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Additional Fibonacci–Bernoulli relations [1]

Abstract. We continue our study on relationships between Fibonacci (Lucas) numbers and Bernoulli numbers and polynomials. The derivations of our results are based on functional equations for the respective generating functions, which in our case are combinations of hyperbolic functions. Special cases and some corollaries will highlight interesting aspects of our findings.
Key words: Bernoulli numbers and polynomials, Fibonacci sequence, Lucas sequence, recurrence, generating function

Анотація. У цій статті ми продовжуємо наші дослідження взаємозв'язків між числами Фібоначчі і Люка та числами (многочленами) Бернуллі. Доведення результатів базується на функціональних рівняннях для відповідних генератрис, які в нашому випадку є комбінаціями гіперболічних функцій.
Ключові слова: Числа і многочлени Бернуллі, послідовність Фібоначчі, послідовність Люка, рекурентне співвідношення, генератриса

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1. Introduction

Fibonacci and Lucas numbers satisfy the linear second-order recurrence $u_n = u_{n-1} + u_{n-2}$ $(n \geq 2)$ with initial conditions $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$, respectively. Both sequences have a long history and are very popular among mathematicians as they appear in important mathematical branches such as number theory, combinatorics and graph theory. They have entries A000045 and A000032 in the On-Line Encyclopedia of Integer Sequences [13]. Excellent references on these sequences are the books [12, 14].

[1]Statements and conclusions made in this article by R. Frontczak are entirely those of the author. They do not necessarily reflect the views of LBBW.

As usual, Bernoulli polynomials are defined by the exponential generating function [11, Chapter 1, Section 1.3]

$$H(x,z) := \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!} = \frac{ze^{xz}}{e^z-1}, \qquad |z|<2\pi.$$

For $n \geq 0$, they satisfy the following relations [11, Chapter 1, Section 1.3]:

$$B_n(1+x) - B_n(x) = nx^{n-1}, \tag{1}$$

$$B_n(1-x) = (-1)^n B_n(x), \tag{2}$$

$$B_n(-x) = (-1)^n\big(B_n(x) + nx^{n-1}\big), \tag{3}$$

$$B_n(mx) = m^{n-1}\sum_{k=0}^{m-1} B_n\Big(x+\frac{k}{m}\Big), \qquad m \geq 1. \tag{4}$$

Also, Bernoulli polynomials have the property

$$B_n(x+z) = \sum_{k=0}^{n}\binom{n}{k}B_{n-k}(x)z^k,$$

from which we get

$$B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_{n-k}x^k, \tag{5}$$

where $B_n = B_n(0)$ are the Bernoulli numbers. The Bernoulli numbers are rational numbers starting with

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30}\,,$$

and $B_{2n+1} = 0$ for $n \geq 1$.

The following identities connect Fibonacci numbers to Bernoulli polynomials and are proved in [7]: for each integers $n \geq 0$, $j \geq 1$, and complex x,

$$\sum_{k=0}^{n}\binom{n}{k}F_{jk}\big(\sqrt{5}F_j\big)^{n-k}B_{n-k}(x) = nF_j\big((\sqrt{5}x+\beta)F_j + F_{j-1}\big)^{n-1},$$

$$\sum_{k=0}^{n}\binom{n}{k}F_{jk}\big(-\sqrt{5}F_j\big)^{n-k}B_{n-k}(x) = nF_j\big((\alpha-\sqrt{5}x)F_j + F_{j-1}\big)^{n-1},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ is the golden ratio and $\beta = \frac{1-\sqrt{5}}{2} = -\frac{1}{\alpha}$. Other results in this direction are contained in [3, 4, 5, 9, 10, 15, 16, 17, 18], among others.

In this paper, we state new relations involving Fibonacci and Lucas numbers and Bernoulli numbers and polynomials. We will work with many exponential generating functions. The results stated are complements of the recent discoveries from [5, 7, 8]. Some of our results were announced in [2].

2. Identities from hidden threefold convolutions

We begin with a known lemma [12, Vol. 1, p. 251].

Lemma 1. *Let n and j be positive integers. Then*

$$\sum_{k=0}^{n}\binom{n}{k}F_{jk}F_{j(n-k)} = \frac{1}{5}\left(2^n L_{jn} - 2L_j^n\right), \tag{6}$$

$$\sum_{k=0}^{n}\binom{n}{k}L_{jk}L_{j(n-k)} = 2^n L_{jn} + 2L_j^n, \tag{7}$$

$$\sum_{k=0}^{n}\binom{n}{k}F_{jk}L_{j(n-k)} = 2^n F_{jn}. \tag{8}$$

Our first main result is the following theorem.

