ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2023, **15** (1), 137–149
doi:10.15330/cmp.15.1.137-149

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2023, Т.15, №1, С.137–149

On some integral inequalities for (h, m)-convex functions in a generalized framework

Kórus P.[1], **Nápoles Valdés J.E.**[2,3]

In this paper, we present some new integral inequalities of Hermite-Hadamard type. To obtain these results, general convex functions of various type are considered such as (h, m)-convex functions. The main results extend some previously known inequalities by taking fractional integral operators.

Key words and phrases: integral inequality, Hermite-Hadamard inequality, (h, m)-convex function, modified (h, m)-convex function, generalized integral operator.

[1] University of Szeged, 10 Hattyas str., 6725, Szeged, Hungary
[2] National University of the Northeast, 5450 Libertad ave., 3400, Corrientes, Argentina
[3] UTN - Facultad Regional, 414 French str., 3500, Resistencia, Argentina
E-mail: korus.peter@szte.hu (Kórus P.), jnapoles@exa.unne.edu.ar (Nápoles Valdés J.E.)

1 Preliminaries

One of the most fruitful notions in current mathematics is that of the convex function.

Definition 1. *A function $\psi : I \to \mathbb{R}$, $I := [\xi_1, \xi_2]$, is said to be convex if*

$$\psi\big(\lambda x + (1-\lambda)y\big) \leq \lambda\psi(x) + (1-\lambda)\psi(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

If the above inequality is reversed, then function ψ is called concave on I. These notions has spread in various directions (the interested reader can consult [30], where a fairly complete overview of the generalizations and extensions of the convex function concept is presented).

For convex functions, the following inequality is known, undoubtedly one of the most famous in mathematics, for its multiple connections and applications

$$\psi\left(\frac{\xi_1 + \xi_2}{2}\right) \leq \frac{1}{\xi_2 - \xi_1}\int_{\xi_1}^{\xi_2} \psi(x)\,dx \leq \frac{\psi(\xi_1) + \psi(\xi_2)}{2}, \tag{1}$$

this is called the Hermite-Hadamard inequality. The interested reader is referred to [1, 6–8, 11–13, 17, 18, 21, 23, 25, 31, 37] and the references therein for more information and other extensions of the Hermite-Hadamard inequality.

G. Toader in [38] defined m-convexity in the following way.

Definition 2. *The function $\psi : [0, \xi_2] \to \mathbb{R}$, $\xi_2 > 0$, is said to be m-convex, where $m \in [0, 1]$, if*

$$\psi\big(tx + m(1-t)y\big) \leq t\psi(x) + m(1-t)\psi(y)$$

holds for all $x, y \in [0, \xi_2]$ and $t \in [0, 1]$.

УДК 517.98
2020 *Mathematics Subject Classification:* 26D10, 26A33, 26A51.

If the above inequality holds in reverse, then we say that the function ψ is m-concave.

The following definitions are successive extensions of the concept of convex function and, as we will see later, they are particular cases of our definition.

Definition 3 ([8,19]). *Let $s \in (0,1]$ be a real number. Function $\psi : [0,\xi_2] \to [0,\infty)$ with $\xi_2 > 0$ is said to be s-convex in the first sense if*

$$\psi\big(tx + (1-t)y\big) \leq t^s\psi(x) + (1-t^s)\psi(y)$$

for all $x, y \in [0,\xi_2]$ and $t \in (0,1)$.

Definition 4 ([8,19]). *Let $s \in (0,1]$ be a real number. Function $\psi : [0,\xi_2] \to [0,\infty)$ with $\xi_2 > 0$ is said to be s-convex in the second sense if*

$$\psi\big(tx + (1-t)y\big) \leq t^s\psi(x) + (1-t)^s\psi(y)$$

for all $x, y \in [0,\xi_2]$ and $t \in (0,1)$.

In [42], the above definition was extended for $s \in [-1,1]$, these functions are called extended s-convex. In [24] and [35], the following classes of generalized convex functions were presented.

Definition 5. *Function $\psi : [0,\xi_2] \to [0,\infty)$ with $\xi_2 > 0$ is said to be (α,m)-convex, where $\alpha, m \in (0,1]$, if for every $x, y \in [0,\xi_2]$ and $t \in [0,1]$ the following inequality holds*

$$\psi\big(tx + m(1-t)y\big) \leq t^\alpha\psi(x) + m(1-t^\alpha)\psi(y).$$

Definition 6. *Function $\psi : [0,\xi_2] \to [0,\infty)$ with $\xi_2 > 0$ is called (s,m)-convex in the second sense, where $s, m \in (0,1]$, if for every $x, y \in [0,\xi_2]$ and $t \in [0,1]$ the following inequality holds*

$$\psi\big(tx + m(1-t)y\big) \leq t^s\psi(x) + m(1-t)^s\psi(y).$$

In [27], the authors presented two classes of s-(α,m)-convex functions ("redefined" in [41]), but the one used there is equivalent to the class of (α,m)-convex functions, therefore we do not go in details. In [23], the following definition is introduced.

Definition 7. *Let $h : [0,1] \to [0,\infty)$ be a function, $h \not\equiv 0$. Function $\psi : [0,\xi_2] \to [0,\infty)$ with $\xi_2 > 0$ is said to be (h,m)-convex on $[0,\xi_2]$ if inequality*

$$\psi\big(tx + m(1-t)y\big) \leq h(t)\psi(x) + mh(1-t)\psi(y)$$

is fulfilled for $m \in [0,1]$, all $x, y \in [0,\xi_2]$ and $t \in [0,1]$.

Remark 1. *In the above definition, if we put*

1) *$h(t) = t^s$, $s \in (0,1]$, then ψ is (s,m)-convex in the second sense on $[0,\xi_2)$,*

2) *$h(t) = t^s$, $s \in [-1,1]$, $m = 1$, then ψ is extended s-convex on $[0,\xi_2)$,*

3) *$h(t) = t$, then ψ is m-convex on $[0,\xi_2)$,*

4) *$h(t) = t$ and $m = 1$, then ψ is convex on $[0,\xi_2)$.*

On the basis of these definitions, we present one more class of functions that will be used in our work.

Definition 8. *Let $h : [0,1] \to [0,\infty)$ be a function, $h \not\equiv 0$. Function $\psi : [0,\xi_2] \to [0,\infty)$ with $\xi_2 > 0$ is said to be modified (h,m)-convex on $[0,\xi_2]$ if*

$$\psi\big(tx + m(1-t)y\big) \le h(t)\psi(x) + m\big(1-h(t)\big)\psi(y)$$

holds for $m \in [0,1]$, all $x, y \in [0,\xi_2]$ and $t \in [0,1]$.

