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A characterization for B-singular integral operator and its commutators on generalized weighted B-Morrey spaces

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We study the maximal operator M_γ and the singular integral operator A_γ, associated with the generalized shift operator. The generalized shift operators are associated with the Laplace-Bessel differential operator. Our analysis is based on two weighted inequalities for the maximal operator, singular integral operators, and their commutators, related to the Laplace-Bessel differential operator in generalized weighted B-Morrey spaces.

Key words and phrases: B-maximal operator, B-singular integral operator, commutator, generalized weighted B-Morrey space.

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Introduction

For $x \in \mathbb{R}^n$ and $r > 0$, we denote the open ball centered at x of radius r by $B(x,r)$. Given $f \in L_1^{\text{loc}}(\mathbb{R}^n)$, we define the maximal operator M by the following formula

$$Mf(x) = \sup_{t>0} \big|B(x,t)\big|^{-1} \int_{B(x,t)} \big|f(y)\big| dy,$$

where $\big|B(x,t)\big|$ is the Lebesgue measure of the ball $B(x,t)$.

For a continuous function $K(x,y)$ defined on $\big\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\big\}$, the Calderon-Zygmund singular integral operator is defined as

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy,$$

where the kernel satisfies the following properties:

$$\big|K(x,y)\big| \le C|x-y|^{-n} \quad \text{for all} \quad x \neq y,$$
$$\big|K(x,y) - K(x,z)\big| \le C\frac{|y-z|^\sigma}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \text{ if } |x-y| > 2|y-z|,$$
$$\big|K(x,y) - K(\xi,y)\big| \le C\frac{|x-\xi|^\sigma}{|x-y|n+\sigma}, \quad \sigma > 0, \text{ if } |x-y| > 2|x-\xi|.$$

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The operators M and T play an important role in real and harmonic analysis (see, for example, [30,34,35]).

In the theory of partial differential equations, Morrey spaces $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ play an important role. They were introduced by C. Morrey in 1938 (see [26]) and defined as follows. For any $0 \le \lambda \le n$, $1 \le p < \infty$, a function f belogs to $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ if $f \in L_p^{\text{loc}}(\mathbb{R}^n)$ and

$$\|f\|_{\mathcal{M}_{p,\lambda}} \equiv \|f\|_{\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty.$$

If $\lambda = 0$, then $\mathcal{M}_{p,\lambda}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, if $\lambda = n$, then $\mathcal{M}_{p,\lambda}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$, if $\lambda < 0$ or $\lambda > n$, then $\mathcal{M}_{p,\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

These spaces appeared to be quite useful in the study of the local behaviour of the solutions to elliptic partial differential equations, apriori estimates, and other topics in the theory of partial differential equations.

Given $f \in WL_p^{\text{loc}}(\mathbb{R}^n)$, $W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ denotes the weak Morrey space, and

$$\|f\|_{W\mathcal{M}_{p,\lambda}} \equiv \|f\|_{W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(\mathbb{R}^n)$ denotes the weak $L_p(\mathbb{R}^n)$ spaces.

F. Chiarenza and M. Frasca [6] studied the boundedness of the maximal operator M in Morrey spaces $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ (see also [2,4,5]). Their results can be summarized as follows.

Theorem 1 ([6]). *Let $0 < \alpha < n$, $0 \le \lambda < n$ and $1 \le p < \infty$.*

i) If $1 < p < \infty$, then M is bounded from $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$.

ii) If $p = 1$, then M is bounded from $\mathcal{M}_{1,\lambda}(\mathbb{R}^n)$ to $W\mathcal{M}_{1,\lambda}(\mathbb{R}^n)$.

If in place of the power function r^λ in the definition of $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ we consider any positive measurable weight function $\omega(r)$, then it becomes generalized Morrey spaces $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$.

Definition 1. *Let $\omega(r)$ be a positive measurable weight function on $(0,\infty)$ and $1 \le p < \infty$. We denote by $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$ the generalized Morrey spaces, the spaces of all functions $f \in L_p^{\text{loc}}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{\mathcal{M}_{p,\omega}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \frac{r^{-\frac{n}{p}}}{\omega(r)} \|f\|_{L_p(B(x,r))}.$$

T. Mizuhara [25], E. Nakai [28,29] and V.S. Guliyev [13] obtained sufficient conditions on weights ω_1 and ω_2 ensuring the boundedness of T from $\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}_{p,\omega_2}(\mathbb{R}^n)$. In [28], the following statement was proved, containing the result of [25] and in the general setting of metric measure spaces obtained in [31,32].

In [13, 25, 28], the authors obtained sufficient conditions on weights ω_1 and ω_2 for the boundedness of the singular integral operator T from $\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}_{p,\omega_2}(\mathbb{R}^n)$. In [28], the following doubling conditions were imposed on $\omega(r)$ such that

$$c^{-1}\omega(r) \le \omega(t) \le c\,\omega(r), \tag{1}$$

whenever $r \le t \le 2r$, where $c \ge 1$ does not depend on t and r, jointly with the condition

$$\int_r^\infty \omega^p(t)\frac{dt}{t} \le C\,\omega^p(r) \tag{2}$$

for the maximal or singular integral operator, where $C > 0$ does not depend on r.

Theorem 2 ([28]). *Let $1 < p < \infty$ and $\omega(r)$ satisfy conditions* (1)–(2). *Then the operators M and singular integral operator T are bounded in $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$.*

The proof of the theorem is given in [28].

The following statement containing the results of [25, 28] was proved in [13]. Note that Theorem 3 do not require condition (1).

Theorem 3 ([13]). *Let $1 < p < \infty$ and $\omega_1(r)$, $\omega_2(r)$ be positive measurable functions satisfying the condition*

$$\int_r^\infty \omega_1(t)\frac{dt}{t} \le C_1\, \omega_2(r) \tag{3}$$

with $C_1 > 0$ not depending on $t > 0$. Then the operators M and singular integral operator T are bounded from $\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}_{p,\omega_2}(\mathbb{R}^n)$.

The maximal operator and singular integral operator associated with the Laplace-Bessel differential operator

$$\Delta_B = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^k \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \qquad \gamma_1 > 0, \dots, \gamma_k > 0$$

have been investigated by many researchers such as B. Muckenhoupt and E. Stein [27], I. Kipriyanov [20], K. Trimeche [38], L. Lyakhov [23], K. Stempak [36], A.D. Gadjiev and I.A. Aliev [11], V.S. Guliyev [15, 16], V.S. Guliyev and J.J. Hasanov [14, 17], J.J. Hasanov [18], A. Serbetci and I. Ekincioglu [8, 9, 33], E.L. Shishkina [37] and others.

In this study, considering the generalized shift operator related to the Laplace-Bessel differential operator Δ_B, the B-maximal operator and B-singular integral operators generated by this operator investigated in generalized weighted B-Morrey spaces.

1 Preliminaries

Let $\mathbb{R}^n$ be the n dimensional Euclidean space with $n \ge 2$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $|x|^2 = \sum_{i=1}^n x_i^2$. Let $1 \le k \le n$. Then we get $x' = (x_1, \dots, x_k) \in \mathbb{R}^k$, $x'' = (x_{k+1}, \dots, x_n) \in \mathbb{R}^{n-k}$, $x = (x', x'') \in \mathbb{R}^n$ and $\mathbb{R}^n_{k,+} = \{x = (x', x'') \in \mathbb{R}^n; x_1 > 0, \dots, x_k > 0\}$.

