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Some remarks on spectrum of nonlinear continuous operators

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In this article, the existence of the spectrum (the eigenvalues) for the nonlinear continuous operators acting in the Banach spaces is investigated. For the study this question it is used a different approach that allows the studying of all eigenvalues of a nonlinear operator relative to another nonlinear operator. Here we show that in nonlinear operators case it is necessary to seek the spectrum of the given nonlinear operator relative to another nonlinear operator satisfying certain conditions. The different examples, for which eigenvalues can be found, are provided. Moreover, the nonlinear problems including parameters are studied.

Key words and phrases: nonlinear continuous operator, spectrum, Banach space, nonlinear differential operator, solvability.

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Introduction

Well-known that the spectral theory for linear operators is one of the most important topics of linear functional analysis. In many cases for the study of a linear operator it is enough to study its spectrum. It should be noted that the spectral theory of linear operators has essential application in many topics of the natural sciences (moreover, the spectral theory is one of the areas that plays a fundamental role in quantum mechanics). The many processes of physics, mechanics, biology, etc. from mathematical point of view usually would be nonlinear problems, consequently, the operators generated by these processes are nonlinear operators acting in Banach spaces.

It needs to be noted that there exist sufficiently many works, devoted to finding the first eigenvalue of the nonlinear continuous operators. In these works, various definitions of the spectra for the specific classes of nonlinear continuous operators were introduced. In many works, the nonlinear equations of type $f(x) - \lambda g(x) = 0$ in the appropriate spaces were considered, where f is the basic operator, g is the compact operator, and an infimum of the parameter λ was found, at which, in general, the existence of a solution of the considered equation studied (see, e.g., [3, 5, 6, 9–11, 24, 39]). In works [6–11, 16, 23, 27] the Strum-Liouville type problem for the perturbed by nonlinear operators of linear operators was investigated. More exactly, the operator of the form $F(\lambda, x) = \lambda Lx + g(\lambda, x)$ was considered, and the bifurcation of solutions to the examined problems was studied.

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It is well-known that the finding of eigenvalues of linear operators allows to study the bifurcations of solutions, which appear under the investigation of the semilinear equations. It should be noted that there are works, where the existence of the first eigenvalue for certain nonlinear smooth operators was studied, and using it the bifurcations of solutions to nonlinear equations with such operators were investigated (see, e.g., [12, 16, 25, 36]).

In works [1, 2, 4, 13–15, 18–22, 26, 28, 29], it has been introduced the definition of the spectra (more exactly, first eigenvalue) using the equation in the form $f(x) - \lambda_1 Ix = 0$, as in the linear operators theory. And also it has been introduced the definition of the spectra for some classes operators: the Frechet differentiable operators, operators that satisfy the Lipschitz condition, operators from a special class of continuous operators, and linearly bounded operators. These approaches supposed that the spectrum of an operator acting in a Banach space can be defined as in the theory of linear operators. These approaches and obtained results in enough form were explained in the book [1] (see also the survey [2]). In the above works, the study used degree theory that requires the condition compactness, this condition in what follows was generalized and used the Kuratowski measure of noncompactness. The definitions of the spectrum introduced in these works, could not satisfy the next requirements since a found parameter λ will be a function of elements of the domain (see provided examples below).

It will be best if one can introduce such a definition of the spectrum of the continuous nonlinear operator that satisfies some basic requirements, which were analogous to properties of the spectrum, existing in the linear operators theory. Then one could be to seek also other spectrums (i.e. eigenvalues and eigenvectors). So, from the explanation below will be seen that in order for the spectrum can to characterize the nonlinear operator, we need to approach another way to the definition of the spectrum of the nonlinear operator.

In this paper, it is proposed a new approach for the study of the spectrum of continuous nonlinear operators in the Banach spaces. In reality, here we find the first eigenvalue of a nonlinear continuous operator in Banach space, and in addition, this approach shows how one can seek the other eigenvalues. Here it is shown that if to use the proposed definition of the spectrum of nonlinear continuous operators in Banach spaces, then the spectra will satisfy the certain properties that are similar to properties, having in the linear operator theory. Moreover, here we investigate also the solvability of nonlinear equations in Banach spaces.

In this paper, we study the spectrum of nonlinear operators acting in Banach spaces, and also the solvability of the nonlinear equations, dependent on parameters, using the general solvability theorems and fixed-point theorems of the works [30–32, 34, 35].

Let X, Y be real Banach spaces over the field $\mathbb{R}$ and X^*, Y^* be of their dual spaces, let Y be reflexive space. Let $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$ be nonlinear continuous operators such that $f(0) = 0$, $g(0) = 0$. For investigation of the spectrum of the continuous nonlinear operators, we will consider the following equation

$$f(x) = \lambda g(x), \quad x \in M \subseteq X,$$

where $f,\ g$ are continuous operators, in particular, g can be the identical operator. In addition, we study the solvability of the equation

$$f_\lambda(x) \equiv f(x) - \lambda g(x) = y, \quad y \in Y,$$

dependent on a parameter λ, where λ is an element of $\mathbb{C}$, generally.

The paper is organized as follows. In Section 1, the definition for the spectrum of the nonlinear continuous operators in the Banach spaces is provided. Also some complementaries to the definition, and examples are provided. Moreover, the general solvability theorems and the fixed-point theorems are proved. Here we show how one can find the first eigenvalue of nonlinear continuous operators relative to another nonlinear continuous operator. Section 2 provides some examples of nonlinear differential operators, for which first eigenvalues relative to other nonlinear differential operators found. The relations between found first eigenvalues and first eigenvalues of linear differential operators are showed. In Section 3, the existence of the first eigenvalues of a fully nonlinear continuous operator relative to other nonlinear continuous operator is studied and some examples are provided.

1 Spectral properties of nonlinear continuous operators

We will consider the nonlinear continuous operators, acting in Banach spaces, and will introduce the concept for the spectrum of the nonlinear continuous operator relative to another nonlinear continuous operator.

Let X and Y be the real Banach spaces, $F : D(f) = X \longrightarrow Y$, $G : X \subseteq D(G) \longrightarrow Y$ be nonlinear bounded continuous operators (for generality) and $\lambda \in \mathbb{C}$ be the number, and $F(0) = 0$, $G(0) = 0$.

So, we will investigate the spectrum of operator F relative to operator G, for this, in the beginning, we will study the solvability of the following equation with a parameter λ

$$f_\lambda(x) \equiv F(x) - \lambda G(x) = 0, \ \text{or} \ \ F(x) = \lambda G(x), \ x \in X. \tag{1}$$

And also we will study the following equation

$$f_\lambda(x) \equiv F(x) - \lambda G(x) = y, \quad y \in Y. \tag{2}$$

Let us introduce concepts that are necessary for this paper.

Definition 1. *The operator $f : D(f) \subseteq X \longrightarrow Y$ is called bounded if there is a continuous function $\mu : \mathbb{R}^1_+ \longrightarrow \mathbb{R}^1_+$ such that*

$$\|f(x)\|_Y \leq \mu(\|x\|_X), \quad \forall\, x \in D(f).$$

We denote the class of such operators by $\mathfrak{B}$ and the class of bounded continuous operators by $\mathfrak{B}C^0$.

Let us introduce the order of relationships in the class of the bounded continuous operators acting in Banach spaces.

Definition 2. *Let X_0, Y_0 be Banach spaces, and $F : D(f) \subseteq X_0 \longrightarrow Y_0$, $G : D(G) \subseteq X_0 \longrightarrow Y_0$ be continuous operators. Denote by $\mathcal{F}_F(Z)$, $\mathcal{F}_G(Z)$ the following sets*

$$\mathcal{F}_F(Z) \equiv \left\{x \in X_0 : \|F(x)\|_Z < \infty\right\} \neq \varnothing, \qquad \mathcal{F}_G(Z) \equiv \left\{x \in X_0 : \|G(x)\|_Z < \infty\right\} \neq \varnothing,$$

which are subsets of X_0 for each Banach space $Z \subseteq Y_0$ satisfying conditions $\operatorname{Im}(F) \cap Z \neq \varnothing$, $\operatorname{Im}(G) \cap Z \neq \varnothing$.

If the inclusion $\mathcal{F}_F(Z) \subset \mathcal{F}_G(Z)$ holds for each Banach space $Z \subseteq Y_0$, then we will say that operator F is greater than operator G, and denote it as $F \succ G$.

Definition 3. *Let $F : X \longrightarrow Y$, $G : X \subseteq D(G) \longrightarrow Y$ and $F \succ G$.*

An element $\lambda \in \mathbb{C}$ is called a regular and belongs to the G-resolvent of the operator F if $f_\lambda^{-1} \equiv (F - \lambda G)^{-1} : F(X) \cap G(X) \subseteq Y \longrightarrow X$ exists and $f_\lambda^{-1} \equiv (F - \lambda G)^{-1} \in \mathfrak{B}C^0$. This subset of $\mathbb{C}$ is denoted by $\rho_G(F) \subseteq \mathbb{C}$, i.e. $\lambda \in \rho_G(F)$, where $f_\lambda(\cdot) \equiv F(\cdot) - \lambda G(\cdot)$.

Consequently, an element $\lambda \in \mathbb{C}$ is called a spectrum if it belongs to the $\mathbb{C} - \rho_G(F)$, which is defined as the G-spectrum of the operator F, and is denoted by $\sigma_G(F)$, i.e. $\lambda \in \sigma_G(F) \equiv \mathbb{C} - \rho_G(F)$.