Remark 2. *In the above definition, if we put*

1) $h(t) = t^\alpha$ with $\alpha \in (0,1]$, then ψ is (α,m)-convex on $[0,\xi_2)$,

2) $h(t) = t^s$, $s \in (0,1]$, $m = 1$, then ψ is s-convex in the first sense on $[0,\xi_2)$,

3) $h(t) = t$, then ψ is m-convex on $[0,\xi_2)$,

4) $h(t) = t$ and $m = 1$, then ψ is convex on $[0,\xi_2)$.

In the last decades, we have witnessed the development of new ways of generalization, some via integrals, which include both fractional and generalized. These integrals are originated from local derivatives and the integral operators that may or may not be fractional. To date, the study of this area has attracted the attention of many researchers, not only in pure mathematics, but in multiple fields of applied science. Between its own theoretical development and the multiplicity of applications, the field has grown rapidly in the recent years, in such a way that a single definition of "fractional derivative or integral" does not exist, or at least is not unanimously accepted. In [5], D. Baleanu and A. Fernandez suggest and justifies the idea of a fairly complete classification of the known operators of the fractional calculus, on the other hand, in the work [4], some reasons are presented why new operators linked to applications and developments theorists appear every day. These operators had been developed by numerous mathematicians with a barely specific formulation, for instance, the Riemann-Liouville (RL), the Weyl, Erdelyi-Kober, Hadamard integrals, and the Liouville and Katugampola fractional operators, while many authors have introduced new fractional operators generated from general classical local derivatives.

In addition, [2, Chapter 1] presents a history of differential operators, both local and global, from Newton to Caputo and presents a definition of local derivative with new parameter, providing a large number of applications, with a difference qualitative between both types of operators, local and global. Most importantly, [2, Section 1.4] concludes: "We can therefore conclude that both the Riemann-Liouville and Caputo operators are not derivatives, and then they are not fractional derivatives, but fractional operators. We agree with the result [39] that, the local fractional operator is not a fractional derivative" (see [2, p. 24]). As we said before, they are new tools that have demonstrated their usefulness and potential in the modeling of different processes and phenomena (see also [3]).

In fractional and generalized calculus, the functions Γ (see [34,36,43,44]) and Γ_k (see [9])

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt, \qquad \Gamma_k(z) = \int_0^\infty t^{z-1}e^{-t^k/k}\,dt, \quad \mathrm{Re}(z) > 0,\ k > 0,$$

are used.

Unmistakably, if $k \to 1$ we have $\Gamma_k(z) \to \Gamma(z)$, moreover $\Gamma_k(z) = k^{z/k-1}\Gamma(z/k)$ and $\Gamma_k(z+k) = z\Gamma_k(z)$.

One of the first operators that can be called fractional is that of Riemann-Liouville fractional derivatives of order $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, defined as follows (see [15]).

Definition 9. *Let $f \in L^1[\xi_1,\xi_2]$, $\xi_1,\xi_2 \in \mathbb{R}$, $\xi_1 < \xi_2$. The right and left side Riemann-Liouville fractional integrals of order α are defined by*

$$
\begin{aligned}
{}^{RL}J^{\alpha}_{\xi_1^+}f(t) &= \frac{1}{\Gamma(\alpha)}\int_{\xi_1}^{t}(t-s)^{\alpha-1}f(s)\,ds, \quad t > \xi_1,\\
{}^{RL}J^{\alpha}_{\xi_2^-}f(t) &= \frac{1}{\Gamma(\alpha)}\int_{t}^{\xi_2}(s-t)^{\alpha-1}f(s)\,ds, \quad t < \xi_2.
\end{aligned}
$$

Their corresponding differential operators are given by

$$
\begin{aligned}
D^{\alpha}_{\xi_1^+}f(t) &= \frac{d}{dt}\left({}^{RL}J^{1-\alpha}_{\xi_1^+}f(t)\right) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{\xi_1}^{t}\frac{f(s)}{(t-s)^{\alpha}}\,ds,\\
D^{\alpha}_{\xi_2^-}f(t) &= -\frac{d}{dt}\left({}^{RL}J^{1-\alpha}_{\xi_2^-}f(t)\right) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{t}^{\xi_2}\frac{f(s)}{(s-t)^{\alpha}}\,ds.
\end{aligned}
$$

In [29], a generalized fractional derivative was defined in the following way (see also [14] and [45]).

Definition 10. *Given a function $f : [0,\infty) \to \mathbb{R}$. Then the N-derivative of f of order α is defined by*

$$
N^{\alpha}_{F}f(t) = \lim_{\varepsilon\to 0}\frac{f\big(t+\varepsilon F(t,\alpha)\big)-f(t)}{\varepsilon}
$$

for all $t > 0$, $\alpha \in (0,1)$, being $F(t,\alpha)$ is some function.

If f is N-differentiable of order α on some interval $(0,t_0)$ and $\lim_{t\to 0^+} N^{\alpha}_{F}f(t)$ exists, then define $N^{\alpha}_{F}f(0) = \lim_{t\to 0^+} N^{\alpha}_{F}f(t)$. Note that if f is differentiable, then $N^{\alpha}_{F}f(t) = F(t,\alpha)f'(t)$, where $f'(t)$ is the ordinary derivative.

It is easy to verify that this operator contains, as particular cases, most of the known local differential operators, both conformable and non-conformable.

Now, we give the definition of a general integral. Throughout the work we will consider that the integral operator kernel F defined below is an absolutely continuous function (the interested reader is referred to [22,40]).

Definition 11. *Let I be an interval $I \subseteq \mathbb{R}$, $\xi_1,\xi_2,t \in I$ and $\alpha \in \mathbb{R}$. The integral operators, right and left, is defined for every locally integrable function f on I as*

$$
\begin{aligned}
J^{\alpha}_{F,\xi_1+}(f)(t) &= \int_{\xi_1}^{t}\frac{f(s)}{F((t-s)/(\xi_2-\xi_1),\alpha)}\,ds, \quad t > \xi_1,\\
J^{\alpha}_{F,\xi_2-}(f)(t) &= \int_{t}^{\xi_2}\frac{f(s)}{F((s-t)/(\xi_2-\xi_1),\alpha)}\,ds, \quad \xi_2 > t.
\end{aligned}
$$

Definition 12. *Let I be an interval $I \subseteq \mathbb{R}$, $\xi_1,\xi_2,t \in I$, $\alpha \in \mathbb{R}$ and $m \in [0,1]$. The m-integral operators, right and left, is defined for every locally integrable function f on I as*

$$
\begin{aligned}
J^{\alpha,m}_{F,\xi_1+}(f)(t) &= \int_{\xi_1}^{t}\frac{f(s)}{F((t-s)/(m\xi_2-\xi_1),\alpha)}\,ds, \quad t > \xi_1,\\
J^{\alpha,m}_{F,\xi_2-}(f)(t) &= \int_{t}^{\xi_2}\frac{f(s)}{F((s-t)/(\xi_2-m\xi_1),\alpha)}\,ds, \quad \xi_2 > t.
\end{aligned}
$$

Definition 13. *We will also use the "central" integral operator defined as follows (see [14, 45])*

$$J_{F,\xi_1}^{\alpha}(f)(\xi_2) = \int_{\xi_1}^{\xi_2} \frac{f(t)}{F(t,\alpha)}\,dt, \quad \xi_2 > \xi_1.$$

To cite just two particular cases of the central integral operator: if we consider $F \equiv 1$ we obtain the classical Riemann integral and if we put $F(t,\alpha) = t^{1-\alpha}$ we obtain the Riemann-Liouville integral of Definition 9. Note that the latter is a fractional integral operator, which can be generated from the conformable differential operator of [20].

