



Fixed sets and fixed points for mappings in generalized Lim-spaces of Fréchet

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In the article, we axiomatically define generalized Lim-spaces (X, Lim) , Cauchy structures, contractive mappings and prove an abstract version of the contraction mapping principle. We also consider ways to specify families of Cauchy sequences and contraction conditions using a base in X^2 , distance-like or sum-like functions with values in some partially ordered set Y . We establish fixed set and fixed point theorems for generalized contractions of the Meir-Keeler and Taylor, Ćirić and Caristi types. The obtained results generalize many known fixed point theorems and are new even in many classical situations.

Key words and phrases: fixed point theorem, family of Cauchy sequences, Fréchet limit space.

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1 Introduction

The theory of fixed point theorems is a well-developed domain of Analysis and Topology (see books [2, 5, 9, 11, 16] and references therein). In a series of papers (see, for example, [1, 14, 20, 23]) the theory of fixed points was developed in abstract L -spaces in the sense of Fréchet.

The classical contraction mapping theorem in metric spaces, which goes back to É. Picard, S. Banach and R. Caccioppoli, was generalized in many various directions, and in most cases, the following scheme was used to prove the obtained theorems. Consider a Picard sequence (an orbit of f) $O(f, x_0) := \{x_0, x_1 = f(x_0), \dots, x_n = f(x_{n-1}), \dots\}$ and using the fact that f is contractive in some sense, establish that it is a Cauchy sequence. Using completeness of the space, obtain a point x such that $x_n \rightarrow x$ as $n \rightarrow +\infty$. Continuity (in some sense) of f gives $f(x) = x$.

The purpose of this note is to discuss the possibility of implementing this scheme in the following abstract situation. Convergence of sequences $\{x_n\} \subset X$ is defined axiomatically, essentially using an approach that goes back to M. Fréchet [10] (see also [21]), but with essentially less requirements on a limiting operator Lim . Continuity (weakly orbital continuity) of a function is then defined as a requirement for operators Lim and f to commute on orbits of f . The family of Cauchy sequences is also defined axiomatically (our definition goes back to [13], but with essentially less requirements). A necessary relation between the family of Cauchy sequences and the Lim operator is given by the (orbital) completeness requirement, namely Cauchy sequences, which are orbit of some $f: X \rightarrow X$ (actually, even not all of them must

be convergent, see, e.g., [24]). A sufficient for fixed point theorems definition of a contractive function is as follows: a function f is locally orbital contractive at a point $x_0 \in X$, if the orbit $O(f, x_0)$ is a Cauchy sequence (this definition will be refined later). Such a definition is motivated by the fact that in the case when the set of Cauchy sequences is determined using a distance-like function, sum-like function or some base in X^2 , it is quite consistent with intuitive ideas about contraction. In these cases, various generalizations of known definitions of contractions can be considered as sufficient conditions for contraction in the above presented general sense and can be written in a form close to the traditional one. The obtained results generalize many known fixed point theorems and are new even in many classical situations.

In Section 2, we give necessary notations and definitions. There we present the definition and some properties of Lim-spaces, the definition of Cauchy structures, and examples of Cauchy structures defined by some base in X^2 , some distance-like or sum-like function taking values in some partially ordered set. An abstract version of a fixed set and point theorem is given in Subsection 3.1. In Subsections 3.2–3.5 fixed set and point theorems are presented for generalized contractions of Meir-Keeler and Taylor, Ćirić, and Caristi type defined in terms of a base in X^2 or distance-like and sum-like functions.

2 Notations and definitions

2.1 General notations

Let X be a nonempty set, $s(X)$ be a set of all sequences $\{x_n\}$ of elements from X , $s^\infty(X)$ be the set of sequences from $s(X)$ with pair-wise different elements, and $\mathcal{P}(X)$ be the family of all subsets from X .

