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On the domain of convergence of general Dirichlet series with complex exponents

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Let (λ_n) be a sequence of the pairwise distinct complex numbers. For a formal Dirichlet series $F(z) = \sum\limits_{n=0}^{+\infty} a_n e^{z\lambda_n}$, $z \in \mathbb{C}$, we denote $G_\mu(F)$, $G_c(F)$, $G_a(F)$ the domains of the existence, of the convergence and of the absolute convergence of maximal term $\mu(z,F) = \max\left\{|a_n|e^{\mathrm{Re}(z\lambda_n)} : n \geq 0\right\}$, respectively. It is well known that $G_\mu(F), G_a(F)$ are convex domains.

Let us denote $\mathcal{N}_1(z) := \{n : \mathrm{Re}(z\lambda_n) > 0\}$, $\mathcal{N}_2(z) := \{n : \mathrm{Re}(z\lambda_n) < 0\}$ and

$$\alpha^{(1)}(\theta) := \varliminf_{\substack{n\to+\infty\\ n\in\mathcal{N}_1(e^{i\theta})}} \frac{-\ln|a_n|}{\mathrm{Re}(e^{i\theta}\lambda_n)}, \qquad \alpha^{(2)}(\theta) := \varlimsup_{\substack{n\to+\infty\\ n\in\mathcal{N}_2(e^{i\theta})}} \frac{-\ln|a_n|}{\mathrm{Re}(e^{i\theta}\lambda_n)}.$$

Assume that $a_n \to 0$ as $n \to +\infty$. In the article, we prove the following statements.

1) If $\alpha^{(2)}(\theta) < \alpha^{(1)}(\theta)$ for some $\theta \in [0,\pi)$ then $\left\{te^{i\theta} : t \in (\alpha^{(2)}(\theta), \alpha^{(1)}(\theta))\right\} \subset G_\mu(F)$ as well as $\left\{te^{i\theta} : t \in (-\infty, \alpha^{(2)}(\theta)) \cup (\alpha^{(1)}(\theta), +\infty)\right\} \cap G_\mu(F) = \varnothing$.

2) $G_\mu(F) = \bigcup\limits_{\theta\in[0,\pi)} \{z = te^{i\theta} : t \in (\alpha^{(2)}(\theta), \alpha^{(1)}(\theta))\}$.

3) If $h := \varliminf\limits_{n\to+\infty} \frac{-\ln|a_n|}{\ln n} \in (1,+\infty)$, then $\left(\frac{h}{h-1} \cdot G_a(F)\right) \supset G_\mu(F) \supset G_c(F)$. If $h = +\infty$ then $G_a(F) = G_c(F) = G_\mu(F)$, therefore $G_c(F)$ is also a convex domain.

Key words and phrases: domain of convergence, abscissa of convergence, Dirichlet series.

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1 Introduction

We consider the class $\mathcal{D}(\Lambda)$ of Dirichlet series of the form

$$F(z) = \sum_{n=0}^{+\infty} a_n e^{z\lambda_n} \tag{1}$$

such that $a_n e^{z_0\lambda_n} \to 0$ as $n \to +\infty$ for some z_0, where a sequence of the exponents $\Lambda = (\lambda_n)$ is a sequence of pairwise distinct complex numbers. In the case $0 < \lambda_n \uparrow +\infty$ as $1 \leq n \uparrow +\infty$ we will write Λ_+ and $\mathcal{D}(\Lambda_+)$. Denote $\mathcal{D} := \bigcup_\Lambda \mathcal{D}(\Lambda)$, $\mathcal{D}_+ := \bigcup_{\Lambda_+} \mathcal{D}(\Lambda_+)$.

Convergence sets of Dirichlet series $F \in \mathcal{D}(\Lambda)$ with complex exponents Λ were studied by W. Schnee [1–3], G.H. Hardy and M. Riesz [4], J.F. Ritt [7], E. Hille [8], J. Micusiński [9], T.M. Gallie [10, 11], G. Peyser [12], etc. In [7], J.F. Ritt considered Dirichlet series with complex exponents in the context of a differential equation of infinite order under the constraint

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that the series $\sum\limits_{k=1}^{+\infty} 1/|\lambda_k|$ should be convergent, but at the same time does not impose any restrictions on the arguments λ_n, which other authors in the field do (see much later research in [9,16,17]). In [9], J. Micusiński considered the Dirichlet series of the form (1) by the condition $\lim\limits_{n\to+\infty} \frac{\operatorname{Re}\lambda_n}{\ln n} = +\infty$. E. Hille [8] and some other authors considered series under the condition $\tau(\Lambda) := \varlimsup\limits_{n\to+\infty} \frac{\ln n}{|\lambda_n|} < +\infty$ and, in particular, in the case $\tau(\Lambda) = 0$ (see also [15], where the series of the form $\sum\limits_{n=0}^{+\infty} a_n z^{m_n} e^{z\lambda_n}$ is considered). Note that the condition $\lim\limits_{n\to+\infty} \frac{\operatorname{Re}\lambda_n}{\ln n} = +\infty$ implies that $\tau(\Lambda) = 0$. The condition $\sum\limits_{k=1}^{+\infty} 1/|\lambda_k| < +\infty$ implies that $\lim\limits_{n\to+\infty} \frac{n}{|\lambda_n|} = 0$, i.e. again $\tau(\Lambda) = 0$.

We will write more about these studies below in the relevant places of the text of the article. Let us just say that throughout the article we try to avoid a priori conditions (to the extent possible at all) on the sequence of exponents. Actually, both in the articles just listed and in the articles of many other authors, various a priori assumptions regarding the sequence of exponents Λ played the main role in the research.

In this article, we will describe the domain of existence of the maximal term of the series of form (1), establish conditions on the coefficients of the series for which the domains of convergence, absolute convergence, and existence of the maximal term coincide. From here, in particular, we get the convexity of the convergence domain without any additional conditions on the sequence of exponents $\Lambda = (\lambda_n)$, $\lambda_n \in \mathbb{C}$, $n \geq 0$. We borrowed the idea of studying the domains of convergence of general Dirichlet series by studying the domains of existence of the maximal term of the series from article [14] (see also articles [18–24]). In the end, we note that both in this introduction and further in the text of the article, we hardly discussed the results of the investigations of the convergence abscissas of Dirichlet series with positive exponents. An overview of this topic can be found, for example, in papers [22, 23].

2 Domain of the existence of maximal term

For a formal Dirichlet series $F \in \mathcal{D}(\Lambda)$ we denote by

$$D_\mu(F) := \Big\{ z \in \mathbb{C} : a_n e^{z\lambda_n} \to 0 \ (n \to +\infty) \Big\}$$

the set of the existence of maximal term $\mu(z, F) = \max\left\{ |a_n| e^{\operatorname{Re}(z\lambda_n)} : n \geq 0 \right\}$ of the series (1), $G_\mu(F) = D_\mu(F) \setminus \partial D_\mu(F)$ is the domain of the existence of maximal term of the series (1).

It is obvious that $G_\mu(F)$ is a convex domain or $G_\mu(F) = \varnothing$, and as simple examples show, it can be both bounded and unbounded in the general case.

Indeed, first consider a formal Dirichlet series $F \in \mathcal{D}(\Lambda)$ of form (1) with a sequence of exponents $\Lambda = (\lambda_n)$ such that $\lambda_n \geq 0$, $n \geq 0$. It is well known that for such Dirichlet series F the value

$$\sigma_\mu(F) := \sup\left\{ \sigma : a_n(\omega) e^{x\lambda_n(\omega)} \to 0 \ \text{ as } \ n \to +\infty \ \ \forall x < \sigma \right\}$$

is the abscissa of the existence of the maximal term of the Dirichlet series and in this case $G_\mu(F) = \Pi_a := \{z : \operatorname{Re} z < a\}$ with $a = \sigma_\mu(F)$ and $\sigma_\mu(F) \neq -\infty$. Thus, the domain $G_\mu(F) = \Pi_a$ with $a = \sigma_\mu(F)$ is a nonempty and unbounded half-plane.