Theorem 1. *Let n and j be positive integers. Then*

$$\sum_{\substack{k=0\\ n-k\equiv 0 \pmod 2}}^{n}\binom{n}{k}\left(2^k L_{jk} - 2L_j^k\right)(\sqrt{5}F_j)^{n-k}\frac{B_{n-k+2}}{n-k+2} = \frac{2^{n+2}L_{j(n+2)} - 2L_j^{n+2}}{5(n+1)(n+2)F_j^2} - L_j^n, \tag{9}$$

$$\sum_{\substack{k=0\\ n-k\equiv 0 \pmod 2}}^{n}\binom{n}{k}\left(2^k L_{jk} + 2L_j^k\right)(\sqrt{5}F_j)^{n-k}\frac{2^{n-k+2}-1}{n-k+2}B_{n-k+2} = L_j^n, \tag{10}$$

$$\sum_{\substack{k=0\\ n-k\equiv 0 \pmod 2}}^{n}\binom{n}{k}2^k F_{jk}(\sqrt{5}F_j)^{n-k}\frac{B_{n-k+2}}{n-k+2} + \frac{2}{\sqrt{5}}\sum_{\substack{k=0\\ n-k\equiv 1 \pmod 2}}^{n-1}\binom{n}{k}L_j^k(\sqrt{5}F_j)^{n-k}\frac{B_{n-k+1}}{n-k+1} = \frac{2^{n+3}F_{j(n+2)}}{5(n+1)(n+2)F_j^2} - \frac{2L_j^{n+1}}{5(n+1)F_j}. \tag{11}$$

Proof. Let $F(z)$ and $L(z)$ denote the exponential generating functions of sequences $(F_{jn})_{n\geq 0}$ and $(L_{jn})_{n\geq 0}$, respectively, with $j \geq 1$. Then, it is easy to derive

$$F^2(z) = \frac{4}{5}e^{L_j z}\sinh^2\left(\frac{\sqrt{5}F_j}{2}z\right), \tag{12}$$

$$L^2(z) = 4e^{L_j z}\cosh^2\left(\frac{\sqrt{5}F_j}{2}z\right). \tag{13}$$

From the power series for the cotangent [11, Chapter 1.3]

$$\coth z = \sum_{n=0}^{\infty} 2^{2n} B_{2n} \frac{z^{2n-1}}{(2n)!}$$

we get

$$-\frac{d}{dz}\coth z = \frac{1}{\sinh^2 z} = \frac{1}{z^2} - \sum_{n=1}^{\infty}(2n-1)2^{2n} B_{2n}\frac{z^{2n-2}}{(2n)!}. \tag{14}$$

Hence, we see that the functional equation (12), using (6) and (14), can be written equivalently as $e^{L_j z} = S_1(z) - S_2(z)$ with

$$S_1(z) = \frac{1}{5F_j^2}\sum_{n=0}^{\infty}\left(2^n L_{jn} - 2L_j^n\right)\frac{z^{n-2}}{n!},$$

$$S_2(z) = \sum_{n=0}^{\infty}\left(2^n L_{jn} - 2L_j^n\right)\frac{z^n}{n!}\cdot\sum_{n=1}^{\infty}(2n-1)5^{n-1}F_j^{2n-2}B_{2n}\frac{z^{2n-2}}{(2n)!}.$$

In the series $S_1(z)$ the first two terms are zero and therefore

$$S_1(z) = \frac{1}{5F_j^2}\sum_{n=0}^{\infty}\frac{2^{n+2}L_{j(n+2)} - 2L_j^{n+2}}{(n+2)(n+1)}\frac{z^n}{n!}.$$

The second series $S_2(z)$ equals

$$S_2(z) = \sum_{n=0}^{\infty}\left(2^n L_{jn} - 2L_j^n\right)\frac{z^n}{n!}\cdot\sum_{n=0}^{\infty}(2n+1)5^n F_j^{2n}B_{2n+2}\frac{z^{2n}}{(2n+2)!}$$

and is a simple Cauchy product. Expanding and comparing the coefficients of z^n proves (9).

Identity (10) follows from the functional equation (10) combined with (7) and

$$\frac{1}{\cosh^2 z} = \frac{d}{dz}\tanh z = \sum_{n=1}^{\infty}(2n-1)2^{2n}(2^{2n}-1)B_{2n}\frac{z^{2n-2}}{(2n)!}.$$

The underlying functional equation for identity (11) is

$$\frac{F(z)L(z)}{\sinh^2\left(\frac{\sqrt{5}F_j}{2}z\right)} = \frac{4}{\sqrt{5}}e^{L_j z}\coth\left(\frac{\sqrt{5}F_j}{2}z\right). \tag{15}$$

By (8), (12) and (13), the LHS of (15) is

$$\begin{aligned}\frac{F(z)L(z)}{\sinh^2\left(\frac{\sqrt{5}F_j}{2}z\right)} &= \frac{4}{5F_j^2}\sum_{n=0}^{\infty}2^n F_{jn}\frac{z^{n-2}}{n!} - 4\sum_{n=0}^{\infty}2^n F_{jn}\frac{z^n}{n!}\cdot\sum_{n=0}^{\infty}\frac{B_{2n+2}5^n F_j^{2n}}{2n+2}\frac{z^{2n}}{(2n)!}\\ &= \frac{8}{5F_j z} + 4\left(\frac{1}{5F_j^2}\sum_{n=0}^{\infty}2^{n+2}\frac{F_{j(n+2)}}{(n+2)(n+1)}\frac{z^n}{n!}\right.\\ &\qquad \left. - \sum_{n=0}^{\infty}2^n F_{jn}\frac{z^n}{n!}\cdot\sum_{n=0}^{\infty}\frac{B_{2n+2}}{2n+2}5^n F_j^{2n}\frac{z^{2n}}{(2n)!}\right),\end{aligned}$$