Definition 14. *Let $p > 0$ and $\alpha \in (0,1]$. We define the space $L_{\alpha}^{p}[\xi_1, \xi_2]$ as the set of functions over $[\xi_1, \xi_2]$ such that $J_{F,\xi_1}^{\alpha}(|f(t)|^p)(\xi_2) < \infty$. Similarly, $L_{\alpha,+}^{p}[\xi_1, \xi_2]$ denotes the functions for wich $J_{F,\xi_1+}^{\alpha}(|f(t)|^p)(\xi_2) < \infty$, $L_{\alpha,-}^{p}[\xi_1, \xi_2]$ the ones satisfy $J_{F,\xi_2-}^{\alpha}(|f(t)|^p)(\xi_1) < \infty$, $L_{\alpha,m,+}^{p}[\xi_1, \xi_2]$ the ones satisfy $J_{F,\xi_1+}^{\alpha,m}(|f(t)|^p)(\xi_2) < \infty$, and $L_{\alpha,m,-}^{p}[\xi_1, \xi_2]$ the ones satisfy $J_{F,\xi_2-}^{\alpha,m}(|f(t)|^p)(\xi_1) < \infty$.*

The following statement is analogous to the one known from the ordinary calculus [14, 16, 45].

Theorem 1. *Let f be an N-differentiable function on (t_0, ∞) for some $t_0 \in \mathbb{R}$ with $\alpha \in (0,1]$. For all $t > t_0$, if $N_F^{\alpha} f \in L_{\alpha}^{1}[t_0, t]$, we have the following.*

(a) If f is differentiable on (t_0, t), then $J_{F,t_0}^{\alpha}(N_F^{\alpha} f(t)) = f(t) - f(t_0)$.

(b) If f is continuous on $[t_0, t]$, then $N_F^{\alpha}(J_{F,t_0}^{\alpha}(f)(t)) = f(t)$.

An important and necessary property in our work is the following established result.

Theorem 2 (Integration by parts)**.** *Let f and g be differentiable functions on (t_0, ∞) with $\alpha \in (0,1]$. Then for all $t > t_0$, if $N_F^{\alpha} f$, $N_F^{\alpha} g \in L_{\alpha}^{1}[t_0, t]$, we have*

$$J_{F,t_0}^{\alpha}((f N_F^{\alpha} g)(t)) = [fg(t) - fg(t_0)] - J_{F,t_0}^{\alpha}((g N_F^{\alpha} f)(t)).$$

In this article, using the notion of modified (h, m)-convex function, we establish new integral inequalities via the generalized integral operators of Definitions 11, 12, and 13.

2 Results

We present our first result.

Lemma 1. *Let $I \subset \mathbb{R}$ be an open interval and $\psi : I \to \mathbb{R}$ be a differentiable function on I, furthermore $\xi_1, \xi_2 \in I$, $m \in [0,1]$ and $\xi_1 < m\xi_2$. If $N_F^{\alpha}\psi \in L_{\alpha,m,+}^{1}[\xi_1, m\xi_2]$ for $\alpha \in (0,1]$, then*

$$\begin{aligned}\frac{\psi(\xi_1) + \psi(m\xi_2)}{2} - \frac{1}{m\xi_2 - \xi_1} J_{F,\xi_1+}^{\alpha,m}(\psi)(m\xi_2) \\ = \frac{m\xi_2 - \xi_1}{2} J_{F,0}^{\alpha}[(1-2t) N_F^{\alpha}\psi(t\xi_1 + m(1-t)\xi_2)](1).\end{aligned} \tag{2}$$

Proof. It is enough to apply Theorem 2 to the integral of the right member of (2), to obtain the left member. □

Remark 3. *One can obtain a similar result in case of taking $\psi(t\xi_2 + m(1-t)\xi_1)$ instead of $\psi(t\xi_1 + m(1-t)\xi_2)$. On the other hand, it is clear that we can work with the ψ function in $t\xi_1 + (1-t)\xi_2$ and $t\xi_2 + (1-t)\xi_1$, getting a result already known for several integral operators (see e.g. [23]).*

Remark 4. *Putting $F \equiv 1$ and $m = 1$ we have [10, Lemma 2.1].*

Hereafter, we will provide various extensions of the Hermite-Hadamard inequality (1), using the above lemma.

Theorem 3. *Under the assumptions of Lemma 1, if $|N_F^\alpha \psi|$ is (h,m)-convex, then we have*

$$\begin{aligned}\Big|\frac{\psi(\xi_1)+\psi(m\xi_2)}{2} - &\frac{1}{m\xi_2-\xi_1} J_{F,\xi_1+}^{\alpha,m}(\psi)(m\xi_2)\Big| \\ &\leq \frac{m\xi_2-\xi_1}{2} J_{F,0}^{\alpha}[|N_F^\alpha\psi(\xi_1)||1-2t|h(t) + m|N_F^\alpha\psi(\xi_2)||1-2t|h(1-t)](1).\end{aligned}$$

Proof. Using Lemma 1 and (h,m)-convexity of $|N_F^\alpha\psi|$ we have

$$\begin{aligned}\Big|\frac{\psi(\xi_1)+\psi(m\xi_2)}{2} - &\frac{1}{m\xi_2-\xi_1} J_{F,\xi_1+}^{\alpha,m}(\psi)(m\xi_2)\Big| \\ &\leq \frac{m\xi_2-\xi_1}{2} J_{F,0}^{\alpha}[|1-2t||N_F^\alpha\psi(t\xi_1+m(1-t)\xi_2)|](1) \\ &\leq \frac{m\xi_2-\xi_1}{2} J_{F,0}^{\alpha}[|N_F^\alpha\psi(\xi_1)||1-2t|h(t) + m|N_F^\alpha\psi(\xi_2)||1-2t|h(1-t)](1),\end{aligned}$$

which is the required inequality. $\square$

Remark 5. *In case of considering $F \equiv 1$, one can obtain [23, Theorem 2.6].*

An analogous theorem could be proved in a similar way.

