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Asymptotic estimates for the widths of classes of functions of high smothness

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We find two-sided estimates for Kolmogorov, Bernstein, linear and projection widths of the classes of convolutions of 2π-periodic functions φ, such that $\|\varphi\|_2 \leq 1$, with fixed generated kernels $\Psi_{\bar{\beta}}$, which have Fourier series of the form $\sum\limits_{k=1}^{\infty} \psi(k)\cos(kt - \beta_k\pi/2)$, where $\psi(k) \geq 0$, $\sum \psi^2(k) < \infty, \beta_k \in \mathbb{R}$. It is shown that for rapidly decreasing sequences $\psi(k)$ (in particular, if $\lim\limits_{k\to\infty} \psi(k+1)/\psi(k) = 0$) the obtained estimates are asymptotic equalities. We establish that asymptotic equalities for widths of this classes are realized by trigonometric Fourier sums.

Key words and phrases: Bernstein width, Kolmogorov width, linear width, projection width, Fourier sum, Weyl-Nagy class, class of the generalized Poisson integrals, $(\psi, \bar{\beta})$-integral, asymptotic equality.

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Introduction

Let L_p, $1 \leq p < \infty$, be the space of 2π-periodic functions f summable to the power p on $[-\pi, \pi)$, in which the norm is given by the formula

$$\|f\|_{L_p} = \|f\|_p = \left(\int_{-\pi}^{\pi} |f(t)|^p dt \right)^{1/p},$$

L_∞ be the space of measurable and essentially bounded 2π-periodic functions f with the norm

$$\|f\|_{L_\infty} = \|f\|_\infty = \operatorname*{ess\,sup}_t |f(t)|,$$

C be the space of continuous 2π-periodic functions f, in which the norm is defined by the equality

$$\|f\|_C = \max_t |f(t)|.$$

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Denote by $C^{\psi}_{\bar{\beta},p}$, $1 \le p \le \infty$, the set of all 2π-periodic functions f, representable as convolution

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(x-t)\Psi_{\bar{\beta}}(t)dt, \quad a_0 \in \mathbb{R}, \quad \varphi \in B_p^0, \tag{1}$$

$$B_p^0 = \{g \in L_p : \ \|g\|_p \le 1, \ g \perp 1\}$$

with a fixed generated kernel $\Psi_{\bar{\beta}} \in L_{p'}$, $1/p + 1/p' = 1$, the Fourier series of which has the form

$$S[\Psi_{\bar{\beta}}](t) = \sum_{k=1}^{\infty} \psi(k) \cos\left(kt - \frac{\beta_k \pi}{2}\right), \quad \beta_k \in \mathbb{R}, \quad \psi(k) \ge 0.$$

A function f in the representation (1) is called $(\psi, \bar{\beta})$-integral of the function φ and is denoted by $\mathcal{J}^{\psi}_{\bar{\beta}}\varphi \left(f = \mathcal{J}^{\psi}_{\bar{\beta}}\varphi\right)$. If $\psi(k) \ne 0$, $k \in \mathbb{N}$, then the function φ in the representation (1) is called $(\psi, \bar{\beta})$-derivative of the function f and is denoted by $f^{\psi}_{\bar{\beta}} \left(\varphi = f^{\psi}_{\bar{\beta}}\right)$. The concepts of $(\psi, \bar{\beta})$-integral and $(\psi, \bar{\beta})$-derivative was introduced by A.I. Stepanets (see, e.g., [30,31]). Since $\varphi \in L_p$ and $\Psi_{\bar{\beta}} \in L_{p'}$, the function f of the form (1) is a continuous function, i.e. $C^{\psi}_{\bar{\beta},p} \subset C$ (see [31, Proposition 3.9.2.]). In the case $\beta_k \equiv \beta$, $\beta \in \mathbb{R}$, the classes $C^{\psi}_{\bar{\beta},p}$ are denoted by $C^{\psi}_{\beta,p}$. For $\psi(k) = k^{-r}, r > 0$, the classes $C^{\psi}_{\bar{\beta},p}$ and $C^{\psi}_{\beta,p}$ are denoted by $W^r_{\bar{\beta},p}$ and $W^r_{\beta,p}$, respectively. The classes $W^r_{\beta,p}$ are the well-known Weyl-Nagy classes (see, e.g., [12,29–31]). In other words, $W^r_{\beta,p}, 1 \le p \le \infty$, are the classes of 2π-periodic functions f, representable as convolutions of the form

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(x-t)B_{r,\beta}(t)dt, \quad a_0 \in \mathbb{R}, \tag{2}$$

the Weyl-Nagy kernels $B_{r,\beta}$ of the form

$$B_{r,\beta}(t) = \sum_{k=1}^{\infty} k^{-r} \cos\left(kt - \frac{\beta\pi}{2}\right), \quad r > 0, \quad \beta \in \mathbb{R}, \tag{3}$$

with functions $\varphi \in B_p^0$. The function φ in the formula (2) is called the Weyl-Nagy derivative of the function f and is denoted by f^r_{β}.

If $r \in \mathbb{N}$ and $\beta = r$, then the functions $B_{r,\beta}$ of the form (3) are the well-known Bernoulli kernels and the corresponding classes $W^r_{\beta,p}$ coincide with the well-known classes W^r_p which consist of 2π-periodic functions f with absolutely continuous derivatives $f^{(k)}$ up to $(r-1)$-th order inclusive and such that $\|f^{(r)}\|_p \le 1$. In addition, $f^{(r)}(x) = f^r_r(x) = \varphi(x)$ for almost all $x \in \mathbb{R}$, where φ is the function from (2).

