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SOME SPECIAL SUMS WITH SQUARED HORADAM NUMBERS AND GENERALIZED TRIBONACCI NUMBERS

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Abstract Some special families of finite sums with squared Horadam numbers and generalized tribonacci numbers are discussed. As special cases of our results sums with squared Fibonacci, Lucas, Pell, Pell-Lucas, Jaconsthal, Jacobsthal-Lucas, balancing, Lucas-balancing, and tribonacci numbers are stated.

1 Introduction and motivation

Let $w_n = w_n(a, b; p, q)$ be a general Horadam sequence, i.e., a second order recurrence

$$w_n = pw_{n-1} - qw_{n-2}, \quad n \geq 2,$$

with nonzero constant p, q and initial values $w_0 = a$, $w_1 = b$. The sequence w_n can be extended to negative subscripts according to

$$w_{-n} = -\frac{1}{q}(pw_{-n+1} - w_{-n+2}), \quad n \geq 1.$$

This family of second order sequences is named after Alwyn Horadam, who studied their properties in the mid-sixties of the last century [9, 10, 11]. Ab initio many researchers became interested in the Horadam sequence. We refer the reader to the survey papers of Larcombe et al. [13] and Larcombe [12]. Both surveys contain condensed information about Horadam numbers and give an account of work subsequently conducted on this sequence. More recent results on Horadam numbers can be found in [1, 2, 3, 6, 7, 8, 14, 15].

The popularity of the Horadam sequence is partially substantiated by its obvious connections to many famous number sequences: $w_n(0, 1; 1, -1) = F_n$ is the Fibonacci sequence, $w_n(0, 1; 2, -1) = P_n$ is the Pell sequence, $w_n(0, 1; 1, -2) = J_n$ is the Jacobsthal sequence, $w_n(0, 1; 3, 2) = M_n$ is the Mersenne sequence, $w_n(0, 1; 6, 1) = B_n$ is the balancing number sequence, $w_n(2, 1; 1, -1) = L_n$ is the Lucas sequence, $w_n(2, 2; 2, -1) = Q_n$ is the Pell-Lucas sequence, $w_n(2, 1; 1, -2) = j_n$ is the Jacobsthal-Lucas sequence, and $w_n(1, 3; 6, 1) = C_n$ is Lucas-balancing number sequence.

The Binet formula of w_n in the non-degenerated case, $p^2 - 4q > 0$, is

$$w_n = \frac{(b - a\beta)\alpha^n - (b - a\alpha)\beta^n}{\alpha - \beta}, \tag{1.1}$$

with

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2}, \qquad \beta = \frac{p - \sqrt{p^2 - 4q}}{2}.$$

Next, we define the generalized tribonacci sequence $v_n = v_n(v_0, v_1, v_2)$. The sequence is a third order recurrence

$$v_n = v_{n-1} + v_{n-2} + v_{n-3}, \quad n \geq 3,$$

with arbitrary initial values v_0, v_1, and v_2 not all being zero. Members of this sequence are $v_n(0,1,1) = T_n$ the tribonacci sequence and $v_n(3,1,3) = K_n$ the tribonacci-Lucas sequence.

This paper is motivated by some recent results on connections between prominent number sequences from [3, 4, 5, 6]. Using generating functions we will be able to express some families of finite sums involving squared Horadam numbers and generalized tribonacci numbers in closed form. Several special cases will complement our search.

2 Main results, Part 1

Generating functions for powers of $(w_n)_{n\geq 0}$ have been studied by some researchers in the past. Horadam himself derived a formula for the functions in 1965 [10]. See also the papers of Mansour [16] and Mezö [17] for alternative expressions. For $k \geq 1$ and $z \in \mathbb{C}$, let

$$W_k(z; a, b; p, q) = W_k(z) = \sum_{n=0}^{\infty} w_n^k z^n.$$

In [10], Horadam derived the following result:

$$W_k(z) = \left(\frac{b - a\beta}{\alpha - \beta}\right)^k \sum_{i=0}^{k} \binom{k}{i} \left(\frac{a\alpha - b}{b - a\beta}\right)^i \left(1 - \alpha^{k-i}\beta^i z\right)^{-1}.$$

As pointed out by Larcombe and Fennessey in [15], the algebraic complexity of $W_k(z)$ increases quickly with k. Even the case $W_2(z)$ requires some effort but is, luckily, derived in detail by the previous authors in [15].