Next, reasoning in a similar way, we will consider the Dirichlet series $F \in \mathcal{D}(\Lambda)$ of form (1) with exponents $\lambda_n = i^n n$, $n \geq 0$, and write

$$F(z) = \sum_{j=1}^{4} F_j(z), \qquad F_j(z) = \sum_{k=0}^{+\infty} a_k^{(j)} e^{\lambda_k^{(j)} z},$$

where $\lambda_k^{(1)} = \lambda_{4k} = 4k$, $\lambda_k^{(2)} = \lambda_{4k+2} = -4k-2$, $\lambda_k^{(3)} = \lambda_{4k+1} = (4k+1)i$, $\lambda_k^{(4)} = \lambda_{4k+3} = -(4k+3)i$, $a_k^{(j)} = e^{-\left|\lambda_k^{(j)}\right|}$. It easy to see that $F_j \in \mathcal{D}(\Lambda)$, $F_j \in \mathcal{D}(\Lambda)$,

$$G_\mu(F_1) = \Pi_1, \quad G_\mu(F_2) = \mathbb{C} \setminus \overline{\Pi}_{-1} = -\Pi_1, \quad G_\mu(F_3) = -i\Pi_1, \quad G_\mu(F_4) = i\Pi_1,$$

and $G_\mu(F) = \bigcap_{j=1}^{4} G_\mu\left(F_j\right)$ is the square $\left\{z = x + iy \colon |x| < 1, |y| < 1\right\}$, that is, the bounded domain.

Denote $\mathcal{N}_1(z) = \mathcal{N}_1(z, F) := \{n : \operatorname{Re}(z\lambda_n) > 1\}$, $\mathcal{N}_2(z) = \mathcal{N}_2(z, F) := \{n : \operatorname{Re}(z\lambda_n) < -1\}$, $\mathcal{N}_3(z) := \{n : -1 \leq \operatorname{Re}(z\lambda_n) \leq 1\}$.

Remark 1. *It easy to see that* $\mathcal{N}_1\left(e^{i(\theta+\pi)}\right) = \mathcal{N}_2\left(e^{i\theta}\right)$ *for* $\theta \in [0, \pi)$.

Proposition 1. *Let* $F \in \mathcal{D}(\Lambda)$ *be of form* (1). *In order to* $G_\mu(F) = \mathbb{C}$, *it is necessary and sufficient that*

$$\forall z \in \mathbb{C}\colon \quad \lim_{\substack{n\to+\infty \\ n\in\mathcal{N}_1(z)}} \frac{-\ln|a_n|}{\operatorname{Re}(z\lambda_n)} = +\infty, \quad \lim_{\substack{n\to+\infty \\ n\in\mathcal{N}_2(z)}} \frac{-\ln|a_n|}{-\operatorname{Re}(z\lambda_n)} = +\infty, \quad \lim_{n\to+\infty} |a_n| = 0.$$

Proof. Let us first assume $\sharp\, \mathcal{N}_1(z) = +\infty$ and $\sharp\, \mathcal{N}_2(z) = +\infty$.

Necessarity. Let $G_\mu(F) = \mathbb{C}$. Then $|a_n| \exp\left(x \operatorname{Re}(z\lambda_n)\right) \to 0$ as $n \to +\infty$ for every $z \in \mathbb{C}$, $x \in \mathbb{R}$, thus $-\ln|a_n| - x\operatorname{Re}(z\lambda_n) \to +\infty$ as $n \to +\infty$. Hence, $-\ln|a_n| \geq x\operatorname{Re}(z\lambda_n)$ for all enough large n. Therefore,

$$\varliminf_{\substack{n\to+\infty \\ n\in\mathcal{N}_1(z)}} \frac{-\ln|a_n|}{\operatorname{Re}(z\lambda_n)} \geq x.$$

It remains to use the arbitrariness of the choice $x > 0$. Similarly,

$$\varlimsup_{\substack{n\to+\infty \\ n\in\mathcal{N}_2(z)}} \frac{-\ln|a_n|}{\operatorname{Re}(z\lambda_n)} \leq x \quad \Longrightarrow \quad \varliminf_{\substack{n\to+\infty \\ n\in\mathcal{N}_2(z)}} \frac{-\ln|a_n|}{-\operatorname{Re}(z\lambda_n)} \geq -x.$$

Again, it remains to use the arbitrariness of choosing $x < 0$.

Sufficiency. From condition it follows that $\ln|a_n| + \operatorname{Re}(z\lambda_n) \to -\infty$ as $n \to +\infty$ with $n \in \mathcal{N}_1(z)$. Indeed, for any $x > 0$ and for all enough large $n \in \mathcal{N}_1(z)$ is fulfilled the inequality $-\ln|a_n| > (1+x)\operatorname{Re}(z\lambda_n)$, thus

$$-\ln|a_n| - \operatorname{Re}(z\lambda_n) > x\operatorname{Re}(z\lambda_n) > x$$

for all enough large $n \in \mathcal{N}_1(z)$ and for every $z \in \mathbb{C}$. Using the arbitrariness of the choice of $x > 0$, we obtain $\ln|a_n| + \operatorname{Re}(z\lambda_n) \to -\infty$ as $n \to +\infty$ with $n \in \mathcal{N}_1(z)$. Hence, $|a_n| e^{\operatorname{Re}(z\lambda_n)} \to 0$ as $n \to +\infty$ with $n \in \mathcal{N}_1(z)$ for every $z \in \mathbb{C}$. Similarly we obtain that $|a_n| e^{\operatorname{Re}(z\lambda_n)} \to 0$ as $n \to +\infty$ with $n \in \mathcal{N}_2(z)$ for every $z \in \mathbb{C}$.

Thus, $z \in D_\mu$ in case $\sharp\,\mathcal{N}_3(z) < +\infty$. But $|a_n|\,e^{\mathrm{Re}(z\lambda_n)} \le e \cdot |a_n| \to 0$ as $n \to +\infty$ with $n \in \mathcal{N}_3(z)$ in case $\sharp\,\mathcal{N}_3(z) = +\infty$. Therefore, $z \in D_\mu$, so $D_\mu = \mathbb{C}$ in case $\sharp\,\mathcal{N}_1(z) = +\infty$, $\sharp\,\mathcal{N}_2(z) = +\infty$. If $\sharp\,\mathcal{N}_1(z) < +\infty$ and $\sharp\,\mathcal{N}_2(z) < +\infty$ for some $z \in \mathbb{C}$ then $\sharp\,\mathcal{N}_3(z) = +\infty$ and $|a_n|e^{\mathrm{Re}(z\lambda_n)} \le e \cdot |a_n| \to 0$ as $n \to +\infty$ with $n \in \mathcal{N}_3(z)$, that is,

$$\lim_{n\to+\infty} |a_n|e^{\mathrm{Re}(z\lambda_n)} = \lim_{\substack{n\to+\infty \\ n\in\mathcal{N}_3(z)}} |a_n|e^{\mathrm{Re}(z\lambda_n)} = 0.$$

Thus, $z \in D_\mu$ and vice versa. Therefore, $z \in D_\mu \iff \lim\limits_{n\to+\infty} a_n = 0$ in case $\sharp\,\mathcal{N}_1(z) < +\infty$, $\sharp\,\mathcal{N}_2(z) < +\infty$. The cases $\sharp\,\mathcal{N}_1(z) = +\infty$, $\sharp\,\mathcal{N}_2(z) < +\infty$ and $\sharp\,\mathcal{N}_1(z) < +\infty$ and $\sharp\,\mathcal{N}_2(z) = +\infty$ are considered similarly as above. Therefore, $D_\mu(F) = G_\mu(F) = \mathbb{C}$. The proof of Proposition 1 is complete. ☐

Lemma 1 ([22, Proposition 2]). *Let $F \in \mathcal{D}(\Lambda)$ be of form* (1)*, where $\Lambda = (\lambda_n)$ such that $\lambda_n \ge 0$, $n \ge 0$, and $\varliminf\limits_{n\to+\infty} \lambda_n := \lambda > 0$. Then*

$$\sigma_\mu(F) = \alpha_0 := \varliminf_{n\to+\infty} \frac{-\ln|a_n|}{\lambda_n}.$$

Remark 2 ([22, Remark 3]). *If $F \in \mathcal{D}(\Lambda)$ is of form* (1)*, where $\Lambda = (\lambda_n)$ such that $\lambda_n \ge 0$, $n \ge 0$, a set $J \subset \mathbb{N}$ is unbounded and $|a_n| \to +\infty$ as $n \to +\infty$ with $n \in J$, then $\lambda_n(\omega) \to +\infty$ as $n \to +\infty$ with $n \in J$, because otherwise $\sigma_\mu(F,\omega) = -\infty$.*