For $\psi(k) = e^{-\alpha k^r}$, $\alpha > 0$, $r > 0$, the classes $C^{\psi}_{\bar{\beta},p}$ and $C^{\psi}_{\beta,p}$ are denoted by $C^{\alpha,r}_{\bar{\beta},p}$ and $C^{\alpha,r}_{\beta,p}$, respectively. The sets $C^{\alpha,r}_{\beta,p}$ are well-known classes of the generalized Poisson integrals (see, e.g., [30,31]), i.e. classes of convolutions

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(x-t)P_{\alpha,r,\beta}(t)dt, \quad a_0 \in \mathbb{R}, \quad \varphi \in B_p^0,$$

with the generalized Poisson kernels

$$P_{\alpha,r,\beta}(t) = \sum_{k=1}^{\infty} e^{-\alpha k^r} \cos\left(kt - \frac{\beta\pi}{2}\right), \quad \alpha > 0, \quad r > 0, \quad \beta \in \mathbb{R}.$$

Let $\mathfrak{N}$ be a some functional class from the space C ($\mathfrak{N} \subset C$). The quantity

$$E_n(\mathfrak{N})_C = \sup_{f \in \mathfrak{N}} E_n(f)_C = \sup_{f \in \mathfrak{N}} \inf_{T_{n-1} \in \mathcal{T}_{2n-1}} \|f - T_{n-1}\|_C$$

is called the best uniform approximation of the class $\mathfrak{N}$ by elements of the subspace $\mathcal{T}_{2n-1}$ of trigonometric polynomials T_{n-1} of the order $n-1$:

$$T_{n-1}(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{n-1} (\alpha_k \cos kx + \beta_k \sin kx), \quad \alpha_k, \beta_k \in \mathbb{R}.$$

The order estimates for the best approximations $E_n(K)_C$ of classes $K = C^{\psi}_{\bar{\beta},p}$, $1 \le p \le \infty$, (and, hence, classes $W^r_{\beta,p}$, $C^{\alpha,r}_{\beta,p}$ and $C^{\psi}_{\beta,p}$) depending on rate of decreasing to zero of sequences $\psi(k)$ were obtained, in particular, in the works of V.N. Temlyakov [35], U.Z. Hrabova and A.S. Serdyuk [5], A.S. Serdyuk and T.A. Stepanyuk [25,26].

If the sequences $\psi(k)$ decrease to zero faster than any geometric progression, then asymptotic equations of the best uniform approximations are even known (see, for example, the authors work [24] and the bibliography available there).

In [24], it was shown that for such classes $C^{\psi}_{\bar{\beta},p}$ the following asymptotic equations take places

$$E_n\left(C^{\psi}_{\bar{\beta},p}\right)_C \sim \mathcal{E}_n\left(C^{\psi}_{\bar{\beta},p}\right)_C \sim \frac{\|\cos t\|_{p'}}{\pi}\psi(n), \quad 1 \le p \le \infty,$$

where

$$\mathcal{E}_n\left(C^{\psi}_{\bar{\beta},p}\right) = \sup_{f \in C^{\psi}_{\bar{\beta},p}} \|f - S_{n-1}(f)\|_C,$$

$S_{n-1}(f)$ is the partial Fourier sum of order $n-1$ of the function f, $\frac{1}{p} + \frac{1}{p'} = 1$, and $A(n) \sim B(n)$ as $n \to \infty$ means that $\lim_{n\to\infty} A(n)/B(n) = 1$.

For $p = \infty$ in the case of $K = W^r_{\bar{\beta},\infty}, r > 0$, and in the cases of $K = C^{\alpha,r}_{\bar{\beta},\infty}, r \ge 1$, and $K = C^{\psi}_{\bar{\beta},\infty}$ $\left(K = C^{\psi}_{\beta,\infty}\right)$ for certain restrictions on sequences ψ and $\bar{\beta}$ the exact values of the best uniform approximations are known thanks to the works of J. Favard [6,7], N.I. Akhiezer and M.G. Krein [1], M.G. Krein [10], B. Sz.-Nagy [12], S.B. Stechkin [29], V.K. Dzyadyk [3,4], Y.-S. Sun [33], A.V. Bushanskij [2], A. Pinkus [13], A.S. Serdyuk [15–19] etc.

For $p = 2$ and for arbitrary $\bar{\beta} = \beta_k \in \mathbb{R}$, $\sum\limits_{k=1}^{\infty} \psi^2(k) < \infty$ the exact values for the quantity $\mathcal{E}_n\left(C^{\psi}_{\bar{\beta},2}\right)_C$ are also known (see [23]).

In this paper, we establish two-sided estimates of Kolmogorov, Bernstein, linear and projection widths of the classes $C^{\psi}_{\bar{\beta},2}$ in the space C, which become into asymptotic equations under certain restrictions on the sequence $\psi(k)$ (in particular, if $\lim_{k\to\infty} \psi(k+1)/\psi(k) = 0$).

Let K be a convex centrally symmetric subset of C and let B be a unit ball of the space C. Let also F_N be an arbitrary N-dimensional subspace of space C, $N \in \mathbb{N}$, and $\mathscr{L}(C, F_N)$ be a set of linear operators from C to F_N. By $\mathscr{P}(C, F_N)$ we denote the subset of projection operators of the set $\mathscr{L}(C, F_N)$, that is, the set of the operators A of linear projection onto the set F_N such that $Af = f$ when $f \in F_N$. The quantities

$$b_N(K, C) = \sup_{F_{N+1}} \sup \left\{\varepsilon > 0 : \varepsilon B \cap F_{N+1} \subset K\right\}, \tag{4}$$

$$d_N(K,C)=\inf_{F_N}\sup_{f\in K}\inf_{u\in F_N}\|f-u\|_C, \qquad (5)$$

$$\lambda_N(K,C)=\inf_{F_N}\inf_{A\in\mathscr{L}(C,F_N)}\sup_{f\in K}\|f-Af\|_C, \qquad (6)$$

$$\pi_N(K,C)=\inf_{F_N}\inf_{A\in\mathscr{P}(C,F_N)}\sup_{f\in K}\|f-Af\|_C \qquad (7)$$

are called Bernstein, Kolmogorov, linear, and projection N-widths of the set K in the space C, respectively.