It is stated as a lemma and will be used in our proofs below.

Lemma 2.1. *The generating function for squared Horadam numbers equals*

$$W_2(z) = \frac{A + Bz + Cz^2}{1 - Dz + Ez^2 - Fz^3}, \tag{2.1}$$

with

$$A = a^2, \quad B = b^2 - a^2(p^2 - q), \quad C = q(b - ap)^2, \quad D = p^2 - q, \quad E = q(p^2 - q), \quad F = q^3.$$

Now, we can state the first result of our study.

Theorem 2.2. *For each $n \geq 0$,*

$$\sum_{i=0}^{n} \big((1-D)v_{n+2-i} + (1+E)v_{n+1-i} + (1-F)v_{n-i}\big) w_i^2$$
$$= Av_{n+3} + Bv_{n+2} + Cv_{n+1} - v_0 w_{n+3}^2 + (Dv_0 - v_1)w_{n+2}^2 + (Dv_1 - v_2 - Ev_0)w_{n+1}^2,$$

where the coefficients $A - F$ are stated in Lemma 2.1.

Proof. Recall that the generating function for $(v_n)_{n\geq 0}$ is

$$V(z) = \sum_{n=0}^{\infty} v_n z^n = \frac{v_0 + (v_1 - v_0)z + (v_2 - v_1 - v_0)z^2}{1 - z - z^2 - z^3}. \tag{2.2}$$

From (2.1) we get that

$$\frac{A + Bz + Cz^2}{W_2(z)} = 1 - Dz + Ez^2 - Fz^3.$$

Hence,

$$\frac{A + Bz + Cz^2}{W_2(z)} + (D-1)z - (E+1)z^2 + (F-1)z^3 = 1 - z - z^2 - z^3.$$

This gives

$$\frac{A + Bz + Cz^2 + \big((D-1)z - (E+1)z^2 + (F-1)z^3\big)W_2(z)}{W_2(z)}$$
$$= \frac{v_0 + (v_1 - v_0)z + (v_2 - v_1 - v_0)z^2}{V(z)}$$

or, equivalently,

$$AV(z) - v_0W_2(z) + BzV(z) - (v_1 - v_0)zW_2(z) + Cz^2V(z) - (v_2 - v_1 - v_0)z^2W_2(z)$$
$$= (1-D)zW_2(z)V(z) + (E+1)z^2W_2(z)V(z) + (1-F)z^3W_2(z)V(z).$$

Now, it is easy (but lengthy) to expand both sides of the equation in power series in z using Cauchy's rule for the multiplication of two power series

$$\sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) z^n.$$

The identity comes out when comparing the coefficients and straightforwardly manipulating the relations. We leave the details to the interested reader. □

Some special instances are given below:

Example 2.3. For $n \geq 0$,

$$\sum_{i=0}^{n} \big(T_{n+3-i} - 3T_{n-i}\big)F_i^2 = F_{n+3}F_n - T_{n+2} + T_{n+1},$$
$$\sum_{i=0}^{n} \big(T_{n+3-i} - 3T_{n-i}\big)L_i^2 = L_{n+3}L_n - 4T_{n+3} + 7T_{n+2} + T_{n+1},$$
$$2\sum_{i=0}^{n} \big(2T_{n+3-i} - 3T_{n-i}\big)P_i^2 = P_n^2 + 4P_nP_{n+1} - T_{n+2} + T_{n+1},$$
$$2\sum_{i=0}^{n} \big(17K_{n+2-i} - 18K_{n+1-i}\big)B_i^2 = 3B_{n+3}^2 - 104B_{n+2}^2 + 73B_{n+1}^2 - K_{n+2} - K_{n+1}.$$

A variant of the sums with even subscripts is stated as our next theorem.