Remark 1. *The above definition of the spectrum is not suitable for all pairs of operators, which are chosen in the independent way, which will be shown below. We will call λ the first eigenvalue of the examined operator relative to another operator as in Definition 3, which is independent of elements from the domain of the examined operators.*

Then the definition will allow seeking, in the above sense, the following eigenvalues of the examined operator relative to another operator.

So, for simplicity, we start to consider the case when $F \succ G$ and when one of these operators has the inverse operator from the class $\mathfrak{B}C^0$. If we assume that operator F is invertible, i.e. operator $F^{-1} : F(X) \subseteq Y \longrightarrow X$ exists, then, using F^{-1}, we get the equation

$$y - \lambda G\Big(F^{-1}(y)\Big) = 0, \quad y = F(x),\ x \in X, \tag{3}$$

that needs to study on the subset $F(X) \subseteq Y$. Thus, we will derive an equation that is equivalent to the examined equation, for which the existence of the first eigenvalue in many works was investigated (see, e.g., [1–4, 6, 9, 13, 18–20, 22, 23, 26, 29] and the references given there). Unlike the usual case, here the operator $G \circ F^{-1}$ is defined on the subset $F(X)$ and acts as $G \circ F^{-1} : F(X) \longrightarrow G(X) \subseteq Y$. If the operator G is invertible then in the same way as above, we get the equation

$$F\Big(G^{-1}(y)\Big) - \lambda y = 0, \quad y = G(x),\ x \in X,$$

where G^{-1} is the inverse operator to G. Consequently, in this case, the obtained equation will need to investigate on the subset $G(X)$ of Y.

Thus, if we assume that the operator F (or G) is invertible, then we obtain the equation

$$\widetilde{f}_\lambda(y) \equiv \lambda^{-1} I y - G\Big(F^{-1}(y)\Big) = 0, \quad \widetilde{f}_\lambda : D\Big(\widetilde{f}_\lambda\Big) \subseteq Y \longrightarrow Y \tag{4}$$

consequently. The finding of a first eigenvalue of the operator F relative to operator G is transformed into the finding of a first eigenvalue of the operator $G \circ F^{-1}$ $\left(\text{or } F \circ G^{-1}\right)$.

It is clear if assume the operator F is the linear continuous operator having the inverse operator F^{-1}, then the equation (3) is equivalent to the equation

$$\lambda^{-1} x - F^{-1} \circ G(x) = 0,$$

consequently, the finding of a first eigenvalue of the operator F relative to operator G is transformed into the finding of a first eigenvalue of the operator $F^{-1} \circ G$. Problems of such types were studied in many articles (see, e.g., [9–11,17,27,38]). We would like to study the problem (1) in the general case. Section 3 will be given explanations relative to the previous cases.

Before starting the investigation of the spectrum of the nonlinear operator relative to other nonlinear operators in the general case, it is necessary to investigate the solvability of the nonlinear equation (2). We will use the general existence and fixed-point theorems of articles [30, 32] to investigate the main equations. At the beginning, we will lead the mentioned results from these articles.

1.1 General solvability results

Let X,Y be real Banach spaces such as above, $f : D(f) \subseteq X \longrightarrow Y$ be an operator, and $B^X_{r_0}(0) \subseteq D(f)$ is the closed ball with a center at $0 \in X$.

Consider the following conditions.

(i) $f : D(f) \subseteq X \longrightarrow Y$ is a nonlinear bounded continuous operator.

(ii) There is a mapping $g : X \subseteq D(g) \longrightarrow Y^*$ such that the conditions $g\left(B^X_{r_0}(0)\right) = B^{Y^*}_{r_1}(0)$ and

$$\langle f(x), \widetilde{g}(x)\rangle \geq \nu\big(\|x\|_X\big) = \nu(r), \quad \forall\, x \in S^X_r(0),$$

hold[1], where $\widetilde{g}(x) \equiv \frac{g(x)}{\|g(x)\|}$, $\nu : \mathbb{R}^1_+ \longrightarrow \mathbb{R}^1$ is a continuous nondecreasing on $[\tau_0, r_0]$ function and $\nu(r_0) \geq \delta_0 > 0$.

(iii) Almost each $x_0 \in \operatorname{Int} B^X_{r_0}(0)$ possesses a neighborhood $V_\varepsilon(x_0)$, $\varepsilon \geq \varepsilon_0 > 0$, such that the following inequality

$$\big\|f(x_2) - f(x_1)\big\|_Y \geq \Phi\big(\|x_2 - x_1\|_X, x_0, \varepsilon\big) \tag{5}$$

holds for any $x_1, x_2 \in V_\varepsilon(x_0) \cap B^X_{r_0}(0)$, where $\Phi(\tau, x_0, \varepsilon) \geq 0$ is a continuous function of τ and $\Phi(\tau, \widetilde{x}, \varepsilon) = 0 \Leftrightarrow \tau = 0$ $\big($in particular, it may be $x_0 = 0$, $\varepsilon = \varepsilon_0 = r_0$ and $V_\varepsilon(x_0) = V_{r_0}(0) \equiv B^X_{r_0}(0)$, consequently $\Phi(\tau, x_0, \varepsilon) \equiv \Phi(\tau, x_0, r_0)$ on $B^X_{r_0}(0)\big)$.

Theorem 1. *Let X,Y be real Banach spaces such as above, $f : D(f) \subseteq X \longrightarrow Y$ be an operator, and $B^X_{r_0}(0) \subseteq D(f)$ is the closed ball centered at $0 \in D(f)$. Assume conditions (i) and (ii) are fulfilled. Then the image $f\left(B^X_{r_0}(0)\right)$ of the ball $B^X_{r_0}(0)$ is contained in an absorbing subset Y and contains an everywhere dense subset of M, which is defined as follows*

$$M \equiv \left\{ y \in Y \mid \big\langle y, \widetilde{g}(x)\big\rangle \leq \big\langle f(x), \widetilde{g}(x)\big\rangle,\ \forall\, x \in S^X_{r_0}(0) \right\}.$$

Furthermore, if the condition (iii) also is fulfilled then the image $f\left(B^X_{r_0}(0)\right)$ of the ball $B^X_{r_0}(0)$ is a bodily subset of Y, moreover $B^Y_{\delta_0}(0) \subseteq M$.

The proof of this theorem, and also its generalization was provided in [30] (see also [32,34,35]). We note that Theorem 2 below is the generalization of such type theorem from [33]. The condition *(iii)* can be generalized, for example, as in the following assertion.

Corollary 1. *Let all conditions of Theorem 1 be fulfilled except for the inequality* (5) *of condition (iii) instead that the following inequality*

$$\big\|f(x_2) - f(x_1)\big\|_Y \geq \Phi\big(\|x_2 - x_1\|_X, x_0, \varepsilon\big) + \psi\big(\|x_1 - x_2\|_Z, x_0, \varepsilon\big) \tag{6}$$

holds, where Z is Banach space and $X \subset Z$ is compact, $\psi(\cdot, x_0, \varepsilon) : \mathbb{R}^1_+ \longrightarrow \mathbb{R}^1$ is a continuous function relatively $\tau \in \mathbb{R}^1_+$ and $\psi(0, x_0, \varepsilon) = 0$. Then the statement of Theorem 1 is true.

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[1] In particular, the mapping g can be a linear bounded operator $g \equiv L : X \longrightarrow Y^*$ satisfying the conditions *(ii)*.

From Theorem 1 the next result immediately follows.

Theorem 2 (Fixed-Point Theorem). *Let X be a real reflexive separable Banach space and $f_1 : D(f_1) \subseteq X \longrightarrow X$ be a bounded continuous operator. Moreover, let on closed ball $B^X_{r_0}(0) \subseteq D(f_1)$, centered at $0 \in D(f_1)$, operators f_1 and $f \equiv Id - f_1$ satisfy the following conditions:*

- *the next inequalities*

$$\|f_1(x)\|_X \leq \mu(\|x\|_X), \qquad \langle f(x), \widetilde{g}(x)\rangle \geq \nu(\|x\|_X), \quad \forall\, x \in B^X_{r_0}(0),$$

  hold, where $f_1\left(B^X_{r_0}(0)\right) \subseteq B^X_{r_0}(0)$, $g : D(g) \subseteq X \longrightarrow X^*$, $D(f_1) \subseteq D(g)$ and the condition (ii) of Theorem 1 is satisfied (in particular, $g \equiv J : X \rightleftarrows X^*$, i.e. g is a duality mapping), μ and ν are such functions as in Theorem 1;

- *almost each $x_0 \in \operatorname{Int} B^X_{r_0}(0)$ possesses a neighborhood $V_\varepsilon(x_0)$, $\varepsilon \geq \varepsilon_0 > 0$, such that the following inequality*

$$\|f(x_2) - f(x_1)\|_X \geq \varphi(\|x_2 - x_1\|_X, x_0, \varepsilon)$$

holds for any $x_1, x_2 \in V_\varepsilon(x_0) \cap B^X_{r_0}(0)$, where the function $\varphi(\tau, x_0, \varepsilon)$ satisfies the similar conditions such as functions from the right hand side of (6)*. Then operator f_1 possesses a fixed-point in the closed ball $B^X_{r_0}(0)$.*

Now we introduce the following concept.

Definition 4. *An operator $f : D(f) \subseteq X \longrightarrow Y$ is said to possess the P-property if each precompact subset $M \subseteq \operatorname{Im} f$ of Y contains (maybe generalized) subsequence $M_0 \subseteq M$ such that $f^{-1}(M_0) \subseteq G$ and $M_0 \subseteq f(G \cap D(f))$, where G is a precompact subset of X.*

Notation 1. *It is easy to see that the condition (iii) of Theorem 1 one can replace by the condition: f possesses the P-property.*

It should be noted that if f^{-1} is the lower or upper semi-continuous mapping then operator $f : D(f) \subseteq X \longrightarrow Y$ possesses the P-property.