Given an $x \in \mathbb{R}^n_{k,+}$ and $r > 0$ we define the open ball at center x and radius r as the set $E(x,r) = \{y \in \mathbb{R}^n_{k,+}; |x - y| < r\}$, and $E_r = E(0,r)$. For $\gamma = (\gamma_1, \dots, \gamma_k)$, $\gamma_1 > 0, \dots, \gamma_k > 0$, we define $|\gamma| = \gamma_1 + \ldots + \gamma_k$ and $(x')^\gamma = x_1^{\gamma_1} \cdots x_k^{\gamma_k}$. For measurable set $E \subset \mathbb{R}^n_{k,+}$, we get

$$\mid E \mid_\gamma = \int_E (x')^\gamma dx,$$

then $|E_r|_\gamma = \omega(n,k,\gamma) r^Q$, $Q = n + |\gamma|$, where

$$\omega(n,k,\gamma) = \int_{E_1} (x')^\gamma dx = \frac{\pi^{\frac{n-k}{2}}}{2^k} \prod_{i=1}^k \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)}.$$

Denote by T^x the generalized shift operator (B-shift operator) acting according to the law

$$T^x f(y) = C_{\gamma,k} \int_0^\pi \dots \int_0^\pi f\big((x',y')_\beta, x''-y''\big)\, d\nu(\beta),$$

where $(x',y')_\beta = ((x_1,y_1)_{\beta_1},\dots,(x_k,y_k)_{\beta_k}) d\nu(\beta) = \prod_{i=1}^{k} \sin^{\gamma_i-1}\beta_i \; d\beta_1\dots d\beta_k$, $1 \le k \le n$, $(x_i,y_i)_{\beta_i} = (x_i^2 - 2x_i y_i \cos\beta_i + y_i^2)^{\frac{1}{2}}$, $1 \le i \le k$, and

$$C_{\gamma,k} = \pi^{-\frac{k}{2}} \prod_{i=1}^{k} \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)} = \frac{2^k}{\pi^k}\omega(2k,k,\gamma).$$

We remark that the generalized shift operator T^x is closely connected with the Bessel differential operator B (for example, $n=k=1$ see [22], $n>1$, $k=1$ see [20] and $n,k>1$ see [23] for details).

The translation operator T^y generates the corresponding B-convolution

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y)\left[T^x g(y)\right](y')^\gamma dy,$$

for which the Young inequality

$$\|f\otimes g\|_{L_{r,\gamma}} \le \|f\|_{L_{p,\gamma}} \|g\|_{L_{q,\gamma}}, \quad 1 \le p,\ q \le r \le \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1,$$

holds.

Let $L_{p,\varphi,\gamma}(\mathbb{R}^n_{k,+})$ be the space of measurable functions on $\mathbb{R}^n_{k,+}$ with finite norm

$$\|f\|_{L_{p,\varphi,\gamma}} = \|f\|_{L_{p,\varphi,\gamma}(\mathbb{R}^n_{k,+})} = \left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \varphi(x)(x')^\gamma dx\right)^{1/p}, \quad 1 \le p < \infty.$$

For $p=\infty$, the space $L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ is defined by means of the usual modification

$$\|f\|_{L_{\infty,\gamma}} = \|f\|_{L_{\infty,\varphi}} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n_{k,+}} \varphi(x)|f(x)|.$$

Definition 2 ([12]). *The weight function φ belongs to the class $A_{p,\gamma}(\mathbb{R}^n_{k,+})$ if*

$$\sup_{x\in\mathbb{R}^n_{k,+},\, r>0} \left(\frac{1}{|E(x,r)|_\gamma}\int_{E(x,r)} \varphi^p(y)(y')^\gamma dy\right)^{\frac{1}{p}} \left(\frac{1}{|E(x,r)|_\gamma}\int_{E(x,r)} \varphi^{-p'}(y)(y')^\gamma dy\right)^{\frac{1}{p'}} < \infty$$

for $1 \le p < \infty$, and φ belongs to $A_{1,\gamma}(\mathbb{R}^n_{k,+})$ if there exists a positive constant C such that

$$|E(x,r)|_\gamma^{-1} \int_{E(x,r)} \varphi(y)(y')^\gamma dy \le C \operatorname*{ess\,sup}_{y\in\mathbb{R}^n_{k,+}} \varphi(y)$$

for any $x \in \mathbb{R}^n_{k,+}$ and $r>0$.

Definition 3. *The weight function* (φ_1, φ_2) *belongs to the class* $\widetilde{A}_{p,\gamma}(\mathbb{R}^n_{k,+})$ *if*

$$\sup_{x\in\mathbb{R}^n_{k,+},\, r>0} \left(\frac{1}{|E(x,r)|_\gamma}\int_{E(x,r)} \varphi_2^p(y)(y')^\gamma dy\right)^{\frac{1}{p}} \left(\frac{1}{|E(x,r)|_\gamma}\int_{E(x,r)} \varphi_1^{-p'}(y)(y')^\gamma dy\right)^{\frac{1}{p'}} < \infty,$$

for $1 < p < \infty$.

Lemma 1. *Let* $E\big((x,0),t\big) = \Big\{(z,\overline{z'}) \in \mathbb{R}^n \times (0,\infty)^k \,:\, \|(x-z,\overline{z'})\| < t\Big\}$. *Then for all* $x \in \mathbb{R}^n_{k,+}$, *the following equality holds*

$$\int_{E_t} T^y g(x)(y')^\gamma dy = \int_{E\left((x,0),t\right)} g\left(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2}, z''\right) d\mu(z,\overline{z'}).$$

Lemma 2. *Let* $0<\theta<1$ *and* ψ *positive measurable weight function. Then for all* $x \in \mathbb{R}^n_{k,+}$, *the following equality holds*

$$\int_{\mathbb{R}^n_{k,+}} T^y g(x)\psi(y)\left(M_\gamma \chi_{E_r}(y)\right)^\theta (y')^\gamma dy$$
$$= \int_{\mathbb{R}^n\times(0,\infty)^k} g\left(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2}, z''\right)\psi(z,\overline{z'})\left(M_\gamma \chi_{E((x,0),r)}(z,\overline{z'})\right)^\theta d\nu(z,\overline{z'}),$$

where $E\big((x,0),t\big) = \Big\{(z,\overline{z'}) \in \mathbb{R}^n \times (0,\infty)^k \,:\, \|(x-z,\overline{z'})\| < t\Big\}$.

For $0 \le \alpha_i < \pi$, $i = 1,\ldots,k$, and $y \in \mathbb{R}^n_{k,+}$, Lemmas 1 and 2 are straightforward via the following substitutions $z'' = x''$, $z_i = y_i\cos\alpha_i$, $\overline{z_i} = y_i \sin\alpha_i$, $\overline{z'} = (\overline{z}_1,\ldots,\overline{z}_k)$ and $(z,\overline{z'}) \in \mathbb{R}^n\times(0,\infty)^k$, $1 \le k \le n$.