Theorem 2.4. *For each $n \geq 0$,*

$$\sum_{i=0}^{n} \big((3-D)v_{2(n+2-i)} + (1+E)v_{2(n+1-i)} + (1-F)v_{2(n-i)}\big)w_i^2$$
$$= Av_{2n+6} + Bv_{2n+4} + Cv_{2n+2}$$
$$-v_0w_{n+3}^2 + (Dv_0 - v_2)w_{n+2}^2 + \big((D-2)v_2 - 2v_1 - (1+E)v_0\big)w_{n+1}^2,$$

with $A - F$ as stated in Lemma 2.1.

Proof. The generating function for $(v_{2n})_{n\geq 0}$ is

$$V^*(z) = \sum_{n=0}^{\infty} v_{2n}z^n = \frac{v_0 + (v_2 - 3v_0)z + (2v_1 - v_2)z^2}{1 - 3z - z^2 - z^3}.$$

Relating $V^*(z)$ to $W_2(z)$ as in the previous proof, we derive the following functional equation

$$AV^*(z) - v_0W_2(z) + BzV^*(z) - (v_2 - 3v_0)zW_2(z) + Cz^2V^*(z) - (2v_1 - v_2)z^2W_2(z)$$
$$= (3-D)zW_2(z)V^*(z) + (E+1)z^2W_2(z)V^*(z) + (1-F)z^3W_2(z)V^*(z).$$

The result follows upon expanding both sides into power series in z and comparing the coefficients. □

Example 2.5. For $n \geq 0$,

$$\sum_{i=0}^{n} \big(T_{2(n+2-i)} - T_{2(n+1-i)} + 2T_{2(n-i)}\big)F_i^2 = T_{2n+4} - T_{2n+2} - F_{n+2}^2 - 2F_{n+1}^2,$$

$$\sum_{i=0}^{n} \big(5T_{2(n+1-i)} - 9T_{2(n-i)}\big)J_i^2 = 2T_{2n+2} - T_{2n+4} + J_{n+2}^2 + J_{n+1}^2,$$

$$2\sum_{i=0}^{n} \big(K_{2(n+2-i)} + 2K_{2(n+1-i)} - K_{2(n-i)}\big)P_i^2 = K_{2n+2} - K_{2n+4} + 3P_{n+3}^2 - 12P_{n+2}^2 - 19P_{n+1}^2,$$

$$4\sum_{i=0}^{n} \big(8T_{2(n+2-i)} - 9T_{2(n+1-i)}\big)B_i^2 = -T_{2n+2} - T_{2n+4} + B_{n+2}^2 - 31B_{n+1}^2.$$

For completeness, we also give the result involving odd subscripted generalized tribonacci numbers.

Theorem 2.6. *For each $n \geq 0$, we have*

$$\sum_{i=0}^{n} \big((3-D)v_{2(n+2-i)+1} + (1+E)v_{2(n+1-i)+1} + (1-F)v_{2(n-i)+1}\big)w_i^2$$
$$= Av_{2n+7} + Bv_{2n+5} + Cv_{2n+3}$$
$$-v_1 w_{n+3}^2 + \big((D-1)v_1 - v_2 + v_0\big)w_{n+2}^2 + \big((D-4)v_2 + (D-3)v_1 + (D-2)v_0 - Ev_1\big)w_{n+1}^2,$$

with $A - F$ as stated in Lemma 2.1.

Proof. First note that by standard methods, we have the generating function for $(v_{2n+1})_{n\geq 0}$ as follows

$$V^-(z) = \sum_{n=0}^{\infty} v_{2n+1}z^n = \frac{v_1 + (v_2 - 2v_1 + v_0)z + (v_2 - v_1 - v_0)z^2}{1 - 3z - z^2 - z^3}.$$

Relating $V^-(z)$ to $W_2(z)$ as in the previous proofs, the following functional equation is derived

$$AV^-(z) - v_1 W_2(z) + BzV^-(z) - (v_2 - 2v_1 + v_0)zW_2(z)$$
$$+ Cz^2V^-(z) - (v_2 - v_1 - v_0)z^2W_2(z)$$
$$= (3-D)zW_2(z)V^-(z) + (E+1)z^2W_2(z)V^-(z) + (1-F)z^3W_2(z)V^-(z).$$

Once more, the result follows upon expanding both sides into power series in z and comparing the coefficients. □