Proposition 2. *If $F \in \mathcal{D}(\Lambda)$ has the form* (1)*, where $\Lambda = (\lambda_n)$ such that $-1 \le \lambda_n \le 1$, $n \ge 0$, then $G_\mu(F) = \mathbb{C}$.*

Indeed, $|a_n|\,e^{x\lambda_n} \le |a_n|\,e^{|x|}$ for all $x \in \mathbb{R}$. For the sequence $\Lambda = (\lambda_n)$ such that $\lambda_n \in \mathbb{R}$, $n \ge 0$, and the Dirichlet series $F \in \mathcal{D}(\Lambda)$ of form (1) we will denote

$$\alpha^{(1)} = \alpha^{(1)}(F) := \varliminf_{\substack{n\to+\infty \\ n\in\mathcal{N}_1}} \frac{-\ln|a_n|}{\lambda_n}, \qquad \alpha^{(2)} = \alpha^{(2)}(F) := \varlimsup_{\substack{n\to+\infty \\ n\in\mathcal{N}_2}} \frac{-\ln|a_n|}{\lambda_n}.$$

Proposition 3. *Let $F \in \mathcal{D}(\Lambda)$ be of form* (1)*, where $\Lambda = (\lambda_n)$ such that $\lambda_n \in \mathbb{R}$, $n \ge 0$.*

1) If $\alpha^{(2)} < \alpha^{(1)}$, then $G_\mu(F) = \{z : \mathrm{Re}\, z \in (\alpha^{(2)}, \alpha^{(1)})\}$.

2) If $\alpha^{(2)} \ge \alpha^{(1)}$, then $G_\mu(F) = \varnothing$.

Proof. Without loss of the generality, for the sake of simplifying, let us assume that $a_n \to 0$ as $n \to +\infty$, that is, $0 \in D_\mu$.

Let us denote $\mathcal{N}_1 := \mathcal{N}_1(1)$, $\mathcal{N}_2 := \mathcal{N}_2(1)$, $\mathcal{N}_3 := \{n : -1 \le \mathrm{Re}\,\lambda_n \le 1\}$,

$$F_j(z) = \sum_{\substack{n=0 \\ n\in\mathcal{N}_j}} a_n e^{z\lambda_n}, \quad j \in \{1,2,3\}, \qquad f(\tau) = F_2(-\tau).$$

By Lemma 1, $\sigma_\mu(F_1) = \alpha^{(1)}$, $\sigma_\mu(f) = -\alpha^{(2)}$. Therefore, $G_\mu(F_1) = \Pi_{\alpha^{(1)}}$, $G_\mu(f) = \Pi_{-\alpha^{(2)}}$, hence $G_\mu(F_2) = \mathbb{C} \setminus \overline{\Pi}_{\alpha^{(2)}}$. Thus, $G_\mu(F_1 + F_2) = \Pi_{\alpha^{(1)}} \setminus \overline{\Pi}_{\alpha^{(2)}} = \{z : \mathrm{Re}\, z \in (\alpha^{(2)}, \alpha^{(1)})\}$.

Therefore, by Proposition 2, $G_\mu(F) = \{z : \mathrm{Re}\, z \in (\alpha^{(2)}, \alpha^{(1)})\}$, because $F = F_1 + F_2 + F_3$ and $G_\mu(F) = G_\mu(F_1 + F_2) \cap G_\mu(F_3) = G_\mu(F_1 + F_2)$.

Let us assume that there exists $z_0 \neq 0$ such that $a_n e^{z_0\lambda_n} \to 0$ as $n \to +\infty$, but $a_n \not\to 0$ as $n \to +\infty$. In this case, we consider the Dirichlet series $F^*(\tau) = \sum_{n=0}^{+\infty} a_n^* e^{\tau\lambda_n}$ with coefficients $a_n^* = a_n e^{z_0\lambda_n}$ instead of a_n. Then $F(z) = F^*(\tau)$ and $\tau \in G_\mu(F^*) \iff z \in G_\mu(F)$ for $z = z_0 + \tau$, and also $\alpha^{(j)}(F^*) = \alpha^{(1)}(F) - \operatorname{Re} z_0$. According to the above proof, we get

$$G_\mu(F^*) = \Big\{\tau\colon \alpha^{(2)}\left(F^*\right) < \operatorname{Re}\tau < \alpha^{(1)}\left(F^*\right)\Big\} = \Big\{\tau\colon \alpha^{(2)}(F) < \operatorname{Re}\tau + \operatorname{Re} z_0 < \alpha^{(1)}(F)\Big\}$$
$$= \Big\{z\colon \alpha^{(2)}(F) < \operatorname{Re} z < \alpha^{(1)}(F)\Big\} = G_\mu(F).$$

□

In what follows, we will assume that $a_n \to 0$ as $n \to +\infty$, that is, $0 \in D_\mu$. If it is false, then instead of a series of the form (1) with coefficients (a_n), we again can consider the series $F^*(\tau) = \sum_{n=0}^{+\infty} a_n^* e^{\tau\lambda_n}$ at $\tau = z - z_0$ with coefficients $a_n^* := a_n e^{z_0\lambda_n}$, where $z_0 \in D_\mu(F)$, i.e. $a_n e^{z_0\lambda_n} \to 0$ as $n \to +\infty$. As above, we have that $z \in G_\mu(F) \iff \tau \in G_\mu\left(F^*\right)$. We will not give the formulations of the statements derived for the general case, both to short the paper and due to the obviousness of these reformulations.

Let us denote

$$\alpha^{(1)}(\theta) = \alpha^{(1)}(\theta, F) := \varliminf_{\substack{n\to+\infty \\ n\in\mathcal{N}_1(e^{i\theta})}} \frac{-\ln|a_n|}{\operatorname{Re}(e^{i\theta}\lambda_n)}, \qquad \alpha^{(2)}(\theta) = \alpha^{(2)}(\theta, F) := \varlimsup_{\substack{n\to+\infty \\ n\in\mathcal{N}_2(e^{i\theta})}} \frac{-\ln|a_n|}{\operatorname{Re}(e^{i\theta}\lambda_n)}$$

for $\theta \in [0, \pi)$, $t \in \mathbb{R}$. Remark $G_\mu\big(f_{(\theta)}\big) \cap \big(e^{i\theta}\cdot\mathbb{R}\big) \subset G_\mu(F)$ and $\alpha^{(j)}(\theta, F) = \alpha^{(j)}\left(f_{\theta,F}\right)$ for $\theta \in [0, \pi)$, because for $t \in \mathbb{R}$ we have $\big|a_n e^{t\operatorname{Re}(e^{i\theta}\lambda_n)}\big| = \big|a_n e^{it\operatorname{Im}(e^{i\theta}\lambda_n)} e^{t\operatorname{Re}(e^{i\theta}\lambda_n)}\big|$ and

$$f_{(\theta,F)}(t) = \sum_{n=0}^{+\infty} a_n e^{it\operatorname{Im}(e^{i\theta}\lambda_n)} e^{t\operatorname{Re}(e^{i\theta}\lambda_n)} = \sum_{n=0}^{+\infty} a_n e^{(te^{i\theta})\lambda_n} = F\big(te^{i\theta}\big), \quad t \in \mathbb{R}.$$

Proposition 4. *Let $F \in \mathcal{D}(\Lambda)$ be of form* (1) *and $z_0 \in D_\mu$. If $\alpha^{(2)}\left(\theta, F^*\right) < \alpha^{(1)}\left(\theta, F^*\right)$ for some $\theta \in [0, \pi)$, then*

$$\Big\{z_0 + te^{i\theta}\colon t \in \big(\alpha^{(2)}\left(\theta, F^*\right), \alpha^{(1)}\left(\theta, F^*\right)\big)\Big\} \subset G_\mu(F),$$

and

$$\Big\{z_0 + te^{i\theta} : t \in \big(-\infty, \alpha^{(2)}\left(\theta, F^*\right)\big) \cup \big(\alpha^{(1)}\left(\theta, F^*\right), +\infty\big)\Big\} \cap G_\mu(F) = \varnothing.$$