The results containing order estimates of the widths (4)–(7) in the case of $K=C^{\psi}_{\bar\beta,p}$ (and, in particular, $W^r_{\beta,p}$ and $C^{\psi}_{\beta,p}$) can be found, for example, in the works of V.M. Tikhomirov [36], A. Pinkus [13], N.P. Kornejchuk [8], A.K. Kushpel' [11], A.S. Romanyuk [14], V.N. Temlyakov [34,35] etc.

1 Main results

The main result of this paper is the following statement.

Theorem 1. *Let $\bar\beta=\{\beta_k\}_{k=1}^{\infty}$, $\beta_k\in\mathbb{R}$, and $\psi(k)>0$ satisfies the condition*

$$\sum_{k=1}^{\infty}\psi^2(k)<\infty. \qquad (8)$$

Then for all $n\in\mathbb{N}$ the following inequalities hold

$$\frac{1}{\sqrt{\pi}}\left(\frac{1}{\psi^2(n)}+2\sum_{k=1}^{n-1}\frac{1}{\psi^2(k)}\right)^{-\frac12}\le P_{2n}\left(C^{\psi}_{\bar\beta,2},C\right)\le P_{2n-1}\left(C^{\psi}_{\bar\beta,2},C\right)\le\frac{1}{\sqrt{\pi}}\left(\sum_{k=n}^{\infty}\psi^2(k)\right)^{\frac12}, \qquad (9)$$

where P_N is any of the widths b_N, d_N, λ_N or π_N.

If, in adition, $\psi(k)$ satisfies the condition

$$\lim_{n\to\infty}\max\left\{\psi(n)\left(\sum_{k=1}^{n-1}\frac{1}{\psi^2(k)}\right)^{\frac12},\frac{1}{\psi(n)}\left(\sum_{k=n+1}^{\infty}\psi^2(k)\right)^{\frac12}\right\}=0, \qquad (10)$$

then the following asymptotic equalities hold

$$\left.\begin{array}{l}P_{2n}(C^{\psi}_{\bar\beta,2},C)\\ P_{2n-1}(C^{\psi}_{\bar\beta,2},C)\end{array}\right\}=\psi(n)\left(\frac{1}{\sqrt{\pi}}+\mathcal{O}(1)\max\left\{\psi(n)\left(\sum_{k=1}^{n-1}\frac{1}{\psi^2(k)}\right)^{\frac12},\frac{1}{\psi(n)}\left(\sum_{k=n+1}^{\infty}\psi^2(k)\right)^{\frac12}\right\}\right), \qquad (11)$$

where $\mathcal{O}(1)$ are the quantities uniformly bounded in all parameters. The equalities (11) *are realized by trigonometric Fourier sums $S_{n-1}(f)$.*

Proof. In the work [23], it was proved that if the condition (8) is satisfied, then the following equality

$$\mathcal{E}\left(C^{\psi}_{\bar\beta,2};S_{n-1}\right)_C=\frac{1}{\sqrt{\pi}}\left(\sum_{k=n}^{\infty}\psi^2(k)\right)^{\frac12},\quad \beta_k\in\mathbb{R},\quad n\in\mathbb{N}, \qquad (12)$$

holds.

Since the operator that assigns to each function $f \in C$ its partial Fourier sum is a linear projector, then by virtue (12)

$$\pi_{2n-1}\left(C^{\psi}_{\bar\beta,2},C\right) \le \mathcal{E}\left(C^{\psi}_{\bar\beta,2};S_{n-1}\right)_C = \frac{1}{\sqrt{\pi}}\left(\sum_{k=n}^{\infty}\psi^2(k)\right)^{\frac12}, \quad \beta_k\in\mathbb{R}, \quad n\in\mathbb{N}. \tag{13}$$

For all $n \in \mathbb{N}$ and $\mathfrak{N} \subset C$

$$P_{2n}(\mathfrak{N},C) \le P_{2n-1}(\mathfrak{N},C),$$

where P_N is any of the widths b_N, d_N, λ_N, and π_N, and, in addition, for all $N \in \mathbb{N}$

$$b_N(\mathfrak{N},C) \le d_N(\mathfrak{N},C) \le \lambda_N(\mathfrak{N},C) \le \pi_N(\mathfrak{N},C). \tag{14}$$

Therefore on the basis of (13) we obtaine an estimate from above for the widths P_N in the formula (9). To obtain a required estimate from below in (9) it suffices to establish that

$$b_{2n}\left(C^{\psi}_{\bar\beta,2},C\right) \ge \frac{1}{\sqrt{\pi}}\left(\frac{1}{\psi^2(n)}+2\sum_{k=1}^{n-1}\frac{1}{\psi^2(k)}\right)^{-\frac12}. \tag{15}$$

In $(2n+1)$-dimensional space $\mathcal{T}_{2n+1}$ of trigonometric polynomials T_n of order n let us consider a ball of the form

$$B_{2n+1} = \left\{T_n \in \mathcal{T}_{2n+1} : \|T_n\|_C \le \frac{1}{\sqrt{\pi}}\left(\frac{1}{\psi^2(n)}+2\sum_{k=1}^{n-1}\frac{1}{\psi^2(k)}\right)^{-\frac12}\right\} \tag{16}$$

and prove the following embedding

$$B_{2n+1} \subset C^{\psi}_{\bar\beta,2}. \tag{17}$$

For any trinometric polynomial

$$T_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k\cos kx + b_k \sin kx) \tag{18}$$

from the ball B_{2n+1} its $(\psi,\bar\beta)$-derivative has a form

$$\begin{aligned}(T_n)^{\psi}_{\bar\beta}(x) &= \sum_{k=1}^{n}\left(\frac{a_k}{\psi(k)}\cos\left(kx+\frac{\beta_k\pi}{2}\right)+\frac{b_k}{\psi(k)}\sin\left(kx+\frac{\beta_k\pi}{2}\right)\right)\\
&= \sum_{k=1}^{n}\left(\frac{a_k\cos\frac{\beta_k\pi}{2}}{\psi(k)}\cos kx - \frac{a_k\sin\frac{\beta_k\pi}{2}}{\psi(k)}\sin kx + \frac{b_k\cos\frac{\beta_k\pi}{2}}{\psi(k)}\sin kx + \frac{b_k\sin\frac{\beta_k\pi}{2}}{\psi(k)}\cos kx\right)\\
&= \sum_{k=1}^{n}\frac{1}{\psi(k)}\left(\left(a_k\cos\frac{\beta_k\pi}{2}+b_k\sin\frac{\beta_k\pi}{2}\right)\cos kx + \left(-a_k\sin\frac{\beta_k\pi}{2}+b_k\cos\frac{\beta_k\pi}{2}\right)\sin kx\right).\end{aligned}$$