whereas the RHS of (15) equals

$$\begin{aligned}
&\frac{4}{\sqrt{5}}e^{L_j z}\coth\Big(\frac{\sqrt{5}F_j}{2}z\Big)\\
&\qquad=\frac{4}{\sqrt{5}}\left(\frac{2}{\sqrt{5}F_j}\sum_{n=0}^{\infty}L_j^n\frac{z^{n-1}}{n!}+\sum_{n=0}^{\infty}L_j^n\frac{z^n}{n!}\cdot 2\sum_{n=1}^{\infty}B_{2n}5^{\frac{2n-1}{2}}F_j^{2n-1}\frac{z^{2n-1}}{(2n)!}\right)\\
&\qquad=\frac{8}{5F_j z}+\frac{8}{\sqrt{5}}\left(\frac{1}{\sqrt{5}F_j}\sum_{n=0}^{\infty}\frac{L_j^{n+1}}{n+1}\frac{z^n}{n!}\right.\\
&\qquad\quad\left.+z\sum_{n=0}^{\infty}L_j^n\frac{z^n}{n!}\cdot\sum_{n=0}^{\infty}\frac{B_{2n+2}}{(2n+2)(2n+1)}5^{\frac{2n+1}{2}}F_j^{2n+1}\frac{z^{2n}}{(2n)!}\right).
\end{aligned}$$

Now, we can apply the Cauchy multiplication theorem on both sides. When simplifying the RHS, use $\binom{n-1}{k}\frac{n}{n-k}=\binom{n}{k}$. The proof is complete.

Formula (9) has been derived recently in [10].

When $j=1$, then the special cases of Theorem 1 reduce to

$$\sum_{\substack{k=0\\ n-k\equiv 0\ (\mathrm{mod}\ 2)}}^{n}\binom{n}{k}\big(2^k L_k-2\big)(\sqrt{5})^{n-k}\frac{B_{n-k+2}}{n-k+2}=\frac{2^{n+2}L_{n+2}-2}{5(n+1)(n+2)}-1\,, \qquad (16)$$

$$\sum_{\substack{k=0\\ n-k\equiv 0\ (\mathrm{mod}\ 2)}}^{n}\binom{n}{k}\big(2^k L_k+2\big)(\sqrt{5})^{n-k}\frac{2^{n-k+2}-1}{n-k+2}B_{n-k+2}=1$$

and

$$\begin{aligned}
&\sum_{\substack{k=0\\ n-k\equiv 0\ (\mathrm{mod}\ 2)}}^{n}\binom{n}{k}2^k F_k(\sqrt{5})^{n-k}\frac{B_{n-k+2}}{n-k+2}\\
&\quad+\frac{2}{\sqrt{5}}\sum_{\substack{k=0\\ n-k\equiv 1\ (\mathrm{mod}\ 2)}}^{n-1}\binom{n}{k}(\sqrt{5})^{n-k}\frac{B_{n-k+1}}{n-k+1}=\frac{2}{5(n+1)}\Big(\frac{2^{n+1}F_{n+2}}{n+2}-1\Big).
\end{aligned}$$

The identity (16) appeared as a problem proposal in [6].

Remark 1. Using the fact that, for any sequence $(a_n)_{n\geq 0}$,

$$\sum_{\substack{k=0\\ n-k\equiv 0\ (\mathrm{mod}\ 2)}}^{n}a_k=\sum_{k=0}^{n}\frac{1+(-1)^{n-k}}{2}a_k\,,$$

$$\sum_{\substack{k=0\\ n-k\equiv 1\ (\mathrm{mod}\ 2)}}^{n}a_k=\sum_{k=0}^{n}\frac{1-(-1)^{n-k}}{2}a_k\,,$$

identities (9)–(11) possess the following equivalent forms without the mod-notation (with n even):

$$\sum_{k=0}^{n/2}\binom{n}{2k}\frac{n-2k-1}{(k+1)(2k+1)}\cdot\frac{L_j^{2k+2}-2^{2k+1}L_{2j(k+1)}}{(5F_j^2)^{k+1}}\,B_{n-2k}=\Big(\frac{L_j}{\sqrt{5}F_j}\Big)^n,$$

$$\sum_{k=0}^{n/2}\binom{n}{2k}\frac{2^{n-2k+2}-1}{n-2k+2}\cdot\frac{2L_j^{2k}+2^{2k}L_{2jk}}{(5F_j^2)^k}\,B_{n-2k+2}=\Big(\frac{L_j}{\sqrt{5}F_j}\Big)^n,$$

$$\sum_{k=0}^{n/2}\binom{n}{2k}\Big(\frac{4}{5F_j^2}\Big)^k\Big(\frac{n-2k-1}{2k+1}F_{j(2k+1)}+\frac{F_jL_j^{2k}}{4^k}\Big)B_{n-2k}=0\,.$$

Theorem 2. *For all positive integers n and j,*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{2^k}{k+1}\left((-1)^k\frac{F_{j(k+1)}}{L_j^k}\Big(\frac{L_j}{\sqrt{5}F_j}\Big)^n-(1+(-1)^n)\frac{2^{k+3}-2}{k+2}F_jB_{k+2}\right)=0\,. \tag{17}$$