For $x_1, \dots, x_n \in X$ by $\langle x_1, \dots, x_n \rangle$ we denote the sequence $\{x_1, \dots, x_n, x_1, \dots, x_n, \dots\}$. For $\{x_n\}, \{y_n\} \in s(X)$ we set

$$\{x_n\} \vee \{y_n\} := \{x_1, y_1, x_2, y_2, \dots, x_n, y_n, \dots\}.$$

For a mapping $f: X \rightarrow X$, $x_0 \in X$, $n \in \mathbb{Z}_+$, we set $O_n(f, x_0) = \{f^n(x_0), f^{n+1}(x_0), \dots\}$, $O(f, x_0) = O_0(f, x_0) = \{f^n(x_0)\}$, and $O(f) = \{O_0(f, x_0) : x_0 \in X\}$.

Finally, let $\mathfrak{N} = \{\{n_k\} \in s(\mathbb{N}) : n_k \rightarrow +\infty \text{ as } k \rightarrow +\infty\}$.

2.2 Limit spaces

Definition 1 (cf. [10, 21]). A pair (X, Lim) of a set X and a mapping

$$\text{Lim}: s(X) \rightarrow \mathcal{P}(X)$$

such that if $\{x_n\} \in s(X)$, then $\text{Lim}\{x_n\} = \text{Lim}\{x_{n+1}\}$, is called a Lim-space. If for some $\{x_n\} \in s(X)$, $\text{Lim}\{x_n\} \neq \emptyset$, then we say that the sequence $\{x_n\}$ is convergent, and write $\{x_n\} \in c(X)$.

Example 1. If X is a metric space and for any $\{x_n\} \in s(X)$, $\text{Lim}\{x_n\}$ is the set of all partial limits of $\{x_n\}$, then (X, Lim) is a Lim-space.

Definition 2 (cf. [10]). Let a Lim-space (X, Lim) and $\mathfrak{A} \subset c(X)$ be given. We say that (X, Lim) is a \mathfrak{A} -Fréchet space, if for each $\{x_n\} \in \mathfrak{A}$ the set $\text{Lim}\{x_n\}$ is a singleton. If $\text{Lim}\{x_n\} = \{x\}$, then we say that $\{x_n\}$ converges to x , and write $\{x_n\} \rightarrow x$ as $n \rightarrow +\infty$.

Definition 3. Let a Lim-space (X, Lim) and a mapping $f: X \rightarrow X$ be given. We say that f is weakly orbital continuous, if for any $\{x_n\} \in O(f) \cap c(X)$

$$\text{Lim} \{f(x_n)\} = f(\text{Lim}\{x_n\}).$$

Remark 1. The definition of orbital continuity of the mapping $f: X \rightarrow X$ was introduced in [7]. The definition of weak orbital continuity is less restrictive.

2.3 Cauchy structures

Definition 4. A set $\mathfrak{C} \subset s(X)$ is called a Cauchy structure (we write $\mathfrak{C} \in CS$), if $\{x_n\} \in \mathfrak{C}$ implies $\{x_{n+1}\} \in \mathfrak{C}$.

A Cauchy structure \mathfrak{C} is called a strong Cauchy structure (we write $\mathfrak{C} \in SCS$), if additionally $\langle x \rangle \in \mathfrak{C}$ for each $x \in X$, and

$$\langle z_1, \dots, z_n \rangle \in \mathfrak{C} \text{ implies } z_1 = z_n \text{ for any } z_1, \dots, z_n \in X. \quad (1)$$

A Lim-space (X, Lim) is called \mathfrak{C} -complete, if arbitrary $\{x_n\} \in \mathfrak{C} \cap s^\infty(X)$ is convergent.

Remark 2. Of course, any classical set of Cauchy sequences forms a Cauchy structure.

Remark 3. For generalized metric spaces, the fact that for completeness of a space it is sufficient to require convergence of only Cauchy sequences with pairwise different elements was noted and used in [24].

Example 2. The set $c(X)$ of all convergent sequences in a Lim-space (X, Lim) can be considered as a Cauchy structure.

Example 3. Let a mapping $f: X \rightarrow X$ be given. The set $\mathfrak{C} = O(f)$ can be considered as a Cauchy structure. In this case \mathfrak{C} -complete Lim-space can be called f -orbital complete (cf. [7]).