Theorem 4. *Under the assumptions of Lemma 1, if $|N_F^\alpha\psi|$ is modified (h,m)-convex, then we have*

$$\begin{aligned}\Big|\frac{\psi(\xi_1)+\psi(m\xi_2)}{2} - &\frac{1}{m\xi_2-\xi_1} J_{F,\xi_1+}^{\alpha,m}(\psi)(m\xi_2)\Big| \\ &\leq \frac{m\xi_2-\xi_1}{2} J_{F,0}^{\alpha}[|N_F^\alpha\psi(\xi_1)||1-2t|h(t) + m|N_F^\alpha\psi(\xi_2)||1-2t|(1-h(t))](1).\end{aligned}$$

Remark 6. *In case of $F \equiv 1$ and $h(t) = t^{\alpha s}$, from this result one can obtain [27, Theorem 2.3], if in addition, $\alpha = m = 1$, then this result implies [26, Theorem 2].*

Theorem 5. *Under the assumptions of Lemma 1, if $|N_F^\alpha\psi|^q$ is (h,m)-convex with $q = p/(p-1)$ and $p > 1$, then we have*

$$\begin{aligned}\Big|\frac{\psi(\xi_1)+\psi(m\xi_2)}{2} - &\frac{1}{m\xi_2-\xi_1} J_{F,\xi_1+}^{\alpha,m}(\psi)(m\xi_2)\Big| \\ &\leq \frac{m\xi_2-\xi_1}{2} (J_{F,0}^{\alpha}|1-2t|^p(1))^{1/p} (J_{F,0}^{\alpha}[|N_F^\alpha\psi(\xi_1)|^q h(t) + m|N_F^\alpha\psi(\xi_2)|^q h(1-t)](1))^{1/q}.\end{aligned}$$

Proof. Using Lemma 1, the (h,m)-convexity of $|N_F^{\alpha}\psi|^q$ and the well-known Hölder integral inequality, we get

$$\begin{aligned}
&\Big|\frac{\psi(\xi_1)+\psi(m\xi_2)}{2}-\frac{1}{m\xi_2-\xi_1}J^{\alpha,m}_{F,\xi_1+}(\psi)(m\xi_2)\Big| \\
&\quad\le \frac{m\xi_2-\xi_1}{2}J^{\alpha}_{F,0}[|1-2t||N_F^{\alpha}\psi(t\xi_1+m(1-t)\xi_2)|](1) \\
&\quad\le \frac{m\xi_2-\xi_1}{2}(J^{\alpha}_{F,0}|1-2t|^p(1))^{1/p}(J^{\alpha}_{F,0}|N_F^{\alpha}\psi(t\xi_1+m(1-t)\xi_2)|^q(1))^{1/q} \\
&\quad\le \frac{m\xi_2-\xi_1}{2}(J^{\alpha}_{F,0}|1-2t|^p(1))^{1/p}(J^{\alpha}_{F,0}[|N_F^{\alpha}\psi(\xi_1)|^q h(t)+m|N_F^{\alpha}\psi(\xi_2)|^q h(1-t)](1))^{1/q},
\end{aligned}$$

that is the desired inequality. □

Remark 7. *If we consider $F\equiv 1$ and $h(t)=t^a$ from this result we obtain [27, Theorem 2.5], if you also have $a=m=1$, then this result reduces to [26, Theorem 4].*

The following theorem can be proved analogously.

Theorem 6. *Under the assumptions of Lemma 1, if $|N_F^{\alpha}\psi|^q$ is modified (h,m)-convex with $q=p/(p-1)$ and $p>1$, then we have*

$$\begin{aligned}
&\Big|\frac{\psi(\xi_1)+\psi(m\xi_2)}{2}-\frac{1}{m\xi_2-\xi_1}J^{\alpha,m}_{F,\xi_1+}(\psi)(m\xi_2)\Big| \\
&\quad\le \frac{m\xi_2-\xi_1}{2}(J^{\alpha}_{F,0}|1-2t|^p(1))^{1/p}(J^{\alpha}_{F,0}[|N_F^{\alpha}\psi(\xi_1)|^q h(t)+m\big|N_F^{\alpha}\psi(\xi_2)\big|^q(1-h(t))](1))^{1/q}.
\end{aligned}$$

Theorem 7. *Under the assumptions of Lemma 1, if $|N_F^{\alpha}\psi|^q$ is (h,m)-convex with $q>1$ and $q=p/(p-1)$, then*

$$\begin{aligned}
&\Big|\frac{\psi(\xi_1)+\psi(m\xi_2)}{2}-\frac{1}{m\xi_2-\xi_1}J^{\alpha,m}_{F,\xi_1+}(\psi)(m\xi_2)\Big| \\
&\le \frac{m\xi_2-\xi_1}{2}(J^{\alpha}_{F,0}|1-2t|(1))^{1/p}(J^{\alpha}_{F,0}[|N_F^{\alpha}\psi(\xi_1)|^q|1-2t|h(t)+m|N_F^{\alpha}\psi(\xi_2)|^q|1-2t|h(1-t)](1))^{1/q}.
\end{aligned}$$

Proof. Using Lemma 1, the (h,m)-convexity of $|N_F^{\alpha}\psi|^q$ and taking into account the well-known power mean inequality, we have

$$\begin{aligned}
&\Big|\frac{\psi(\xi_1)+\psi(m\xi_2)}{2}-\frac{1}{m\xi_2-\xi_1}J^{\alpha,m}_{F,\xi_1+}(\psi)(m\xi_2)\Big| \\
&\quad\le \frac{m\xi_2-\xi_1}{2}J^{\alpha}_{F,0}[|1-2t||N_F^{\alpha}\psi(t\xi_1+m(1-t)\xi_2)|](1) \\
&\quad\le \frac{m\xi_2-\xi_1}{2}(J^{\alpha}_{F,0}|1-2t|(1))^{1/p}(J^{\alpha}_{F,0}[|1-2t||N_F^{\alpha}\psi(t\xi_1+m(1-t)\xi_2)|^q](1))^{1/q} \\
&\quad\le \frac{m\xi_2-\xi_1}{2}(J^{\alpha}_{F,0}|1-2t|(1))^{1/p} \\
&\qquad\times(J^{\alpha}_{F,0}[|N_F^{\alpha}\psi(\xi_1)|^q|1-2t|h(t)+m|N_F^{\alpha}\psi(\xi_2)|^q|1-2t|h(1-t)](1))^{1/q},
\end{aligned}$$

that completes the proof. □

Remark 8. *If as before, we consider $F\equiv 1$ and $h(t)=t^{as}$ from this result we obtain [27, Theorem 2.7], if additionally $a=m=1$, then this result reduces to [26, Theorem 6]. If we consider the kernel $F\equiv 1$ and work with convex functions, that is, h is the identity function and $m=1$, this result becomes [33, Theorem 1].*

The following theorem can be proved analogously.