Example 2.7. For $n \geq 0$,

$$\sum_{k=0}^{n} \big(T_{2(n+2-k)+1} - T_{2(n+1-k)+1} + 2T_{2(n-k)+1}\big)F_k^2 = T_{2n+5} - T_{2n+3} - F_{n+3}^2 - F_{n+1}^2,$$

$$\sum_{k=0}^{n} \big(T_{2(n+2-k)+1} - T_{2(n+1-k)+1} + 2T_{2(n-k)+1}\big)L_k^2 = 4T_{2n+7} - 7T_{2n+5} - T_{2n+3} - L_{n+3}^2 - L_{n+1}^2,$$

$$\sum_{k=0}^{n} \big(5K_{2(n+1-k)+1} - 9K_{2(n+2-k)+1}\big)J_k^2 = T_{2n+3} - T_{2n+5} + J_{n+3}^2 + 4J_{n+2}^2 - 6J_{n+1}^2,$$

$$4\sum_{k=0}^{n} \big(8T_{2(n+2-k)+1} - 9T_{2(n+1-k)+1}\big)B_k^2 = -T_{2n+5} - T_{2n+3} + B_{n+3}^2 - 33B_{n+2}^2 - 28B_{n+1}^2.$$

3 Main Results, Part 2

In this section, we present a pair of connections between squared odd and even subscripted Horadam numbers and generalized tribonacci numbers. In the following lemma we derive expression for sequences $(w_{2n+1}^2)_{n\geq 0}$ and $(w_{2n}^2)_{n\geq 0}$.

Lemma 3.1. *The generating functions for squared odd (even) subscripted Horadam numbers are given by*

$$\omega_1(z)=\sum_{n=0}^{\infty} w_{2n+1}^2 z^n=\frac{A_1+B_1z+C_1z^2}{1-D_0z+E_0z^2-F_0z^3}, \tag{3.1}$$

$$\omega_2(z)=\sum_{n=0}^{\infty} w_{2n}^2 z^n=\frac{A_2+B_2z+C_2z^2}{1-D_0z+E_0z^2-F_0z^3}, \tag{3.2}$$

where

$$A_1=b^2,\quad B_1=q(a^2p^2q-2abp^3+2abpq+2b^2p^2-2b^2q),\quad C_1=q^4(ap-b)^2,$$
$$A_2=a^2,\quad B_2=(aq-bp)^2-a^2(p^2-q)(p^2-3q),\quad C_2=(ap^2-bp-aq)^2q^2,$$
$$D_0=(p^2-q)(p^2-3q),\quad E_0=q^2(p^2-q)(p^2-3q),\quad F_0=q^6.$$

Proof. We provide a proof of formula (3.1). Adapting the featured proof will yield the second identity, the details of which we leave to the reader.

Using Binet's formula (1.1), we have

$$w_{2n+1}^2=\frac{(b-a\beta)^2}{p^2-4q}\alpha^{4n+2}-2\frac{(b-a\beta)(b-a\alpha)}{p^2-4q}q^{2n+1}+\frac{(b-a\alpha)^2}{p^2-4q}\beta^{4n+2}.$$

Thus,

$$\begin{aligned}&\sum_{n=0}^{\infty} w_{2n+1}^2 x^n\\
&=\frac{(b-a\beta)^2}{p^2-4q}\alpha^2\sum_{n=0}^{\infty}(\alpha^4x)^n-2q\frac{(b-a\beta)(b-a\alpha)}{p^2-4q}\sum_{n=0}^{\infty}(q^2x)^n+\frac{(b-a\alpha)^2}{p^2-4q}\beta^2\sum_{n=0}^{\infty}(\beta^4x)^n\\
&=\frac{1}{p^2-4q}\left(\frac{\alpha^2(b-a\beta)^2}{1-\alpha^4x}-\frac{2q(b-a\beta)(b-a\alpha)}{1-q^2x}+\frac{\beta^2(b-a\alpha)^2}{1-\beta^4x}\right).\end{aligned}$$

The result follows after simple algebra manipulations. □

A proof comparable to the one given for Theorem 2.6 yields the following relations between squared odd (even) subscripted Horadam numbers and generalized tribonacci numbers.

