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Characterizing Riesz bases via biorthogonal Bessel sequences

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Recently D.T. Stoeva proved that if two Bessel sequences in a separable Hilbert space $\mathcal{H}$ are biorthogonal and one of them is complete in $\mathcal{H}$, then both sequences are Riesz bases for $\mathcal{H}$. This improves a well known result where completeness is assumed on both sequences.

In this note we present an alternative proof of Stoeva's result which is quite short and elementary, based on the notion of Riesz-Fischer sequences.

Key words and phrases: Riesz-Fischer sequence, Bessel sequence, Riesz sequence, Riesz basis, biorthogonal sequence, completeness.

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Introduction

Let $\mathcal{H}$ be a separable Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\| \cdot \|$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of vectors in $\mathcal{H}$. We say that:

(i) $\{f_n\}_{n=1}^{\infty}$ is a complete sequence if the closed span of $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ is equal to $\mathcal{H}$;

(ii) $\{f_n\}_{n=1}^{\infty}$ is minimal if each f_n does not belong to the closed span of $\{f_k\}_{k\neq n}$ in $\mathcal{H}$;

(iii) $\{f_n\}_{n=1}^{\infty}$ is exact if it is both complete and minimal.

It is well known that $\{f_n\}_{n=1}^{\infty}$ is a minimal sequence in $\mathcal{H}$ if and only if it has a biorthogonal sequence $\{g_n\}_{n=1}^{\infty}$ in $\mathcal{H}$, that is

$$\langle f_n, g_m \rangle = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases}$$

Remark 1. *An exact sequence in $\mathcal{H}$ has a unique biorthogonal sequence.*

We also say that

(iv) $\{f_n\}_{n=1}^{\infty}$ is a Bessel sequence if

$$\sum_{n=1}^{\infty} \big\| \langle f, f_n \rangle \big\|^2 < \infty \qquad \forall\, f \in \mathcal{H};$$

(v) $\{f_n\}_{n=1}^{\infty}$ is a Riesz sequence (see [2, p. 68] and [4, Lemma 3.2]), if there are some positive constants A and B, $A \leq B$, so that for any finite scalar sequence $\{\beta_n\}$ we have

$$A \sum_{n=1}^{\infty} |\beta_n|^2 \leq \Big| \sum_{n=1}^{\infty} \beta_n f_n \Big|^2 \leq B \sum_{n=1}^{\infty} |\beta_n|^2;$$

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(vi) $\{f_n\}_{n=1}^{\infty}$ is a Riesz basis for $\mathcal{H}$ if $f_n = U(e_n)$, where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and U is a bounded bijective operator from $\mathcal{H}$ onto $\mathcal{H}$.

Remark 2. *A Riesz sequence is a Riesz basis for the closure of its linear span in $\mathcal{H}$ (see [2, p. 68]). Therefore, a complete Riesz sequence in $\mathcal{H}$ is a Riesz basis for $\mathcal{H}$.*

There are many equivalences of Riesz bases (see, e.g., [5, Theorem 1.1]). One of them states that a sequence $\{f_n\}_{n=1}^{\infty}$ is a Riesz basis for $\mathcal{H}$ if and only if $\{f_n\}_{n=1}^{\infty}$ is a complete Bessel sequence having a complete biorthogonal Bessel sequence $\{g_n\}_{n=1}^{\infty}$ in $\mathcal{H}$.

Recently D.T. Stoeva [5] improved the above by assuming completeness on just one (anyone) of the two $\{f_n\}_{n=1}^{\infty}$, $\{g_n\}_{n=1}^{\infty}$ sequences.

Theorem A ([5, Theorem 2.5])**.** *Let two sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ be biorthogonal. If both of them are Bessel sequences and one of them is complete in $\mathcal{H}$, then they are Riesz bases for $\mathcal{H}$.*

Our goal in this note is to offer an alternative proof of Theorem A, which is quite short and elementary. We only need to use the notion of Riesz-Fischer sequences introduced below and a result by P. Casazza et al. [1].

1 Riesz-Fischer sequences and Bessel sequences

Following R.M. Young (see [7, Chapter 4, Section 2]), we say that a sequence of vectors $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ is a Riesz-Fischer sequence if the moment problem

$$\langle f, f_n \rangle = c_n$$

has at least one solution $f \in \mathcal{H}$ for every sequence $\{c_n\}_{n=1}^{\infty}$ in the space $l^2(\mathbb{N})$.

In [7, Chapter 4, Section 2, Theorem 3], we find the following two theorems, attributed to N. Bari, which provide a necessary and sufficient condition so that a sequence in $\mathcal{H}$ is either a Riesz-Fischer sequence or a Bessel sequence.

(A) $\{f_n\}_{n=1}^{\infty}$ is a Riesz-Fischer sequence in $\mathcal{H}$ if and only if there exists a positive number A so that for any finite scalar sequence $\{\beta_n\}$ we have

$$A \sum_{n=1}^{\infty} |\beta_n|^2 \leq \left\| \sum_{n=1}^{\infty} \beta_n f_n \right\|^2. \tag{1}$$

(B) $\{f_n\}_{n=1}^{\infty}$ is a Bessel sequence in $\mathcal{H}$ if and only if there exists a positive number B so that for any finite scalar sequence $\{\beta_n\}$ we have

$$\left\| \sum_{n=1}^{\infty} \beta_n f_n \right\|^2 \leq B \sum_{n=1}^{\infty} |\beta_n|^2. \tag{2}$$

Remark 3. *Hence, a Riesz sequence is a Bessel sequence and a Riesz-Fischer sequence simultaneously.*

It easily follows from (1) that a Riesz-Fischer sequence is also a minimal sequence hence it has at least one biorthogonal sequence. As stated by P. Casazza et al. [1], one of them is a Bessel sequence.