Proof. For a given $\theta \in [0, \pi)$, applying Proposition 3 to the Dirichlet series $f_{(\theta)}(t)$ with a sequence of exponents $\big(\operatorname{Re}(e^{i\theta}\lambda_n)\big)$, which is constructed according to the series F^* instead of the series F, we obtain the assertion of Proposition 4. □

Proposition 5. *Let $F \in \mathcal{D}(\Lambda)$ be of form* (1) *and $z_0 \in D_\mu$. Then*

$$G_\mu(F) = \bigcup_{\theta\in[0,\pi)} \Big\{z = z_0 + te^{i\theta}\colon t \in \big(\alpha^{(2)}\left(\theta, F^*\right), \alpha^{(1)}\left(\theta, F^*\right)\big)\Big\}.$$

Proof. The assertion of Proposition 5 follows directly from Proposition 4. □

Remark 3. *Since $\alpha^{(2)}(\theta, F^*) = -\infty$ in case $\sharp\, \mathcal{N}_2\left(e^{i\theta}\right) < +\infty$, the set of the convergence $D_\mu(F)$ is unbounded. Since $\mathcal{N}_1\left(e^{i(\theta+\pi)}\right) = \mathcal{N}_2\left(e^{i\theta}\right)$, for $\theta \in [0, \pi)$ we have*

$$\alpha^{(2)}(\theta, F^*) = \varlimsup_{\substack{n\to+\infty\\ n\in\mathcal{N}_2\left(e^{i\theta}\right)}} \frac{-\ln|a_n^*|}{-\operatorname{Re}\left(e^{i(\theta+\pi)}\lambda_n\right)} = \varliminf_{\substack{n\to+\infty\\ n\in\mathcal{N}_1\left(e^{i(\theta+\pi)}\right)}} \frac{-\ln|a_n^*|}{-\operatorname{Re}\left(e^{i(\theta+\pi)}\lambda_n\right)} = -\alpha^{(1)}(\theta+\pi, F^*).$$

Hence, for $\theta \in [\pi, 2\pi)$ we get

$$z - z_0 = \alpha^{(1)}(\theta, F^*)\, e^{i\theta} = -\alpha^{(2)}(\theta-\pi, F^*)\, e^{i\theta} = \alpha^{(2)}(\theta-\pi, F^*)\, e^{i(\theta-\pi)}, \quad \theta - \pi \in [0, \pi).$$

Therefore, Proposition 5 implies

$$\partial G_\mu(F) = \left\{z = z_0 + \alpha^{(1)}(\theta, F^*)\, e^{i\theta} : \theta \in [0, 2\pi]\right\}.$$

Remark 4. *It easy to see that*

$$\alpha^{(1)}(\theta, F^*) = \varliminf_{\substack{n\to+\infty\\ n\in\mathcal{N}_1\left(e^{i\theta}\right)}} \frac{-\ln|a_n| - \operatorname{Re}(z_0\lambda_n)}{\operatorname{Re}\left(e^{i\theta}\lambda_n\right)}, \qquad \alpha^{(2)}(\theta, F^*) = \varlimsup_{\substack{n\to+\infty\\ n\in\mathcal{N}_2\left(e^{i\theta}\right)}} \frac{-\ln|a_n| - \operatorname{Re}(z_0\lambda_n)}{\operatorname{Re}\left(e^{i\theta}\lambda_n\right)}.$$

In the case $\{\lambda_n\} \subset \mathbb{R}$,

$$\alpha^{(1)}(\theta, F^*) = \frac{\alpha^{(1)}(F) - \operatorname{Re} z_0}{\cos\theta}, \qquad \alpha^{(2)}(\theta, F^*) = \frac{\alpha^{(2)}(F) - \operatorname{Re} z_0}{\cos\theta}.$$

Let us formulate some simple corollaries.

Corollary 1. *Let $\Lambda = (\lambda_n)$, $\lambda_n \geq 0$, $n \geq 0$. If $F \in \mathcal{D}(\Lambda)$ of form (1), then $\sigma_\mu(F) = \alpha_0$.*

The statement of Corollary 1 was previously proved in the master's thesis of A.Yu. Bodnarchuk, Ivan Franko Lviv National University, 2021 (see also [24]).

Proof. Again, without loss of the generality, assume that $a_n \to 0$ as $n \to +\infty$, i.e. $z_0 \in D_\mu$. Since $\alpha^{(2)}(\theta) = -\infty$ for $\theta \in [0, \pi/2)$, $\alpha^{(1)}(\theta) = +\infty$ for $\theta \in [\pi/2, \pi)$ and $\operatorname{Re}(e^{i\theta}\lambda_n) = \lambda_n \cos\theta$,

$$\alpha^{(1)}(\theta) = \varliminf_{n\to+\infty} \frac{-\ln|a_n|}{\lambda_n \cos\theta} = \frac{\alpha_0}{\cos\theta} \qquad \text{for } \theta \in [0, \pi/2)$$

and

$$\alpha^{(2)}(\theta) = \varlimsup_{n\to+\infty} \frac{-\ln|a_n|}{\lambda_n \cos\theta} = \frac{\alpha_0}{\cos\theta} \qquad \text{for } \theta \in (\pi/2, \pi].$$

In addition, it is obvious that $|a_n|\, e^{t\operatorname{Re}\left(e^{i\theta}\lambda_n\right)} = |a_n| \to 0$ as $n \to +\infty$ at $\theta = \pi/2$ for all $t \in \mathbb{R}$. Hence, by Proposition 5,

$$\partial G_\mu(F) = \bigcup_{\theta\in[0,\pi)} \left\{z = \alpha_0 e^{i\theta}/\cos\theta : \theta \in [0, \pi)\right\} = \{z : \operatorname{Re} z = \alpha_0\}.$$

Therefore, $\sigma_\mu = \alpha_0$. $\square$

Corollary 2. *Let $\Lambda = (\lambda_n)$, $\{\lambda_n\} = \bigsqcup_{j=1}^{4} \{\lambda_n^{(j)}\}$, $\lambda_n^{(1)} \geq 0$, $\lambda_n^{(2)} < 0$, $i\lambda_n^{(3)} < 0$, $i\lambda_n^{(4)} > 0$, $n \geq 0$. Let $F \in \mathcal{D}(\Lambda)$ be of form* (1) *and $a_n \to 0$ as $n \to +\infty$.*

1. *If $\alpha^{(2)}(\pi) < \alpha^{(1)}(0)$ and $\alpha^{(2)}(-\pi/2) < \alpha^{(1)}(\pi/2)$, then*
$$G_\mu(F) = \{z = x + iy : \alpha^{(2)}(\pi) < x < \alpha^{(1)}(0),\ \alpha^{(2)}(-\pi/2) < y < \alpha^{(1)}(\pi/2)\}.$$
2. *If $\alpha^{(2)}(\pi) \geq \alpha^{(1)}(0)$ or $\alpha^{(2)}(-\pi/2) \geq \alpha^{(1)}(\pi/2)$, then $G_\mu(F) = \varnothing$.*

Proof. Let us write
$$F(z) = \sum_{j=1}^{4} F_j(z), \qquad F_j(z) = \sum_{k=0}^{+\infty} a_k^{(j)} e^{\lambda_k^{(j)} z},$$
where $a_k^{(j)} = a_n$ in the case $\lambda_k^{(j)} = \lambda_n$. We get successively
$$\alpha^{(1)}(0) = \varliminf_{\substack{n\to+\infty\\ n\in\mathcal{N}_1(0)}} \frac{-\ln|a_n|}{\operatorname{Re}\lambda_n} = \varliminf_{k\to+\infty} \frac{-\ln\left|a_k^{(1)}\right|}{\lambda_k^{(1)}},$$
$$\alpha^{(2)}(\pi) = \varlimsup_{\substack{n\to+\infty\\ n\in\mathcal{N}_2(\pi)}} \frac{-\ln|a_n|}{-\operatorname{Re}\lambda_n} = \varlimsup_{k\to+\infty} \frac{-\ln\left|a_k^{(2)}\right|}{\left|\lambda_k^{(2)}\right|},$$
$$\alpha^{(1)}\left(\frac{\pi}{2}\right) = \varliminf_{\substack{n\to+\infty\\ n\in\mathcal{N}_1(\pi/2)}} \frac{-\ln|a_n|}{\operatorname{Re}(i\lambda_n)} = \varliminf_{k\to+\infty} \frac{-\ln\left|a_k^{(4)}\right|}{i\lambda_k^{(4)}},$$
$$\alpha^{(2)}\left(-\frac{\pi}{2}\right) = \varlimsup_{\substack{n\to+\infty\\ n\in\mathcal{N}_2(-\frac{\pi}{2})}} \frac{-\ln|a_n|}{-\operatorname{Re}(i\lambda_n)} = \varlimsup_{k\to+\infty} \frac{-\ln\left|a_k^{(3)}\right|}{-i\lambda_k^{(3)}}.$$
It remains to apply Proposition 4 twice, first to $F_1(z) + F_2(z)$, and then to $F_3(z/i) + F_4(z/i)$. ☐