In the above results, condition *(iii)* is required for the completeness of the image of considered operator f. One can bring also other sufficient conditions on f, at which $\mathfrak{R}(f)$ will be the closed subset (see, e.g. [32,34,35]). In particular, the following results are true.

Lemma 1. *Let X, Y be Banach spaces such as above, $f : D(f) \subseteq X \longrightarrow Y$ be a bounded continuous operator, and $D(f)$ is a weakly closed subset of the reflexive space X. Let f has a weakly closed graph and for each bounded subset $M \subset Y$ the subset $f^{-1}(M)$ is a bounded subset of X. Then f is a weakly closed operator.*

Note that the graph of operator f is weakly closed if and only if from $x_m \stackrel{X}{\rightharpoonup} x_0 \in D(f)$ and $f(x_m) \stackrel{Y}{\rightharpoonup} y_0 \in Y$ the equality $f(x_0) \equiv y_0 \in \mathfrak{R}(f) \subset Y$ follows (for the general case see [32,34]).

For the proof it is enough to note that if $\{y_m\}_{m=1}^{\infty} \subset \mathfrak{R}(f) \subset Y$ is a weakly convergent sequence of Y then $f^{-1}(\{y_m\}_{m=1}^{\infty})$ is a bounded subset of X. Consequently, it has a subsequence $\{x_m\}_{m=1}^{\infty}$ such that $x_m \in f^{-1}(y_m)$ and $x_m \stackrel{X}{\rightharpoonup} x_0 \in D(f)$ for some element $x_0 \in D(f)$ by virtue of the reflexivity of X.

Lemma 2. *Let X, Y be reflexive Banach spaces, $f : D(f) \subseteq X \longrightarrow Y$ be a bounded continuous mapping that satisfies the condition: if $G \subseteq D(f)$ is a closed convex subset of X then $f(G)$ is a weakly closed subset of Y. Then if $G \subseteq D(f)$ is a bounded closed convex subset of X then $f(G)$ is a closed subset of Y.*

For the proof it is enough to use the reflexivity of the space X and properties of a bounded closed convex subset of X (see, e.g., [30,32]).

Lemma 3. *Let X be a Banach space such as above, $f : X \longrightarrow X^*$ be a monotone operator satisfying conditions of Theorem 1, and $r \geq \tau_1$ be some number. Then $f(G)$ is a bounded closed subset containing a ball $B_{r_1}^{X^*}(f(0))$ for every bounded closed convex body $G \subset X$ such that $B_r^X(0) \subset G$, where $r_1 = r_1(r) \geq \delta_1 > 0$.*

1.2 Investigation of equations (1), (2) and existence of spectra

We start with the study of the equation (2), in order to understand the role of the parameter λ. Let X, Y be real reflexive Banach spaces, $F : X \longrightarrow Y$, $G : X \subseteq D(G) \longrightarrow Y$ be nonlinear operators and $B_{r_0}^X(0)$, $r_0 > 0$, be a closed ball, centered at $0 \in X$, that belongs to $D(F)$. Since in this work we will consider only operators acting in real spaces, we will seek real numbers λ_0, under which the considered equation may be solvable.

Assume that on the ball $B_{r_0}^X(0)$ the following conditions are fulfilled:

1) let $F : B_{r_0}^X(0) \longrightarrow Y$, $G : B_{r_0}^X(0) \longrightarrow Y$ be bounded continuous operators, i.e. there exist continuous functions $\mu_j : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, $j = 1, 2$, such that inequalities

$$\left\|F(x)\right\|_Y \leq \mu_1\left(\|x\|_X\right), \quad \left\|G(x)\right\|_Y \leq \mu_2\left(\|x\|_X\right),$$

hold for any $x \in B_{r_0}^X(0)$, in addition $F \succ G$;

2) let $f_\lambda \equiv F - \lambda G$ be the operator from (2); assume there exists a parameter $\lambda_0 \in \mathbb{R}_+$ such that for each (y^*, r, λ) there exists $x \in S_r^X(0)$ such that the following inequality

$$\left\langle f_\lambda(x), y^* \right\rangle \geq \nu_\lambda\left(\|x\|_X\right), \quad \exists\, x \in S_r^X(0), \quad g(x) = y^*,$$

holds, where $\left(y^*, r, |\lambda|\right) \in S_1^{Y^*}(0) \times (0, r_0] \times (0, \lambda_0]$ and $\nu_\lambda : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is the continuous function satisfying condition (ii) of Theorem 1, in this case, $\delta_0 = \delta_{0\lambda} \searrow 0$ if $|\lambda| \nearrow |\lambda_0|$;

3) assume for almost every point x_0 from $B_{r_0}^X(0)$ there exist numbers $\varepsilon \geq \varepsilon_0 > 0$ and continuous on τ functions $\varphi_\lambda(\tau, x_0, \varepsilon) \geq 0$, $\psi_\lambda(\tau, x_0, \varepsilon)$ such that the following inequality

$$\left\|f_\lambda(x_1) - f_\lambda(x_2)\right\|_Y \geq \varphi_\lambda\left(\|x_1 - x_2\|_X, x_0, \varepsilon\right) + \psi_\lambda\left(\|x_1 - x_2\|_Z, x_0, \varepsilon\right)$$

holds for any $x_1, x_2 \in B_\varepsilon^X(x_0)$, where $\varphi(\tau, x_0, \varepsilon) = 0 \Leftrightarrow \tau = 0$, $\psi_\lambda(\cdot, x_0, \varepsilon) : \mathbb{R}_+ \longrightarrow \mathbb{R}$, $\psi_\lambda(0, x_0, \varepsilon) = 0$ for any (x_0, ε) and Z is the Banach space such that $X \subset Z$ is compact.

Theorem 3. *Let the conditions 1), 2) and 3) be fulfilled on the closed ball $B_{r_0}^X(0) \subset X$. Then equation (2) is solvable for all $\widetilde{y} \in V_\lambda \subset Y$ and each λ, $0 \leq |\lambda| \leq \lambda_0$, moreover, the condition 2) implies the inclusion $B_{\delta_0}^Y(0) \subseteq f_\lambda\left(B_{r_0}^X(0)\right)$ holds for $\delta_0 \equiv \delta_0(\lambda) > 0$, where V_λ is defined as follows*

$$V_\lambda \equiv \left\{ \widetilde{y} \in Y \middle|\ \left\langle \widetilde{y}, g(x) \right\rangle \leq \left\langle f_\lambda(x), g(x) \right\rangle, \quad \forall\, x \in S_{r_0}^X(0) \right\}.$$

For the proof it is sufficient to note that all conditions of Theorem 1 are fulfilled for each fixed λ, $|\lambda| < \lambda_0$, due to conditions of Theorem 3. Therefore, applying Theorem 1, we get the correctness of Theorem 3.

Consequently, the equation (4) also is solvable in $B_r^X(0)$ under the conditions on $\widetilde{f}_\lambda$ of the above type that depends on λ_0, e.g.

$$\left\| G\left(F^{-1}(y_1)\right) - G\left(F^{-1}(y_2)\right)\right\|_Y \leq C(x_0,\varepsilon)\|y_1-y_2\|_Y + \psi_\lambda(\|y_1-y_2\|_Z, y_0, \varepsilon),$$

where $C(x_0,\varepsilon)\lambda_0 < 1$ and the inclusion $Y \subset Z$ is compact.

Whence, using Theorem 2, one can obtain the solvability of the equation (2). Indeed, let $Y = X^*$ and closed ball $B_{r_0}^X(x_0)$ ($r_0 > 0$) belongs to $D(F)$. Let condition *1)* is fulfilled on ball $B_{r_0}^X(x_0)$. Assume the following conditions are fulfilled:

2') there exists a parameter $\lambda_1 \in \mathbb{R}$ such that $\lambda_1 G\left(F^{-1}\left(F\left(B_{r_0}^X(x_0)\right)\right)\right) \subseteq B_{r_0}^X(x_0)$ and for each $x^* \in S_1^{X^*}(0)$ there exists an $x \in S_r^X(x_0)$ for each $r \in (0, r_0]$ such that the following inequality

$$\left\langle \widetilde{f}_{\lambda_1}(x), x^*\right\rangle \geq \nu_{\lambda_1}(\|x-x_0\|_X) = \nu_{\lambda_1}(r), \quad x \in S_r^X(x_0) \subset B_{r_0}^X(x_0),$$

holds, where $\nu_{\lambda_1} : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is the continuous function that satisfies the condition (ii) of Theorem 1;

3') for almost every $\widehat{x} \in B_{r_0}^X(x_0)$ there are numbers $\varepsilon \geq \varepsilon_0 > 0$ and continuous functions $\Phi_{\lambda_1}(\cdot, \widehat{x}, \varepsilon) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, $\varphi_\lambda(\cdot, \widehat{x}, \varepsilon) : \mathbb{R}_+ \longrightarrow \mathbb{R}$ for each $(\widehat{x}, \varepsilon)$ such that the following inequality

$$\left\| \widetilde{f}_{\lambda_1}(x_1) - \widetilde{f}_{\lambda_1}(x_2)\right\|_X \geq \Phi_{\lambda_1}(\|x_1-x_2\|_X, \widehat{x}, \varepsilon) + \varphi_{\lambda_1}(\|x_1-x_2\|_Z, \widehat{x}, \varepsilon)$$

holds for any $x_1, x_2 \in U_\varepsilon(\widehat{x}) \cap B_{r_0}^X(x_0)$, where $\Phi_{\lambda_1}(\tau, \widehat{x}, \varepsilon) \geq 0$ and $\Phi_{\lambda_1}(\tau, \widehat{x}, \varepsilon) = 0 \Leftrightarrow \tau = 0$, $\varphi_{\lambda_1}(0, \widehat{x}, \varepsilon) = 0$, and Z is the Banach space such that $X \subset Z$ is compact.