By virtue of Parseval equality, from the above equalities we get

$$\begin{aligned}\left\|(T_n)^{\psi}_{\bar\beta}\right\|_2 &= \sqrt{\pi}\left(\sum_{k=1}^{n}\frac{1}{\psi^2(k)}\left(\left(a_k\cos\frac{\beta_k\pi}{2}+b_k\sin\frac{\beta_k\pi}{2}\right)^2\right.\right.\\
&\qquad\left.\left.+\left(-a_k\sin\frac{\beta_k\pi}{2}+b_k\cos\frac{\beta_k\pi}{2}\right)^2\right)\right)^{\frac12}\\
&= \sqrt{\pi}\left(\sum_{k=1}^{n}\frac{1}{\psi^2(k)}\left(a_k^2+b_k^2\right)\right)^{\frac12}.\end{aligned} \tag{19}$$

By Parseval equality for the polynomial T_n of the form (18) we obtaine

$$\frac{a_0^2}{2}+\sum_{k=1}^{n}\left(a_k^2+b_k^2\right)=\frac{1}{\pi}\int_{-\pi}^{\pi}T_n^2(x)dx.$$

Therefore we have a chain of inequalities

$$a_k^2+b_k^2\leq\sum_{k=1}^{n}\left(a_k^2+b_k^2\right)\leq\frac{1}{\pi}\int_{-\pi}^{\pi}\|T_n\|_C^2\,dx=2\,\|T_n\|_C^2\,,$$

and, consequently, we obtaine an estimate for $\sqrt{a_k^2+b_k^2}$ of the following form

$$\sqrt{a_k^2+b_k^2}\leq\sqrt{2}\,\|T_n\|_C\,,\quad k=\overline{1,n}. \tag{20}$$

In the case of $k=n$ this estimate can be improved. To do this, let us consider a trigonometric polynomial

$$\begin{aligned}\tau_n(x):=\frac{T_n(x)}{\sqrt{a_n^2+b_n^2}}&=\frac{1}{\sqrt{a_n^2+b_n^2}}\left(\frac{a_0}{2}+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx)\right)\\&=\frac{1}{\sqrt{a_n^2+b_n^2}}\left(\frac{a_0}{2}+\sum_{k=1}^{n}\sqrt{a_k^2+b_k^2}\left(\frac{a_k}{\sqrt{a_k^2+b_k^2}}\cos kx+\frac{b_k}{\sqrt{a_k^2+b_k^2}}\sin kx\right)\right)\\&=\frac{\rho_0}{2}+\sum_{k=1}^{n}\rho_k\cos(kx+\theta_k),\end{aligned} \tag{21}$$

where

$$\rho_0=\frac{a_0}{\sqrt{a_n^2+b_n^2}},\quad\rho_k=\frac{\sqrt{a_k^2+b_k^2}}{\sqrt{a_n^2+b_n^2}},\quad k=\overline{1,n-1},\quad\rho_n=1,$$

and θ_k are such that

$$\begin{cases}\cos\theta_k=\dfrac{a_k}{\sqrt{a_k^2+b_k^2}}, & k=\overline{1,n},\\[2mm]\sin\theta_k=\dfrac{-b_k}{\sqrt{a_k^2+b_k^2}}, & k=\overline{1,n}.\end{cases}$$

As it follows from [9, Statement 2.9.1] for all $p\in[1,\infty]$ the following inequality

$$\left\|\tau_n(\cdot)\right\|_p\geq\left\|\cos n(\cdot)\right\|_p,$$

holds and, consequently, for $p=\infty$

$$\|\tau_n\|_C\geq 1. \tag{22}$$

From (21) and (22) we get

$$\sqrt{a_n^2+b_n^2}=\frac{\|T_n\|_C}{\|\tau_n\|_C}\leq\|T_n\|_C\,. \tag{23}$$

Using the equations (19) and the estimates (20) and (23) we have

$$\begin{aligned}\left\|(T_n)_{\bar\beta}^{\psi}\right\|_2&=\sqrt{\pi}\left(\sum_{k=1}^{n-1}\frac{a_k^2+b_k^2}{\psi^2(k)}+\frac{a_n^2+b_n^2}{\psi^2(n)}\right)^{\frac12}\\&\leq\sqrt{\pi}\left(2\sum_{k=1}^{n-1}\frac{\|T_n\|_C^2}{\psi^2(k)}+\frac{\|T_n\|_C^2}{\psi^2(n)}\right)^{\frac12}=\sqrt{\pi}\left(\frac{1}{\psi^2(n)}+2\sum_{k=1}^{n-1}\frac{1}{\psi^2(k)}\right)^{\frac12}\|T_n\|_C\,.\end{aligned} \tag{24}$$

Since the polynomials T_n belongs to the ball B_{2n+1} of the form (16), from (24) it follows that

$$\left\|(T_n)^{\psi}_{\bar\beta}\right\|_2 \leq 1.$$

The embedding (17) is proved.