Definition 5 (cf. [13]). Setting $\mathfrak{C} \text{Lim}\{x_n\} := \{x \in X: \langle x \rangle \vee \{x_n\} \in \mathfrak{C}\}$, we transform a Cauchy space (X, \mathfrak{C}) to a Lim-space $(X, \mathfrak{C} \text{Lim})$.

If $\mathfrak{C} \in SCS$, and additionally the following condition is satisfied: for all $\{x_n\}, \{z_n\} \in s(X)$ and $\{y_n\} \in \mathfrak{C}$,

$$\{x_n\} \vee \{y_n\}, \{y_n\} \vee \{z_n\} \in \mathfrak{C} \text{ imply } \{x_n\} \vee \{z_n\} \in \mathfrak{C}, \quad (2)$$

then, as it is easily seen, the space $(X, \mathfrak{C} \text{Lim})$ is a \mathfrak{C} -Fréchet space. Condition (2) is a generalization of well-known Fréchet-Wilson conditions (see, e.g., [26]).

2.4 Cauchy structures defined by a base in X^2

Generalizing the definition of a Cauchy sequence in a uniform space (see, e.g., [15, Chapter 6]) and, in particular, in metric or distance spaces, we obtain the following example of a Cauchy structure in the sense of Definition 4. Recall (see, e.g., [27, Chapter 3.2]) that a family \mathcal{U} of non-empty subsets of a set X is called a base in X , if for any $U, V \in \mathcal{U}$ there exists $W \in \mathcal{U}$ such that $W \subset U \cap V$.

Definition 6. Let X be a nonempty set and \mathcal{U} be a base in X^2 . The class

$$\left\{ \{x_n\} \in s(X) : \text{for any } U \in \mathcal{U} \text{ there exists } N \in \mathbb{N} \text{ such that } n > m \geq N \text{ implies } (x_m, x_n) \in U \right\}$$

forms a Cauchy structure $\mathfrak{C}_{\mathcal{U}}$. It is clear that

$$\mathfrak{C}_{\mathcal{U}} \text{Lim}\{x_n\} = \left\{ x \in X : \{x_n\} \in \mathfrak{C}_{\mathcal{U}} \text{ and for all } U \in \mathcal{U} \text{ there exists } N \in \mathbb{N} \text{ such that } (x_n, x), (x, x_n) \in U \text{ for any } n \geq N \right\}.$$

The following lemma contains a Fréchet-Wilson type condition for the space $(X, \mathfrak{C}_{\mathcal{U}} \text{Lim})$.

Lemma 1. Suppose that $\bigcap \{U : U \in \mathcal{U}\} \subset \Delta(X) = \{(x, x) : x \in X\}$. If for arbitrary $U \in \mathcal{U}$ there exist $V, W \in \mathcal{U}$ such that

$$(x, z) \in V \text{ and } (z, y) \in W \text{ imply } (x, y) \in U, \tag{3}$$

then the space $(X, \mathfrak{C}_{\mathcal{U}} \text{Lim})$ is $\mathfrak{C}_{\mathcal{U}}$ -Fréchet.

Proof. For any $U \in \mathcal{U}$ let $V, W \in \mathcal{U}$ be such that (3) is satisfied. Assume that $x, y \in \mathfrak{C}_{\mathcal{U}} \text{Lim}\{x_n\}$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$(x, x_n) \in V \text{ and } (x_n, y) \in W.$$

Condition (3) implies $(x, y) \in U$, and hence due to arbitrariness of U ,

$$(x, y) \in \bigcap \{U : U \in \mathcal{U}\} \subset \Delta(X).$$

Thus $x = y$. □

2.5 Cauchy structures defined by distance functions

Let X be a set and (Y, Lim_Y) be a Lim-space.

Definition 7. A mapping $d: X^2 \rightarrow Y$ is called an Y -valued distance, if for all $x, y \in X$

$$d(x, y) = d(y, x) = d(x, x) = d(y, y) \text{ implies } x = y. \tag{4}$$

Definition 8. The family \mathfrak{C}_d that consists of all sequences $\{x_n\} \in s(X)$ such that

$$\{d(x_{m_k}, x_{n_k})\} \in c(Y)$$

for any sequences $\{m_k\}, \{n_k\} \in \mathfrak{N}$, and the set $\text{Lim}_Y \{d(x_{m_k}, x_{n_k})\}$ does not depend on $\{m_k\}$ and $\{n_k\}$, is a Cauchy structure.