In particular,

$$\sum_{k=1}^{2n}(-1)^k\binom{2n-1}{k-1}\frac{2^kF_{jk}}{kL_j^k}=0\,.$$

Similarly,

$$\sum_{k=0}^{n}\binom{n}{k}2^k\left((-1)^k\frac{L_{jk}}{L_j^k}\Big(\frac{L_j}{\sqrt{5}F_j}\Big)^n-\frac{1+(-1)^n}{n-k+1}B_k\right)=1+(-1)^n\,, \tag{18}$$

$$\sum_{k=0}^{2n-1}(-1)^k\binom{2n-1}{k}\frac{2^kL_{jk}}{L_j^k}=0\,.$$

Proof. From the basic identity $\sinh z=\tanh z\cdot\cosh z$ we get

$$\frac{\sqrt{5}}{2}F(z)e^{-\frac{L_j}{2}z}=\tanh\Big(\frac{\sqrt{5}F_j}{2}z\Big)\cosh\Big(\frac{\sqrt{5}F_j}{2}z\Big). \tag{19}$$

Then

$$\begin{aligned}\text{LHS of (19)}&=\frac{\sqrt{5}}{2}\sum_{n=1}^{\infty}F_{jn}\frac{z^n}{n!}\cdot\sum_{n=0}^{\infty}\Big(-\frac{L_j}{2}\Big)^n\frac{z^n}{n!}\\&=\frac{\sqrt{5}}{2}\sum_{n=0}^{\infty}F_{j(n+1)}\frac{z^{n+1}}{(n+1)!}\cdot\sum_{n=0}^{\infty}\Big(-\frac{L_j}{2}\Big)^n\frac{z^n}{n!}\\&=\frac{\sqrt{5}}{2}z\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{F_{j(k+1)}}{k+1}\Big(-\frac{L_j}{2}\Big)^{n-k}\frac{z^n}{n!}\,,\end{aligned}$$

whereas

$$\begin{aligned}
\text{RHS of (19)} &= \sum_{n=1}^{\infty} 2^{2n}(2^{2n}-1)\Big(\frac{\sqrt{5}F_j}{2}\Big)^{2n-1} B_{2n}\frac{z^{2n-1}}{(2n)!}\cdot\sum_{n=0}^{\infty}\Big(\frac{\sqrt{5}F_j}{2}\Big)^{2n}\frac{z^{2n}}{(2n)!}\\
&= \sum_{n=0}^{\infty} 2^{2n+2}(2^{2n+2}-1)\Big(\frac{\sqrt{5}F_j}{2}\Big)^{2n+1} B_{2n+2}\frac{z^{2n+1}}{(2n+2)!}\cdot\sum_{n=0}^{\infty}\Big(\frac{\sqrt{5}F_j}{2}\Big)^{2n}\frac{z^{2n}}{(2n)!}\\
&= z\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1+(-1)^n}{2}\cdot\frac{1+(-1)^k}{2}\,2^{k+2}(2^{k+2}-1)\\
&\quad\times\Big(\frac{\sqrt{5}F_j}{2}\Big)^{k+1}\frac{B_{k+2}}{(k+2)!}\Big(\frac{\sqrt{5}F_j}{2}\Big)^{n-k}\frac{z^n}{(n-k)!}\\
&= z\Big(\frac{\sqrt{5}F_j}{2}\Big)^{n+1}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1+(-1)^n}{2}\cdot\frac{1+(-1)^k}{2}\\
&\quad\times\binom{n}{k}2^{k+2}(2^{k+2}-1)\frac{B_{k+2}}{(k+2)(k+1)}\frac{z^n}{n!}\\
&= z\Big(\frac{\sqrt{5}F_j}{2}\Big)^{n+1}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1+(-1)^n}{2}\binom{n}{k}2^{k+2}(2^{k+2}-1)\frac{B_{k+2}}{(k+2)(k+1)}\frac{z^n}{n!}.
\end{aligned}$$

Note that above we used

$$\sum_{n=0}^{\infty} a_{2n}z^{2n}\cdot\sum_{n=0}^{\infty} b_{2n}z^{2n} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1+(-1)^n}{2}\frac{1+(-1)^k}{2}a_k b_{n-k}z^n.$$

Comparing the coefficients of z^n after some simple manipulations we have (17).

Identity (18) follows from $\cosh z = \coth z\cdot\sinh z$ which gives

$$\frac{1}{2}L(z)e^{-\frac{L_j}{2}z} = \coth\Big(\frac{\sqrt{5}F_j}{2}z\Big)\sinh\Big(\frac{\sqrt{5}F_j}{2}z\Big). \tag{20}$$

Proceeding as before,

$$\begin{aligned}
\text{LHS of (20)} &= \frac{1}{2}\sum_{n=0}^{\infty}L_{jn}\frac{z^n}{n!}\cdot\sum_{n=0}^{\infty}\Big(-\frac{L_j}{2}\Big)^n\frac{z^n}{n!}\\
&= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}L_{jk}\Big(-\frac{L_j}{2}\Big)^{n-k}\frac{z^n}{n!},
\end{aligned}$$

and

$$\begin{aligned}
\text{RHS of (20)} &= \sum_{n=0}^{\infty}4^n\Big(\frac{\sqrt{5}F_j}{2}\Big)^{2n-1}B_{2n}\frac{z^{2n-1}}{(2n)!}\cdot\sum_{n=0}^{\infty}\Big(\frac{\sqrt{5}F_j}{2}\Big)^{2n+1}\frac{z^{2n+1}}{(2n+1)!}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{1+(-1)^n}{2}\frac{1+(-1)^k}{2}2^k\Big(\frac{\sqrt{5}F_j}{2}\Big)^{k-1}\\
&\quad\times\frac{B_k}{n-k+1}\Big(\frac{\sqrt{5}F_j}{2}\Big)^{n-k+1}\frac{z^n}{n!}
\end{aligned}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{1+(-1)^n}{2}2^k\left(\frac{\sqrt{5}F_j}{2}\right)^{k-1}\frac{B_k}{n-k+1}\left(\frac{\sqrt{5}F_j}{2}\right)^{n-k+1}\frac{z^n}{n!}.$$

Comparing the coefficients of z^n after some simple manipulations we have (18). The theorem is proved.