Theorem 8. *Under the assumptions of Lemma 1, if $|N_F^\alpha \psi|^q$ is modified (h,m)-convex with $q>1$ and $q=p/(p-1)$, then*

$$\Big|\frac{\psi(\xi_1)+\psi(m\xi_2)}{2} - \frac{1}{m\xi_2-\xi_1} J^{\alpha,m}_{F,\xi_1+}(\psi)(m\xi_2)\Big|$$
$$\le \frac{m\xi_2-\xi_1}{2}(J^\alpha_{F,0}|1-2t|(1))^{1/p}(J^\alpha_{F,0}[|N_F^\alpha\psi(\xi_1)|^q|1-2t|h(t)+m|N_F^\alpha\psi(\xi_2)|^q|1-2t|(1-h(t))](1))^{1/q}.$$

Theorem 9. *Let $\psi:[0,\infty)\to\mathbb{R}$ be an (h,m)-convex function with $m\in(0,1]$. If $0\le\xi_1<\xi_2<\infty$, $\psi(t)\in L^1_{\alpha,+}[\xi_1,\xi_2]\cap L^1_{\alpha,-}[\xi_1/m,\xi_2/m]$ and $h(t),h(1-t)\in L^1_\alpha[0,1]$, $h(1/2)\neq 0$, then we have the following inequality*

$$\begin{aligned}\frac{\mathbb{F}}{h(1/2)}\psi\Big(\frac{\xi_1+\xi_2}{2}\Big) &\le \frac{1}{\xi_2-\xi_1}\Big(J^\alpha_{F,\xi_1+}(\psi)(\xi_2)+m^2 J^\alpha_{F,\frac{\xi_2}{m}-}(\psi)\Big(\frac{\xi_1}{m}\Big)\Big)\\ &\le \Big(\psi(\xi_1)+m^2\psi\Big(\frac{\xi_2}{m^2}\Big)\Big)J^\alpha_{F,0}(h(t))(1)m\Big(\psi\Big(\frac{\xi_1}{m}\Big)+\psi\Big(\frac{\xi_2}{m}\Big)\Big)J^\alpha_{F,0}(h(1-t))(1)\end{aligned} \tag{3}$$

with

$$\mathbb{F}=\int_0^1\frac{dt}{F(t,\alpha)}.$$

Proof. For $x,y\in[0,\infty)$ and $t=1/2$, we have

$$\psi\Big(\frac{x+y}{2}\Big)\le h\Big(\frac12\Big)\psi(x)+mh\Big(\frac12\Big)\psi\Big(\frac{y}{m}\Big).$$

If we choose $x=t\xi_1+(1-t)\xi_2$ and $y=t\xi_2+(1-t)\xi_1$ with $t\in[0,1]$, we get

$$\psi\Big(\frac{\xi_1+\xi_2}{2}\Big)\le h\Big(\frac12\Big)\Big(\psi(t\xi_1+(1-t)\xi_2)+m\psi\Big(t\frac{\xi_2}{m}+(1-t)\frac{\xi_1}{m}\Big)\Big). \tag{4}$$

After dividing the members of the previous inequality by $F(t,\alpha)$ and integrating with respect to t on $[0,1]$, and changing variables brings us to the first inequality of (3).

Using the (h,m)-convexity of ψ again, we obtain

$$\begin{aligned}&h\Big(\frac12\Big)\Big(\psi(t\xi_1+(1-t)\xi_2)+m\psi\Big(t\frac{\xi_2}{m}+(1-t)\frac{\xi_1}{m}\Big)\Big)\\ &\le h\Big(\frac12\Big)\Big(h(t)\psi(\xi_1)+mh(1-t)\psi\Big(\frac{\xi_2}{m}\Big)+mh(1-t)\psi\Big(\frac{\xi_1}{m}\Big)+m^2h(t)\psi\Big(\frac{\xi_2}{m^2}\Big)\Big).\end{aligned}$$

After division by $F(t,\alpha)$ and integration with respect to t, between 0 and 1, we obtain the second inequality of (3). □

Remark 9. *If in the previous Theorem we consider the Riemann integral, or what is the same, $F\equiv 1$ and ψ is a convex function $\big(h(t)=t$ and $m=1\big)$, then from* (4) *we obtain the classic Hermite-Hadamard inequality* (1). *This result also extends and corrects [32, Theorem 9].*

As we will see, the following result complements the previous one.

Theorem 10. *Let $\psi : [0,\infty) \to \mathbb{R}$ be an (h,m)-convex function with $m \in (0,1]$. If $0 \leq \xi_1 < m\xi_2 < \infty$, $\psi(t) \in L^1_{\alpha,m,+}[\xi_1, m\xi_2]$ and $h(t), h(1-t) \in L^1_\alpha[0,1]$, then we have the following inequality*

$$\frac{1}{m\xi_2 - \xi_1}[J^{\alpha,m}_{F,\xi_1+}(\psi)(m\xi_2)] \leq \psi(\xi_1) J^\alpha_{F,0}(h(t))(1) + m\psi(\xi_2) J^\alpha_{F,0}(h(1-t))(1). \tag{5}$$

Proof. Using the (h,m)-convexity of ψ, we have

$$\psi(t\xi_1 + m(1-t)\xi_2) \leq h(t)\psi(\xi_1) + mh(1-t)\psi(\xi_2).$$

After dividing the members of the previous inequality by $F(t,\alpha)$ and integrating with respect to t on $[0,1]$, we obtain

$$\int_0^1 \frac{\psi(t\xi_1 + m(1-t)\xi_2)}{F(t,\alpha)}\,dt \leq \psi(\xi_1)\int_0^1 \frac{h(t)}{F(t,\alpha)}\,dt + m\psi(\xi_2)\int_0^1 \frac{h(1-t)}{F(t,\alpha)}\,dt.$$

The required result is easily obtained by changing variables in the first integral. $\square$

Remark 10. *It is easy to verify that if $F \equiv 1$ and ψ is a convex function, then from* (5) *we get the right member of the classic Hermite-Hadamard inequality* (1). *Also, Theorem 10 extends [32, Theorem 10], and taking $F \equiv 1$, we obtain [23, Theorem 2.1], while [23, Remark 2.1] is also still valid. If, on the contrary, we consider the kernel $F(t,\alpha) = t^{1-\alpha}/\Gamma(\alpha)$, the following inequality not reported in the literature, is valid for Riemann-Liouville fractional integrals*

$$\frac{1}{(m\xi_2 - \xi_1)^{1-\alpha}}[^{RL}J^\alpha_{\xi_1+}\psi(m\xi_2)] \leq \psi(\xi_1)[^{RL}J^\alpha_{0+}(h(t))(1)] + m\psi(\xi_2)[^{RL}J^\alpha_{0+}(h(1-t))(1)].$$

Naturally, if we consider different kernels, we will get new variants of (5).