The inequality (15) follows from the definition of the Berstein width $b_{2n}\left(C^{\psi}_{\bar\beta,2}, C\right)$ and the embedding (17). The relations (13)–(15) prove the inequalities (9). To prove the asymptotic equations (11) under satisfying the condition (10) first of all we note that

$$\left(\sum_{k=n}^{\infty}\psi^2(k)\right)^{\frac{1}{2}} \leq \psi(n) + \left(\sum_{k=n+1}^{\infty}\psi^2(k)\right)^{\frac{1}{2}} \tag{25}$$

and

$$\left(\frac{1}{\psi^2(n)} + 2\sum_{k=1}^{n-1}\frac{1}{\psi^2(k)}\right)^{\frac{1}{2}} \leq \frac{1}{\psi(n)} + \left(2\sum_{k=1}^{n-1}\frac{1}{\psi^2(k)}\right)^{\frac{1}{2}}. \tag{26}$$

From (26) we get

$$\left(\frac{1}{\psi^2(n)} + 2\sum_{k=1}^{n-1}\frac{1}{\psi^2(k)}\right)^{-\frac{1}{2}} \geq \frac{1}{\frac{1}{\psi(n)} + \left(2\sum_{k=1}^{n-1}\frac{1}{\psi^2(k)}\right)^{\frac{1}{2}}}$$

$$= \psi(n) - \left(\frac{1}{\frac{1}{\psi(n)}} - \frac{1}{\frac{1}{\psi(n)} + \left(2\sum_{k=1}^{n-1}\frac{1}{\psi^2(k)}\right)^{\frac{1}{2}}}\right) = \psi(n) - \frac{\left(2\sum_{k=1}^{n-1}\frac{1}{\psi^2(k)}\right)^{\frac{1}{2}}}{\frac{1}{\psi(n)}\left(\frac{1}{\psi(n)} + \left(2\sum_{k=1}^{n-1}\frac{1}{\psi^2(k)}\right)^{\frac{1}{2}}\right)}$$

$$= \psi(n)\left(1 - \frac{\left(2\sum_{k=1}^{n-1}\frac{1}{\psi^2(k)}\right)^{\frac{1}{2}}}{\frac{1}{\psi(n)} + \left(2\sum_{k=1}^{n-1}\frac{1}{\psi^2(k)}\right)^{\frac{1}{2}}}\right) \geq \psi(n)\left(1 - \psi(n)\left(2\sum_{k=1}^{n-1}\frac{1}{\psi^2(k)}\right)^{\frac{1}{2}}\right).$$

So, as it follows from (9) and (25), on the one hand,

$$P_{2n}\left(C^{\psi}_{\bar\beta,2}, C\right) \leq P_{2n-1}\left(C^{\psi}_{\bar\beta,2}, C\right) \leq \psi(n)\left(\frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi}\psi(n)}\left(\sum_{k=n+1}^{\infty}\psi^2(k)\right)^{\frac{1}{2}}\right), \tag{27}$$

and, on the other hand, by virtue of (9)

$$P_{2n}\left(C^{\psi}_{\bar\beta,2}, C\right) \geq \psi(n)\left(\frac{1}{\sqrt{\pi}} - \sqrt{\frac{2}{\pi}}\psi(n)\left(\sum_{k=1}^{n-1}\frac{1}{\psi^2(k)}\right)^{\frac{1}{2}}\right). \tag{28}$$

The combination of (27) and (28) allows us to write equations

$$P_{2n}\left(C^{\psi}_{\bar\beta,2}, C\right) = \psi(n)\left(\frac{1}{\sqrt{\pi}} + \gamma_n^{(1)}\right), \tag{29}$$

$$P_{2n-1}\left(C^{\psi}_{\bar\beta,2}, C\right) = \psi(n)\left(\frac{1}{\sqrt{\pi}} + \gamma_n^{(2)}\right), \tag{30}$$

in which for $\gamma_n^{(i)}, i = 1, 2$, the following double inequalities

$$-\sqrt{\frac{2}{\pi}}\psi(n)\left(\sum_{k=1}^{n-1}\frac{1}{\psi^2(k)}\right)^{\frac{1}{2}} \leq \gamma_n^{(i)} \leq \frac{1}{\sqrt{\pi}\psi(n)}\left(\sum_{k=n+1}^{\infty}\psi^2(k)\right)^{\frac{1}{2}} \tag{31}$$

hold. If the condition (10) is satisfied, then by virtue of (29)–(31) the asymptotic equations (11) take place. Theorem 1 is proved. □

We note that the condition

$$\lim_{n\to\infty}\frac{1}{\psi^2(n)}\sum_{k=n+1}^{\infty}\psi^2(k) = 0$$

is satisfied if $\psi(k)$ satisfies the condition

$$\lim_{k\to\infty}\frac{\psi(k+1)}{\psi(k)} = 0. \tag{32}$$

To make sure of this, let us put

$$\varepsilon_n = \sup_{k\geq n}\frac{\psi(k+1)}{\psi(k)}.$$

By virtue of (32) $\varepsilon_n \downarrow 0$ as $n \to \infty$. So, we get

$$\sum_{k=n+1}^{\infty}\psi^2(k) = \psi^2(n)\left(\frac{\psi^2(n+1)}{\psi^2(n)} + \frac{\psi^2(n+2)}{\psi^2(n+1)}\frac{\psi^2(n+1)}{\psi^2(n)} + \dots\right)$$
$$\leq \psi^2(n)\left(\varepsilon_n^2 + \varepsilon_n^4 + \dots\right) = \psi^2(n)\frac{\varepsilon_n^2}{1-\varepsilon_n^2} = o\Big(\psi^2(n)\Big).$$

Let us show that for strictly decreasing sequences ψ the fulfillment of condition of the form (32) ensures the truth of the following equality

$$\lim_{n\to\infty}\psi^2(n)\sum_{k=1}^{n-1}\frac{1}{\psi^2(k)} = 0. \tag{33}$$

To do this, we use Stoltz's theorem, according to which the relation (33) is followed from the following equality

$$\lim_{n\to\infty}\frac{\sum\limits_{k=1}^{n-1}\frac{1}{\psi^2(k)} - \sum\limits_{k=1}^{n-2}\frac{1}{\psi^2(k)}}{\frac{1}{\psi^2(n)} - \frac{1}{\psi^2(n-1)}} = 0. \tag{34}$$

Since

$$\frac{\sum\limits_{k=1}^{n-1}\frac{1}{\psi^2(k)} - \sum\limits_{k=1}^{n-2}\frac{1}{\psi^2(k)}}{\frac{1}{\psi^2(n)} - \frac{1}{\psi^2(n-1)}} = \frac{\frac{1}{\psi^2(n-1)}}{\frac{1}{\psi^2(n)} - \frac{1}{\psi^2(n-1)}} = \frac{\frac{\psi^2(n)}{\psi^2(n-1)}}{1-\frac{\psi^2(n)}{\psi^2(n-1)}},$$

then (34) follows from (32). In view of the above, we have the following statement.