In view of the binomial theorem and the Binet formula, (18) is equivalent to

$$\big(1+(-1)^n\big)\sum_{k=0}^{n}\binom{n}{k}\frac{2^k B_k}{n-k+1}=0\,,$$

so that we have

$$\sum_{k=0}^{n}\binom{n}{k}\frac{2^k B_k}{n-k+1}=0\,,\qquad n \text{ even}.$$

Theorem 3. *For all positive integers n and j,*

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}(5F_j^2)^k\left(\frac{F_{j(n-2k+1)}}{n-2k+1}B_{2k}-\frac{F_jL_j^{n-2k}}{2^n}\right)=0\,, \tag{21}$$

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\frac{(5F_j^2)^k}{2k+1}\left(\frac{4^{k+1}-1}{k+1}L_{j(n-2k)}B_{2k+2}-\frac{L_j^{n-2k}}{2^n}\right)=0\,. \tag{22}$$

Proof. Use the exponential generating functions from (12) and (13) in conjunction with

$$\sum_{n=0}^{\infty}a_nz^n\cdot\sum_{n=0}^{\infty}b_{2n}z^{2n}=\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor}a_{n-2k}b_{2k}z^n$$

and $\sinh z=\frac{\cosh z}{\coth z}$, $\cosh z=\frac{\sinh z}{\tanh z}$. The theorem is proved.

On account of the identity

$$2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}x^{n-2k}z^{2k}=(x+z)^n+(x-z)^n\,,$$

formula (21) can also be written as

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\frac{(5F_j^2)^kF_{j(n-2k+1)}}{n-2k+1}B_{2k}=\frac{F_jL_{nj}}{2}.$$

3. Special Bernoulli polynomial identities

The properties of the Bernoulli polynomials stated in Lemmas 2 and 3 below are direct consequences of the functional relations (1) and (2).

Lemma 2. *If n is any non-negative integer, then*

$$B_n(1+x) \pm B_n(1+y) = B_n(x) \pm B_n(y) + n(x^{n-1} \pm y^{n-1}),$$
$$B_n(1+x) \pm B_n(1+y) = (-1)^n \big(B_n(1-x) \pm B_n(1-y)\big) + n(x^{n-1} \pm y^{n-1}),$$
$$B_n(-x) \pm B_n(-y) = (-1)^n \big(B_n(x) \pm B_n(y) + n(x^{n-1} \pm y^{n-1})\big). \tag{23}$$

Lemma 3. *Let n be any non-negative integer. If $x - y = 1$, then*

$$B_n(x) - B_n(y) = ny^{n-1}, \tag{24}$$

$$B_n(-x) - B_n(-y) = n(-1)^n x^{n-1};$$

while if $x + y = 1$, then

$$B_n(x) - (-1)^n B_n(y) = 0, \tag{25}$$

$$B_n(1+x) - B_n(1+y) = \begin{cases} n(x^{n-1} - y^{n-1}), & \text{if } n \text{ is even;} \\ -2B_n(y) + n(x^{n-1} - y^{n-1}), & \text{otherwise.} \end{cases}$$

Lemma 4. *For real or complex z, let a given well-behaved function $h(z)$ have in its domain the representation $h(z) = \sum_{k=c_1}^{c_2} v_k z^{w_k}$, where v_k and w_k are given real sequences and $-\infty \le c_1 < c_2 \le +\infty$. Let i and m be integers. Then*

$$\sum_{k=c_1}^{c_2} F_{iw_k+m} v_k z^{w_k} = \frac{1}{\sqrt{5}} \big(\alpha^m h(\alpha^i z) + \beta^m h(\beta^i z)\big), \tag{26}$$

$$\sum_{k=c_1}^{c_2} L_{iw_k+m} v_k z^{w_k} = \alpha^m h(\alpha^i z) - \beta^m h(\beta^i z)\,. \tag{27}$$

We can find Lemma 4, in a slightly different form, in [1, Theorem 1].

Theorem 4. *Let j and m be integers and n a non-negative integer. Then*

$$\sum_{k=0}^{n} \binom{n}{k} F_{jk+m} B_{n-k}(x) z^k = \frac{1}{\sqrt{5}} \big(\alpha^m B_n(x+\alpha^j z) - \beta^m B_n(x+\beta^j z)\big), \tag{28}$$

$$\sum_{k=0}^{n} \binom{n}{k} L_{jk+m} B_{n-k}(x) z^k = \alpha^m B_n(x+\alpha^j z) + \beta^m B_n(x+\beta^j z). \tag{29}$$

Proof. Use (26) and (27) with $h(z) = B_n(x+z) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(x) z^k$, so that $w_k = k$, $v_k = \binom{n}{k} B_{n-k}(x)$, $c_1 = 0$ and $c_2 = n$. The proof is complete.