Proposition ([1, Proposition 2.3, (ii)]). *The Riesz-Fischer sequences in $\mathcal{H}$ are precisely the families for which a biorthogonal Bessel sequence exists.*

For the sake of completeness, we present a proof of one of the two directions of the above result.

Lemma 1. *Suppose that a Bessel sequence $\{f_n\}_{n=1}^{\infty}$ is biorthogonal to a sequence $\{g_n\}_{n=1}^{\infty}$ in $\mathcal{H}$. Then $\{g_n\}_{n=1}^{\infty}$ is a Riesz-Fischer sequence.*

Proof. Consider a finite scalar sequence $\{\beta_n\}$. Due to biorthogonality, the Cauchy-Schwartz inequality and since $\{f_n\}_{n=1}^{\infty}$ is a Bessel sequence, thus (2) holds, there is some positive constant A so that

$$\begin{aligned}\left(\sum_{n=1}^{\infty}|\beta_n|^2\right)^2 &= \left(\left\langle \sum_{n=1}^{\infty}\beta_n\cdot f_n, \sum_{n=1}^{\infty}\beta_n\cdot g_n\right\rangle\right)^2 \\ &\leq \left\|\sum_{n=1}^{\infty}\beta_n\cdot f_n\right\|^2\cdot\left\|\sum_{n=1}^{\infty}\beta_n\cdot g_n\right\|^2 \\ &\leq A\cdot\sum_{n=1}^{\infty}|\beta_n|^2\cdot\left\|\sum_{n=1}^{\infty}\beta_n\cdot g_n\right\|^2.\end{aligned}$$

It is clearly now, that (1) holds, therefore $\{g_n\}_{n=1}^{\infty}$ is a Riesz-Fischer sequence. □

2 Proof of Theorem A and an application

Consider the assumptions of Theorem A. Then by Lemma 1 the biorthogonal Bessel sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are also Riesz-Fischer sequences in $\mathcal{H}$. Therefore, both of them are Riesz sequences in $\mathcal{H}$. If one of them, say $\{f_n\}_{n=1}^{\infty}$, is complete in $\mathcal{H}$, then it follows from Remark 2 that $\{f_n\}_{n=1}^{\infty}$ is a Riesz basis for $\mathcal{H}$. Biorthogonality yields that $\{g_n\}_{n=1}^{\infty}$ is a Riesz basis for $\mathcal{H}$ as well. The proof of Theorem A is now complete.

As an application of Theorem A, consider an exact Bessel sequence $\{f_n\}_{n=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ such that it is not a Riesz basis for $\mathcal{H}$. Since it is exact, it has a unique biorthogonal sequence $\{g_n\}_{n=1}^{\infty}$. By Lemma 1, $\{g_n\}_{n=1}^{\infty}$ is a Riesz-Fischer sequence. However, $\{g_n\}_{n=1}^{\infty}$ is not a Bessel sequence: if it were, then by Theorem A the families $\{g_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ would be Riesz bases for $\mathcal{H}$.

As an example, consider the exponential system $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$, where

$$\lambda_n = \begin{cases} n+\frac{1}{4}, & n>0, \\ 0, & n=0, \\ n-\frac{1}{4}, & n<0. \end{cases}$$

The system $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is not only minimal but also uniformly minimal in $L^2(-\pi,\pi)$ (see [3, Theorem 5]). In fact it is exact in $L^2(-\pi,\pi)$ (see [7, Chapter 3, Section 2, Theorem 4]). However, this exponential system is not a Riesz basis for $L^2(-\pi,\pi)$ (see [3, Theorem 4]). Nevertheless, since the frequencies λ_n are uniformly separated ($\lambda_{n+1}-\lambda_n \geq 1$ for all $n\in\mathbb{Z}$), the system is a Bessel sequence in every $L^2(-A,A)$ space, $A>0$ (see [7, Chapter 4, Section 3, Theorem 4]). But clearly, the system is not a Riesz-Fischer sequence in $L^2(-\pi,\pi)$, otherwise it would be a Riesz basis. Moreover, the various properties of this exponential system, Lemma 1 and Theorem A, imply that the system has a unique biorthogonal sequence $\{g_n\}_{n\in\mathbb{Z}}$ in $L^2(-\pi,\pi)$, which is a Riesz-Fischer sequence but not a Bessel one. In addition, it follows from [6], that $\{g_n\}_{n\in\mathbb{Z}}$ is also exact in $L^2(-\pi,\pi)$, a property enjoyed by biorthogonal families to exact exponential systems $\{e^{i\mu_n t}\}_{n\in\mathbb{Z}}$ in $L^2(-\pi,\pi)$.

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Недавно Д.Т. Стоєва довела, що якщо дві послідовності Бесселя в сепарабельному гільбертовому просторі $\mathcal{H}$ є біортогональними та одна з них є повною в $\mathcal{H}$, то обидві послідовності є базами Рісса для $\mathcal{H}$. Це покращує добре відомий результат, коли передбачається повнота обох послідовностей.

У цій замітці ми представляємо альтернативне доведення результату Стоєвої, яке є досить коротким і елементарним, та ґрунтується на понятті послідовностей Рісса-Фішера.

Ключові слова і фрази: послідовність Рісса-Фішера, послідовність Бесселя, послідовність Рісса, базис Рісса, біортогональна послідовність, повнота.