Definition 4 ([15])**.** *Let* $1 \le p < \infty$ *and* $0 \le \lambda \le Q$. *We denote by* $\mathcal{M}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ *Morrey space (≡ B-Morrey space), associated with the Laplace-Bessel differential operator, the set of locally integrable functions* $f(x)$, $x \in \mathbb{R}^n_{k,+}$, *with the finite norm*

$$\|f\|_{\mathcal{M}_{p,\lambda,\gamma}} = \sup_{t>0,\, x\in\mathbb{R}^n_{k,+}} \left(\int_{E_t} T^y[|f|]^p(x)(y')^\gamma dy\right)^{1/p}.$$

Let ω and φ positive measurable weight functions. The norms in spaces $\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ and $\mathcal{M}_{p,\omega,\varphi,\gamma}(\mathbb{R}^n_{k,+})$, respectively, defined by

$$\|f\|_{\mathcal{M}_{p,\omega,\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} \frac{t^{-\frac{Q}{p}}}{\omega(t)}\left(\int_{E_t} T^y[|f|]^p(x)(y')^\gamma dy\right)^{1/p},$$
$$\|f\|_{\mathcal{M}_{p,\omega,\varphi,\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} \frac{t^{-\frac{Q}{p}}}{\omega(t)}\left(\int_{E_t} T^y[|f|]^p(x)\varphi(y)(y')^\gamma dy\right)^{1/p}.$$

If $\omega(t) \equiv t^{-\frac{Q}{p}}$, then $\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}^n_{k,+}) \equiv L_{p,\gamma}(\mathbb{R}^n_{k,+})$, and if $\omega(t) \equiv t^{\frac{\lambda-Q}{p}}$ and $0 \le \lambda < Q$, then $\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}^n_{k,+}) \equiv \mathcal{M}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

B-BMO space $BMO_\gamma(\mathbb{R}^n_{k,+})$ is defined as the space of locally integrable functions f with finite norm

$$\|f\|_{BMO_\gamma} = \sup_{t>0,\, x\in\mathbb{R}^n_{k,+}} \big|E(0,t)\big|_\gamma^{-1} \int_{E(0,t)} \big|T^y f(x) - f_{E(0,t)}(x)\big| (y')^\gamma dy < \infty,$$

or

$$\|f\|_{BMO_\gamma} = \inf_C \sup_{t>0,\, x\in\mathbb{R}^n_{k,+}} \big|E(0,t)\big|_\gamma^{-1} \int_{E(0,t)} \big|T^y f(x) - C\big| (y')^\gamma dy < \infty,$$

where $f_{E(0,t)}(x) = \big|E(0,t)\big|_\gamma^{-1} \int_{E(0,t)} T^y f(x) (y')^\gamma dy$.

The following theorem was proved in [1].

Theorem 4.

i) Let $f \in L^{loc}_{1,\gamma}(\mathbb{R}^n_{k,+})$. If

$$\sup_{t>0,\, x\in\mathbb{R}^n_{k,+}} \left(\big|E(0,t)\big|_\gamma^{-1} \int_{E(0,t)} \big|T^y f(x) - f_{E(0,t)}(x)\big|^p (y')^\gamma dy \right)^{1/p} = \|f\|_{BMO_{p,\gamma}} < \infty,$$

then for any $1 < p < \infty$ we have

$$\|f\|_{BMO_\gamma} \le \|f\|_{BMO_{p,\gamma}} \le A_p \|f\|_{BMO_\gamma},$$

where the constant A_p depends only on p.

ii) Let $f \in BMO_\gamma(\mathbb{R}^n_{k,+})$. Then there is a constant $C > 0$ such that

$$\big|f_{E(0,r)} - f_{E(0,t)}\big| \le C\|f\|_{BMO_\gamma} \ln \frac{t}{r}, \qquad 0 < 2r < t,$$

where C is independent of f, x, r and t.

Lemma 3. *Let $1 < p < \infty$, $\varphi \in A_{p,\gamma}(\mathbb{R}^n_{k,+})$ and $b \in BMO_\gamma(\mathbb{R}^n_{k,+})$. Then*

$$\|b\|_{BMO_\gamma} \approx \sup_{x\in\mathbb{R}^n_{k,+}, r>0} \frac{\|T^{\cdot} b(x) - b_{E(0,r)}\|_{L_{p,\varphi,\gamma}(E(0,r))}}{\|\varphi\|_{L_{p,\gamma}(E(0,r))}}.$$

Proof. From Hölder's inequality, we get

$$\|b\|_{BMO_\gamma} \lesssim \sup_{x\in\mathbb{R}^n_{k,+}, r>0} \frac{\|T^{\cdot} b(x) - b_{E(0,r)}\|_{L_{p,\varphi,\gamma}(E(0,r))}}{\|\varphi\|_{L_{p,\gamma}(E(0,r))}}.$$

Now, we obtain that

$$\sup_{x\in\mathbb{R}^n, r>0} \frac{\|T^{\cdot} b(x) - b_{E(0,r)}\|_{L_{p,\varphi,\gamma}(E(0,r))}}{\|\varphi\|_{L_{p,\gamma}(E(0,r))}} \lesssim \|b\|_{BMO_\gamma}.$$

We can assume without loss of generality that $\|b\|_{BMO_\gamma} = 1$; otherwise, we replace b by $b/\|b\|_{BMO_\gamma}$. It follows that

$$\int_{E(0,r)} \left(\frac{|T^{\cdot} b(x) - b_{E(0,r)}|\varphi(y)}{\|b\|_{BMO_\gamma}} \right)^p dy = \int_{E(0,r)} \Big(|T^{\cdot} b(x) - b_{B(x,r)}|\varphi(y) \Big)^p dy \lesssim 1.$$

□

2 Two-weighted inequalities for B-maximal operator and B-maximal commutators in the spaces $\mathcal{M}_{p,\omega,\varphi,\gamma}(\mathbb{R}^n_{k,+})$

First, consider B-maximal operator

$$M_\gamma f(x) = \sup_{r>0} \mid E_r \mid_\gamma^{-1} \int_{E_r} T^y [|f|\,](x)(y')^\gamma dy.$$

Homogeneous type maximal function defined by

$$M_\nu f(x) = \sup_{r>0} \nu\big(E(x,r)\big)^{-1} \int_{E(x,r)} \|f(y)\| d\nu(y).$$

Also, in the works [21,24] it was proved the following assertion.

Proposition 1. *Let $1 \le p < \infty$, $0 < \delta < 1$ and $(\varphi, \varphi_1) \in \widetilde{A}_p(Y)$. Then M_ν is bounded from $L_{p,\varphi_1^\delta}(Y)$ to $L_{p,\varphi_2^\delta}(Y)$, where (Y, d, ν) homogeneous type space.*

Theorem 5. *Let $1 \le p < \infty$, $0 < \delta < 1$, $(\varphi_1, \varphi_2) \in \widetilde{A}_{p,\gamma}(\mathbb{R}^n_{k,+})$ and $\omega_1(r)$, $\omega_2(r)$ be positive measurable functions satisfying the condition*

$$r^Q \omega_1^p(r) + C \int_r^\infty t^{Q-1} \omega_1^p(t) dt \le C_1 r^Q \omega_2^p(r), \tag{4}$$

where $C > 0$ and $C_1 > 0$ does not depend on r. Then M_γ is bounded from $\mathcal{M}_{p,\omega_1,\varphi_1^\delta,\gamma}(\mathbb{R}^n_{k,+})$ to $\mathcal{M}_{p,\omega_2,\varphi_2^\delta,\gamma}$.