Remark 5. *In particular, under the conditions of the Corollary 2, in the case $\alpha^{(2)}(\pi) = -\infty$, $\alpha^{(1)}(0) < +\infty$ or $\alpha^{(2)}(\pi) > -\infty$, $\alpha^{(1)}(0) = +\infty$ we obtain horizontal semi-strips*
$$G_\mu(F) = \{z = x + iy : x < \alpha^{(1)}(0), \alpha^{(2)}(-\pi/2) < y < \alpha^{(1)}(\pi/2)\}$$
or
$$G_\mu(F) = \{z = x + iy : x > \alpha^{(2)}(\pi), \alpha^{(2)}(-\pi/2) < y < \alpha^{(1)}(\pi/2)\},$$
respectively.

3 Domain of absolute convergence

For a formal Dirichlet series $F \in \mathcal{D}(\Lambda)$ of form (1) we denote $D_c(F)$, $D_a(F)$ the set of the convergence and the set of the absolute convergence of the series (1), respectively. Let $G_c(F) = D_c(F) \setminus \partial D_c(F)$ and $G_a(F) = D_a(F) \setminus \partial D_a(F)$ be the domains of the convergence and the absolute convergence, respectively.

By Hölder's inequality (see also [7, p. 47], [8, 9]) the set $D_a(F) \neq \varnothing$ is convex, hence the domain $G_a(F)$ is convex or $G_a(F) = \varnothing$.

It easy to see that $D_a(F) \subset D_c(F) \subset D_\mu(F)$, $G_a(F) \subset G_c(F) \subset G_\mu(F)$. The set $D_c(F)$ of convergence of a Dirichlet series can be not convex (see [8]). There are examples of Dirichlet series that have this property, that is, their convergence set is not convex. For example (see [8]), an ordinary trigonometric series $f(t) = \sum\limits_{k=0}^{+\infty} f_k e^{ikt}$, $t \in \mathbb{R}$, is a Dirichlet series with exponents $\lambda_k = ik$. The set of convergence of such a series may reduce to a set of points of measure zero on the real axis [13], that is, a Dirichlet series has a set of the convergence on the real axis of measure zero. W. Schnee [3] gave an example of a Dirichlet series that had an isolated point of convergence in addition to the usual half-plane. At the same time, he noted that when considering the problem, it is necessary to narrow the class of sequences of exponents. E. Hille [8] investigated only absolute convergence. He proved without any restrictions on the sequence (λ_n) that the set of absolute convergence is a convex set. His proof now looks elementary given that Hölder's inequality, which he first proves and then uses, is now well known. He attempt to determine the set of absolute convergence in terms of (a_n) and (λ_n) under various assumptions regarding the rate of increase of $\big(|\lambda_n|\big)$. If $\ln n/\lambda_n \to 0$ as $n \to +\infty$, we can determine the precise domain. In any case, we can find a maximal domain and a minimal one. J. Micusiński [9] considered the Dirichlet series of the form (1) by condition

$$\lim_{n\to+\infty} \frac{\operatorname{Re}\lambda_n}{\ln n} = +\infty. \tag{2}$$

He proved that if condition (2) is fulfilled and $x_0 = \varliminf\limits_{n\to+\infty} \frac{-\ln|a_n|}{\operatorname{Re}\lambda_n}$, then the Dirichlet series $F \in \mathcal{D}(\Lambda)$ of form (1) convereges absolutely for real $z < x_0$, converges uniformly for $z \in (-\infty, x_1]$, where x_1 is arbitrary such that $x_1 < x_0$, and diverges for all real $z > x_0$. J. Micusiński [9] considerred also case

$$-\frac{\pi}{2} < \delta_1 = \varliminf_{n\to+\infty} \arg\lambda_n \le \varlimsup_{n\to+\infty} \arg\lambda_n = \delta_2 < \frac{\pi}{2}. \tag{3}$$

If $\delta_1 < \delta_2$, then without loss of generality we can assume that $0 < -\delta_1 = \delta_2 = \delta < \frac{\pi}{2}$. J. Micusiński [9] proved that if a series of form (1) converges at a point $z = z_0$, it does so absolutely at any point $z = z_0 + \varrho e^{i\varphi}$ for $\varrho > 0$ and $|\pi - \varphi| < \frac{\pi}{2} - \delta$.

In addition, J. Micusiński [9] proved that by conditions (2), (3), the domain G_c of convergence of a series of form (1) is convex, and at every interior point of this domain the series is absolutely convergent, and uniformly convergent on every bounded and closed set $E \subset G_c$.

First, we consider some relationships between sets of convergence, absolute convergence, and the existence of the maximum term of a Dirichlet series. In our article, we will use the properties of the domains of existence of the maximal term, which are considered in Section 2, to obtain new statements about domains of convergence and absolute convergence. Our statements will in some sense be similar to the statements about domains of absolute convergence from articles [8,9,15]. The fundamental difference, in particular, will be that instead of the condition on the sequence of exponents $\ln n = o\,(\lambda_n)$ as $n \to +\infty$ and some other conditions on it only the following restriction on the sequence of the coefficients $\ln n = o\big(\ln|a_n|\big)$ as $n \to +\infty$ will be used.

Let us denote

$$aG + b := \{az + b \colon z \in G\}$$

for $G \subset \mathbb{C}$ and $a, b \in \mathbb{C}$. Note that $-\Pi_a = \mathbb{C} \setminus \overline{\Pi}_{-a}$. In the article [20], it is proved the following statement (see also [21,22,24], where this statement is formulated in almost the same form).

Proposition 6. *Let* $\Lambda = (\lambda_n)$, $\lambda_n \geq 0$, $n \geq 0$. *If* $F \in \mathcal{D}$ *has the form* (1) *and* $\delta \in \mathbb{R}$, $\gamma > 0$ *such that*

$$\sum_{n=0}^{+\infty} |a_n|^{1-\gamma} e^{-\delta\lambda_n} < +\infty, \tag{4}$$

then $\sigma_a(F) \leq \sigma_c(F) \leq \sigma_\mu(F) = \alpha_0$ *and* $\sigma_a(F) \geq \gamma\alpha_0 - \delta \geq \gamma\sigma_c(F) - \delta$, *that is,*

$$\Pi_{\sigma_a(F)} \subset \Pi_{\sigma_c(F)} \subset \Pi_{\sigma_\mu(F)} = \Pi_{\alpha_0} \subset \frac{1}{\gamma}\Pi_{\sigma_a(F)} + \frac{\delta}{\gamma}. \tag{5}$$

Proof. The inequalities $\sigma_a(F) \leq \sigma_c(F) \leq \sigma_\mu(F)$ are obvious, and by Corollary 1, $\alpha_0 = \sigma_\mu(F)$, so we get $\sigma_a(F) \leq \sigma_c(F) \leq \sigma_\mu(F) = \alpha_0$. Therefore, $\Pi_{\sigma_a(F)} \subset \Pi_{\sigma_c(F)} \subset \Pi_{\sigma_\mu(F)} = \Pi_{\alpha_0}$. An almost verbatim repetition of the arguments from the proof in the article [20] proves the inequality $\sigma_a(F) \geq \gamma\alpha_0 - \delta$. So, we get $\Pi_{\sigma_\mu(F)} = \Pi_{\alpha_0} \subset \frac{1}{\gamma}\Pi_{\sigma_a(F)} + \frac{\delta}{\gamma}$. □