A more general result than the previous one, in which two (h,m)-convex functions are involved, is as follows.

Theorem 11. *Let ψ_1 be an (h_1,m)-convex function and ψ_2 an (h_2,m)-convex function with $m \in (0,1]$. If $0 \leq \xi_1 < m\xi_2 < \infty$, $\psi_1(t)\psi_2(t) \in L^1_{\alpha,m,+}[\xi_1, m\xi_2] \cap L^1_{\alpha,m,-}[m\xi_1, \xi_2]$, and*

$$h_1(t)h_2(t), h_1(1-t)h_2(1-t), h_1(t)h_2(1-t), h_1(1-t)h_2(t) \in L^1_\alpha[0,1],$$

then we have the following inequality

$$\begin{aligned}
&\frac{1}{m\xi_2 - \xi_1} J^{\alpha,m}_{F,\xi_1+}(\psi_1\psi_2)(m\xi_2) + \frac{1}{\xi_2 - m\xi_1} J^{\alpha,m}_{F,\xi_2-}(\psi_1\psi_2)(m\xi_1)\\
&\qquad \leq (\psi_1(\xi_1)\psi_2(\xi_1) + \psi_1(\xi_2)\psi_2(\xi_2))(J^\alpha_{F,0}[h_1(t)h_2(t)](1) + mJ^\alpha_{F,0}[h_1(1-t)h_2(1-t)](1))\\
&\qquad\quad + m(\psi_1(\xi_1)\psi_2(\xi_2) + \psi_1(\xi_2)\psi_2(\xi_1))J^\alpha_{F,0}[h_1(t)h_2(1-t) + h_1(1-t)h_2(t)](1).
\end{aligned}$$

Proof. Since ψ_1 and ψ_2 are (h_1,m)-convex and (h_2,m)-convex, respectively, we have

$$\begin{aligned}
\psi_1(t\xi_1 &+ m(1-t)\xi_2)\psi_2(t\xi_1 + m(1-t)\xi_2)\\
&\leq (h_1(t)\psi_1(\xi_1) + mh_1(1-t)\psi_1(\xi_2))(h_2(t)\psi_2(\xi_1) + mh_2(1-t)\psi_2(\xi_2))
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
\psi_1(t\xi_2 &+ m(1-t)\xi_1)\psi_2(t\xi_2 + m(1-t)\xi_1)\\
&\leq (h_1(t)\psi_1(\xi_2) + mh_1(1-t)\psi_1(\xi_1))(h_2(t)\psi_2(\xi_2) + mh_2(1-t)\psi_2(\xi_1)).
\end{aligned} \tag{7}$$

After multiplying and ordering from (6) and (7) we get

$$\begin{aligned}\psi_1(t\xi_1+&m(1-t)\xi_2)\psi_2(t\xi_1+m(1-t)\xi_2)+\psi_1(t\xi_2+m(1-t)\xi_1)\psi_2(t\xi_2+m(1-t)\xi_1)\\ &\le h_1(t)h_2(t)(\psi_1(\xi_1)\psi_2(\xi_1)+\psi_1(\xi_2)\psi_2(\xi_2))\\ &\quad+mh_1(t)h_2(1-t)\psi_1(\xi_1)\psi_2(\xi_2)+mh_1(1-t)h_2(t)\psi_1(\xi_2)\psi_2(\xi_1)\\ &\quad+mh_1(t)h_2(1-t)\psi_1(\xi_2)\psi_2(\xi_1)+mh_1(1-t)h_2(t)\psi_1(\xi_1)\psi_2(\xi_2)\\ &\quad+m^2h_1(1-t)h_2(1-t)\big(\psi_1(\xi_1)\psi_2(\xi_1)+\psi_1(\xi_2)\psi_2(\xi_2)\big).\end{aligned}$$

The sought inequality is obtained after dividing the members of the previous inequality by $F(t,\alpha)$ and then integrating with respect to t between 0 and 1. □

Remark 11. *In the Theorem 11, if we put $F \equiv 1$ and consider only* (6), *we can obtain [23, Theorem 2.2]. If we consider the kernel $F(t,\alpha) = t^{1-\alpha}$, we can obtain new inequalities under the Riemann-Liouville fractional integrals. If we use another kernel F, we can obtain inequalities not reported in the literature.*

Another general conclusion is given in the following result.

Theorem 12. *Let ψ_1 be an (h_1,m_1)-convex function and ψ_2 an (h_2,m_2)-convex function with $m_1,m_2\in(0,1]$. If $0\le\xi_1<\xi_2<\infty$, $\psi_1(t)\psi_2(t)\in L^1_{\alpha,+}[\xi_1,\xi_2]\cap L^1_{\alpha,-}[\xi_1,\xi_2]$ and*

$$h_1(t)h_2(t), h_1(1-t)h_2(1-t), h_1(t)h_2(1-t), h_1(1-t)h_2(t)\in L^1_\alpha[0,1],$$

then the following inequality is satisfied

$$\begin{aligned}\frac{1}{\xi_2-\xi_1}&[J^\alpha_{F,\xi_1+}(\psi_1\psi_2)(\xi_2)+J^\alpha_{F,\xi_2-}(\psi_1\psi_2)(\xi_1)]\le(\psi_1(\xi_1)\psi_2(\xi_1)+\psi_1(\xi_2)\psi_2(\xi_2))J^\alpha_{F,0}(h_1h_2)(1)\\ &+m_1\left[\psi_1\Big(\frac{\xi_2}{m_1}\Big)\psi_2(\xi_1)+\psi_1\Big(\frac{\xi_1}{m_1}\Big)\psi_2(\xi_2)\right]J^\alpha_{F,0}[h_1(1-t)h_2(t)](1)\\ &+m_2\left[\psi_1(\xi_1)\psi_2\Big(\frac{\xi_2}{m_2}\Big)+\psi_1(\xi_2)\psi_2\Big(\frac{\xi_1}{m_2}\Big)\right]J^\alpha_{F,0}[h_1(t)h_2(1-t)](1)\\ &+m_1m_2\left(\psi_1\Big(\frac{\xi_1}{m_1}\Big)\psi_2\Big(\frac{\xi_1}{m_2}\Big)+\psi_1\Big(\frac{\xi_2}{m_1}\Big)\psi_2\Big(\frac{\xi_2}{m_2}\Big)\right)J^\alpha_{F,0}[h_1(1-t)h_2(1-t)](1).\end{aligned}$$