Lemma 2. Let (Y, Lim_Y) be such that $\langle \alpha \rangle, \langle \beta \rangle \in c(Y)$ for any $\alpha, \beta \in Y$, and

$$\text{Lim}_Y \langle \alpha \rangle = \text{Lim}_Y \langle \beta \rangle \text{ implies } \alpha = \beta. \tag{5}$$

Then $\mathfrak{C}_d \in \text{SCS}$.

Proof. Indeed, $\langle x \rangle \in \mathfrak{C}_d$, and if $\langle z_1, z_2, \dots, z_n \rangle \in \mathfrak{C}_d$, then

$$\text{Lim}_Y \{d(z_1, z_n)\} = \text{Lim}_Y \{d(z_1, z_1)\} = \text{Lim}_Y \{d(z_n, z_n)\} = \text{Lim}_Y \{d(z_n, z_1)\}.$$

Due to (5), we obtain $d(z_1, z_n) = d(z_1, z_1) = d(z_n, z_n) = d(z_n, z_1)$, and due to (4), $z_1 = z_n$. Hence \mathfrak{C}_d satisfies (1). \square

Remark 4. The Lim-spaces from Example 1 satisfy property (5).

It follows from Definition 5 that for any $\{x_n\} \in \mathfrak{C}_d$ we have

$$\begin{aligned} \mathfrak{C}_d \text{Lim}\{x_n\} \subset \bigcap_{\{m_k\}, \{n_k\} \in \mathfrak{N}} \{x \in X: \text{Lim}_Y \{d(x_{m_k}, x)\} = \text{Lim}_Y \{d(x, x_{n_k})\} \\ = \text{Lim}_Y \{d(x_{n_k}, x_{m_k})\} = \text{Lim}_Y \langle d(x, x) \rangle\}, \end{aligned}$$

which is consistent with the definition of convergence in almost all previously considered spaces, in particular in usual metric spaces, in partial metric spaces [18], dualistic partial metric spaces [22], dislocated metric spaces [12], metric-like spaces [3], distance spaces [16], metric and distance spaces with more general than \mathbb{R} sets of values of a distance function (K -metric spaces, cone metric spaces, M -distance spaces, probabilistic metric spaces, fuzzy metric spaces, and others; see [4] and references therein).

2.6 Another way to define Cauchy structures

Let (Y, \leq) be a partially ordered set and $\psi: \bigcup_{n \in \mathbb{N}} Y^n \rightarrow Y$ be a function such that for any $y_1, y_2, \dots, y_n \in Y$

$$\psi(y_2, \dots, y_n) \leq \psi(y_1, y_2, \dots, y_n). \tag{6}$$

Example 4. In the case $Y = \mathbb{R}_+$, the function $\psi(y_1, y_2, \dots, y_n) = \sum_{k=1}^n y_k$ satisfies (6). Moreover, it is coordinate-wise monotone, i.e.

$$\psi(y_1, y_2, \dots, y_n) \leq \psi(z_1, z_2, \dots, z_n),$$

provided $y_i \leq z_i, i = 1, \dots, n, n \in \mathbb{N}$, and for all $y_1, \dots, y_{m+n} \in Y$,

$$\psi(y_1, \dots, y_m, y_{m+1}, \dots, y_{m+n}) = \psi(y_1, \dots, y_m, \psi(y_{m+1}, \dots, y_{m+n})).$$

We say that a set $A \subset Y$ is bounded from above, if there exists $\alpha \in Y$ such that $a \leq \alpha$ for all $a \in A$.

Definition 9. The following set

$$\mathfrak{C}_\psi = \left\{ \{x_n\} \in s(X): \{\psi(x_1, x_2, \dots, x_n)\} \text{ is bounded from above} \right\}$$

forms a Cauchy structure.

A partial case of the family \mathfrak{C}_ψ can be obtained as follows.