Corollary 3. *Let* $\Lambda = (\lambda_n)$, $\lambda_n \geq 0$, $n \geq 0$. *If* $F \in \mathcal{D}(\Lambda)$ *has the form* (1) *and* $\tau(\Lambda) = \overline{\lim\limits_{n\to+\infty}} \frac{\ln n}{\lambda_n} < +\infty$, *then* $\sigma_a(F) \leq \sigma_c(F) \leq \sigma_\mu(F) = \alpha_0 \leq \sigma_a(F) + \tau(\Lambda)$, *i.e.*

$$\Pi_{\sigma_a(F)} \subset \Pi_{\sigma_c(F)} \subset \Pi_{\sigma_\mu(F)} = \Pi_{\alpha_0} \subset \Pi_{\sigma_a(F)} + \tau(\Lambda).$$

Proof. The proof uses only Proposition 6 and is exactly the same as in the article [20] (under the additional condition $\underline{\lim\limits_{n\to+\infty}} \lambda_n > 0$). Condition (4) of Proposition 6 follows from the condition $\tau(\Lambda) < +\infty$ at $\gamma = 1$ and $\tau(\Lambda) + \varepsilon$ instead δ for arbitrary $\varepsilon > 0$. Note $\ln n < \big(\tau(\Lambda) + \varepsilon/2\big)\lambda_n$, $n \geq n_0$, $\exp\{-\delta\lambda_n\} < \exp\{-\big(\tau(\Lambda)+\varepsilon\big)/\big(\tau(\Lambda)+\varepsilon/2\big)\cdot \ln n\}$. Therefore, $\sum\limits_{n=n_0}^{+\infty} e^{-\delta\lambda_n} < +\infty$. By Proposition 6, we now obtain $\sigma_c(F) \leq \sigma_\mu(F) = \alpha_0 \leq \sigma_a(F) + \tau(\Lambda) + \varepsilon$. Due to the arbitrariness of $\varepsilon > 0$, we get

$$\sigma_c(F) \leq \sigma_\mu(F) = \alpha_0 \leq \sigma_a(F) + \tau(\Lambda), \qquad \text{i.e.} \qquad \Pi_{\sigma_c(F)} \subset \Pi_{\sigma_\mu(F)} = \Pi_{\alpha_0} \subset \Pi_{\sigma_a(F)} + \tau(\Lambda).$$

□

Proposition 7. *Let* $\Lambda = (\lambda_n)$, $\lambda_n \leq 0$, $n \geq 0$. *If* $F \in \mathcal{D}$ *has the form* (1) *and* $\delta \in \mathbb{R}$, $\gamma > 0$ *are such that condition* (4) *is fulfilled, then* $G_\mu(F) = \mathbb{C} \setminus \overline{\Pi}_{\alpha_0^{(2)}}$ *and*

$$\frac{1}{\gamma}G_a(F) + \frac{\delta}{\gamma} \supset \mathbb{C} \setminus \overline{\Pi}_{\alpha_0^{(2)}} \supset G_c(F) \supset G_a(F).$$

Proof. Let us consider a Dirichlet series $F^*(z) = \sum\limits_{n=0}^{+\infty} a_n e^{z\lambda_n^*}$, where $\lambda_n^* = -\lambda_n$. From condition (4) for the sequences (a_n) and (λ_n) it follows that for the sequences (a_n) and (λ_n^*) we have $\sum\limits_{n=0}^{+\infty} |a_n|^{1-\gamma} e^{-\delta_1\lambda_n^*} < +\infty$, where $\delta_1 = -\delta$. Then, by Proposition 6, we get

$$\sigma_a(F^*) \leq \sigma_c(F^*) \leq \sigma_\mu(F^*) = \underline{\lim_{n\to+\infty}} \frac{-\ln|a_n|}{\lambda_n^*} = -\overline{\lim_{n\to+\infty}} \frac{-\ln|a_n|}{\lambda_n} := -\alpha_0^{(2)}$$

and $\sigma_a(F^*) \geq -\gamma\alpha_0^{(2)} - \delta_1 \geq \gamma\sigma_c(F^*) - \delta_1$. Thus, $\sigma_a(F^*) \geq -\gamma\alpha_0^{(2)} + \delta \geq \gamma\sigma_c(F^*) + \delta$, that is, $\frac{1}{\gamma}\sigma_a(F^*) - \frac{\delta}{\gamma} \geq -\alpha_0^{(2)} \geq \sigma_c(F^*)$. It is obvious that $z \in G_\mu(F) \iff -z \in G_\mu(F^*)$. Therefore

$G_\mu(F) = \mathbb{C} \setminus \overline{\Pi}_{-\sigma_\mu(F^*)} = \mathbb{C} \setminus \overline{\Pi}_{\alpha_0^{(2)}}$. In addition, $z \in G_a(F) \iff -z \in G_a(F^*)$, $z \in G_c(F) \iff -z \in G_c(F^*)$. Hence,

$$\frac{1}{\gamma} G_a(F) + \frac{\delta}{\gamma} = \frac{1}{\gamma}\mathbb{C} \setminus \overline{\Pi}_{\sigma_a(F^*)} + \frac{\delta}{\gamma} \supset \mathbb{C} \setminus \overline{\Pi}_{\alpha_0^{(2)}} \supset \mathbb{C} \setminus \overline{\Pi}_{\sigma_c(F^*)} = G_c(F).$$

$\square$

Remark 6. *If in condition* (4) $\gamma = 1$*, then in the case* $\lambda_n \geq 0$, $n \geq 0$*, it is necessary* $\delta > 0$*, and in the case* $\lambda_n \leq 0$, $n \geq 0$*, it is necessary* $\delta < 0$.

Corollary 4. *Let* Λ *be a sequence of real numbers* λ_n *such that* $\sharp \mathcal{N}_1 = +\infty$, $\sharp \mathcal{N}_2 = +\infty$*, and* $F \in \mathcal{D}(\Lambda)$ *be of form* (1)*. If* $\tau_j(\Lambda) = \varlimsup\limits_{\substack{n\to+\infty \\ n\in\mathcal{N}_j}} \frac{\ln n}{|\lambda_n|} < +\infty$, $j \in \{1,2\}$, $G_a(F) \neq \varnothing$ *and* $\alpha^{(2)} < \alpha^{(1)}$*, then*

$$G_\mu(F) = \Pi_{\alpha^{(1)}} \setminus \overline{\Pi}_{\alpha^{(2)}} \quad \text{and} \quad G_a(F) - \tau_2(\Lambda) \subset G_\mu(F) \subset G_a(F) + \tau_1(\Lambda).$$

Proof. We put $F = F_1 + F_2$, $F_j(z) = \sum_{n\in\mathcal{N}_j} a_n e^{\lambda_n z}$, $j \in \{1,2\}$. By Corollary 3, we obtain the inclusion $G_\mu(F_1) \subset G_a(F_1) + \tau_1(\Lambda)$. The inclusion $G_a(F_2) - \tau_2(\Lambda) \subset G_\mu(F_2)$ follows from the Proposition 7 in the same way as in the proof of Corollary 3 the previous inclusion from Proposition 6. Actually, it is enough to take $\gamma = 1$ and $(-\tau_2(\Lambda) - \varepsilon)$ instead δ for arbitrary $\varepsilon > 0$.

It remains to note that $G_\mu(F) = G_\mu(F_1) \cap G_\mu(F_2)$, $G_a(F) = G_a(F_1) \cap G_a(F_2)$ and $G_a(F_1) = \Pi_{\sigma_a(F_1)}$, $G_a(F_2) = \mathbb{C} \setminus \overline{\Pi}_{\sigma_a(F_2)}$, $G_\mu(F_1) = \Pi_{\alpha^{(1)}}$, $G_\mu(F_2) = \mathbb{C} \setminus \overline{\Pi}_{\alpha^{(2)}}$. $\square$

Applying Propositions 6 and 7 to the function $f_{(\theta)}$, we obtain the following statement.