Whence this implies that for defined above λ_1 all conditions of Theorem 1 are fulfilled for the operator $\widetilde{f}_{\lambda_1}$ on the closed ball $B_{r_0}^X(x_0)$. Consequently, $\widetilde{f}_{\lambda_1}\left(B_{r_0}^X(x_0)\right)$ contains a closed absorbing subset of X (at least, $0 \in X$) by virtue of the Theorem 1. In the other words, $0 \in \widetilde{f}_{\lambda_1}\left(B_{r_0}^X(x_0)\right)$ and therefore there exists an element $\widetilde{x} \in B_{r_0}^X(x_0)$ for which $f_{\lambda_1}(\widetilde{x}) = 0$ holds, i.e. $F(\widetilde{x}) = \lambda_1 G(\widetilde{x})$.

The obtained result one can formulate as follows.

Corollary 2. *Let F, G be above determined operators, $F \succ G$, $D(F) \subseteq D(G)$, and there exists a number λ_1 that conditions 1), 2'), 3') are fulfilled on the closed ball $B_r^X(x_0) \subseteq D(F) \subseteq X$. Then there exists an element $\widetilde{x} \in B_r^X(x_0)$ such that $F(\widetilde{x}) = \lambda_1 G(\widetilde{x})$ or $\lambda_1 G\left(F^{-1}(\cdot)\right)$ has fixed point.*

Let X, Y be Banach spaces, and $B_{r_0}^X(0) \subseteq D(F) \subset X$, $r_0 > 0$, $F \succ G$, $F(0) = 0$, $G(0) = 0$ be bounded operators. Assume that there are continuous functions $\nu_F, \nu_G : \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfying condition (ii) of Theorem 1 such that for each $y^* \in S_1^{Y^*}(0)$ there exists $x \in S_r^X(0)$, for which the inequalities

$$\langle F(x), y^*\rangle \geq \nu_F(\|x\|_X), \quad \langle G(x), y^*\rangle \geq \nu_G(\|x\|_{X_1})$$

hold, where X_1 is the Banach space such that $X \subseteq X_1$ (we will denote the relation between x and $y^* \in S_1^{Y^*}(0)$ by $g : S_r^X(0) \longrightarrow S_1^{Y^*}(0)$, $0 < r \leq r_0$, so that $g(x) = y^*$). Then according to condition *2')*, one may expect that spectrum of the operator $F : D(F) \subset X \longrightarrow Y$ relative to operator $G : D(G) \subseteq X \longrightarrow Y$ can be defined in the following way

$$\lambda = \inf \left\{ \frac{\langle F(x), g(x) \rangle}{\langle G(x), g(x) \rangle} : x \in B_{r_0}^X(0) \setminus \{0\} \right\}, \quad r_0 > 0. \tag{7}$$

Now we will clarify when the determined by (7) λ is the spectrum of operator $G \circ F^{-1}$ or the spectrum of operator F relative to operator G. Generally speaking, one cannot name it since the composition $G \circ F^{-1}$ can be nonlinear and λ_1 may be a function $\lambda_1 = \lambda_1(x_1)$, unlike the linear case, where x_1 is the element on which the relation (7) attains the infimum. Moreover, if we define the subspace $\Gamma_{\lambda_1} = \{\alpha x_1 : \alpha \in \mathbb{R}\} \subset X$ then for $\alpha x_1 \in D(F)$, generally, we have $\alpha\lambda_1 x_1 \neq G \circ F^{-1}(\alpha x_1)$ since $G \circ F^{-1}$ is the nonlinear operator.

Indeed, if the power of nonlinearity of the operator F is greater than the power of nonlinearity of operator G, or the inverse of its, then obviously, it will be the case $\lambda_1 = \lambda_1(x_1)$. For example, let operators F and G be defined in the following way

$$F(u) = -\nabla \circ \left(|\nabla u|^{p_0-2} \nabla u \right), \quad G(u) = |u|^{p_1-2} u, \quad Y = W^{-1,q}(\Omega),$$

where $X = W_0^{1,p_0}(\Omega) \cap L^{p_1}(\Omega)$, $\Omega \subset \mathbb{R}^n$, $n \geq 1$, with sufficiently smooth boundary $\partial\Omega$ and $p = \max\{p_0, p_1\}$, $p_0, p_1 > 2$, $q = p' = \frac{p}{p-1}$. Assume $p_0 \neq p_1$ and $F : D(F) = W^{1,p_0}(\Omega) \longrightarrow W^{-1,q_0}(\Omega)$, $G : D(G) = L^{p_1}(\Omega) \longrightarrow L^{q_1}(\Omega)$. Then using (7) we get

$$\lambda = \inf \left\{ \frac{\langle F(u), u \rangle}{\langle G(u), u \rangle} : u \in B_{r_0}^X(0) \setminus \{0\} \right\} = \inf \left\{ \frac{\|\nabla u\|_{L^{p_0}}^{p_0}}{\|u\|_{L^{p_1}}^{p_1}} : u \in B_{r_0}^{W_0^{1,p_0} \cap L^{p_1}(\Omega)}(0) \setminus \{0\} \right\}.$$

Whence we have if $p_0 > p_1$, then

$$\lambda = \inf \left\{ \left(\frac{\|\nabla u\|_{L^{p_0}}}{\|u\|_{L^{p_1}}} \right)^{p_1} \|\nabla u\|_{L^{p_0}}^{p_0 - p_1} : \ u \in B_{r_0}^{W_0^{1,p_0} \cap L^{p_1}(\Omega)}(0) \setminus \{0\} \right\}$$

and if $p_0 < p_1$, then

$$\lambda = \inf \left\{ \left(\frac{\|\nabla u\|_{L^{p_0}}}{\|u\|_{L^{p_1}}} \right)^{p_0} \|u\|_{L^{p_1}}^{p_0 - p_1} : u \in B_{r_0}^{W_0^{1,p_0} \cap L^{p_1}(\Omega)}(0) \setminus \{0\} \right\}.$$

Consequently, λ will be a function $\lambda = \lambda(u_1)$, where $u_1 \in X$ is the element of the domain, on which the above-mentioned expression attained the infimum (see examples in Section 3).

Remark 2. *It follows that in order to λ_0 could not be a function of x, the main parts of operators F and G must have a common degree of nonlinearity. Theorem 1 follows that the defined number λ_0 is the number assumed to exist in the conditions of this theorem. Moreover, the finding number allows us in mentioned theorem to state the existence of solutions for each λ, $0 \leq |\lambda| \leq \lambda_0$, if the element on the right hand side is from the determined subset.*

The spectrum of an operator usually must characterize the examined operator, but the found λ does not satisfy this. Therefore, another way is used here, different from the above-mentioned works.

Due to the above explanations, we get that in order to the found λ not be a function on x, it is necessary the existence of some relations between the operators F and G.

So, we assume one of the following conditions are fulfilled: F and G are homogeneous with common exponent $p > 0$ or a common function $\phi(\cdot)$, i.e. $F(\mu x) \equiv \mu^p F(x)$, $G(\mu x) \equiv \mu^p G(x)$ or $F(\mu x) \equiv \phi(\mu) F(x)$, $G(\mu x) \equiv \phi(\mu) G(x)$ for any $\mu > 0$.

Let us investigate the problem locally, i.e. we study the problem on the closed ball $B_r^X(0) \subseteq D(f_\lambda)$ for selected $r > 0$ and seek λ in the form $\lambda \equiv \lambda(r)$.

We start to study the first case, i.e. when F and G are homogeneous with exponent $p > 0$. It follows that (7) defines the number λ independent of x. Hence, if we denote this minimum by λ_1 and the element, at which the minimum is attained, by x_1, then (7) will be fulfilled for all $x \in \Gamma_{\lambda_1} \cap D(F)$. Consequently, in this case, one can define x_1 as the first eigenvector and λ_1 as the first eigenvalue of the operator F relative to the operator G (as in the linear case). In other words, $y_1 = F(x_1)$ is the fixed point of operator $\lambda G \circ F^{-1}$.

Consider the second case. If the orders of homogeneity of F and G are different, i.e. given by different functions, e.g. by polynomial functions with exponents $p_F \neq p_G$, then there are possible two subcases: *(a)* $p_F > p_G$ and *(b)* $p_F < p_G$.

Consider the subcase *(a)*. If any $x \in X_0$ we write as $x \equiv r\widetilde{x}$, where $\|x\|_{X_0} \equiv r$ and $\widetilde{x} = \frac{x}{r} \in S_1^{X_0}(0) \subset X_0$, then $F(x) = r^{p_F} F(\widetilde{x})$ and $G(x) = r^{p_G} G(\widetilde{x})$. Hence due to Theorem 3, we get that G can be the perturbation of operator F, therefore this case not is essential.