Definition 10. Let an arbitrary function $d: X^2 \rightarrow Y$ be given. The following set

$$\mathfrak{C}_{\psi,d} = \left\{ \{x_n\} \in s(X): \text{the sequence } \{\psi(d(x_1, x_2), \dots, d(x_{n-1}, x_n))\} \text{ is bounded from above} \right\}$$

forms a Cauchy structure.

3 Fixed sets and points theorems

3.1 A general fixed set and point theorem

Definition 11. A set $A \subset X$ is called a fixed set of a mapping $f: X \rightarrow X$, if $f(A) = A$. If a singleton $A = \{x\}$ is a fixed set, then we say that x is a fixed point of the mapping f .

Before stating the theorem, we discuss the question of how the contractiveness of a mapping could be understood in an abstract situation. Let us start with the observation that contractiveness usually allows one to establish the fact that an orbit $O(f, x_0)$ is a Cauchy sequence. On the other hand, if the property of being a Cauchy sequence is defined using some distance function, then it means that the elements of the sequence become arbitrarily close as their indices increase. This is quite consistent with intuitive ideas (at least ours) about contractiveness. This is also observed in other cases (see, for example, Definition 6). Therefore, in our opinion, in the abstract situation under consideration, it is natural to adopt the following definition.

Definition 12. Let a mapping $f: X \rightarrow X$ and $x_0 \in X$ be such that

$$O(f, x_0) \in (\mathfrak{C} \cap s^\infty(X)) \cup (s(X) \setminus s^\infty(X)).$$

Then f is called locally orbital contractive at the point x_0 .

Theorem 1. Let a Lim-space (X, Lim) , a set $\mathfrak{C} \in CS$ be given, and (X, Lim) be \mathfrak{C} -complete. Let also a weakly orbital continuous and locally orbital contractive at some point $x_0 \in X$ mapping $f: X \rightarrow X$ be given. Then f has a fixed set.

If $\mathfrak{C} \in SCS$, the space (X, Lim) is a \mathfrak{C} -Fréchet space, and $O(f, x_0) \in \mathfrak{C}$, then f has a fixed point. If $\{f^n(x)\} \vee \{f^n(y)\} \in \mathfrak{C}$ for any $x, y \in X$, then the fixed point is unique.

Proof. Let $\mathfrak{C} \in CS$. Assume $\{f^n(x_0)\} = O(f, x_0) \in \mathfrak{C} \cap s^\infty(X)$. Since the space (X, Lim) is \mathfrak{C} -complete, $\text{Lim} \{f^n(x_0)\} \neq \emptyset$. Due to weakly orbital continuity of f , we obtain

$$\text{Lim} \{f^n(x_0)\} = \text{Lim} \{f^{n+1}(x_0)\} = \text{Lim} \{f(f^n(x_0))\} = f(\text{Lim} \{f^n(x_0)\}).$$

Thus $\text{Lim} \{f^n(x_0)\}$ is a fixed set of f .

If $\{f^n(x_0)\} \in s(X) \setminus s^\infty(X)$, then for some $k < l$ one has $f^k(x_0) = f^l(x_0)$ and hence $\{f^k(x_0), f^{k+1}(x_0), \dots, f^{l-1}(x_0)\}$ is a fixed set.

Let now $\mathfrak{C} \in SCS$, and let $\{f^n(x_0)\} \in \mathfrak{C}$. If $\{f^n(x_0)\} \in \mathfrak{C} \cap s^\infty(X)$, then there exists $x \in X$ such that $\text{Lim} \{f^n(x_0)\} = \{x\}$, and hence x is a fixed point of f . If $\{f^n(x_0)\} \in \mathfrak{C} \setminus s^\infty(X)$ and $f^k(x_0) = f^l(x_0), k < l$, then in the case $l = k + 1$, $f^k(x_0)$ is a fixed point of f . If $l > k + 1$, then the sequence $\langle f^k(x_0), f^{k+1}(x_0), \dots, f^{l-1}(x_0) \rangle$ also belongs to \mathfrak{C} . By property (1), we have

$$f^l(x_0) = f^k(x_0) = f^{l-1}(x_0),$$

i.e. $f^{l-1}(x_0)$ is a fixed point of f .