Proof. We need to introduce the maximal operator defined on a space of homogeneous type (Y, d, ν). By this we mean a topological space $Y = \mathbb{R}^n \times (0,\infty)^k$ equipped with a continuous pseudometric d and a positive measure ν satisfying

$$\nu\Big(E\big((x,\overline{x'}),2r\big)\Big) \le C_1 \nu\Big(E\big((x,\overline{x'}),r\big)\Big) \tag{5}$$

with a constant C_1 independent of $(x,\overline{x'})$ and $r > 0$.

Here

$$E\big((x,\overline{x'}),r\big) = \big\{(y,\overline{y'}) \in Y \;:\; d\big((x,\overline{x'}),(y,\overline{y'})\big) < r\big\},$$

$$d\big((x,\overline{x'}),(y,\overline{y'})\big) = \big|(x,\overline{x'}) - (y,\overline{y'})\big| \equiv \Big(|x-y|^2 + (\overline{x'} - \overline{y'})^2\Big)^{\frac{1}{2}}.$$

Let (Y, d, ν) be a space of homogeneous type. Define

$$M_\nu \overline{f}(x,\overline{x'}) = \sup_{r>0} \nu\Big(E\big((x,\overline{x'}),r\big)\Big)^{-1} \int_{E\left((x,\overline{x'}),r\right)} \|\overline{f}(y,\overline{y'})\| d\nu(y),$$

where

$$\overline{f}(x,\overline{x'}) = f\left(\sqrt{x_1^2 + \overline{x}_1^2}, \ldots, \sqrt{x_k^2 + \overline{x}_k^2}, x''\right).$$

It is well known that the fractional maximal operator M_ν is bounded from $L_{p,\psi_1^\delta}(Y,d\nu)$ to $L_{p,\psi_2^\delta}(Y,d\nu)$ for $1<p<\infty$, $(\psi_1,\psi_2)\in \widetilde{A}_p(Y)$ (see [21]). Here we are concerned with the fractional maximal operator defined by $d\nu(y,\overline{y'})=(\overline{y'})^{\gamma-1}dy\,d\overline{y'}$. It is clear that this measure satisfies the doubling condition (5).

It can be proved that

$$M_\gamma f\left(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2},z''\right)=M_\nu\overline{f}\left(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2},z'',0\right) \tag{6}$$

and

$$M_\gamma f(x)=M_\nu\overline{f}(x,0). \tag{7}$$

Indeed, from Lemma 2 and

$$\begin{aligned}\psi_1^\delta(y)&=\varphi_1^\delta(y)\big(M_\nu\chi_{E((x,0),r)}(y)\big)^\theta,\\ \psi_2^\delta(y)&=\varphi_2^\delta(y)\big(M_\nu\chi_{E((x,0),r)}(y)\big)^\theta,\end{aligned}$$

for any $0<\theta<1$ and $(\psi_1,\psi_2)\in\widetilde{A}_p(Y)$, we have that

$$\begin{aligned}&\int_{\mathbb{R}^n_{k,+}} T^y|f(x)|^p\varphi_1^\delta(y)\Big(M_\gamma\chi_{E_r}(y)\Big)^\theta(y')^\gamma\,dy\\ &\quad=\int_Y\left|\overline{f}\Big(\sqrt{y_1^2+\overline{y}_1^2},\ldots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\Big)\right|^p\varphi_1^\delta(y,\overline{y'})\big(M_\nu\chi_{E((x,0),r)}(y,\overline{y'})\big)^\theta\,d\nu(y,\overline{y'})\end{aligned}$$

and

$$|E_r|_\gamma=\nu E\Big(\Big(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2},z'',0\Big),r\Big)$$

imply (6).

Furthermore, taking $\overline{z}_k=0$ in (6), we get (7). Using Lemma 2 and equality (6), we have

$$\begin{aligned}&\int_{E_r} T^y(M_\gamma f(x))^p\varphi_2^\delta(y)(y')^\gamma\,dy\\ &\quad\le\int_{\mathbb{R}^n_{k,+}} T^y(M_\gamma f(x))^p\varphi_2^\delta(y)(M_\gamma\chi_{E_r}(y))^\theta(y')^\gamma\,dy\\ &\quad=\int_{\mathbb{R}^n\times(0,\infty)^k}\Big(M_\gamma f\Big(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2},z''\Big)\Big)^p\varphi_2^\delta(z,\overline{z'})(M_\gamma\chi_{E((x,0),r)}(z,\overline{z'}))^\theta\,d\nu(z,\overline{z'})\\ &\quad=\int_Y\Big(M_\nu\overline{f}\Big(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2},z'',0\Big)\Big)^p\varphi_2^\delta(z,\overline{z'})(M_\nu\chi_{E((x,0),r)}(z,\overline{z'}))^\theta\,d\nu(z,\overline{z'}).\end{aligned}$$

By the Proposition 1, we have

$$
\begin{aligned}
&\left(\int_{E_r} T^y(M_\gamma f(x))^p \varphi_2^\delta(y)(y')^\gamma dy\right)^{\frac{1}{p}}\\
&\leq \left(\int_Y \left(M_\nu \overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\ldots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)\right)^p \varphi_2^\delta(y,\overline{y'})(M_\nu\chi_{E((x,0),r)}(y,\overline{y'}))^\theta\, d\nu(y,\overline{y'})\right)^{\frac{1}{p}}\\
&= \left(\int_Y \left(M_\nu \overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\ldots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)\right)^p \psi_2^\delta(y,\overline{y'})\, d\nu(y,\overline{y'})\right)^{\frac{1}{p}}\\
&\leq C_2\left(\int_Y \left|\overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\ldots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)\right|^p \psi_1^\delta(y,\overline{y'})\, d\nu(y,\overline{y'})\right)^{\frac{1}{p}}\\
&= C_2\left(\int_Y \left|\overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\ldots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)\right|^p \varphi_1^\delta(y,\overline{y'})(M_\nu\chi_{E((x,0),r)}(y,\overline{y'}))^\theta\, d\nu(y,\overline{y'})\right)^{\frac{1}{p}}\\
&= C_2\left(\int_Y \left|f\left(\sqrt{y_1^2+\overline{y}_1^2},\ldots,\sqrt{y_k^2+\overline{y}_k^2},y''\right)\right|^p \varphi_1^\delta(y,\overline{y'})(M_\nu\chi_{E((x,0),r)}(y,\overline{y'}))^\theta\, d\nu(y,\overline{y'})\right)^{\frac{1}{p}}\\
&= C_2\left(\int_{\mathbb{R}^n_{k,+}} T^y\left[|f|\right]^p(x)\varphi_1^\delta(y)(M_\gamma\chi_{E_r}(y))^\theta(y')^\gamma dy\right)^{\frac{1}{p}}\\
&\leq C_2\left(\int_{E_r} T^y\left[|f|\right]^p(x)\varphi_1^\delta(y)(y')^\gamma\, dy + \sum_{j=1}^\infty \int_{E_{2^{j+1}r}\setminus E_{2^j r}} T^y\left[|f|\right]^p(x)\varphi_1^\delta(y)(M_\gamma\chi_{E_r}(y))^\theta(y')^\gamma\, dy\right)^{\frac{1}{p}}\\
&\leq C_2\left(\int_{E_r} T^y\left[|f|\right]^p(x)\varphi_1^\delta(y)(y')^\gamma\, dy + \sum_{j=1}^\infty \int_{E_{2^{j+1}r}\setminus E_{2^j r}} T^y\left[|f|\right]^p(x)\varphi_1^\delta(y)\frac{r^{Q\theta}}{(|y|+r)^{Q\theta}}(y')^\gamma dy\right)^{\frac{1}{p}}\\
&\leq C_3\|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1^\delta,\gamma}}\left(r^Q\omega_1^p(r) + \sum_{j=1}^\infty \frac{1}{(2^j+1)^{Q\theta}}(2^{j+1}r)^Q\omega_1^p(2^{j+1}r)\right)^{\frac{1}{p}}\\
&\leq C_3\|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1^\delta,\gamma}}\left(r^Q\omega_1^p(r) + C\int_r^\infty t^{Q-1}\omega_1^p(t)\,dt\right)^{\frac{1}{p}} \leq C_4 r^{\frac{Q}{p}}\omega_2(r)\|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1^\delta,\gamma}}.
\end{aligned}
$$