In the subcase *(b)*, if there exist λ_0 and x_0 such that $F(x_0) = \lambda_0 G(x_0)$, then implications

$$F(x_0) = r_0^{p_F} F(\widetilde{x}_0), \; G(x) = r_0^{p_G} G(\widetilde{x}_0) \Longrightarrow r_0^{p_F} F(\widetilde{x}_0) = \lambda_0 r_0^{p_F} G(\widetilde{x}_0)$$
$$\Longrightarrow F(\widetilde{x}_0) = \lambda(\lambda_0, r_0) G(\widetilde{x}_0) \Longrightarrow \lambda(\lambda_0, r_0) = \lambda_0 r_0^{p_G - p_F}$$

hold. Hence, if we change $x_0 \equiv r_0 \widetilde{x}_0$ to $x_1 \equiv r_1 \widetilde{x}_0$, then λ will be changed to $\lambda = \lambda_0 r_1^{p_G - p_F}$. In other words, if $p_F \neq p_G$, then any existing number λ will depend on element $x \in X$, i.e. $\lambda = \lambda(r)$ on the line $\{x \in X : \; x = r\,\widetilde{x}_0, r \in \mathbb{R}\}$. The previous discussion shows, that there are two variants: either $p_F = p_G$ or $\lambda_0 = \lambda_0(x_0)$, and investigating these cases will be sufficient.

So, here we will study the posed question mainly in the case when condition $p_F = p_G$ holds.

Consequently, the concept defined in the articles [1–3, 11, 13, 18, 19, 22, 26, 28, 29] of the semi-linear spectral set is special case of the Definition 3 by virtue of (3) and (4).

2 Some application of general results

Consider the following problems

$$-\nabla \circ \left(\left|\nabla u\right|^{p-2}\nabla u\right) - \lambda\left|u\right|^{p_0-2} u\left|\nabla u\right|^{p_1} = 0, \quad u\mid_{\partial\Omega} = 0, \;\; \lambda \in \mathbb{C}, \tag{8}$$

$$-\nabla \circ \left(\left|u\right|^{p-2} \nabla u\right) - \lambda\left|u\right|^{p_0-2} u = 0, \quad u\mid_{\partial\Omega} = 0, \;\; \lambda \in \mathbb{C}, \tag{9}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain with sufficiently smooth boundary $\partial\Omega$, $n \geq 1$, $p_0 + p_1 = p$ and $\nabla \equiv (D_1, \ldots, D_n)$. Denote by f_0 the operator generated by (8) which acts from $W_0^{1,p}(\Omega)$ to $W^{-1,q}(\Omega)$. It is easy to see that $f_0 : W_0^{1,p}(\Omega) \longrightarrow W^{-1,q}(\Omega)$ is a continuous operator, and

$$0 = \left\langle f_0(u), u\right\rangle \equiv \left\langle -\nabla \circ \left(\left|\nabla u\right|^{p-2}\nabla u\right) - \lambda\left|u\right|^{p_0-2} u\left|\nabla u\right|^{p_1}, u\right\rangle$$
$$= \left\|\nabla u\right\|_p^p - \int_\Omega \lambda\left|u\right|^{p_0}\left|\nabla u\right|^{p_1} dx \Longrightarrow \left\|\nabla u\right\|_p^p = \lambda \int_\Omega \left|u\right|^{p_0}\left|\nabla u\right|^{p_1} dx$$

holds for any $u \in W_0^{1,p}(\Omega)$. It follows that $\lambda \geq 0$, because both of these expressions are positive.

(1) We will investigate the problem (8) by using the Theorem 1 or Theorem 3. But we interest to study the question on the spectrum, therefore here we will use Corollary 2. According to the previous section, we can introduce the following denotations

$$F(u) = -\nabla\left(|\nabla u|^{p-2}\nabla u\right), \quad F : W_0^{1,p}(\Omega) \longrightarrow W^{-1,q}(\Omega),$$

$$G(u) = |u|^{p_0-2}u|\nabla u|^{p_1}, \quad G : W_0^{1,p}(\Omega) \longrightarrow W^{-1,q}(\Omega).$$

So, we need to seek the minimal value of λ and a function $u_\lambda(x)$ (if it exists) for which the equality

$$\|\nabla u\|_p^p = \lambda \int_\Omega |u|^{p_0} |\nabla u|^{p_1} dx, \quad u \in B_1^{W_0^{1,p}(\Omega)}(0)$$

or

$$\lambda = \frac{\|\nabla u\|_p^p}{\int_\Omega |u|^{p_0} |\nabla u|^{p_1} dx} = \int_\Omega \left(\frac{\|\nabla u\|_p}{|u|}\right)^{p_0} \left(\frac{\|\nabla u\|_p}{|\nabla u|}\right)^{p_1} dx$$

holds. Consequently, we need to find the following number

$$\lambda_1 = \inf\left\{\frac{\|\nabla u\|_p^p}{\int_\Omega |u|^{p_0}|\nabla u|^{p_1}\, dx} : u \in B_1^{W_0^{1,p}(\Omega)}(0)\right\}.$$

It is clear that λ_1 exists and $\lambda_1 > 0$. It follows that

$$\lambda_1 \geq \left(\frac{\|\nabla u\|_p}{\|u\|_p}\right)^{p_0} \Longrightarrow \lambda_1^{\frac{1}{p_0}} \geq \frac{\|\nabla u\|_p}{\|u\|_p} \tag{10}$$

for any $u \in W_0^{1,p}(\Omega)$, $u(x) \neq 0$, that is assumed and in what follows.

We denote by λ_{p_0,p_1} the first spectrum of the posed problem that one can define as

$$\lambda_{p_0,p_1} = \inf\left\{\|\nabla u\|_p \left[\int_\Omega |u|^{p_0} |\nabla u|^{p_1}\, dx\right]^{-\frac{1}{p}} : u \in S_1^{W_0^{1,p}(\Omega)}(0)\right\}. \tag{11}$$

From (10) we obtain

$$\lambda_{p_0,p_1} \geq \lambda_1^{\frac{1}{p_0}} = \inf\left\{\frac{\|\nabla u\|_p}{\|u\|_p} : u \in W_0^{1,p}(\Omega)\right\}$$

that is well-known, $\lambda_1^{\frac{1}{p_0}} = \lambda_1\left(-\Delta_p\right)$ was defined as the first spectrum of p-Laplacian (see, e.g., [23]) that is

$$\lambda_1(-\Delta_p) = \inf\left\{\frac{\|\nabla u\|_p}{\|u\|_p} : u \in W_0^{1,p}(\Omega)\right\},$$

consequently, that inequality shows that λ_{p_0,p_1} is comparable with the spectrum $\lambda_1(-\Delta_p)$ of the p-Laplacian, i.e. λ_{p_0,p_1} satisfy the inequality $\lambda_{p_0,p_1} \geq \lambda_1\left(-\Delta_p\right)$.

(2) Now we will consider the problem (9). Then we get

$$\int_{\Omega} |u|^{p-2} |\nabla u|^2 \, dx = \lambda \int_{\Omega} |u|^{p_0} \, dx \qquad \text{or} \qquad \frac{4}{p^2} \left\| \nabla \left(|u|^{\frac{p-2}{2}} u \right) \right\|_2^2 = \lambda \left\| |u|^{\frac{p_0-2}{2}} u \right\|_2^2,$$

here if we assume $p_0 = p$ and $|u|^{\frac{p-2}{2}} u \equiv v$, then we get

$$\frac{4}{p^2} \|\nabla v\|_2^2 = \frac{4}{p^2} \left\| \nabla \left(|u|^{\frac{p-2}{2}} u \right) \right\|_2^2 = \lambda \left\| |u|^{\frac{p-2}{2}} u \right\|_2^2 = \lambda \|v\|_2^2 .$$

It follows that the first eigenvalue $\lambda_1(p)$ of the operator $-\nabla \cdot \left(|u|^{p-2} \nabla u \right)$ relative to the operator $|u|^{p-2} u$ can be defined using the first eigenvalue $\lambda_1 (-\Delta)$ of the Laplacian, that is it can be defined by expression

$$\lambda_1 (-\Delta) = \inf \left\{ \frac{\|\nabla v\|_2}{\|v\|_2} : v \in W_0^{1.2}(\Omega) \right\} .$$

Consequently, the first eigenvalue $\lambda_1 (p)$ of the operator $-\nabla \cdot \left(|u|^{p-2} \nabla u \right)$ relative to the operator $|u|^{p-2} u$ (with respect to the problem (9)) one can define by the equality $\lambda_1 (p) = \left(\frac{2}{p} \lambda_1 (-\Delta) \right)^2$.

Thus we get the following result.

Proposition 1. *(1) Let Δ_p be the p-Laplacian operator with homogeneous boundary conditions on the bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $p_0 + p_1 = p$. Then the first eigenvalue $\lambda_{p_0 p_1}$ of the operator $-\Delta_p$ relative to the operator $G : G(u) \equiv |u|^{p_0-2} u \left|\nabla u\right|^{p_1}$ exists and it can be defined by the equality* (11).

(2) If F and G are operators, generated by the problem (9)*, and $p_0 = p$, then the first eigenvalue of the operator F relative to the operator G is determined as $\lambda_1 (p) = \left(\frac{2}{p} \lambda_1 (L) \right)^2$, where $\lambda_1 (-\Delta)$ is the first eigenvalue of the Laplacian.*

Remark 3. *It should be noted that by the same way one can define the spectrum of operator Δ_p (and also of the operator $F : F(u) \equiv \sum_{i=1}^{n} D_i \left(|D_i u|^{p-2} D_i u \right)$) relative to the operator $G_0 : G_0 (u) \equiv \sum_{i=1}^{n} |u|^{p_0-2} u \, |D_i u|^{p_1}$.*

Now, using the previous results we will investigate the solvability of the problem on the open bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$ with the following homogeneous boundary condition

$$f_\lambda (u) \equiv -\nabla \left(\left|\nabla u\right|^{p-2} \nabla u \right) - \lambda \, |u|^{p_0-2} u \left|\nabla u\right|^{p_1} = h (x) . \tag{12}$$

For this problem the following result holds.