Let for any $x, y \in X$, $\{f^n(x)\} \vee \{f^n(y)\} \in \mathfrak{C}$. Assume that x, y are fixed points of the function f . Using property (1), we obtain

$$\{f^n(x)\} \vee \{f^n(y)\} = \langle x \rangle \vee \langle y \rangle = \langle x, y \rangle \in \mathfrak{C},$$

hence $x = y$, i.e. the fixed point is unique. □

3.2 A fixed set and point theorem in $\mathfrak{C}_{\mathcal{U}}$ -complete Lim-spaces

Below we present a result of Meir-Keeler [19] and Taylor [25] type. Let \mathcal{U} be a base in X^2 and a mapping $\zeta: \mathcal{U}^2 \rightarrow \mathcal{U}$ be such that

$$\text{for all } U, V \in \mathcal{U} \quad (u, v) \in U \text{ and } (v, w) \in V \text{ imply } (u, w) \in \zeta(U, V). \quad (7)$$

An important example of ζ for which (7) is satisfied is

$$\zeta(U, V) = U \circ V := \left\{ (u, v) : \text{there exists } w \text{ such that } (u, w) \in U \text{ and } (w, v) \in V \right\}.$$

In uniform spaces, contraction condition (i) in the theorem below with such ζ coincides with the one used by Taylor [25].

Theorem 2. *Let a space (X, Lim) be $\mathfrak{C}_{\mathcal{U}}$ -complete. Assume $f: X \rightarrow X$ satisfies the following properties:*

- (i) *for all $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $(x, y) \in \zeta(U, V)$ implies $(f(x), f(y)) \in U$;*
- (ii) *there exists $x \in X$ such that for all $U \in \mathcal{U}$ there exists $N \in \mathbb{N}$ for which $n \geq N$ implies $(f^n(x), f^{n+1}(x)) \in U$.*

Then f has a fixed set. If the space (X, Lim) is $\mathfrak{C}_{\mathcal{U}}$ -Fréchet, then f has a fixed point.

Proof. We show that $\{f^n(x)\} \in \mathfrak{C}_{\mathcal{U}}$. For a fixed $U \in \mathcal{U}$ choose $V \in \mathcal{U}$ according to condition (i). Due to (ii), there exists N such that for all $n \geq N$ one has

$$(f^n(x), f^{n+1}(x)) \in V \text{ and } (f^n(x), f^{n+1}(x)) \in U. \quad (8)$$

Let $n > N$ be fixed. We show by induction on k that for all $k \in \mathbb{N}$ one has

$$(f^n(x), f^{n+k}(x)) \in U. \quad (9)$$

For $k = 1$ inclusion (9) holds due to (8). Assume that it is true for some $k \geq 1$. Then due to (8), $(f^{n-1}(x), f^n(x)) \in V$ and $(f^n(x), f^{n+k}(x)) \in U$ by the inductive assumption. Due to (7), $(f^{n-1}(x), f^{n+k}(x)) \in \zeta(U, V)$, and hence in virtue of condition (i), $(f^n(x), f^{n+k+1}(x)) \in U$, which completes the induction step.

Thus for arbitrary $U \in \mathcal{U}$ there exists $N \in \mathbb{N}$ such that for all $n > m \geq N$, inclusion $(f^m(x), f^n(x)) \in U$ holds, and hence we obtain $\{f^n(x)\} \in \mathfrak{C}_{\mathcal{U}}$.

Application of Theorem 1 finishes the proof. \square

3.3 A fixed set and point theorem in \mathfrak{C}_d -complete Lim-spaces

Let Y be a Lim_Y -space and at the same time a partially ordered set with a partial order \leq . We assume that the partial order and the operator Lim_Y agree in the following sense: if two sequences $\{\alpha_n\}, \{\beta_n\} \in c(Y)$ have equal limits, and $\alpha_n \leq \gamma_n \leq \beta_n$ for each $n \in \mathbb{N}$, then $\{\gamma_n\} \in c(Y)$ and

$$\text{Lim}_Y\{\gamma_n\} = \text{Lim}\{\alpha_n\} = \text{Lim}\{\beta_n\}.$$

We say that a set $A \subset X$ is bounded, if there exist $\alpha, \beta \in Y$ such that $\alpha \leq d(x, y) \leq \beta$ for all $x, y \in A$.