Setting $x = 0$ in (28) and (29) yield the following Fibonacci–Bernoulli and Lucas–Bernoulli relations.

Corollary 1. *Let j and m be integers and n non-negative integer. Then*

$$\sum_{k=0}^{n}\binom{n}{k}F_{jk+m}B_{n-k}z^k=\frac{1}{\sqrt{5}}\left(\alpha^m B_n(\alpha^j z)-\beta^m B_n(\beta^j z)\right), \tag{30}$$

$$\sum_{k=0}^{n}\binom{n}{k}L_{jk+m}B_{n-k}z^k=\alpha^m B_n(\alpha^j z)+\beta^m B_n(\beta^j z). \tag{31}$$

Theorem 5. *Let j and m be integers and n non-negative integer. Then*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{F_{jk+m}}{L_j^k}B_{n-k}=\begin{cases}F_m B_n\Big(\frac{\alpha^j}{L_j}\Big), & \text{if } n \text{ is even;}\\ \frac{L_m}{\sqrt{5}}B_n\Big(\frac{\alpha^j}{L_j}\Big), & \text{if } n \text{ is odd,}\end{cases} \tag{32}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{L_{jk+m}}{L_j^k}B_{n-k}=\begin{cases}L_m B_n\Big(\frac{\alpha^j}{L_j}\Big), & \text{if } n \text{ is even;}\\ \sqrt{5}F_m B_n\Big(\frac{\alpha^j}{L_j}\Big), & \text{if } n \text{ is odd.}\end{cases}$$

Proof. Choose $x=\frac{\alpha^j}{L_j}$ in (2) and use the Binet formula $L_j=\alpha^j+\beta^j$ to obtain

$$B_n\Big(\frac{\beta^j}{L_j}\Big)=(-1)^n B_n\Big(\frac{\alpha^j}{L_j}\Big). \tag{33}$$

Now use this information in Corollary 1 with $z=\frac{1}{L_j}$.

Lemma 5. *Let a, b, c and d be rational numbers and λ an irrational number. Then $a+\lambda b=c+\lambda d$ if and only if $a=c$ and $b=d$.*

Corollary 2. *Let j be an integer and n a non-negative integer. Then*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{F_{jk-1}}{L_j^k}B_{n-k}=B_n\Big(\frac{\alpha^j}{L_j}\Big),\qquad n \text{ even}, \tag{34}$$

$$\sum_{k=1}^{n}\binom{n}{k}\frac{F_{jk}}{L_j^k}B_{n-k}=0,\qquad n \text{ even}, \tag{35}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{L_{jk-1}}{L_j^k}B_{n-k}=\sqrt{5}\,B_n\Big(\frac{\alpha^j}{L_j}\Big),\qquad n \text{ odd}, \tag{36}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{L_{jk}}{L_j^k}B_{n-k}=0,\qquad n \text{ odd}. \tag{37}$$

Proof. Since the expression on the left side of (32) is rational, being the finite sum of rational numbers, it follows that $B_n\big(\frac{\alpha^j}{L_j}\big)$ is a rational number for even

n. Now, using (5) and relation $\alpha^s = \alpha F_s + F_{s-1}$, we have

$$B_n\Big(\frac{\alpha^j}{L_j}\Big) = \sum_{k=0}^{n}\binom{n}{k}\frac{B_k\alpha^{j(n-k)}}{L_j^{n-k}}$$
$$= \alpha\sum_{k=0}^{n}\binom{n}{k}\frac{B_kF_{j(n-k)}}{L_j^{n-k}} + \sum_{k=0}^{n}\binom{n}{k}\frac{B_kF_{j(n-k)-1}}{L_j^{n-k}},$$

from which identities (34) and (35) follow when we invoke Lemma 5. The proof of (36) and (37) is similar.

Remark 2. We observe from (32) that $\sqrt{5}B_n\big(\frac{\alpha^j}{L_j}\big)$ is rational for n odd.

Theorem 6. *Let j and m be integers and n non-negative integer. Then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{F_{jk+m}}{L_j^k}B_{n-k} = \begin{cases} F_mB_n\big(\frac{\alpha^j}{L_j}\big) + \frac{nF_{j(n-1)+m}}{L_j^{n-1}}, & \text{if } n \text{ is even;}\\ -\frac{L_m}{\sqrt{5}}B_n\big(\frac{\alpha^j}{L_j}\big) - \frac{nF_{j(n-1)+m}}{L_j^{n-1}}, & \text{otherwise.}\end{cases}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{L_{jk+m}}{L_j^k}B_{n-k} = \begin{cases} L_mB_n\big(\frac{\alpha^j}{L_j}\big) + \frac{nF_{j(n-1)+m}}{L_j^{n-1}}, & \text{if } n \text{ is even;}\\ -\sqrt{5}F_mB_n\big(\frac{\alpha^j}{L_j}\big) - \frac{nL_{j(n-1)+m}}{L_j^{n-1}}, & \text{otherwise,}\end{cases}$$