Proof. Inequalities (6) and (7) can be rewritten as follows

$$\begin{aligned}\psi_1(t\xi_1+&(1-t)\xi_2)\psi_2(t\xi_1+(1-t)\xi_2)\\ &=\psi_1\Big(t\xi_1+m_1(1-t)\frac{\xi_2}{m_1}\Big)\psi_2\Big(t\xi_1+m_2(1-t)\frac{\xi_2}{m_2}\Big)\\ &\le\Big(h_1(t)\psi_1(\xi_1)+m_1h_1(1-t)\psi_1\Big(\frac{\xi_2}{m_1}\Big)\Big)\Big(h_2(t)\psi_2(\xi_1)+m_2h_2(1-t)\psi_2\Big(\frac{\xi_2}{m_2}\Big)\Big)\end{aligned}\tag{8}$$

and

$$\begin{aligned}\psi_1(t\xi_2+&(1-t)\xi_1)\psi_2(t\xi_2+(1-t)\xi_1)\\ &=\psi_1\Big(t\xi_2+m_1(1-t)\frac{\xi_1}{m_1}\Big)\psi_2\Big(t\xi_2+m_2(1-t)\frac{\xi_1}{m_2}\Big)\\ &\le\Big(h_1(t)\psi_1(\xi_2)+m_1h_1(1-t)\psi_1\Big(\frac{\xi_1}{m_1}\Big)\Big)\Big(h_2(t)\psi_2(\xi_2)+m_2h_2(1-t)\psi_2\Big(\frac{\xi_1}{m_2}\Big)\Big).\end{aligned}$$

Adding up the members yields

$$\begin{aligned}
\psi_1(t\xi_1+&(1-t)\xi_2)\psi_2(t\xi_1+(1-t)\xi_2)+\psi_1(t\xi_2+(1-t)\xi_1)\psi_2(t\xi_2+(1-t)\xi_1)\\
&\le h_1(t)h_2(t)\big(\psi_1(\xi_1)\psi_2(\xi_1)+\psi_1(\xi_2)\psi_2(\xi_2)\big)\\
&\quad+m_2h_1(t)h_2(1-t)\psi_1(\xi_1)\psi_2\Big(\frac{\xi_2}{m_2}\Big)+m_1h_1(1-t)h_2(t)\psi_1\Big(\frac{\xi_2}{m_1}\Big)\psi_2(\xi_1)\\
&\quad+m_2h_1(t)h_2(1-t)\psi_1(\xi_2)\psi_2\Big(\frac{\xi_1}{m_2}\Big)+m_1h_1(1-t)h_2(t)\psi_1\Big(\frac{\xi_1}{m_1}\Big)\psi_2(\xi_2)\\
&\quad+m_1m_2h_1(1-t)h_2(1-t)\Big(\psi_1\Big(\frac{\xi_1}{m_1}\Big)\psi_2\Big(\frac{\xi_1}{m_2}\Big)+\psi_1\Big(\frac{\xi_2}{m_1}\Big)\psi_2\Big(\frac{\xi_2}{m_2}\Big)\Big).
\end{aligned}$$

Proceeding as in the proof of Theorem 11, we obtain the required inequality. □

Remark 12. *If we consider the kernel $F\equiv 1$ and use only* (8)*, we get an analogous result to [23, Theorem 2.3]. Using different kernels, we obtain new integral inequalities, in particular, for the Riemann-Liouville fractional integrals.*

3 Conclusions

In this paper, using a generalized integral operator, we have obtained various integral inequalities, which generalize several interesting results reported in the literature and which open up new work possibilities, depending on the kernel used: for example, if we use that of the generalized fractional integral of Hilfer (see [28]), we can derive new inequalities not yet published.

Finally, we want to emphasize that the results presented contain generalized inequalities valid for various functional classes such as convex, h-convex functions, m-convex functions, and s-convex functions, defined on a closed interval of non-negative real numbers. The problem of extending these results to other types of generalized convex functions remains open.

References

[1] Ali M.A., Nápoles V.J.E., Kashuri A., Zhang Z. *Fractional non conformable Hermite-Hadamard inequalities for generalized ϕ-convex functions*. Fasc. Math. 2020, **64**, 5–16. doi:10.21008/j.0044-4413.2020.0007

[2] Atangana A. Derivative with a New Parameter: Theory, Methods and Applications. Academic Press, 2015.

[3] Atangana A. *Extension of rate of change concept: from local to nonlocal operators with applications*. Results in Physics 2020, **19**, 103515. doi:10.1016/j.rinp.2020.103515

[4] Baleanu D. *Comments on: "The failure of certain fractional calculus operators in two physical models" by M. Ortigueira, V. Martynyuk, M. Fedula and J.A.T. Machado*. Fract. Calc. Appl. Anal. 2020, **23** (1), 292–297. doi:10.1515/fca-2020-0012

[5] Baleanu D., Fernandez A. *On Fractional Operators and Their Classifications*. Mathematics 2019, **7** (9), 830. doi:10.3390/math7090830

[6] Bermudo S., Kórus P., Nápoles V.J.E. *On q-Hermite–Hadamard inequalities for general convex functions*. Acta Math. Hungar. 2020, **162**, 364–374. doi:10.1007/s10474-020-01025-6

[7] Bessenyei M., Páles Z. *On generalized higher-order convexity and Hermite–Hadamard-type inequalities*. Acta Sci. Math. (Szeged) 2004, **70** (1–2), 13–24.

[8] Breckner W.W. *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Räumen*. Publ. Inst. Math. (Beograd) (N.S.) 1978, **23** (37), 13–20.

[9] Díaz R., Pariguan E. *On hypergeometric functions and Pochhammer k-symbol*. Divulg. Mat. 2007, **15** (2), 179–192.

[10] Dragomir S.S., Agarwal R.P. *Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula*. Appl. Math. Lett. 1998, **11** (5), 91–95. doi:10.1016/S0893-9659(98)00086-X

[11] Dragomir S.S., Fitzpatrik S. *The Hadamard inequalities for s-convex functions in the second sense*. Demonstr. Math. 1999, **32** (4), 687–696. doi:10.1515/dema-1999-0403

[12] Dragomir S.S., Pearce C.E.M. *Selected Topics on Hermite-Hadamard Inequalities and Applications*. Science Direct Working Paper No S1574-0358(04)70845-X.

[13] Dragomir S.S., Pecaric J., Persson L.E. *Some inequalities of Hadamard type*. Soochow J. Math. 1995, **21** (3), 335–341.

[14] Fleitas A., Nápoles V.J.E., Rodríguez J.M., Sigarreta J.M. *On the generalized fractional derivative*. Revista de la UMA, to appear.

[15] Gorenflo R., Mainardi F. Fractional Calculus: Integral and Differential Equations of Fractional Order. Springer Verlag, Wien and New York, 1997.

[16] Guzmán P.M., Lugo L.M., Nápoles V.J.E., Vivas M. *On a New Generalized Integral Operator and Certain Operating Properties*. Axioms 2020, **9** (2), 69. doi:10.3390/axioms9020069

[17] Guzmán P.M., Nápoles V.J.E., Gasimov Y. *Integral inequalities within the framework of generalized fractional integrals*. Fract. Differ. Calc., to appear.