Theorem 3.2. *For $n\geq 0$, the following identities hold:*

$$\begin{aligned}&\sum_{k=0}^{n}\big((1-D_0)v_{n+2-k}+(1+E_0)v_{n+1-k}+(1-F_0)v_{n-k}\big)w_{2k+1}^2\\
&=A_1v_{n+3}+B_1v_{n+2}+C_1v_{n+1}-v_0w_{2n+7}^2-(v_1-D_0v_0)w_{2n+5}^2-(v_2-D_0v_1+E_0v_0)w_{2n+3}^2\end{aligned}$$

and

$$\begin{aligned}&\sum_{k=0}^{n}\big((1-D_0)v_{n+2-k}+(1+E_0)v_{n+1-k}-(1-F_0)v_{n-k}\big)w_{2k}^2\\
&=A_2v_{n+3}+B_2v_{n+2}+C_2v_{n+1}-v_0w_{2n+6}^2-(v_1-D_0v_0)w_{2n+4}^2-(v_2-D_0v_1+E_0v_0)w_{2n+2}^2,\end{aligned}$$

where all constants are defined as in Lemma 3.1.

Proof. The formulas are essentially the consequence of the relations

$$\begin{aligned}&\big(v_0 + (v_1 - v_0)z + (v_2 - v_1 - v_0)z^2\big)\omega_1(z)\\&= (A_1 + B_1 z + C_1 z^2)V(z) + \big(D_0 - 1 - (E_0 + 1)z + (F_0 - 1)z^2\big)z\omega_1(z)V(z)\end{aligned}$$

and

$$\begin{aligned}&\big(v_0 + (v_1 - v_0)z + (v_2 - v_1 - v_0)z^2\big)\omega_2(z)\\&= (A_2 + B_2 z + C_2 z^2)V(z) + (D_0 - 1 - (E_0 + 1)z + (F_0 - 1)z^2)z\omega_2(z)V(z)\end{aligned}$$

which we derived from (2.2), (3.1) and (2.2), (3.2), respectively. □

Example 3.3. For $n \geq 0$,

$$\sum_{i=0}^{n} \big(7T_{n+2-i} - 9T_{n+1-i}\big)F_{2i+1}^2 = 5T_{n+2} - T_{n+4} + F_{2n+5}^2 - 7F_{2n+3}^2,$$

$$\sum_{i=0}^{n} \big(7K_{n+2-i} - 9K_{n+1-i}\big)L_{2i+1}^2 = -K_{n+4} - 7K_{n+2} + 3L_{2n+7}^2 - 23L_{2n+5}^2 + 19L_{2n+3}^2,$$

$$2\sum_{i=0}^{n} \big(17T_{n+2-i} - 18T_{n+1-i}\big)P_{2i}^2 = -4T_{n+2} - 4T_{n+1} + P_{2n+4}^2 - 34P_{2n+2}^2,$$

$$\sum_{i=0}^{n} \big(20K_{n+2-i} - 85K_{n+1-i} + 63K_{n-i}\big)M_{2i}^2 = -9K_{n+2} - 36K_{n+1} - 62M_{2n+4}^2 + 234M_{2n+2}^2.$$

4 Another extended identity

In this section, we outline how generalizations of the identities from the previous sections can be derived. As a showcase, we offer a generalization of Theorem 2.2 that is able to produce many interesting sum identities, including alternating sums.

Theorem 4.1. *Let the coefficients $A - F$ be defined as in Lemma 2.1 and $x \in \mathbb{C}$. Then, we have for each $n \geq 0$*

$$\begin{aligned}&\sum_{k=0}^{n} x^k\big((1 - Dx)v_{n+2-k} + (1 + Ex^2)v_{n+1-k} + (1 - Fx^3)v_{n-k}\big)w_k^2\\&= Av_{n+3} + Bxv_{n+2} + Cx^2v_{n+1}\\&-v_0x^{n+3}w_{n+3}^2 + (Dxv_0 - v_1)x^{n+2}w_{n+2}^2 + (Dxv_1 - Ex^2v_0 - v_2)x^{n+1}w_{n+1}^2.\end{aligned}$$

