{i(k)}}{1}$, і теореми про продовження збіжності із уже відомої малої області до більшої. Отримано, що об'єднання перетинів параболічних і кругових областей є областю збіжності багатовимірного S-дробу з нерівнозначними змінними, а об'єднання параболічних областей — областю збіжності оберненого до нього гіллястого ланцюгового дробу з нерівнозначними змінними.

Ключові слова і фрази: багатовимірний S-дріб з нерівнозначними змінними, збіжність.