[18] Hernández Hernández J.E. *On Some New Integral Inequalities Related With The Hermite-Hadamard Inequality via h-Convex Functions*. MAYFEB J. Math. 2017, **4**, 1–12.

[19] Hudzik H., Maligranda L. *Some remarks on s-convex functions*. Aequationes Math. 1994, **48**, 100–111. doi:10.1007/BF01837981

[20] Khalil R., Al Horani M., Yousef A., Sababheh M. *A new definition of fractional derivative*. J. Comput. Appl. Math. 2014, **264**, 65–70. doi:10.1016/j.cam.2014.01.002

[21] Klaričić Bakula M., Neuman E., Pečarić J., Šimić V. *Hermite–Hadamard's inequalities for multivariate g-convex functions*. Math. Inequal. Appl. 2005, **8** (2), 305–316. doi:10.7153/mia-08-28

[22] Kórus P., Lugo L.M., Nápoles V.J.E. *Integral inequalities in a generalized context*. Studia Sci. Math. Hungar. 2020, **57** (3), 312–320. doi:10.1556/012.2020.57.3.1464

[23] Matloka M. *On some integral inequalities for (h, m)-convex functions*. Math. Econ. 2013, **9** (16), 55–70. doi:10.15611/me.2013.9.05

[24] Mihesan V.G. *A generalization of the convexity*. Seminar on Functional Equations Approx. and Convex., Cluj-Napoca, Romania, 1993.

[25] Moslehian M.S. *Matrix Hermite-Hadamard type inequalities*. Houston J. Math. 2013, **39** (1), 177–189.

[26] Muddassar M., Bhatti M.I., Irshad W. *Some new s-Hermite Hadamard type ineqalities for differentiable functions and their applications*. Proc. Pak. Acad. Sci. A 2012, **49** (1), 9–17.

[27] Muddassar M., Bhatti M.I., Irshad W. *Generalisation of integral inequalities of Hermite-Hadamard type through convexity*. Bull. Aust. Math. Soc. 2013, **88** (2), 320–330. doi:10.1017/S0004972712000937

[28] Nápoles Valdés J.E. *Generalized fractional Hilfer integral and derivative*. Contrib. Math. 2020, **2**, 55–60. doi:10.47443/cm.2020.0036

[29] Nápoles J.E., Guzmán P.M., Lugo L.M., Kashuri A. *The local generalized derivative and Mittag Leffler function*. Sigma J. Eng. & Nat. Sci. 2020, **38** (2), 1007–1017.

[30] Nápoles Valdés J.E., Rabossi F., Samaniego A.D. *Convex functions: Ariadne's thread or Charlotte's Spiderweb?* Advanced Mathematical Models & Applications 2020, **5** (2), 176–191.

[31] Nápoles Valdés J.E., Rodríguez J.M., Sigarreta J.M. *On Hermite–Hadamard type inequalities for non-conformable integral operators*. Symmetry 2019, **11** (9), 1108. doi:10.3390/sym11091108

[32] Özdemir M.E., Akdemir A.O., Set E. *On $(h - m)$-convexity and Hadamard-type inequalities*. Transylv. J. Math. Mech. 2016, **8** (1), 51–58. doi:10.48550/arXiv.1103.6163

[33] Pearce C.E.M., Pečarić J. *Inequalities for differentiable mappings with application to special means and quadrature formulae*. Appl. Math. Lett. 2000, **13** (2), 51–55. doi:10.1016/S0893-9659(99)00164-0

[34] Qi F., Guo B.-N. *Integral representations and complete monotonicity of remainders of the Binet and Stirling formulas for the gamma function*. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM 2017, **111** (2), 425–434. doi:10.13140/2.1.2733.3928

[35] Park J. *Generalization of Ostrowski-type inequalities for differentiable real (s, m)-convex mappings*. Far East J. Math. Sci. 2011, **49** (2), 157–171.

[36] Rainville E.D. Special Functions. The Macmillan Co., New York, 1960.

[37] Sarikaya M.Z., Saglam A., Yildirim H. *On Some Hadamard-type-Inequalities for h-convex Functions*. J. Math. Inequal. 2008, **2** (3), 335–341. doi:10.7153/jmi-02-30

[38] Toader G. Some generalizations of the convexity. Proceedings of the Colloquium on Approximation and Optimization, University Cluj-Napoca, Cluj-Napoca, 1985, 329–338.

[39] Umarov S., Steinberg S. *Variable order differential equations with piecewise constant order-function and diffusion with changing modes*. Z. Anal. Anwend. 2009, **28** (4), 431–450. doi:10.4171/ZAA/1392

[40] Vivas-Cortez M., Kórus P., Nápoles Valdés J.E. *Some generalized Hermite-Hadamard-Fejér inequality for convex functions*. Adv. Difference Equ. 2021, **2021**, 199. doi:10.1186/s13662-021-03351-7

[41] Xi B.-Y., Gao D.-D., Qi F. *Integral inequalities of Hermite-Hadamard type for (α, s)-convex and (α, s, m)-convex functions*. Ital. J. Pure Appl. Math. 2020, **44**, 499–510.

[42] Xi B.-Y., Qi F. *Inequalities of Hermite-Hadamard type for extended s-convex functions and applications to means*. J. Nonlinear Convex Anal. 2015, **16** (5), 873–890.

[43] Yang Z.-H., Tian J.-F. *Monotonicity and inequalities for the gamma function*. J. Inequal. Appl. 2017, **2017**, 317. doi:10.1186/s13660-017-1591-9

[44] Yang Z.-H., Tian J.-F. *Monotonicity and sharp inequalities related to gamma function*. J. Math. Inequal. 2018, **12** (1), 1–22. doi:10.7153/jmi-2018-12-01

[45] Zhao D., Luo M. *General conformable fractional derivative and its physical interpretation*. Calcolo 2017, **54**, 903–917. doi:10.1007/s10092-017-0213-8

Received 29.06.2021
Revised 06.08.2021

Корус П., Наполес Вальдес Х.Е. *Про деякі інтегральні нерівності для (h, m)-опуклих функцій у рамках певного узагальнення* // Карпатські матем. публ. — 2023. — Т.15, №1. — С. 137–149.

У цій статті розглядаються загальні опуклі функції різного типу, такі як (h, m)-опуклі функції. Отримано деякі нові інтегральні нерівності типу Ерміта-Адамара. Показано, що основні результати розширюють деякі раніше відомі нерівності у термінах операторів дробового інтегрування.

Ключові слова і фрази: інтегральна нерівність, нерівність Ерміта-Адамара, (h, m)-опукла функція, модифікована (h, m)-опукла функція, узагальнений інтегральний оператор.