Proof. Identities (3) and (33) give

$$B_n\Big(-\frac{\alpha^j}{L_j}\Big) + B_n\Big(-\frac{\beta^j}{L_j}\Big) = \begin{cases} 2B_n\Big(\frac{\alpha^j}{L_j}\Big) + \frac{nL_{j(n-1)}}{L_j^{n-1}}, & \text{if } n \text{ is even;}\\ -\frac{nL_{j(n-1)}}{L_j^{n-1}}, & \text{otherwise,}\end{cases}$$

$$B_n\Big(-\frac{\alpha^j}{L_j}\Big) - B_n\Big(-\frac{\beta^j}{L_j}\Big) = \begin{cases} \frac{\sqrt{5}nF_{j(n-1)}}{L_j^{n-1}}, & \text{if } n \text{ is even;}\\ -2B_n\Big(\frac{\alpha^j}{L_j}\Big) - \frac{\sqrt{5}nF_{j(n-1)}}{L_j^{n-1}}, & \text{otherwise.}\end{cases}$$

Use these in Corollary 1 with $z = -\frac{1}{L_j}$. Note the use of the known identities $F_rL_s + F_sL_r = 2F_{r+s}$ and $L_rL_s + 5F_sF_r = 2L_{r+s}$ (see [12, Vol. 1, p. 111]).

Theorem 7. *Let j be integer and n non-negative integer. Then*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{2^kF_{jk}}{L_j^k}B_{n-k} = \frac{n}{\sqrt{5}}\Big(\frac{\sqrt{5}F_j}{L_j}\Big)^{n-1}, \qquad n \text{ even,} \tag{38}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{2^kL_{jk}}{L_j^k}B_{n-k} = n\Big(\frac{\sqrt{5}F_j}{L_j}\Big)^{n-1}, \qquad n \text{ odd.} \tag{39}$$

Proof. Setting $x = \frac{2\alpha^j}{L_j} - 1$ in (1) yields

$$B_n\Big(\frac{2\alpha^j}{L_j}\Big) - B_n\Big(\frac{F_j\sqrt{5}}{L_j}\Big) = n\Big(\frac{F_j\sqrt{5}}{L_j}\Big)^{n-1}. \tag{40}$$

Setting $x = \frac{2\beta^j}{L_j}$ in (2) gives

$$B_n\Big(\frac{2\beta^j}{L_j}\Big) - B_n\Big(\frac{F_j\sqrt{5}}{L_j}\Big) = 0, \qquad n \text{ even}. \tag{41}$$

From (40) and (41) we find

$$B_n\Big(\frac{2\alpha^j}{L_j}\Big) - B_n\Big(\frac{2\beta^j}{L_j}\Big) = n\Big(\frac{F_j\sqrt{5}}{L_j}\Big)^{n-1}, \qquad n \text{ even},$$

from which, upon use in (30), with $m = 0$ and $z = \frac{2}{L_j}$, identity (38) follows.

Using $x = \frac{2\beta^j}{L_j}$ in (2) gives

$$B_n\Big(\frac{2\beta^j}{L_j}\Big) + B_n\Big(\frac{F_j\sqrt{5}}{L_j}\Big) = 0, \qquad n \text{ odd}. \tag{42}$$

Addition of (40) and (42) produces

$$B_n\Big(\frac{2\alpha^j}{L_j}\Big) + B_n\Big(\frac{2\beta^j}{L_j}\Big) = n\Big(\frac{F_j\sqrt{5}}{L_j}\Big)^{n-1}, \qquad n \text{ odd},$$

from which, upon use in (31), with $m = 0$ and $z = \frac{2}{L_j}$, identity (39) follows.

The result of the next theorem exhibits strong similarity to the polynomial identities from Introduction.

Theorem 8. *The following identity is valid for all $n \geq 0$, $j \geq 1$, and complex x:*

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}2^k F_{jk}(\pm\sqrt{5}F_j)^{n-k}B_{n-k}(x)\\ = nF_j\Big((\pm\sqrt{5}F_j x + L_j)^{n-1} + (\pm\sqrt{5}F_j(x-1) + L_j)^{n-1}\Big).\end{aligned} \tag{43}$$

Proof. Since

$$H(x, \sqrt{5}F_j z) = \frac{\sqrt{5}F_j e^{\frac{\sqrt{5}F_j}{2}(2x-1)z}}{2\sinh\big(\frac{\sqrt{5}F_j}{2}z\big)}z,$$

we get the relation

$$F(z)L(z)H(x, \sqrt{5}F_j z) = 2F_j z e^{L_j z} e^{\frac{\sqrt{5}F_j}{2}(2x-1)z}\cosh\Big(\frac{\sqrt{5}F_j}{2}z\Big).$$

Hence,

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}2^k F_{jk}(\sqrt{5}F_j)^{n-k}B_{n-k}(x)\frac{z^n}{n!}\\ &= F_j z e^{\left(\frac{\sqrt{5}F_j}{2}(2x-1)+L_j\right)z}\Big(e^{\frac{\sqrt{5}F_j}{2}z} + e^{-\frac{\sqrt{5}F_j}{2}z}\Big)\\ &= F_j z\Big(e^{(\sqrt{5}F_j x + L_j)z} + e^{(\sqrt{5}F_j(x-1)+L_j)z}\Big).\end{aligned}$$

This proves (43) with the positive root. The second follows upon replacing x by $1-x$ and using (2).

Setting $x=0$ in Theorem 2, we have the following.