Proof. Let $W_2^*(z)$ be defined as

$$W_2^*(z) = \sum_{n=0}^{\infty} x^n w_n^2 z^n.$$

Then, obviously,

$$W_2^*(z) = \frac{A + Bxz + Cx^2z^2}{1 - Dxz + Ex^2z^2 - Fx^3z^3}. \tag{4.1}$$

Proceeding as before, we arrive at the functional equation

$$\begin{aligned}&AV(z) - v_0W_2^*(z) + BxzV(z) - (v_1 - v_0)zW_2^*(z) + Cx^2z^2V(z) - (v_2 - v_1 - v_0)z^2W_2^*(z)\\&= (1 - Dx)zW_2^*(z)V(z) + (1 + Ex^2)z^2W_2^*(z)V(z) + (1 - Fx^3)z^3W_2^*(z)V(z),\end{aligned}$$

with $V(z)$ given in (2.2). The result follows from the equation. Details are omitted. □

Focusing on Fibonacci and Lucas numbers we get the next corollary.

Corollary 4.2. *For each $x \in \mathbb{C}$ and $n \geq 0$, we have*

$$\sum_{k=0}^{n} x^k \big((1-2x)T_{n+2-k} + (1-2x^2)T_{n+1-k} + (1+x^3)T_{n-k}\big)F_k^2$$
$$= xT_{n+2} - x^2T_{n+1} - x^{n+2}F_{n+2}^2 + (2x-1)x^{n+1}F_{n+1}^2$$

and

$$\sum_{k=0}^{n} x^k \big((1-2x)T_{n+2-k} + (1-2x^2)T_{n+1-k} + (1+x^3)T_{n-k}\big)L_k^2$$
$$= 4T_{n+3} - 7xT_{n+2} - x^2T_{n+1} - x^{n+2}L_{n+2}^2 + (2x-1)x^{n+1}L_{n+1}^2.$$

We conclude with the following evaluations:

Example 4.3.

$$\sum_{k=1}^{n}(-1)^k\big(3T_{n+2-k} - T_{n+1-k}\big)F_k^2 = (-1)^{n+1}F_{n+2}^2 + 3(-1)^nF_{n+1}^2 - T_{n+2} - T_{n+1},$$

$$\sum_{k=0}^{n}(-1)^k\big(3T_{n+2-k} - T_{n+1-k}\big)L_k^2 = (-1)^{n+1}L_{n+2}^2 + 3(-1)^nL_{n+1}^2 + 4T_{n+3} + 7T_{n+2} - T_{n+1},$$

$$\sum_{k=1}^{n}\big(4T_{n+1-k} + 9T_{n-k}\big)\frac{F_k^2}{2^k} = 4T_{n+2} - 2T_{n+1} - \frac{F_{n+2}^2}{2^{n-1}},$$

$$\sum_{k=0}^{n}\big(4T_{n+1-k} + 9T_{n-k}\big)\frac{L_k^2}{2^k} = 32T_{n+3} - 28T_{n+2} - 2T_{n+1} - \frac{L_{n+2}^2}{2^{n-1}}.$$

More special sums of this kind can be derived from Theorem 4.1.

5 Summary and concluding remarks

In this paper, closed forms for special families of non-alternating and alternating finite series involving squared Horadam numbers and tribonacci numbers were stated. The method of proof is to derive functional relations between ordinary generating functions, from which the results follow fairly straightforwardly. The main results have been illustrated by a variety of examples, which are interesting on their own. For instance, let $S_m(n)$, $m = 1, 2, 3, 4$, denote the four sums on the left in Example 2.3. These sums form new integer sequences and their first entries are

$$(S_1(n))_{n\geq 0} = \{0, 2, 3, 13, 33, \ldots\},$$
$$(S_2(n))_{n\geq 0} = \{8, 6, 35, 71, 205, \ldots\},$$
$$(S_3(n))_{n\geq 0} = \{0, 4, 21, 262, 1.530, \ldots\},$$
$$(S_4(n))_{n\geq 0} = \{0, 66, 2.506, 85.652, 2.910.616, \ldots\}.$$

In our future work we are going to investigate more relations for mixed sequences. Specifically, we study finite sums of non-convolutional type of products of Horadam numbers and tribonacci numbers. We also work on identities involving more than two different sequences.

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