Theorem 4. *Let numbers p, p_0, $p_1 \geq 0$ be such that $p_0 + p_1 = p \geq 2$, and λ_{p_0,p_1} be the number defined in* (11)*. Then if $\lambda < \lambda_{p_0,p_1}$, then the posed problem is solvable in $W_0^{1,p} (\Omega)$ for each $h \in W^{-1,q} (\Omega)$.*

Proof. Let

$$f_\lambda(u) \equiv -\nabla\left(\left|\nabla u\right|^{p-2}\nabla u\right) - \lambda\left|u\right|^{p_0-2}u\left|\nabla u\right|^{p_1}$$

be the operator, generated by the posed problem for (12), acting as $f_\lambda : X \longrightarrow Y$, where $X \equiv W_0^{1,p}(\Omega)$, $Y \equiv W^{-1,q}(\Omega)$ and fulfill the above conditions. We will use the Corollary 1 for the proof of solvability of this problem.

It follows that the inequality

$$\begin{aligned}\left\langle f_\lambda(u), u\right\rangle &\equiv \left\|\nabla u\right\|_p^p - \lambda\int_\Omega |u|^{p_0}\left|\nabla u\right|^{p_1}dx \geq \left\|\nabla u\right\|_p^p - \lambda\left\|u\right\|_p^{p_0}\left\|\nabla u\right\|_p^{p_1}\\ &= \left\|\nabla u\right\|_p^{p_1}\left(\left\|\nabla u\right\|_p^{p_0} - \lambda\left\|u\right\|_p^{p_0}\right) = \left\|\nabla u\right\|_p^p\left(1 - \frac{\lambda}{\lambda_{p_0,p_1}}\right) = \lambda_{p_0,p_1}^{-1}\left(\lambda_{p_0,p_1} - \lambda\right)\left\|\nabla u\right\|_p^p\end{aligned}$$

holds for any $u \in W_0^{1,p}(\Omega)$ under conditions of Theorem 4.

Consequently, if $\lambda < \lambda_{p_0,p_1}$, then f_λ satisfies the condition *(ii)* of the Theorem 1, moreover it is fulfilled for $x_0 = 0$ and $g \equiv Id$. The realization of the condition *(i)* of Theorem 1 for f_λ is obvious.

The following inequalities show the fulfillment of condition *(iii)* of the Theorem 1 (Corollary 1) for this problem. It is not difficult to see

$$\begin{aligned}\left\langle f(u) - f(v), u - v\right\rangle &\equiv \left\langle\left(\left|\nabla u\right|^{p-2}\nabla u - \left|\nabla v\right|^{p-2}\nabla v\right), \nabla(u-v)\right\rangle\\ &\quad - \lambda\left\langle\left(|u|^{p_0-2}\left|\nabla u\right|^{p_1}u - |v|^{p_0-2}\left|\nabla v\right|^{p_1}v\right), u - v\right\rangle\\ &\geq c_0\left\|\nabla(u-v)\right\|_p^p - \lambda\left\langle |u|^{p_0-2}u\left(\left|\nabla u\right|^{p_1} - \left|\nabla v\right|^{p_1}\right), u - v\right\rangle\\ &\quad - \lambda\left\langle\left|\nabla v\right|^{p_1}\left(|u|^{p_0-2}u - |v|^{p_0-2}v\right), u - v\right\rangle\end{aligned}$$

hold for any $u, v \in W_0^{1,p}(\Omega)$. Here the second term of the right side one can estimate as

$$\begin{aligned}&\left|\left\langle |u|^{p_0-2}u\left(\left|\nabla u\right|^{p_1} - \left|\nabla v\right|^{p_1}\right), u - v\right\rangle\right|\\ &\qquad\leq c_1\left\langle |u|^{p_0-1}\left|\nabla\widetilde{u}\right|^{p_1-1}\left|\nabla u - \nabla v\right|, |u - v|\right\rangle\\ &\qquad\leq \varepsilon\left\|\nabla(u-v)\right\|_p^p + C(\varepsilon)\left\|u\right\|_p^{(p_0-1)p'}\left\|\nabla\widetilde{u}\right\|_p^{(p_1-1)p'}\left\|u - v\right\|_p^{p'},\end{aligned}$$

where $p' = \frac{p}{p-1}$.

Thus we get that all of the conditions of Theorem 1 (the case of Corollary 1) are fulfilled for the problem (12). Consequently, applying Theorem 1 we get the correctness of Theorem 4. □

3 Fully nonlinear operator

Now we will study the question on the existence of the spectrum of the fully nonlinear operator. Let X, Y, Z are real Banach spaces, the inclusion $Y \subset Z^*$ is continuous and dense, where Z^* is the dual space of Z. Let $L : D(L) \subseteq X \longrightarrow Y$ be the linear operator, where $D(L)$ is dense in X. Let $f : D(f) \subseteq Y \longrightarrow Z$ and $g : X \subseteq D(f) \longrightarrow Z$ be nonlinear operators. Assume $L\left(D(L)\right) \subseteq D(f)$.

We wish to define the spectrum of the operator $f \circ L : D(L) \subseteq X \longrightarrow Z$ with respect to the operator g, where $D(L) \subseteq D(g)$. We should note that the nonlinearity of the operator g depends on the nature of the nonlinearity of the operator f.

Consider the problem

$$f_\lambda(x) \equiv f(Lx) - \lambda g(x) = h, \quad h \in Z, \tag{13}$$

where $\lambda \in \mathbb{R}$ is a parameter and h is an element of Z.

We will investigate the existence of the spectrum of the operator $f \circ L$ relative to the operator g, and also the solvability of the equation (13) with a parameter.

We will call a $\lambda \in \mathbb{C}$ the spectrum of the nonlinear operator if it characterizes the examined operator similarly to the linear operator theory according to Definition 3. It follows that it is necessary to assume the identical homogeneity of the nonlinearities of operators f and g according to the above explanations.

So, we study the following particular case that can explain the general case. We use the general results of articles [30,32] to investigate posed problems. Let $B^X_{r_0}(0) \subset D(L)$, $r_0 > 0$.

Consider the following conditions.

1) There are constants $c_1, c_2 > 0$ such that $c_1 \|x\|_X \geq \|Lx\|_Y \geq c_2 \|x\|_X$ for any $x \in D(L) \subseteq X$, where X and Y are reflexive spaces and the inverse to L is a compact operator.

2) The operator $f \circ L$ is greater than the operator g, i.e. $f \circ L \succ g$; f and g are continuous as the functions and satisfy the following conditions: $f(t) \cdot t > 0$ for all $t \in \mathbb{R} \setminus \{0\}$; $f(0) = 0$, $g(0) = 0$.

3) There is a number $\lambda_0 > 0$ such that for each $z^* \in S^{Z^*}_1(0)$ there exist an $x(z^*) \in S^X_r(0)$, $0 \leq r \leq r_0$, such that the inequality
$$\langle f_\lambda(x), z^* \rangle \equiv \langle f(Lx) - \lambda g(x), z^* \rangle \geq \nu(\|Lx\|_Y, \lambda)$$
holds for all λ, $|\lambda| < \lambda_0$, where $\nu : \mathbb{R}_+ \longmapsto \mathbb{R}$ is a continuous function, and there exists $\delta_0(\lambda) > 0$ such that $\nu(t, \lambda) \geq \delta_0(\lambda)$ for all $t = \|Lx\|_Y$, when the variable x moves over the sphere $S^X_{r_0}(0)$.

4) There exist an $\varepsilon_0 > 0$ and a neighborhood $U_\varepsilon(x)$ of a.e. $x \in B^X_{r_0}(0) \subseteq X$ such that the following inequalities
$$\langle f(Lx_1) - f(Lx_2), Lx_1 - Lx_2 \rangle \geq l(x_1, x_2) \|Lx_1 - Lx_2\|^2_Y,$$
$$\|g(x_1) - g(x_2)\|_Z \leq l_1(x_1, x_2) \|x_1 - x_2\|_X$$
hold for all $x_1, x_2 \in U_\varepsilon(x)$, where $l(x_1, x_2) > 0$, $l_1(x_1, x_2) > 0$ are functionals that are bounded in the sense, similar to the Definition 1.