Corollary 3. *For $n \geq 0$ and $j \geq 1$,*

$$\sum_{k=0}^{n} \binom{n}{k} 2^k F_{jk} (\pm\sqrt{5}F_j)^{n-k} B_{n-k} = nF_j \Big(L_j^{n-1} + (\mp\sqrt{5}F_j + L_j)^{n-1} \Big).$$

Corollary 4. *For $n \geq 0$ and $j \geq 1$,*

$$\sum_{k=0}^{n} \binom{n}{k} 2^k F_{jk} (\sqrt{5}F_j)^{n-k} B_{n-k}(\alpha) = nF_j 2^{1-n} \Big((\sqrt{5}F_j + L_{j+3})^{n-1} + (-\sqrt{5}F_j + L_{j+3})^{n-1} \Big),$$

where α is the golden ratio. Also, for $j \geq 3$, we have the analog identity

$$\sum_{k=0}^{n} \binom{n}{k} 2^k F_{jk} (-\sqrt{5}F_j)^{n-k} B_{n-k}(\alpha) = nF_j 2^{1-n} \Big((\sqrt{5}F_j - L_{j-3})^{n-1} + (-\sqrt{5}F_j - L_{j-3})^{n-1} \Big). \tag{44}$$

Proof. Set $x=\alpha$ in Theorem 2 and simplify using $5F_n = L_{n+1} + L_{n-1}$.

We mention the special case of (44) for $j=3$:

$$\sum_{k=0}^{n} \binom{n}{k} (-\sqrt{5})^{n-k} F_{3k} B_{n-k}(\alpha) = (-1)^{n-1} n L_{n-1}.$$

Also, inserting β in (43) and setting $j=1$ we can state the identity

$$\sum_{k=0}^{n} \binom{n}{k} 2^k F_k (\sqrt{5})^{n-k} B_{n-k}(\beta) = (-1)^{n-1} n L_{2n-2}.$$

Corollary 5. *Let n, j and q be integers with $n, j \geq 1$ and $q \geq 2$. Then*

$$\sum_{k=0}^{n} \binom{n}{k} 2^k F_{jk} (\pm\sqrt{5}F_j)^{n-k} \big(q^{1-(n-k)} - 1 \big) B_{n-k} = nF_j q^{1-n} \sum_{r=1}^{q-1} \Big((\pm\sqrt{5}F_j r + qL_j)^{n-1} + (\pm\sqrt{5}F_j (r-q) + qL_j)^{n-1} \Big).$$

Proof. Formula (4) gives

$$\big(q^{1-n} - 1 \big) B_n = \sum_{r=1}^{q-1} B_n \Big(\frac{r}{q} \Big).$$

Therefore, we can write

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}&2^k F_{jk}(\pm\sqrt{5}F_j)^{n-k}\big(q^{1-(n-k)}-1\big)B_{n-k}\\&= nF_j\sum_{r=1}^{q-1}\Big(\big(\pm\sqrt{5}F_j\,\frac{r}{q}+L_j\big)^{n-1}+\big(\pm\sqrt{5}F_j\,\big(\frac{r}{q}-1\big)+L_j\big)^{n-1}\Big)\\&= nF_jq^{1-n}\sum_{r=1}^{q-1}\Big((\pm\sqrt{5}F_jr+qL_j)^{n-1}+(\pm\sqrt{5}F_j(r-q)+qL_j)^{n-1}\Big).\end{aligned}$$

The proof is complete.

We proceed with some examples. The special case $q=2$ takes the form

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}&2^k F_{jk}(\sqrt{5}F_j)^{n-k}\big(2^{1-(n-k)}-1\big)B_{n-k}\\&= nF_j2^{1-n}\Big((L_j+2\alpha^j)^{n-1}+(L_j+2\beta^j)^{n-1}\Big)\\&= nF_j2^{1-n}\sum_{m=0}^{n-1}\binom{n-1}{m}2^mL_{jm}L_j^{n-1-m}.\end{aligned}$$

For $j=1$ the left-hand side can be expressed in closed-form and we obtain after some manipulations

$$\sum_{k=0}^{n}\binom{n}{k}\Big(\frac{\sqrt{5}}{4}\Big)^k\big(2-2^k\big)F_{n-k}B_k=\frac{nL_{3(n-1)}}{2^{2n-1}}.$$

Similarly the case $q=3$ is treated. The calculations are lengthy and omitted. The result is

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}&6^k\big(1-3^{n-k-1}\big)F_{jk}(\sqrt{5}F_j)^{n-k}B_{n-k}\\&= nF_j\sum_{m=0}^{n-1}x\binom{n-1}{m}\big(2^{n-1}+4^m\big)L_j^{n-1-m}L_{jm}.\end{aligned}$$

For $j=1$ we get

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}&6^k(\sqrt{5})^{n-k}\big(1-3^{n-k-1}\big)F_kB_{n-k}\\&= n2^{n-1}L_{2n-2}+\sum_{m=1}^{n}\binom{n}{m}m4^{m-1}L_{m-1}.\end{aligned}$$

4. Conclusion

In this paper, we have discovered new identities relating Bernoulli polynomials (numbers) to Fibonacci and Lucas numbers. In our future papers, we will discuss the analogous results for Euler polynomials (numbers) and Fibonacci and Lucas numbers as well as identities connecting Bernoulli polynomials (numbers) with Jacobsthal, Pell and balancing numbers.

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