Then, we get

$$
\|M_\gamma f\|_{\mathcal{M}_{p,\omega_2,\varphi_1^\delta,\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\, t>0}\frac{t^{-\frac{Q}{p}}}{\omega_2(t)}\left\|T^{\cdot}(M_\gamma f(x))\right\|_{L_{q,\varphi_2^\delta,\gamma}(E_t)} \leq C_4\|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1^\delta,\gamma}}.
$$

Theorem 5 is proved. □

If $\omega_1(r)=\omega_2(r)=r^{-\frac{Q}{p}}$, then from Theorem 5 we get the following result.

Corollary 1. *Let $1<p<\infty$, $0<\delta<1$ and $(\varphi_1,\varphi_2)\in\widetilde{A}_{p,\gamma}(\mathbb{R}^n_{k,+})$, then the operator M_γ is bounded from $L_{p,\varphi_1^\delta,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{p,\varphi_2^\delta,\gamma}(\mathbb{R}^n_{k,+})$.*

If $\omega_1(r)=\omega_2(r)=r^{-\frac{Q}{p}}$ and $\varphi_1^\delta(x)=\varphi_2^\delta(x)=\varphi(x)$, then from Theorem 5 we get the following result.

Corollary 2 ([3])**.** *Let $1<p<\infty$, $\varphi\in A_{p,\gamma}(\mathbb{R}^n_{k,+})$. Then M_γ is bounded on the space $L_{p,\varphi,\gamma}(\mathbb{R}^n_{k,+})$.*

For a given suitable function b, the commutator generated by the B-maximal operator M_γ is formally defined by $[M_\gamma, b]f = M_\gamma(bf) - bM_\gamma(f)$ and for a given measurable function b, the B-maximal commutator is defined by

$$M_{b,\gamma}(f)(x) := \sup_{r>0} \left|E(0,r)\right|_\gamma^{-1} \int_{E(0,r)} T^y \Big|\big(b(x)-b(y)\big)f(x)\Big| \big(y'\big)^\gamma dy \quad \text{for all} \quad x \in \mathbb{R}^n_{k,+}.$$

Lemma 4 ([19]). *Let* $1 < s < \infty$, $b \in BMO(\mathbb{R}^n_{k,+})$. *Then there exists* $C > 0$ *such that for all* $x \in \mathbb{R}^n_{k,+}$ *the inequality* $M_\gamma(M_{b,\gamma}f)(x) \le C\|b\|_{BMO_\gamma}\left(\left(M_\gamma(M_\gamma f)^s\right)^{\frac{1}{s}}(x) + M_\gamma\left(M_\gamma|f|^s\right)^{\frac{1}{s}}(x)\right)$ *holds.*

Theorem 6. *Let* $1 < p < \infty$, $0 < \delta < 1$, $b \in BMO_\gamma(\mathbb{R}^n_{k,+})$ *and* $(\varphi_1, \varphi_2) \in \widetilde{A}_{p,\gamma}(\mathbb{R}^n_{k,+})$, $\varphi_1 \in A_{p,\gamma}(\mathbb{R}^n_{k,+})$ *and* $\omega_1(r)$, $\omega_2(r)$ *be positive measurable functions satisfying the condition* (4). *Then* $M_{b,\gamma}$ *is bounded from* $\mathcal{M}_{p,\omega_1,\varphi_1^\delta,\gamma}(\mathbb{R}^n_{k,+})$ *to* $\mathcal{M}_{p,\omega_2,\varphi_2^\delta,\gamma}$.

Proof. Let $f \in \mathcal{M}_{p,\omega_1,\varphi_1^\delta,\gamma}(\mathbb{R}^n_{k,+})$. By Lemma 4 and Theorem 5, we get

$$\|M_{b,\gamma}f\|_{\mathcal{M}_{p,\omega_2,\varphi_2^\delta,\gamma}} \le C\|b\|_{BMO}\left\|\left(M_\gamma(M_\gamma f)^s\right)^{\frac{1}{s}}\right\|_{\mathcal{M}_{p,\omega_2,\varphi_2^\delta,\gamma}} + C\|b\|_{BMO}\left\|M_\gamma\left(M_\gamma|f|^s\right)^{\frac{1}{s}}\right\|_{\mathcal{M}_{p,\omega_2,\varphi_2^\delta,\gamma}}$$

$$\le C\|b\|_{BMO_\gamma}\left\|(M_\gamma f)^s\right\|_{\mathcal{M}_{\frac{p}{s},\omega_1,\varphi_1^\delta,\gamma}} + C\|b\|_{BMO_\gamma}\left\|\left(M_\gamma|f|^s\right)^{\frac{1}{s}}\right\|_{\mathcal{M}_{p,\omega_1,\varphi_1^\delta,\gamma}} \le C_1\|b\|_{BMO_\gamma}\|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1^\delta,\gamma}}.$$

□

Corollary 3. *Let* $1 < p < \infty$, $0 < \delta < 1$, $b \in BMO_\gamma(\mathbb{R}^n_{k,+})$ *and* $(\varphi_1, \varphi_2) \in \widetilde{A}_{p,\gamma}(\mathbb{R}^n_{k,+})$, $\varphi_1 \in A_{p,\gamma}(\mathbb{R}^n_{k,+})$, *then the operator* $M_{b,\gamma}$ *is bounded from* $L_{p,\varphi_1^\delta,\gamma}(\mathbb{R}^n_{k,+})$ *to* $L_{p,\varphi_2^\delta,\gamma}(\mathbb{R}^n_{k,+})$.