Theorem 5. *Let conditions 1)–4) be fulfilled, $D(L) = X$, and $|\lambda| \leq \lambda_0$. Then equation* (13) *is solvable for all $h \in Z$ and satisfies the condition: for any $z^* \in S^{Z^*}_1(0)$ there exists an $x(z^*) \in S^X_{r_0}(0)$ such that the following relation*

$$|\langle h, z^* \rangle| \leq \langle f_\lambda(x), z^* \rangle$$

holds, in particular, $B^Z_{\delta_0(\lambda)}(0) \subseteq M(\lambda)$ or (13) *is solvable for all $h \in M(\lambda) \subseteq Z$, where the subset $M(\lambda)$ is determined as follows*

$$M(\lambda) \equiv \left\{ z \in Z : |\langle z, z^* \rangle| \leq \nu(\|Lx\|_Y, \lambda),\ \forall z^* \in S^{Z^*}_1(0)\ \exists x(z^*) \in S^X_{r_0}(0) \right\}.$$

Proof. For the proof, it is sufficient to show that the examined operator satisfies all conditions of the general result of Subsection 1.1. It is clear that the operator $F_\lambda(x) \equiv f(Lx) - \lambda g(x)$ satisfies conditions (*i*), (*ii*) of the general results with $x_0 = 0$, according to conditions 1)–4) (since $F_\lambda(0) = 0$). Then it remains to show fulfillment of condition *(iii)*, and for this it is sufficient to investigate the following expression

$$\left\|F_\lambda(x_1) - F_\lambda(x_2)\right\|_Z = \left\|\left(f(Lx_1) - \lambda g(x_1)\right) - \left(f(Lx_2) - \lambda g(x_2)\right)\right\|_Z .$$

Let us prove that this expression satisfies the following inequality

$$\left\|F_\lambda(x_1) - F_\lambda(x_2)\right\|_Z \geq c\Big(l(x_1,x_2)\left\|x_1 - x_2\right\|_X, \lambda\Big) - c_1\Big(l_1(x_1,x_2)\left\|x_1 - x_2\right\|_{X_0}, \lambda\Big).$$

Set the following expression

$$\left\langle F_\lambda(x_1) - F_\lambda(x_2), Lx_1 - Lx_2\right\rangle = \left\langle (f(Lx_1) - f(Lx_2)), Lx_1 - Lx_2\right\rangle - \lambda\left\langle g(x_1) - g(x_2), Lx_1 - Lx_2\right\rangle,$$

that is defined correctly, since $F : D(L) \subseteq X \longrightarrow Z$ and $L : D(L) \subseteq X \longrightarrow Y \subset Z^*$. Hence, by carrying out certain necessary operations and considering the conditions of this section, we get

$$\begin{aligned} &\left\langle F_\lambda(x_1) - F_\lambda(x_2), Lx_1 - Lx_2\right\rangle \\ &\qquad = \left\langle f(Lx_1) - f(Lx_2), Lx_1 - Lx_2\right\rangle - \left\langle \lambda g(x_1) - \lambda g(x_2), Lx_1 - Lx_2\right\rangle \\ &\qquad \geq l(x_1,x_2)\left\|Lx_1 - Lx_2\right\|_Y^2 - |\lambda|\left\|g(x_1) - g(x_2)\right\|_Z \left\|L(x_1 - x_2)\right\|_Y \\ &\qquad \geq l(x_1,x_2)\left\|L(x_1 - x_2)\right\|_Y^2 - |\lambda|\left\|g(x_1) - g(x_2)\right\|_Z \left\|L(x_1 - x_2)\right\|_Y \end{aligned} \tag{14}$$

according to condition 1). Now, again taking into account the condition 1), we get

$$\left|\left\langle F_\lambda(x_1) - F_\lambda(x_2), Lx_1 - Lx_2\right\rangle\right| \leq \left\|F_\lambda(x_1) - F_\lambda(x_2)\right\|_Z \cdot \left\|Lx_1 - Lx_2\right\|_Y . \tag{15}$$

Thus, using inequalities (14) and (15) and condition 4), we obtain the following estimate

$$\begin{aligned} \left\|F_\lambda(x_1) - F_\lambda(x_2)\right\|_Z &\geq l(x_1,x_2)\left\|L(x_1 - x_2)\right\|_Y - |\lambda|\left\|g(x_1) - g(x_2)\right\|_Z \\ &\geq l(x_1,x_2)\left\|L(x_1 - x_2)\right\|_Y - |\lambda|\, l_1(x_1,x_2)\left\|x_1 - x_2\right\|_X . \end{aligned}$$

So, conditions (*i*)–(*iii*) of the general theorem are fulfilled under conditions of this theorem. From the above inequality it follows the fulfillment of condition *(iv)*, that ensures the closedness of the image of $F\left(B_{r_0}^X(0)\right)$. Consequently, the correctness of Theorem 5 follows from the general theorem. □

Remark 4. *It should be noted that the defined in Theorem 5 subset $M(\lambda)$ decreases by increasing the number $|\lambda| \nearrow \lambda_0$. Moreover, the above-mentioned articles actually sought numbers of the type λ_0 the existence assumed in the previous theorem.*

Here we investigate the discovery of such numbers that are independent of elements of the domain. Then, the founded number in such a way can be called the first eigenvalue of the examined operator relative to another operators as in Definition 3.

In what follows we use some results from the article M.S. Berger [6] (see, also [37]), therefore here we provide these results.

Definition 5 ([6]). *Let $A : X \longrightarrow X^*$ be a variational operator. Then A is of class I if:*

(i) A is bounded, i.e. $\|A(x)\| \leq \mu(\|x\|)$;

(ii) A is continuous from the strong topology of X to the weak topology of X^*;

(iii) A is odd, i.e. $A(-x) = -A(x)$;

iv) $\int_0^1 \langle A(sx), x\rangle \, ds \nearrow \infty$ as $\|x\|_X \nearrow \infty$;

(v) $\langle A(x_1) - A(x_2), x_1 - x_2\rangle > 0$ for any $x_1, x_2 \in X$.

Lemma 4 ([6]). *Let A be a variational operator of class I, then*

$$\partial A_R = \left\{ x \in X : \int_0^1 \langle A(sx), x\rangle \, ds = R \right\}$$

is a closed and bounded set in X. Furthermore, $\|x\|_X \geq k(R) > 0$ and ∂A_R is a weakly closed and bounded convex set, where $k(R)$ is a constant independent of $x \in \partial A_R$.

So, consider the homogeneous equation (13) in order to investigate the existence of the necessary number λ_0.

Proposition 2. *Let $X \subset Y$ and it is dense in Y, $Z = Y^*$. Let conditions 1), 2), 4) of Theorem 5 are fulfilled for this case and operators f, g as the functions are monotone odd functions. Then there exist a $\lambda_0 > 0$ and an $x_{\lambda_0} \in \partial E^{B^X_{R_0}(0)} \subset X$ such that $F_{\lambda_0}(x_{\lambda_0}) \equiv f(Lx_{\lambda_0}) - \lambda_0 g(x_{\lambda_0}) = 0$ holds for some number $R_0 \gg 1$, where $\partial E^{B^X_{R_0}(0)}$ is defined as follows*

$$\partial E^{B^X_{R_0}(0)} = \left\{ x \in B^X_{R_0}(0) \subset X : \int_0^1 \langle f(sLx), Lx\rangle \, ds = R_0 \right\},$$
$$E^{B^X_{R_0}(0)} = \left\{ x \in B^X_{R_0}(0) \subset X : \int_0^1 \langle f(sLx), Lx\rangle \, ds \leq R_0 \right\}.$$

Moreover, the condition similar to condition 3) of the above Theorem 5 is satisfied.

Proof. It is clear that $L\big(B^X_{R_0}(0)\big)$ is convex due to the linearity of the operator L. From above Lemma 4 it follows that $E^{B^X_{R_0}(0)}$ is a weakly closed, bounded convex set and $\|x\|_X \geq k(r_0) > 0$, where $k(r_0)$ is a constant independent of $x \in E^{B^X_{R_0}(0)}$, as the operator $f \circ L$ satisfies all conditions of Lemma 4.

Consequently, Lemma 4 implies that $E^{B^X_{R_0}(0)}$ and $\partial E^{B^X_{R_0}(0)}$ are weakly closed bounded convex sets. Consequently, these are closed convex sets due to Mazur Theorem.

Now, consider the expression $\langle g(x), Lx\rangle$ and note that there exists a constant M such that $0 < \sup\left\{ \|g(x)\|_{Y^*} : x \in \partial E^{B^X_{R_0}(0)} \right\} = M < \infty$, according to the conditions 1), 2) and boundedness of the norm $\|Lx\|_Y$, i.e. $0 < \|Lx\|_{Y^*} < M_1 < \infty$, $x \in B^X_{R_0}(0)$.

Consequently, there exist a constant $\lambda_0 = \lambda_0(M)$, where $0 < \lambda_0 < \infty$, and appriopreate x_{λ_0} such that $F_{\lambda_0}(x_{\lambda_0}) = 0$. $\square$

So, we provide the result on the spectrum of the operator $f \circ L$ relative to the operator g.

Theorem 6. *Let f and g be homogeneous functions with equal order of nonlinearity, which is the continuous function φ, i.e. for any $\tau \in \mathbb{R}_+$ equalities $f(\tau \cdot y) = \varphi(\tau) \cdot f(y)$, $g(\tau \cdot y) = \varphi(\tau) \cdot g(y)$ hold. Assume all conditions of the above Proposition 2 are fulfilled. Then operator $f \circ L$ has a spectrum relative to operator g, which is a function of the spectrum of operator L.*

Proof. From Proposition 2 it follows the existence of a $\lambda \in \mathbb{R}_+$ and an element $x \in X$ such that the equation $F_\lambda(\widetilde{x}_\lambda) \equiv f(L\widetilde{x}_\lambda) - \lambda g(\widetilde{x}_\lambda) = 0$ is solvable. Then using the well-known approach it is necessary to seek elements $\lambda_0 \in \mathbb{R}_+$ and $x_0 \in X$, which satisfy the following equality

$$\lambda = \inf\left\{\frac{\langle f(Lx), Lx\rangle}{\langle g(x), Lx\rangle} : x \in X\right\}. \tag{16}$$

Due to the conditions of this theorem, it is enough to study the above question only for $x \in S_1^X(0)$. We can take into account that f is an N-function and the expression $\langle f(Lx), Lx\rangle$ generates a functional $\Phi(Lx)$ according to the condition on f. Note that in the case when L is the differential operator, $\langle f(Lx), Lx\rangle$ is a function of the norm $\|Lx\|_{L_\Phi}$ on some Lebesgue or Orlicz space, where Φ is an N-function.