Proposition 8. *Let* $\Lambda = (\lambda_n)$, $\lambda_n \in \mathbb{C}$, $n \geq 0$*. If* $F \in \mathcal{D}$ *has the form* (1), $F = F_1 + F_2$*, where*

$$F_j(z) = \sum_{n\in\mathcal{N}_j(e^{i\theta})} a_n e^{\lambda_n z},$$

and $\delta_j = \delta_j(\theta) \in \mathbb{R}$, $\gamma_j = \gamma_j(\theta) > 0$, $j \in \{1,2\}$, $\theta \in [0,\pi)$*, are such that*

$$\sum_{n\in\mathcal{N}_j(e^{i\theta})} |a_n|^{1-\gamma_j} e^{-\delta_j \operatorname{Re}(e^{i\theta}\lambda_n)} < +\infty, \qquad j \in \{1,2\}, \tag{6}$$

then $G_\mu(F_1) \cap (e^{i\theta} \cdot \mathbb{R}) = (-\infty, \alpha^{(1)}(\theta)) \cdot e^{i\theta}$, $G_\mu(F_2) \cap (e^{i\theta} \cdot \mathbb{R}) = (\alpha^{(2)}(\theta), +\infty) \cdot e^{i\theta}$, $G_\mu(F) \cap (e^{i\theta} \cdot \mathbb{R}) = \left(\alpha^{(2)}(\theta), \alpha^{(1)}(\theta)\right) e^{i\theta}$ *and*

$$\frac{1}{\gamma_j(\theta)}\Big(G_a(F_j) \cap (e^{i\theta} \cdot \mathbb{R})\Big) + \frac{\delta_j(\theta)}{\gamma_j(\theta)} \cdot e^{i\theta} \supset G_\mu(F_j) \cap (e^{i\theta} \cdot \mathbb{R}) \supset G_c(F_j) \cap (e^{i\theta} \cdot \mathbb{R}) \tag{7}$$

for $j \in \{1,2\}$.

Proof. Note that $\mathcal{N}_1(e^{i\theta}, f_{(\theta,F)}) = \mathcal{N}_1(e^{i\theta}, F_1)$, $\mathcal{N}_2(e^{i\theta}, f_{(\theta,F)}) = \mathcal{N}_2(e^{i\theta}, F_2)$, $\alpha^{(j)}(\theta, F) = \alpha^{(j)}(f_{\theta,F})$. Let us now apply Propositions 6 and 7 to the Dirichlet series F_1 and F_2, respectively, with exponents $(e^{i\theta}\lambda_n)$ instead (λ_n). Thus, from Propositions 6 and 7 we obtain inclusions (7).

Directly from Proposition 3 we get the equalities $G_\mu(F_1) \cap (e^{i\theta} \cdot \mathbb{R}) = (-\infty, \alpha^{(1)}(\theta)) \cdot e^{i\theta}$, $G_\mu(F_2) \cap (e^{i\theta} \cdot \mathbb{R}) = (\alpha^{(2)}(\theta), +\infty) \cdot e^{i\theta}$ and $G_\mu(F) \cap (e^{i\theta} \cdot \mathbb{R}) = (\alpha^{(2)}(\theta), \alpha^{(1)}(\theta)) \cdot e^{i\theta}$. $\square$

Applying Corollary 4 to the Dirichlet series $f_{(\theta)}$ with exponents $\left(\mathrm{Re}\left(e^{i\theta}\lambda_n\right)\right)$ with fixed $\theta \in [0,\pi)$, we obtain the following statement. Let Λ be a sequence of the complex numbers λ_n and $F \in \mathcal{D}(\Lambda)$ be of the form (1).

Corollary 5. *If $\theta \in [0,\pi)$ is such that $\tau_j(\Lambda,\theta) = \varlimsup\limits_{\substack{n\to+\infty\\ n\in\mathcal{N}_j\left(e^{i\theta}\right)}} \ln n/\mathrm{Re}\left(e^{i\theta}\lambda_n\right) < +\infty$, $j \in \{1,2\}$, $G_a(F) \neq \varnothing$ and $\alpha^{(2)}(\theta) < \alpha^{(1)}(\theta)$, then*

$$G_\mu(F) \cap \left(e^{i\theta}\cdot\mathbb{R}\right) = \left(\alpha^{(2)}(\theta), \alpha^{(1)}(\theta)\right)\cdot e^{i\theta}$$

and

$$G_a(F)\cap\left(e^{i\theta}\cdot\mathbb{R}\right) - \left(\tau_2(\Lambda,\theta)e^{i\theta}\cdot\mathbb{R}\right) \subset G_\mu(F)\cap\left(e^{i\theta}\cdot\mathbb{R}\right) \subset G_a(F)\cap\left(e^{i\theta}\cdot\mathbb{R}\right) + \left(\tau_1(\Lambda,\theta)e^{i\theta}\cdot\mathbb{R}\right).$$

Proof. Since $G_\mu(F)\cap\left(e^{i\theta}\cdot\mathbb{R}\right) = G_\mu\big(f_{(\theta)}\big)$, $G_a(F)\cap\left(e^{i\theta}\cdot\mathbb{R}\right) = G_a\big(f_{(\theta)}\big)$, from Corollary 4 applied to the Dirichlet series $f_{(\theta)}$ with exponents $\left(\mathrm{Re}\left(e^{i\theta}\lambda_n\right)\right)$ we obtain the conclusion of Corollary 5. □

From Propositions 5 and 8 we obtain also the following statement.

Corollary 6. *Let $\Lambda = (\lambda_n)$, $\lambda_n \in \mathbb{C}$, $n \geq 0$. If $F \in \mathcal{D}$ has the form* (1) *and $\gamma > 0$ is such that*

$$\sum_{n=0}^{+\infty} |a_n|^{1-\gamma} < +\infty, \tag{8}$$

then $G_\mu(F) = \bigcup\limits_{\theta\in[0,\pi)} \left\{z = te^{i\theta}\colon t \in \left(\alpha^{(2)}(\theta), \alpha^{(1)}(\theta)\right)\right\}$ and

$$\frac{1}{\gamma}\cdot G_a(F) \supset G_\mu(F) \supset G_c(F). \tag{9}$$

Proof. Since the condition (8) implies the conditions (6) at $\delta_j(\theta) \equiv 0$, $\gamma_j(\theta) \equiv \gamma$, then from the relations (7) in Proposition 8 we get the inclusions (9). Directly from Proposition 5 we obtain the equality for $G_\mu(F)$. □

Corollary 7. *Let $\Lambda = (\lambda_n)$, $\lambda_n \in \mathbb{C}$, $n \geq 0$. Let $F \in \mathcal{D}$ be of form* (1)*. If condition* (8) *is fulfilled for each $\gamma \in (0,1)$ and $\alpha^{(2)}(\theta) < \alpha^{(1)}(\theta)$ for all $\theta \in [0,\pi)$, then*

$$G_a(F) = G_c(F) = G_\mu(F) = \bigcup_{\theta\in[0,\pi)} \left\{z = te^{i\theta}\colon t \in \left(\alpha^{(2)}(\theta), \alpha^{(1)}(\theta)\right)\right\}.$$

In particular, the domain of convergence $G_c(F)$ is also a convex domain.

Proof. To prove the assertion it is enough to use Corollary 6 for arbitrary $\gamma \in (0,1)$ and $\theta \in [0,\pi)$. □

Let us denote

$$h := \varliminf_{n\to+\infty} \frac{-\ln|a_n|}{\ln n}.$$

Remark 7. *If $h = +\infty$, then condition* (8) *is fulfilled for each $\gamma \in (0, 1)$ and by Corollary 7 we have*

$$G_a(F) = G_c(F) = G_\mu(F) = \bigcup_{\theta\in[0,\pi)} \Big\{ z = te^{i\theta} : t \in \big(\alpha^{(2)}(\theta), \alpha^{(1)}(\theta)\big) \Big\}.$$

In particular, the domain of convergence $G_c(F)$ is a convex domain.