3 Two-weighted inequalities B-singular integral operators and its commutators in the spaces $\mathcal{M}_{p,\omega,\varphi,\gamma}(\mathbb{R}^n_{k,+})$

We consider the B-singular integral operators

$$A_\gamma f(x) = \int_{\mathbb{R}^n_{k,+}} T^y f(x) K(y) \big(y'\big)^\gamma dy$$

such that the kernel K satisfy the following conditions

$$\left|\int_{\{x\in\mathbb{R}^n_{k,+}:\varepsilon<|x|<r\}} K(x)(x')^\gamma\, dx\right| \le C, \quad 0<\varepsilon<r<\infty,$$

$$\int_{\{x\in\mathbb{R}^n_{k,+}:r<|x|<4r\}} \big|K(x)\big|(x')^\gamma dx \le C, \quad 0<r<\infty,$$

$$\int_{\{x\in\mathbb{R}^n_{k,+}:|x|\ge 4|y|\}} \big|T^y K(x) - K(x)\big|(x')^\gamma dx \le C, \quad |y| < \frac{1}{4},$$

and we assume additionally that

$$A_\gamma f(x) = \lim_{\varepsilon\to 0+} A_{\varepsilon,\gamma} f(x) \tag{8}$$

for $\varepsilon > 0$, where

$$A_{\varepsilon,\gamma} f(x) = \int_{\{y\in\mathbb{R}^n_{k,+}:|y|>\varepsilon\}} T^y f(x) K(y) (y')^\gamma\, dy.$$

Theorem 7. *Let* $1 < p < \infty$, $0 < \delta < 1$, $(\varphi_1, \varphi_2) \in \widetilde{A}_{p,\gamma}(\mathbb{R}^n_{k,+})$ *and* $\omega_1(r)$, $\omega_2(r)$ *be positive measurable functions satisfying the condition* (4). *Then the singular integral operator* A_γ *exists almost everywhere in* $\mathbb{R}^n_{k,+}$ *and operator* A_γ *is bounded from* $\mathcal{M}_{p,\omega_1,\varphi_1^\delta,\gamma}(\mathbb{R}^n_{k,+})$ *to* $\mathcal{M}_{p,\omega_2,\varphi_2^\delta,\gamma}(\mathbb{R}^n_{k,+})$.

Proof. We need to introduce the maximal operator defined on a space of homogeneous type (Y, d, ν). By this we mean a topological space $Y = \mathbb{R}^n \times (0,\infty)^k$ equipped with a continuous pseudometric d and a positive measure ν satisfying (5).

Let (Y, d, ν) be a space of homogeneous type. Define

$$T_\nu \overline{f}(x, \overline{x'}) = \int_{E((x,\overline{x'}),r)} \overline{f}(x-y, \overline{x'}-\overline{y'}) K(y, \overline{y'}) d\nu(y),$$

where $\overline{f}(x, \overline{x'}) = f\left(\sqrt{x_1^2+\overline{x}_1^2}, \ldots, \sqrt{x_k^2+\overline{x}_k^2}, x''\right)$. It can be proved that

$$A_\gamma f\left(\sqrt{z_1^2+\overline{z}_1^2}, \ldots, \sqrt{z_k^2+\overline{z}_k^2}, z''\right) = T_\nu \overline{f}\left(\sqrt{z_1^2+\overline{z}_1^2}, \ldots, \sqrt{z_k^2+\overline{z}_k^2}, z'', 0\right)$$

and

$$A_\gamma f(x) = T_\nu \overline{f}(x, 0).$$

Also, in the work [7] it was proved the following assertion.

Proposition 2. *Let* $1 \le p < \infty$, $0 < \delta < 1$ *and* $(\varphi, \varphi_1) \in \widetilde{A}_p(Y)$. *Then singular integral operator* T_ν *exists almost everywhere in* Y *and the operator* T_ν *is bounded from* $L_{p,\varphi_1^\delta}(Y)$ *to* $L_{p,\varphi_2^\delta}(Y)$, *where* (Y, d, ν) *homogeneous type space.*

Indeed, from Lemma 2 and $\psi_1^\delta(y) = \varphi_1^\delta(y)\left(M_\nu \chi_{E((x,0),r)}(y)\right)^\theta$, $(\psi_1, \psi_2) \in \widetilde{A}_p(Y)$ and $\psi_2^\delta(y) = \varphi_2^\delta(y)\left(M_\nu \chi_{E((x,0),r)}(y)\right)^\theta$, we have

$$\begin{aligned}
&\int_{E_r} T^y \left(A_\gamma f(x)\right)^p \varphi_2^\delta(y)(y')^\gamma \, dy \\
&\quad \le \int_{\mathbb{R}^n_{k,+}} T^y \left(A_\gamma f(x)\right)^p \varphi_2^\delta(y)\left(M_\gamma \chi_{E_r}(y)\right)^\theta (y')^\gamma \, dy \\
&\quad = \int_{\mathbb{R}^n \times (0,\infty)^k} \left(A_\gamma f\left(\sqrt{z_1^2+\overline{z}_1^2}, \ldots, \sqrt{z_k^2+\overline{z}_k^2}, z''\right)\right)^p \varphi_2^\delta(z, \overline{z'})\left(M_\gamma \chi_{E((x,0),r)}(z, \overline{z'})\right)^\theta d\nu(z, \overline{z'}) \\
&\quad = \int_Y \left(T_\nu \overline{f}\left(\sqrt{z_1^2+\overline{z}_1^2}, \ldots, \sqrt{z_k^2+\overline{z}_k^2}, z'', 0\right)\right)^p \varphi_2^\delta(z, \overline{z'})\left(M_\nu \chi_{E((x,0),r)}(z, \overline{z'})\right)^\theta d\nu(z, \overline{z'}).
\end{aligned}$$

By the Proposition 1, we have

$$\left(\int_{E_r} T^y\left(A_\gamma f(x)\right)^p \varphi_2^\delta(y)\left(y'\right)^\gamma dy\right)^{\frac{1}{p}}$$

$$\leq \left(\int_Y \left(T_\nu \overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\ldots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)\right)^p \varphi_2^\delta(y,\overline{y'})\left(M_\nu \chi_{E((x,0),r)}(y,\overline{y'})\right)^\theta d\nu(y,\overline{y'})\right)^{\frac{1}{p}}$$

$$= \left(\int_Y \left(T_\nu \overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\ldots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)\right)^p \psi_2^\delta(y,\overline{y'})\, d\nu(y,\overline{y'})\right)^{\frac{1}{p}}$$

$$\leq C_2\left(\int_Y \left|\overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\ldots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)\right|^p \psi_1^\delta(y,\overline{y'})\, d\nu(y,\overline{y'})\right)^{\frac{1}{p}}$$

$$= C_2\left(\int_Y \left|\overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\ldots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)\right|^p \varphi_1^\delta(y,\overline{y'})\left(M_\nu \chi_{E((x,0),r)}(y,\overline{y'})\right)^\theta d\nu(y,\overline{y'})\right)^{\frac{1}{p}}$$

$$= C_2\left(\int_Y \left|f\left(\sqrt{y_1^2+\overline{y}_1^2},\ldots,\sqrt{y_k^2+\overline{y}_k^2},y''\right)\right|^p \varphi_1^\delta(y,\overline{y'})\left(M_\nu \chi_{E((x,0),r)}(y,\overline{y'})\right)^\theta d\nu(y,\overline{y'})\right)^{\frac{1}{p}}$$

$$= C_2\left(\int_{\mathbb{R}^n_{k,+}} T^y\left[|f|\right]^p(x)\varphi_1^\delta(y)\left(M_\gamma \chi_{E_r}(y)\right)^\theta (y')^\gamma\, dy\right)^{\frac{1}{p}}$$