It follows that it is enough to seek the number λ in the following way

$$\lambda = \inf\left\{\frac{\|f(Lx)\|_Z}{\|g(x)\|_Z} : x \in S_1^X(0)\right\}.$$

From the above expression it follows the existence of number $\lambda > 0$.

Thus one can state that there exists a number λ such that condition 3) of the Theorem 5 is fulfilled for $F_\lambda(\widetilde{x}_\lambda)$ with the mentioned number λ. Since according to conditions of this theorem other conditions of Theorem 5, i.e. conditions 1), 2), 4), are fulfilled. Consequently, using Theorem 5, we get the existence of an element $x_0 \in S_1^X(0)$ and an appropriate number λ_0, that is the element on which the expression (16) attained the infimum λ_0. □

Notation 2. *In particular, if we assume that $x_1 \in S_1^X(0)$ is the first eigenfunction and λ_1 is the first eigenvalue of the operator L, then we have*

$$\lambda_0 \leq \frac{\varphi(\lambda_1)\|f(x_1)\|_Z}{\|g(x_1)\|_Z}.$$

Now we provide some examples of operators related to the above theorems.

1. Let $L : W^{m,p}(\Omega) \longrightarrow L_p(\Omega)$ be a linear differential operator with the spectrum $P(L) \subset \mathbb{R}_+$, the operator f is the function $f(\tau) = |\tau|^{p-2}\tau$ and $g \equiv f$. So, it needs to be defined the first eigenfunction and eigenvalue of the operator $f(L\circ)$ relative to operator $g(\circ)$. Then using the expression (16) we get

$$\lambda_f = \inf\left\{\frac{\langle f(Lu), Lu\rangle}{\langle g(u), Lu\rangle} : u \in W^{m,p}(\Omega)\right\} = \inf\left\{\frac{\|Lu\|_{L_p}^p}{\int_\Omega |u|^{p-2}u\, Lu\, dx} : u \in W^{m,p}(\Omega)\right\}$$
$$\geq \inf\left\{\frac{\|Lu\|_{L_p}^{p-1}}{\|u\|_{L_p}^{p-1}} : u \in W^{m,p}(\Omega)\right\} = \inf\left\{\left(\frac{\|Lu\|_{L_p}}{\|u\|_{L_p}}\right)^{p-1} : u \in S_1^{W^{m,p}(\Omega)}(0)\right\}.$$

Whence we arrive that $\lambda_{f1} \geq \lambda_{L1}^{p-1}$, where λ_{L1} is the first eigenvalue and the function $u_1 \in S_1^{W^{m,p}(\Omega)}(0)$ is the first eigenfunction of the operator L.

2. We study the spectral property of the fully nonlinear operator in the following two special cases

$$-\left|\Delta u\right|^{p-2}\Delta u=\lambda\left|\nabla u\right|^{\mu-2}u,\quad x\in\Omega,\quad u\big|_{\partial\Omega}=0, \tag{17}$$

$$-\left|\Delta u\right|^{p-2}\Delta u=\lambda\left|u\right|^{\nu}u,\quad x\in\Omega,\quad u\big|_{\partial\Omega}=0, \tag{18}$$

i.e. we seek the spectrum of the operator $-\left|\Delta u\right|^{p-2}\Delta u$ relative to operators $\left|\nabla u\right|^{\mu-2}u$ and $\left|u\right|^{\nu}u$ separately.

2 (a). Consider the problem (17). We use the following equality

$$\left\langle -\left|\Delta u\right|^{p-2}\Delta u,-\Delta u\right\rangle=\left\langle \lambda\left|\nabla u\right|^{\mu-2}u,-\Delta u\right\rangle .$$

Then we get $\left\|\Delta u\right\|_p^p=\lambda\left(\mu-1\right)^{-1}\left\|\nabla u\right\|_\mu^\mu$. Hence,

$$\lambda_1(p,\mu)=(\mu-1)\inf\left\{\frac{\|\Delta u\|_p^p}{\|\nabla u\|_\mu^\mu}:u\in W^{2,p}\cap W_0^{1,p}\right\}=(\mu-1)\inf\left\{\frac{\|\Delta u\|_p}{\|\nabla u\|_\mu^{\mu/p}}:u\in W^{2,p}\cap W_0^{1,p}\right\}.$$

It follows that to find λ, satisfying the assumed condition, we must select the corresponded exponent μ.

Consequently, we need to assume $\mu=p$, then we get

$$\lambda_1\left(p,p\right)=\left(p-1\right)\inf\left\{\frac{\|\Delta u\|_p}{\left\|\nabla u\right\|_p}:u\in W^{2,p}\left(\Omega\right)\cap W_0^{1,p}\left(\Omega\right)\right\}. \tag{19}$$

It is well-known that $\left\|\nabla u\right\|_p\leq c\left(p,\Omega\right)\left\|\Delta u\right\|_p$ under the condition $u\big|_{\partial\Omega}=0$, consequently, $\lambda_1\left(p,p\right)\leq\left(p-1\right)c\left(p,\Omega\right)$.

Proposition 3. *Let $f_0:W^{2,p}\left(\Omega\right)\cap W_0^{1,p}\left(\Omega\right)\longrightarrow L^q\left(\Omega\right)$ has the form $f_0\left(u\right)=-\left|\Delta u\right|^{p-2}\Delta u$ and $f_1:W_0^{1,p}\left(\Omega\right)\longrightarrow L^q\left(\Omega\right)$ has the form $f_1\left(u\right)=\left|\nabla u\right|^{p-2}u$. Then the operator f_0 has the first eigenvalue relative to operator f_1, which is defined by* (19).

2 (b). Consider the problem (18) for $\nu=p-2$. Then we have that

$$\left\langle -\left|\Delta u\right|^{p-2}\Delta u,-\Delta u\right\rangle=\left\langle \lambda\left|u\right|^{p-2}u,-\Delta u\right\rangle$$

implies

$$\left\|\Delta u\right\|_p^p=\lambda\frac{4\left(p-1\right)}{p^2}\left\|\nabla\left(\left|u\right|^{\frac{p-2}{2}}u\right)\right\|_2^2\qquad\text{or}\qquad\left\|\Delta u\right\|_p^p=\lambda\left(p-1\right)\left\|\left(\left|u\right|^{p-2}\left|\nabla u\right|^2\right)\right\|_1.$$

Thus we get

$$\begin{aligned}\widetilde{\lambda}_1(p)&=\frac{1}{p-1}\inf\left\{\frac{\|\Delta u\|_p^p}{\left\|\left|u\right|^{\frac{p-2}{2}}\left|\nabla u\right|\right\|_2^2}:u\in W^{2,p}\cap W_0^{1,p}\right\}\\&=\frac{p}{2(p-1)}\inf\left\{\frac{\left\|\left|\Delta u\right|^{\frac{p}{2}}\right\|_2}{\left\|\nabla\left(\left|u\right|^{\frac{p-2}{2}}\left|u\right|\right)\right\|_2}:u\in W^{2,p}\cap W_0^{1,p}\right\}.\end{aligned} \tag{20}$$

Hence we obtain

$$\widetilde{\lambda}_1(p) \geq \frac{1}{p-1} \frac{\|\Delta u\|_p^p}{\|u\|_p^{p-2} \|\nabla u\|_p^2} \tag{21}$$

according to the following inequality

$$\left\| \left(|u|^{p-2} |\nabla u|^2 \right) \right\|_1 \leq \|u\|_p^{p-2} \|\nabla u\|_p^2 .$$

So, we arrive at the following result.

Proposition 4. *Let $f_0 : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \longrightarrow L^q(\Omega)$ has the form $f_0(u) = -|\Delta u|^{p-2} \Delta u$ and $f_1 : L^p(\Omega) \longrightarrow L^q(\Omega)$ has the form $f_1(u) = |u|^{p-2} u$. Then the operator f_0 has the first eigenvalue relative to operator f_1, which is defined by* (20) *and it satisfies the inequation* (21).

Remark 5. *In the previous case, it would be to use the following equality*

$$-\int_\Omega |\Delta u|^{p-2} \Delta u \, dx = \lambda \int_\Omega |u|^{p-2} u \, dx.$$

It implies $\|\Delta u\|_{p-1} = \lambda \|u\|_{p-1}$. Then we get

$$\lambda = \inf \left\{ \frac{\|\Delta u\|_{p-1}}{\|u\|_{p-1}} : u \in W^{2,p} \cap W_0^{1,p} \right\},$$

since in the conditions of this section the operator $-\Delta$ is positive.

4 Conclusions

To seek the eigenvalues of a nonlinear continuous operator in a Banach space it is necessary to choose the other operator in such a way that the order of nonlinearity should be identical with the order of nonlinearity of the examined operator. If one uses the proposed approach, then it is possible to find the other eigenvalues of this operator. The importance of the knowledge of the eigenvalues of the operators in the study of the bifurcation of solutions of nonlinear equations is showed in many articles dedicated to studying the bifurcation of the solutions of nonlinear equations (see Section 3 and, e.g., articles [12, 16, 36]).

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У цій статті досліджено існування спектру (власних значень) для нелінійних неперервних операторів, що діють у банахових просторах. Для дослідження цього питання використовується інший підхід, що дозволяє вивчення всіх власних значень нелінійного оператора відносно іншого нелінійного оператора. Тут показано, що у випадку нелінійних операторів необхідно шукати спектр даного нелінійного оператора відносно іншого нелінійного оператора, що задовольняє певні умови. Наведено різні приклади, для яких можна знайти власні значення. Більше того, у статті досліджуються нелінійні задачі з параметрами.

Ключові слова і фрази: нелінійний неперервний оператор, спектр, банаховий простір, нелінійний диференціальний оператор, розв'язність.