Denote by $\Lambda(Y)$ the set of all non-decreasing mappings $\lambda: Y \rightarrow Y$ such that for some non-empty set $\text{Lim}(\lambda) \subset Y$ and arbitrary $\alpha \in Y$,

$$\text{Lim}_Y \{ \lambda^n(\alpha) \} = \text{Lim}(\lambda).$$

Observe that if Y is an ordinary metric space and λ is a contractive in the usual sense mapping, then $\lambda \in \Lambda(Y)$, and $\text{Lim}(\lambda)$ is a singleton.

Theorem 3. Assume that a Lim-space (X, Lim) and Y -valued distance function d in X be such that (X, Lim) is \mathfrak{C}_d -complete. Suppose that for a weakly orbitally continuous function $f: X \rightarrow X$ there exists a point $x_0 \in X$ such that the orbit $O(f, x_0)$ is bounded, and two mappings $\lambda_1, \lambda_2 \in \Lambda(Y)$ are such that $\text{Lim}(\lambda_1) = \text{Lim}(\lambda_2)$. If for arbitrary $x, y \in O_n(f, x_0)$ one can find $x', y', x'', y'' \in O_{n-1}(f, x_0)$ for which

$$\lambda_1(d(x', y')) \leq d(x, y) \leq \lambda_2(d(x'', y'')), \tag{10}$$

then the mapping f has a fixed set. If the space (X, Lim) is \mathfrak{C}_d -Fréchet, then f has a fixed point.

Remark 5. Contraction conditions (10) generalize the Ćirić conditions [6, 8].

Proof. For arbitrary sequences $\{m_k\}, \{n_k\}$ that tend to infinity, from condition (10) and boundedness of $O(f, x_0)$ we obtain

$$\lambda_1^{\min(m_k, n_k)}(\alpha) \leq d(x_{m_k}, x_{n_k}) \leq \lambda_2^{\min(m_k, n_k)}(\beta),$$

where $\alpha, \beta \in Y$ are the elements from the definition of boundedness.

Since $\min(m_k, n_k) \rightarrow +\infty$ whenever $k \rightarrow +\infty$, we obtain

$$\text{Lim} \{ d(x_{m_k}, x_{n_k}) \} = \text{Lim}(\lambda_1) = \text{Lim}(\lambda_2),$$

and hence $O(f, x_0) \in \mathfrak{C}_d$. Application of Theorem 1 finishes the proof. □

3.4 A fixed set and point theorem in \mathfrak{C}_ψ -complete Lim-spaces

The following theorem is a variant of the Caristi theorem (see, e.g., [17] and references therein).

Theorem 4. Let Y be a partially ordered set, and a Lim-space (X, Lim) be \mathfrak{C}_ψ -complete, where ψ is a mapping satisfying property (6). Assume that there exists a function $\bar{\psi}: (\bigcup_n X^n) \times Y \rightarrow Y$ and such that for arbitrary $x_1, \dots, x_n \in X$ and $y, z \in Y$,

$$\psi(x_1, \dots, x_n) \leq \bar{\psi}(x_1, \dots, x_n, y), \quad \bar{\psi}(x_1, \dots, x_n, y) \leq \bar{\psi}(x_1, \dots, x_{n-1}, \bar{\psi}(x_n, y)), \tag{11}$$

and $y \leq z$ implies $\bar{\psi}(x_1, \dots, x_n, y) \leq \bar{\psi}(x_1, \dots, x_n, z)$. If $f: X \rightarrow X$ is a weakly orbitally continuous function and there exists $\phi: X \rightarrow Y$ such that for some $x_0 \in X$

$$\bar{\psi}(x, \phi(f(x))) \leq \phi(x) \quad \text{for all } x \in O(f, x_0), \tag{12}$$

then f has a fixed set. If the space (X, Lim) is \mathfrak{C}_d -Fréchet, then f has a fixed point.