Corollary 1. *Let $\bar{\beta} = \{\beta_k\}_{k=1}^{\infty}, \beta_k \in \mathbb{R}$ and the sequence $\psi(k) > 0$ is strictly decreasing and satisfies the condition of the form* (32)*. Then the asymptotic equalities* (11) *hold as $n \to \infty$.*

We give the corollaries of Theorem 1 in some important special cases.

Theorem 2. *Let $\bar{\beta} = \{\beta_k\}_{k=1}^{\infty}, \beta_k \in \mathbb{R}$ and $n \in \mathbb{N}$. Then for all $r \geq \frac{n+1}{2}$ the following inequalities*

$$\frac{1}{\sqrt{\pi}} n^{-r} \left(1 - \frac{4\big(1-\frac{1}{n}\big)^{2r}}{1+4\big(1-\frac{1}{n}\big)^{2r}}\right)^{\frac{1}{2}} \leq P_{2n}\left(W^r_{\bar{\beta},2}, C\right) \leq P_{2n-1}\left(W^r_{\bar{\beta},2}, C\right) \leq \frac{1}{\sqrt{\pi}} n^{-r}\left(1 + \frac{2+\frac{1}{n}}{\big(1+\frac{1}{n}\big)^r}\right)^{\frac{1}{2}}, \tag{35}$$

hold, where P_N is any of the widths b_N, d_N, λ_N or π_N.

Proof. Let us put $\psi(k) = k^{-r}, r > 1$. Obviously, the condition (8) is satisfied. Since for $2r \geq n+1$, $n \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{k=n+1}^{\infty} \frac{1}{k^{2r}} &< \frac{1}{(n+1)^{2r}} + \int_{n+1}^{\infty} \frac{dt}{t^{2r}} = \frac{1}{(n+1)^{2r}} + \frac{1}{(2r-1)(n+1)^{2r-1}} = \frac{1}{(n+1)^{2r}} \frac{2r+n}{2r-1} \\ &\leq \frac{1}{(n+1)^{2r}} \frac{4r-1}{2r-1} \leq \frac{1}{(n+1)^{2r}}\left(2 + \frac{1}{2r-1}\right) \leq \frac{1}{n^{2r}} \frac{2+\frac{1}{n}}{(1+\frac{1}{n})^{2r}},\end{aligned} \tag{36}$$

then according to the right-hand side of the equality (9) of Theorem 1 we obtain the estimate

$$P_{2n-1}\left(W^r_{\bar{\beta},2}, C\right) \leq \frac{1}{\sqrt{\pi}}\left(\sum_{k=n}^{\infty} \frac{1}{k^{2r}}\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{\pi}} n^{-r}\left(1 + \frac{2+\frac{1}{n}}{\big(1+\frac{1}{n}\big)^{2r}}\right)^{\frac{1}{2}}. \tag{37}$$

On the other hand, for $r \geq \dfrac{n+1}{2}$ and $\psi(k) = k^{-r}$ we have

$$\begin{aligned}\frac{1}{\psi^2(n)} + 2\sum_{k=1}^{n-1} \frac{1}{\psi^2(k)} &= n^{2r} + 2\sum_{k=1}^{n-1} k^{2r} \leq n^{2r} + 2\left((n-1)^{2r} + \int_1^{n-1} t^{2r} dt\right) \\ &= n^{2r} + 2\left((n-1)^{2r} + \frac{(n-1)^{2r+1}}{2r+1} - \frac{1}{2r+1}\right) \\ &< n^{2r} + 2\left((n-1)^{2r} + \frac{(n-1)^{2r+1}}{n+2}\right) \\ &< n^{2r} + 4(n-1)^{2r} = n^{2r}\left(1 + 4\Big(1-\frac{1}{n}\Big)^{2r}\right).\end{aligned} \tag{38}$$

By virtue of the left part of the inequality (9) of Theorem 1 and the formula (38) we get the estimate

$$P_{2n}\left(W^r_{\bar{\beta},2}, C\right) \geq \frac{1}{\sqrt{\pi}} n^{-r}\left(\frac{1}{1+4\big(1-\frac{1}{n}\big)^{2r}}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{\pi}} n^{-r}\left(1 - \frac{4\big(1-\frac{1}{n}\big)^{2r}}{1+4\big(1-\frac{1}{n}\big)^{2r}}\right)^{\frac{1}{2}}. \tag{39}$$

Combining the estimates (37) and (39) we obtaine (35). Theorem 2 is proved. □

Note that if the condition

$$\lim_{n\to\infty}\frac{r}{n}=\infty \tag{40}$$

is satisfied, then for $\psi(k)=k^{-r}$ the condition (32) is also satisfied, because

$$\frac{\psi(k+1)}{\psi(k)}=\left(\frac{k}{k+1}\right)^{r}=\left(1+\frac{1}{k}\right)^{-r}=\left(\left(1+\frac{1}{k}\right)^{k+1}\right)^{-\frac{r}{k+1}}\le e^{-\frac{r}{k+1}}\to 0,\quad k\to\infty.$$

Taking the limit as $n\to\infty$ in the relations (35), we obtain the following statement.

Theorem 3. *Let $\bar\beta=\{\beta_k\}_{k=1}^{\infty},\beta_k\in\mathbb{R}$, $n\in\mathbb{N}$, and the condition (40) is satisfied. Then the following asymptotic equalities*

$$\left.\begin{array}{l}P_{2n}(W^r_{\bar\beta,2},C)\\ P_{2n-1}(W^r_{\bar\beta,2},C)\end{array}\right\}=n^{-r}\left(\frac{1}{\sqrt{\pi}}+\mathcal{O}(1)\left(1+\frac{1}{n}\right)^{-r}\right) \tag{41}$$

hold, where P_N is any of the widths b_N,d_N,λ_N or π_N, and $\mathcal{O}(1)$ are the quantities uniformly bounded in all parameters.