$$\leq C_2\left(\int_{E_r} T^y\left[|f|\right]^p(x)\varphi_1^\delta(y)(y')^\gamma dy + \sum_{j=1}^{\infty}\int_{E_{2^{j+1}r}\setminus E_{2^j r}} T^y\left[|f|\right]^p(x)\varphi_1^\delta(y)\left(M_\gamma \chi_{E_r}(y)\right)^\theta (y')^\gamma\, dy\right)^{\frac{1}{p}}$$

$$\leq C_2\left(\int_{E_r} T^y\left[|f|\right]^p(x)\varphi_1^\delta(y)(y')^\gamma dy + \sum_{j=1}^{\infty}\int_{E_{2^{j+1}r}\setminus E_{2^j r}} T^y\left[|f|\right]^p(x)\varphi_1^\delta(y)\frac{r^{Q\theta}}{(|y|+r)^{Q\theta}}(y')^\gamma\, dy\right)^{\frac{1}{p}}$$

$$\leq C_3\|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1^\delta,\gamma}}\left(r^Q\omega_1^p(r) + \sum_{j=1}^{\infty}\frac{1}{(2^j+1)^{Q\theta}}(2^{j+1}r)^Q\omega_1^p(2^{j+1}r)\right)^{\frac{1}{p}}$$

$$\leq C_3\|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1^\delta,\gamma}}\left(r^Q\omega_1^p(r) + C\int_r^\infty t^{Q-1}\omega_1^p(t)dt\right)^{\frac{1}{p}} \leq C_4\, r^{\frac{Q}{p}}\omega_2(r)\|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1^\delta,\gamma}}.$$

Then we get

$$\|A_\gamma f\|_{\mathcal{M}_{p,\omega_2,\varphi_2^\delta,\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\, t>0}\frac{t^{-\frac{Q}{p}}}{\omega_2(t)}\left\|T^{\cdot}\left(A_\gamma f(x)\right)\right\|_{L_{p,\varphi_2^\delta,\gamma}(E_t)} \leq C_4\|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1^\delta,\gamma}}.$$

Thus, Theorem 7 is proved. □

Lemma 5 ([10]). *Let* $1<s<\infty$, $b\in BMO(Y)$. *Then there exists* $C>0$ *such that the inequalities*

$$\left|[b,T_\nu]f\right|(x) \leq M_\nu\left(\left|[b,T_\nu]f\right|(x)\right) \leq C\|b\|_{BMO}\left(\left(M_\nu|T_\nu f|^s\right)^{\frac{1}{s}}(x) + \left(M_\nu\mid f\mid^s\right)^{\frac{1}{s}}(x)\right)$$

hold for all $x\in Y$.

Theorem 8. *Let* $1 < p < \infty$, $0 < \delta < 1$, $b \in BMO(Y)$ *and* $(\varphi_1, \varphi_2) \in \widetilde{A}_p(Y)$, $\varphi_1 \in A_p(Y)$. *Then the operator* $[b, T_\nu]$ *is bounded from* $L_{p,\varphi_1^\delta}(Y)$ *to* $L_{p,\varphi_2^\delta}(Y)$.

Proof. Let $f \in L_{p,\varphi_1^\delta}(Y)$, $b \in BMO(Y)$ and $(\varphi_1, \varphi_2) \in \widetilde{A}_p(Y)$, $\varphi_1 \in A_p(Y)$. From Lemma 5, Corollary 1 and Proposition 2, we get

$$\begin{aligned}
\big\|[b, T_\nu]f\big\|_{L_{p,\varphi_2^\delta}(Y)} &\le \big\|M_\nu\big([b, T_\nu]f\big)\big\|_{L_{p,\varphi_2^\delta}(Y)} \\
&\le C\|b\|_{BMO}\Big\|\big(M_\nu|T_\nu f|^s\big)^{\frac{1}{s}} + \big(M_\nu \mid f \mid^s\big)^{\frac{1}{s}}\Big\|_{L_{p,\varphi_2^\delta}(Y)} \\
&\le C\|b\|_{BMO}\Big[\Big\|\big(M_\nu \mid T_\nu f \mid^s\big)^{\frac{1}{s}}\Big\|_{L_{p,\varphi_2^\delta}(\mathbb{R}^n)} + \Big\|\big(M_\nu|f|^s\big)^{\frac{1}{s}}\Big\|_{L_{p,\varphi_2^\delta}(Y)}\Big] \\
&\le C\|b\|_{BMO}\Big[\Big\|\big(|T_\nu f|^s\big)^{\frac{1}{s}}\Big\|_{L_{p,\varphi_1^\delta}(Y)} + \Big\|\big(|f|^s\big)^{\frac{1}{s}}\Big\|_{L_{p,\varphi_1^\delta}(Y)}\Big] \\
&\le C\|b\|_{BMO}\|f\|_{L_{p,\varphi_1^\delta}(Y)}.
\end{aligned}$$

□

Theorem 9. *Let* $1 < p < \infty$, $0 < \delta < 1$, $b \in BMO_\gamma(\mathbb{R}^n_{k,+})$, $(\varphi_1, \varphi_2) \in \widetilde{A}_{p,\gamma}(\mathbb{R}^n_{k,+})$, $\varphi_1 \in A_{p,\gamma}(\mathbb{R}^n_{k,+})$ *and* $\omega_1(r)$, $\omega_2(r)$ *be positive measurable functions satisfying the condition* (4). *Then the commutator of the* B*-singular integral operator* $[b, A_\gamma]$ *is bounded from* $\mathcal{M}_{p,\omega_1,\varphi_1^\delta,\gamma}(\mathbb{R}^n_{k,+})$ *to* $\mathcal{M}_{p,\omega_2,\varphi_2^\delta,\gamma}(\mathbb{R}^n_{k,+})$.

Proof. We need to introduce a specific maximal operator, defined on a homogeneous type space (Y, d, ν). We mean a topological space $Y = \mathbb{R}^n \times (0, \infty)^k$ equipped with a continuous pseudometric d and a positive measure ν satisfying (5).

Let us define

$$[b, T_\nu\overline{f}](x, \overline{x'}) = \int_{E^*\left((x,\overline{x'}),r\right)} b(y, \overline{y'}) - b(x, \overline{x'})\overline{f}(x - y, \overline{x'} - \overline{y'})K(y, \overline{y'})d\nu(y),$$

where $\overline{f}(x, \overline{x'}) = f\Big(\sqrt{x_1^2 + \overline{x}_1^2}, \ldots, \sqrt{x_k^2 + \overline{x}_k^2}, x''\Big)$.