Corollary 8. *Let $\Lambda = (\lambda_n)$, $\lambda_n \in \mathbb{C}$, $n \geq 0$. If $F \in \mathcal{D}$ has the form* (1) *and $h \in (1, +\infty]$, then*

$$\left(\frac{h}{h-1} \cdot G_a(F)\right) \supset G_\mu(F) \supset G_c(F). \tag{10}$$

Proof. To prove the assertion it is enough to use Corollary 6 for arbitrary $\varepsilon > 0$ such that $h - 2\varepsilon > 1$. Then $-\ln|a_n| > (h-\varepsilon)\ln n$ for enough large n. Hence, we obtain

$$|a_n|^{1-\gamma} < \exp\big\{-(1-\gamma)(h-\varepsilon)\ln n\big\} = \exp\left\{-\frac{h-\varepsilon}{h-2\varepsilon}\ln n\right\}$$

at $\gamma = 1/(h-2\varepsilon) \in (0, 1)$. So, $\sum_n |a_n|^{1-\gamma} < +\infty$. Since, $1/\gamma = (h-2\varepsilon)/(h-2\varepsilon-1)$, by Corollary 6, we get

$$\left(\frac{h-2\varepsilon}{h-2\varepsilon-1} \cdot G_a(F)\right) \supset G_\mu(F) \supset G_c(F).$$

Since $\varepsilon > 0$ is arbitrary, we finally have (10). □

4 Some more examples about sharpness of estimates

Let $\Lambda_+ = (\lambda_n)$ be a sequence of nonnegative numbers increasing to $+\infty$ such that

$$\tau(\Lambda_+) = \varlimsup_{n\to+\infty} \frac{\ln n}{\lambda_n} < +\infty.$$

It is well known (see, for example, [5,6,14,20]) that for each $F \in \mathcal{D}(\Lambda_+)$

$$\sigma_a(F) \leq \sigma_c(F) \leq \sigma_\mu(F) \leq \sigma_a(F) + \tau(\Lambda_+)$$

and (see [14, Theorem A]) for each $a, m, c \in [-\infty, +\infty]$ such that $a \leq c \leq m \leq a + \tau(\Lambda_+)$ there exists $F \in \mathcal{D}(\Lambda_+)$ with the following properties $\sigma_a(F) = a$, $\sigma_c(F) = c$, $\sigma_\mu(F) = m$. This statement allows us to discuss the finality of the statements obtained in this article. Let us start with Corollary 4.

Let Λ be a sequence of real numbers such that $\sharp\mathcal{N}_j = +\infty$, $j \in \{1, 2\}$. Let $F \in \mathcal{D}(\Lambda)$ be of form (1) and

$$F_1(z) = \sum_{n\in\mathcal{N}_1} a_n e^{z\lambda_n}, \quad F_2(z) = \sum_{n\in\mathcal{N}_2} a_n e^{z\lambda_n}, \quad \text{i.e.} \quad F = F_1 + F_2, \quad F^*(z) = F_2(-z).$$

Denote $\Lambda^{(1)} = \big(\lambda_n^{(1)}\big) = \big(\lambda_n\big)_{n\in\mathcal{N}_1}$, $\Lambda^{(2)} = \big(\lambda_n^{(2)}\big) = \big(-\lambda_n\big)_{n\in\mathcal{N}_2}$ and assume that $\tau_j(\Lambda) = \varlimsup_{n\to+\infty} \frac{\ln n}{\lambda_n^{(j)}} < +\infty$, $j \in \{1, 2\}$. By Corollary 4, we have

$$-\sigma_a(F_2^*) - \tau_2\big(\Lambda^{(2)}\big) \leq -\sigma_\mu(F_2^*) \leq -\sigma_a(F_2^*) \leq \sigma_a(F_1) \leq \sigma_\mu(F_1) \leq \sigma_a(F_1) + \tau_1\big(\Lambda^{(1)}\big).$$

Let us make an additional assumption that the sequences $\Lambda^{(j)}$ are monotonically increasing to $+\infty$ and the numbers a_j, m_j, c_j such that

$$-a_2 - \tau_2\big(\Lambda^{(2)}\big) \le -m_2 \le -a_2 \le a_1 \le m_1 \le a_1 + \tau_1\big(\Lambda^{(1)}\big).$$

According to [14, Theorem A], for each $a_j, m_j, c_j \in [-\infty, +\infty]$ such that

$$a_j \le c_j \le m_j \le a_j + \tau_j\big(\Lambda_+^{(j)}\big)$$

there exist two Dirichlet series $F_1 \in \mathcal{D}\big(\Lambda_+^{(1)}\big)$ and $F_2^* \in \mathcal{D}\big(\Lambda_+^{(2)}\big)$ with the following properties $\sigma_a(F_j) = a_j$, $\sigma_c(F_j) = c_j$, $\sigma_\mu(F_j) = m_j$.

But,

$$G_c(F_1) = \Pi_{\sigma_c(F_1)} = \Pi_{c_1}, \quad G_c(F_2) = \mathbb{C} \setminus \overline{\Pi}_{-\sigma_c(F_2^*)} = \mathbb{C} \setminus \overline{\Pi}_{-c_2}, \quad G_a(F_1) = \Pi_{\sigma_a(F_1)} = \Pi_{a_1},$$
$$G_a(F_2) = \mathbb{C} \setminus \overline{\Pi}_{-\sigma_a(F_2^*)} = \mathbb{C} \setminus \overline{\Pi}_{-a_2}, \quad G_\mu(F) = \Pi_{\sigma_\mu(F_1)} \setminus \overline{\Pi}_{-\sigma_\mu(F_2^*)} = \Pi_{m_1} \setminus \overline{\Pi}_{-m_2}.$$

Note that examples of the finality of statements that can be obtained from Corollary 5 in the case when the domains of existence of the maximum term and absolute convergence are convex polygons with a finite number of sides can be obtained similarly by just considered scheme of application of [14, Theorem A].

The same can be obtained in the case of a sequence of exponents, that satisfy the conditions of Mikusinsky's theorem mentioned above.

Clearly, this indicates that in general the statement of Corollary 5 is sharp. However, the question of constructing a general example for the finality of the statement of Corollary 5 in the cases of arbitrary convex domains of convergence and existence of maximal term remains open.

A similar question regarding the inclusions (10) from Corollary 8 also remains open.

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Куриляк М.Р., Скасків О.Б. *Області збіжності загальних рядів Діріхле з комплексними показниками* // Карпатські матем. публ. — 2023. — Т.15, №2. — С. 594–607.

Нехай (λ_n) — послідовність попарно різних комплексних чисел. Для формального ряду Діріхле $F(z) = \sum\limits_{n=0}^{+\infty} a_n e^{z\lambda_n}, z \in \mathbb{C}$, через $G_\mu(F), G_c(F), G_a(F)$ позначимо області існування, збіжності та абсолютної збіжності максимального члена $\mu(z,F) = \max\{|a_n| e^{\mathrm{Re}(z\lambda_n)} : n \geq 0\}$, відповідно. Позначимо $\mathcal{N}_1(z) := \{n : \mathrm{Re}(z\lambda_n) > 0\}, \mathcal{N}_2(z) := \{n : \mathrm{Re}(z\lambda_n) < 0\}$,

$$\alpha^{(1)}(\theta) := \varliminf_{\substack{n\to+\infty \\ n\in\mathcal{N}_1(e^{i\theta})}} \frac{-\ln|a_n|}{\mathrm{Re}(e^{i\theta}\lambda_n)}, \qquad \alpha^{(2)}(\theta) := \varlimsup_{\substack{n\to+\infty \\ n\in\mathcal{N}_2(e^{i\theta})}} \frac{-\ln|a_n|}{\mathrm{Re}(e^{i\theta}\lambda_n)}.$$

Припустимо, що $a_n \to 0$ при $n \to +\infty$. У статті, зокрема, доведено наступні твердження.

1) Якщо $\alpha^{(2)}(\theta) < \alpha^{(1)}(\theta)$ для деякого $\theta \in [0, \pi)$, то $\big\{te^{i\theta} : t \in (\alpha^{(2)}(\theta), \alpha^{(1)}(\theta))\big\} \subset G_\mu(F)$, а також $\big\{te^{i\theta} : t \in (-\infty, \alpha^{(2)}(\theta)) \cup (\alpha^{(1)}(\theta), +\infty)\big\} \cap G_\mu(F) = \varnothing$.

2) $G_\mu(F) = \bigcup\limits_{\theta\in[0,\pi)} \{z = te^{i\theta} \colon t \in (\alpha^{(2)}(\theta), \alpha^{(1)}(\theta))\}$.

3) Якщо $h := \varliminf\limits_{n\to+\infty} \frac{-\ln|a_n|}{\ln n} \in (1, +\infty]$, то $\Big(\frac{h}{h-1} \cdot G_a(F)\Big) \supset G_\mu(F) \supset G_c(F)$. Якщо $h = +\infty$, то $G_a(F) = G_c(F) = G_\mu(F)$, тому $G_c(F)$ також опукла область.

Ключові слова і фрази: область збіжності, абсциса збіжності, ряд Діріхле.