Note also that the equalities (41) are easy obtained from the formula (11) and estimates (36) and (38).

Theorem 4. *Let $\bar\beta=\{\beta_k\}_{k=1}^{\infty},\beta_k\in\mathbb{R}$, $\alpha>0$, $r>1$, $n\in\mathbb{N}$ and be such that*

$$(n-1)^r>\frac{1}{\alpha}, \tag{42}$$

then the following inequalities

$$\frac{1}{\sqrt{\pi}}e^{-\alpha n^r}\left(1-\frac{2\gamma_{\alpha,r,n}e^{-2\alpha r(n-1)^{r-1}}}{1+2\gamma_{\alpha,r,n}e^{-2\alpha r(n-1)^{r-1}}}\right)^{\frac12}\le P_{2n}\left(C^{\alpha,r}_{\bar\beta,2},C\right)\le P_{2n-1}\left(C^{\alpha,r}_{\bar\beta,2},C\right)$$
$$\le\frac{1}{\sqrt{\pi}}e^{-\alpha n^r}\left(1+e^{-2\alpha r n^{r-1}}\left(1+\frac{1}{2\alpha r n^{r-1}}\right)\right)^{\frac12} \tag{43}$$

hold, where P_N is any of the widths b_N,d_N,λ_N or π_N and

$$\gamma_{\alpha,r,n}=\left(1+\frac{1}{\alpha r(n-1)^{r-1}}+e^{-2\alpha(n-1)^r}\max\left\{e^{4\alpha},\frac{e^2}{\alpha^{1+1/r}}\right\}\right). \tag{44}$$

Proof. First of all, note that if $\alpha>0$, $r>1$, $n\in\mathbb{N}$ and satisfy the condition (42), then for a quantity of the form

$$I_{n-1}:=\int_1^{n-1}e^{2\alpha t^r}dt,\quad \alpha>0,\ r>1,$$

the following inequality

$$I_{n-1}\le\frac{e^{2\alpha(n-1)^r}}{\alpha r(n-1)^{r-1}}+\max\left\{e^{4\alpha},\frac{e^2}{\alpha^{1+1/r}}\right\} \tag{45}$$

holds. Indeed, integrating by parts we have

$$I_{n-1}=\frac{1}{2\alpha r}\int_1^{n-1}t^{1-r}de^{2\alpha t^r}=\frac{1}{2\alpha r}\left(\frac{e^{2\alpha(n-1)^r}}{(n-1)^{r-1}}-e^{2\alpha}\right)+\frac{r-1}{2\alpha r}\int_1^{n-1}\frac{e^{2\alpha t^r}}{t^r}dt. \tag{46}$$

For $0<2\alpha<1$, taking into account (42), we obtain

$$\int_1^{n-1}\frac{e^{2\alpha t^r}}{t^r}dt=\int_1^{\alpha^{-1/r}}\frac{e^{2\alpha t^r}}{t^r}dt+\int_{\alpha^{-1/r}}^{n-1}\frac{e^{2\alpha t^r}}{t^r}dt<\frac{e^2}{\alpha^{1/r}}+\alpha\int_{\alpha^{-1/r}}^{n-1}e^{2\alpha t^r}dt\le\frac{e^2}{\alpha^{1/r}}+\alpha I_{n-1}. \quad (47)$$

From (46) and (47) under condition (42) we get

$$I_{n-1}<\frac{e^{2\alpha(n-1)^r}}{2\alpha r(n-1)^{r-1}}+\frac{e^2}{2\alpha^{1+1/r}}+\frac{1}{2}I_{n-1}$$

and

$$I_{n-1}<\frac{e^{2\alpha(n-1)^r}}{\alpha r(n-1)^{r-1}}+\frac{e^2}{\alpha^{1+1/r}},\quad 0<2\alpha<1. \quad (48)$$

For $2\alpha\ge 1$ we obtain

$$\begin{aligned}\int_1^{n-1}\frac{e^{2\alpha t^r}}{t^r}dt&=\int_1^{2^{1/r}}\frac{e^{2\alpha t^r}}{t^r}dt+\int_{2^{1/r}}^{n-1}\frac{e^{2\alpha t^r}}{t^r}dt\le\int_1^{2^{1/r}}e^{2\alpha t^r}dt+\frac{1}{2}\int_{2^{1/r}}^{n-1}e^{2\alpha t^r}dt\\&=\frac{1}{2}\int_1^{2^{1/r}}e^{2\alpha t^r}dt+\frac{1}{2}I_{n-1}<\frac{2^{1/r}-1}{2}e^{4\alpha}+\frac{1}{2}I_{n-1}<\frac{e^{4\alpha}}{2}+\frac{1}{2}I_{n-1}.\end{aligned} \quad (49)$$

From (46) and (49) under condition (42) we have

$$I_{n-1}<\frac{e^{2\alpha(n-1)^r}}{2\alpha r(n-1)^{r-1}}+\frac{e^{4\alpha}}{4\alpha}+\frac{1}{4\alpha}I_{n-1}<\frac{e^{2\alpha(n-1)^r}}{2\alpha r(n-1)^{r-1}}+\frac{e^{4\alpha}}{2}+\frac{1}{2}I_{n-1}$$

and

$$I_{n-1}<\frac{e^{2\alpha(n-1)^r}}{\alpha r(n-1)^{r-1}}+e^{4\alpha},\quad 2\alpha\ge 1. \quad (50)$$