Proof. Let $x_n = f(x_{n-1}), n \in \mathbb{N}$. We prove by induction on n that

$$\bar{\psi}(x_0, x_1, \dots, x_{n-1}, \phi(x_n)) \leq \phi(x_0).$$

For $n = 1$ the inequality follows from (12). Assume it holds for some $n = k \geq 1$. Then for $n = k + 1$, we obtain

$$\begin{aligned} \bar{\psi}(x_0, x_1, \dots, x_k, \phi(x_{k+1})) &\leq \bar{\psi}(x_0, x_1, \dots, x_{k-1}, \bar{\psi}(x_k, \phi(x_{k+1}))) \\ &\leq \bar{\psi}(x_0, x_1, \dots, x_{k-1}, \phi(x_k)) \leq \phi(x_0), \end{aligned}$$

which finishes the induction step. Inequality (11) implies that $O(f, x_0) \in \mathfrak{C}_\psi$, and in order to finish the proof, it is sufficient to apply Theorem 1. □

3.5 Fixed set and point theorems in $\mathfrak{C}_{\psi,d}$ -complete Lim-spaces

The following theorem uses a generalization of the Ćirić-type contraction condition.

Theorem 5. *Let a Lim-space (X, Lim) be $\mathfrak{C}_{\psi,d}$ complete, where ψ is a coordinate-wise non-decreasing mapping that satisfies property (6), $d: X^2 \rightarrow Y$ is some function, and Y is a partially ordered set. Assume that there exists a non-decreasing $\lambda: Y \rightarrow Y$ such that for each $y \in Y$ the sequence*

$$\{\psi(y, \lambda(y), \dots, \lambda^n(y))\}$$

is bounded from above, $x_0 \in X$, and a weakly orbital continuous mapping $f: X \rightarrow X$ are such that for all $n > 1$ and all $x, y \in O_n(f, x_0)$ there exist $x', y' \in O_{n-1}(f, x_0)$ that satisfy the inequality

$$d(x, y) \leq \lambda(d(x', y')). \quad (13)$$

If there exists $\alpha \in Y$ such that $d(x, y) \leq \alpha$ for all $x, y \in O(f, x_0)$, then f has a fixed set. If the space (X, Lim) is \mathfrak{C}_d -Fréchet, then f has a fixed point.

Proof. Let $x_n = f(x_{n-1})$, $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, consecutively applying inequality (13), we obtain that for some $y, z \in O(f, x_0)$ and α from the statement of the theorem,

$$d(x_n, x_{n+1}) \leq \lambda^n(d(y, z)) \leq \lambda^n(\alpha).$$

Thus for all $n \in \mathbb{N}$ we have

$$\psi(d(x_0, x_1), d(x_1, x_2), \dots, d(x_n, x_{n+1})) \leq \psi(\alpha, \lambda(\alpha), \dots, \lambda^n(\alpha)),$$

which implies boundedness of the sequence $\{\psi(d(x_0, x_1), \dots, d(x_n, x_{n+1}))\}$.

Hence $O(f, x_0) \in \mathfrak{C}_{\psi,d}$, and it is enough to apply Theorem 1 in order to finish the proof. \square

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У статті ми аксіоматично означаємо узагальнені Lim-простори (X, Lim) , множини послідовностей Коші, стискаючі відображення та доводимо абстрактну версію принципу стискаючих відображень. Ми також розглядаємо різні способи задання множин послідовностей Коші та умов стиску за допомогою бази у X^2 , функцій, що подібні до відстані чи суми, зі значеннями у деякій частково впорядкованій множині Y . Ми доводимо теореми про нерухомі множини і нерухомі точки для узагальнених стискаючих відображень типів Меїра-Кілера і Тейлора, Чіріча та Карісті. Отримані результати узагальнюють багато відомих теорем про нерухомі точки і є новими навіть у багатьох класичних ситуаціях.

Ключові слова і фрази: теорема про нерухому точку, множина послідовностей Коші, Lim-простір Фреше.