It can be easily proved that

$$[b, A_\gamma f]\left(\sqrt{z_1^2 + \overline{z}_1^2}, \ldots, \sqrt{z_k^2 + \overline{z}_k^2}, z''\right) = [b, T_\nu\overline{f}]\left(\sqrt{z_1^2 + \overline{z}_1^2}, \ldots, \sqrt{z_k^2 + \overline{z}_k^2}, z'', 0\right),$$

$$[b, A_\gamma f](x) = [b, T_\nu\overline{f}](x, 0).$$

Indeed, from Lemma 2 and

$$\psi_1^\delta(y) = \varphi_1^\delta(y)\big(M_\nu\chi_{E((x,0),r)}(y)\big)^\theta, \quad \psi_2^\delta(y) = \varphi_2^\delta(y)\big(M_\nu\chi_{E((x,0),r)}(y)\big)^\theta, \quad (\psi_1, \psi_2) \in \widetilde{A}_p(Y)$$

we have

$$\begin{aligned}
&\int_{E_r} T^y\big|[b, A_\gamma f](x)\big|^p\varphi_2^\delta(y)(y')^\gamma\, dy \\
&\quad\le \int_{\mathbb{R}^n_{k,+}} T^y\big|[b, A_\gamma f](x)\big|^p\varphi_2^\delta(y)\big(M_\gamma\chi_{E_r}(y)\big)^\theta(y')^\gamma\, dy \\
&\quad= \int_{\mathbb{R}^n\times(0,\infty)^k}\Big|[b, A_\gamma f]\Big(\sqrt{z_1^2 + \overline{z}_1^2}, \ldots, \sqrt{z_k^2 + \overline{z}_k^2}, z''\Big)\Big|^p\varphi_2^\delta(z, \overline{z'})\big(M_\gamma\chi_{E((x,0),r)}(z, \overline{z'})\big)^\theta d\nu(z, \overline{z'}) \\
&\quad= \int_Y\Big|[b, T_\nu\widetilde{f}]\Big(\sqrt{z_1^2 + \overline{z}_1^2}, \ldots, \sqrt{z_k^2 + \overline{z}_k^2}, z'', 0\Big)\Big|^p\varphi_2^\delta(z, \overline{z'})\big(M_\nu\chi_{E((x,0),r)}(z, \overline{z'})\big)^\theta d\nu(z, \overline{z'}).
\end{aligned}$$

By the Theorem 8, we have

$$\left(\int_{E_r} T^y \big|[b, A_\gamma f](x)\big|^p \varphi_2^\delta(y)(y')^\gamma \, dy\right)^{\frac{1}{p}}$$
$$\leq \left(\int_Y \Big|[b, T_\nu \overline{f}]\Big(\sqrt{y_1^2+\overline{y}_1^2}, \dots, \sqrt{y_k^2+\overline{y}_k^2}, y'', 0\Big)\Big|^p \varphi_2^\delta(y, \overline{y'}) \big(M_\nu \chi_{E((x,0),r)}(y, \overline{y'})\big)^\theta \, d\nu(y, \overline{y'})\right)^{\frac{1}{p}}$$
$$= \left(\int_Y \Big|[b, T_\nu \overline{f}]\Big(\sqrt{y_1^2+\overline{y}_1^2}, \dots, \sqrt{y_k^2+\overline{y}_k^2}, y'', 0\Big)\Big|^p \psi_2^\delta(y, \overline{y'}) \, d\nu(y, \overline{y'})\right)^{\frac{1}{p}}$$
$$\leq C_2\|b\|_{BMO}\left(\int_Y \Big|\overline{f}\Big(\sqrt{y_1^2+\overline{y}_1^2}, \dots, \sqrt{y_k^2+\overline{y}_k^2}, y'', 0\Big)\Big|^p \psi_1^\delta(y, \overline{y'}) \, d\nu(y, \overline{y'})\right)^{\frac{1}{p}}$$
$$= C_2\|b\|_{BMO}\left(\int_Y \Big|\overline{f}\Big(\sqrt{y_1^2+\overline{y}_1^2}, \dots, \sqrt{y_k^2+\overline{y}_k^2}, y'', 0\Big)\Big|^p \varphi_1^\delta(y, \overline{y'}) \big(M_\nu \chi_{E((x,0),r)}(y, \overline{y'})\big)^\theta d\nu(y, \overline{y'})\right)^{\frac{1}{p}}$$
$$= C_2\|b\|_{BMO}\left(\int_Y \Big|f\Big(\sqrt{y_1^2+\overline{y}_1^2}, \dots, \sqrt{y_k^2+\overline{y}_k^2}, y''\Big)\Big|^p \varphi_1^\delta(y, \overline{y'}) \big(M_\nu \chi_{E((x,0),r)}(y, \overline{y'})\big)^\theta \, d\nu(y, \overline{y'})\right)^{\frac{1}{p}}$$
$$= C_2\|b\|_{BMO_\gamma}\left(\int_{\mathbb{R}^n_{k,+}} T^y\big[|f|\big]^p(x)\varphi_1^\delta(y)\big(M_\gamma \chi_{E_r}(y)\big)^\theta (y')^\gamma \, dy\right)^{\frac{1}{p}}$$
$$\leq C_2\|b\|_{BMO_\gamma}\Bigg(\int_{E_r} T^y\big[|f|\big]^p(x)\varphi_1^\delta(y)(y')^\gamma dy$$
$$+\sum_{j=1}^{\infty}\int_{E_{2^{j+1}r}\setminus E_{2^j r}} T^y\big[|f|\big]^p(x)\varphi_1^\delta(y)\big(M_\gamma \chi_{E_r}(y)\big)^\theta (y')^\gamma \, dy\Bigg)^{\frac{1}{p}}$$
$$\leq C_2\|b\|_{BMO_\gamma}\Bigg(\int_{E_r} T^y\big[|f|\big]^p(x)\varphi_1^\delta(y)(y')^\gamma dy$$
$$+\sum_{j=1}^{\infty}\int_{E_{2^{j+1}r}\setminus E_{2^j r}} T^y\big[|f|\big]^p(x)\varphi_1^\delta(y)\frac{r^{Q\theta}}{(|y|+r)^{Q\theta}}(y')^\gamma \, dy\Bigg)^{\frac{1}{p}}$$
$$\leq C_3\|b\|_{BMO_\gamma}\|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1^\delta,\gamma}}\left(r^Q\omega_1^p(r)+\sum_{j=1}^{\infty}\frac{1}{(2^j+1)^{Q\theta}}(2^{j+1}r)^Q\omega_1^p(2^{j+1}r)\right)^{\frac{1}{p}}$$
$$\leq C_3\|b\|_{BMO_\gamma}\|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1^\delta,\gamma}}\left(r^Q\omega_1^p(r)+C\int_r^\infty t^{Q-1}\omega_1^p(t)\,dt\right)^{\frac{1}{p}}$$
$$\leq C_4\|b\|_{BMO_\gamma} r^{\frac{Q}{p}}\omega_2(r)\|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1^\delta,\gamma}}.$$

Then we obtain

$$\big\|[b, A_\gamma f]\big\|_{\mathcal{M}_{p,\omega_2,\varphi_2^\delta,\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\, t>0}\frac{t^{-\frac{Q}{p}}}{\omega_2(t)}\Big\|T^{\cdot}\big([b, A_\gamma f](x)\big)\Big\|_{L_{p,\varphi_2^\delta,\gamma}(E_t)} \leq C_4\|b\|_{BMO_\gamma}\|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1^\delta,\gamma}}.$$

□

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Ми вивчаємо максимальний оператор M_γ та сингулярний інтегральний оператор A_γ, пов'язаний з узагальненим оператором зсуву. Узагальнені оператори зсуву пов'язані з диференціальним оператором Лапласа-Бесселя. Наш аналіз ґрунтується на двох зважених нерівностях для максимального оператора, сингулярних інтегральних операторів та їхніх комутаторів, пов'язаних із диференціальним оператором Лапласа-Бесселя в узагальнених зважених просторах B-Моррі.

Ключові слова і фрази: B-максимальний оператор, B-сингулярний інтегральний оператор, комутатор, узагальнений зважений простір B-Моррі.