The inequality (45) follows from (48) and (50). For $\psi(k)=e^{-\alpha k^r}$, $\alpha>0$, $r>1$, under condition (42), taking into account that for $r>1$ and $n\in\mathbb{N}$ the inequality

$$\left(1+\frac{1}{n}\right)^r-1>\frac{r}{n}$$

holds, we obtain

$$\begin{aligned}\psi^2(n)\sum_{k=1}^{n-1}\frac{1}{\psi^2(k)}&=e^{-2\alpha n^r}\sum_{k=1}^{n-1}e^{2\alpha k^r}\le e^{-\alpha n^r}\left(e^{2\alpha(n-1)^r}+\int_1^{n-1}e^{2\alpha t^r}dt\right)\\&\le e^{-2\alpha n^r}\left(e^{2\alpha(n-1)^r}\left(1+\frac{1}{\alpha r(n-1)^{r-1}}+e^{-2\alpha(n-1)^r}\max\left\{e^{4\alpha},\frac{e^2}{\alpha^{1+1/r}}\right\}\right)\right)\\&\le e^{-2\alpha(n-1)^r((1+\frac{1}{n-1})^r-1)}\gamma_{\alpha,r,n}\le\gamma_{\alpha,r,n}e^{-2\alpha r(n-1)^{r-1}}.\end{aligned} \quad (51)$$

Thus, by virtue of the left part of the inequality (9) of Theorem 1 and (51), we obtain a required estimate from below for widths $P_{2n}\left(C^{\alpha,r}_{\beta,2},C\right)$, $\alpha>0,\ r>1$, under condition (42)

$$P_{2n}(C^{\alpha,r}_{\bar\beta,2},C)\ge\frac{1}{\sqrt{\pi}}\left(e^{2\alpha n^r}+2\sum_{k=1}^{n-1}e^{2\alpha k^r}\right)^{-\frac{1}{2}}\ge\frac{1}{\sqrt{\pi}}e^{-\alpha n^r}\left(1-\frac{2\gamma_{\alpha,r,n}e^{-2\alpha r(n-1)^{r-1}}}{1+2\gamma_{\alpha,r,n}e^{-2\alpha r(n-1)^{r-1}}}\right)^{\frac{1}{2}}. \quad (52)$$

As was shown in [30, p. 163–164],

$$\sum_{k=n+1}^{\infty} e^{-\alpha k^r} < e^{-\alpha n^r} \left(1 + \frac{1}{\alpha r n^{r-1}}\right) e^{-\alpha r n^{r-1}}, \quad r > 1, \alpha > 0, n \in \mathbb{N}.$$

Therefore,

$$\frac{1}{\psi^2(n)} \sum_{k=n+1}^{\infty} \psi^2(k) = e^{2\alpha n^r} \sum_{k=n+1}^{\infty} e^{-2\alpha k^r} < \left(1 + \frac{1}{2\alpha r n^{r-1}}\right) e^{-2\alpha r n^{r-1}}. \tag{53}$$

Thus, by virtue of the right part of the inequality (9) of Theorem 1 and the formula (53) we get the estimate

$$P_{2n}(W^r_{\bar\beta,2}, C) \le \frac{1}{\sqrt{\pi}} e^{-\alpha n^r} \left(1 + e^{-2\alpha r n^{r-1}} \left(1 + \frac{1}{2\alpha r n^{r-1}}\right)\right)^{\frac{1}{2}}. \tag{54}$$

Combining the estimates (52) and (54) we obtaine (43). Theorem 4 is proved. □

Taking the limit as $n \to \infty$ in the relations (43), we obtain the following statement.

Theorem 5. *Let* $\bar\beta = \{\beta_k\}_{k=1}^{\infty}, \beta_k \in \mathbb{R}$, $\alpha > 0$, $r > 1$, $n \in \mathbb{N}$ *and the condition* (42) *is satisfied. Then as* $n \to \infty$ *the following asymptotic equalities*

$$\left.\begin{array}{l} P_{2n}(C^{\alpha,r}_{\bar\beta,2}, C) \\ P_{2n-1}(C^{\alpha,r}_{\bar\beta,2}, C) \end{array}\right\} = e^{-\alpha n^r} \left(\frac{1}{\sqrt{\pi}} + \mathcal{O}(1)\gamma_{\alpha,r,n} e^{-\alpha r (n-1)^{r-1}}\right) \tag{55}$$

hold, where P_N *is any of the widths* b_N, d_N, λ_N *or* π_N *and* $\gamma_{\alpha,r,n}$ *is defined by* (44) *and* $\mathcal{O}(1)$ *are the quantities uniformly bounded in all parameters.*

Note that the Theorem 5 complements the results of the works [17,21,22,28,32], which contain exact estimates for the widths of the classes of convolutions with classical or generalized Poisson kernels. As it follows from the proofs of Theorems 4 and 5 the asymptotic equalities for widths in (55) are realized by trigonometric Fourier sums. The asymptotic equalities for deviations of Fourier sums on classes of generalized Poisson integrals $C^{\alpha,r}_{\beta,p}$ in the uniform metric are seen, for example, in [20,27,31] and others.

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Знайдено двосторонні оцінки колмогоровських, берштейнівських, лінійних та проекційних поперечників класів у просторі згорток 2π-періодичних функцій φ таких, що $\|\varphi\|_2 \leq 1$, із довільними твірними ядрами $\Psi_{\bar\beta}$, ряд Фур'є яких має вигляд $\sum\limits_{k=1}^{\infty} \psi(k)\cos(kt-\beta_k\pi/2)$, де $\psi(k)\geq 0$, $\sum\psi^2(k)<\infty$, $\beta_k\in\mathbb{R}$. Показано, що для швидко спадних послідовностей $\psi(k)$ (зокрема, таких, що $\lim\limits_{k\to\infty}\psi(k+1)/\psi(k)=0$) одержані оцінки є асимптотичними рівностями. Встановлено, що асимптотичні рівності для поперечників зазначених класів реалізують тригонометричні суми Фур'є.

Ключові слова і фрази: берштейнівський поперечник, колмогоровський поперечник, лінійний поперечник, проекційний поперечник, сума Фур'є, клас Вейля-Надя, клас узагальнених інтегралів Пуассона, $(\psi,\bar\beta)$-інтеграл, асимптотична рівність.