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Fibonacci and Lucas numbers via the determinants of tridiagonal matrix

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Abstract: Applying the apparatus of triangular matrices, we proved new recurrence formulas for the Fibonacci and Lucas numbers with even (odd) indices by tridiagonal determinants.
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1 Triangular matrix and parapermanents of triangular matrix

The functions of triangular matrices are widely used in algebra, combinatorics, number theory and other branches of mathematics [9, 11, 12].

Definition 1.1. [11]. *A triangular number table*

$$A_n = \begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \tag{1}$$

is called a nth-order triangular matrix.

Note that a matrix (1) is not a triangular matrix in the usual sense of this term as it is not a square matrix.

The product $a_{ij}\, a_{i,j+1} \cdots a_{ii}$ is denoted by $\{a_{ij}\}$ and is called a *factorial product* of the element a_{ij}.

Definition 1.2. [11]. *The parapermanent* $\operatorname{pper}(A_n)$ *of a triangular matrix* (1) *is the number*

$$\operatorname{pper}(A_n) \equiv \begin{bmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_n = \sum_{r=1}^{n} \sum_{p_1+\ldots+p_r=n} \prod_{s=1}^{r} \{a_{p_1+\ldots+p_s,\, p_1+\ldots+p_{s-1}+1}\}, \qquad (2)$$

where $p_1, p_2, \ldots, p_r$ *are positive integers,* $\{a_{ij}\}$ *is the factorial product of the element* a_{ij}*.*

Example 1.3. *The parapermanent of a 4-th order matrix:*

$$\operatorname{pper}(A_4) = \begin{bmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ a_{31} & a_{32} & a_{33} & \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} =$$

$$= a_{41}a_{42}a_{43}a_{44} + a_{31}a_{32}a_{33}a_{44} + a_{21}a_{22}a_{43}a_{44} + a_{21}a_{22}a_{33}a_{44} +$$
$$+ a_{11}a_{42}a_{43}a_{44} + a_{11}a_{32}a_{33}a_{44} + a_{11}a_{22}a_{43}a_{44} + a_{11}a_{22}a_{33}a_{44}.$$

To each element a_{ij} of a matrix (1) we associate the triangular table of elements of matrix A_n that has a_{ij} in the bottom left corner. We call this table a *corner* of the matrix and denote it by $R_{ij}(A_n)$. Corner $R_{ij}(A_n)$ is a triangular matrix of order $(i - j + 1)$, and it contains only elements a_{rs} of matrix (1) whose indices satisfy the inequalities $j \leqslant s \leqslant r \leqslant i$.

Theorem 1.4. [11] (Decomposition of a parapermanent $\operatorname{pper}(A_n)$ by elements of the last row). *The following formula are valid:*

$$\operatorname{pper}(A_n) = \sum_{s=1}^{n} \{a_{ns}\} \operatorname{pper}(R_{s-1,1}(A_n)), \qquad (3)$$

where $\operatorname{pper}(R_{0,1}(A_n)) \equiv 1$.

Example 1.5. *Decomposition of a parapermanent* $\operatorname{pper}(A_4)$ *by elements of the last row:*

$$\operatorname{pper}(A_4) = a_{44}\operatorname{pper}(A_3) + a_{43}a_{44}\operatorname{pper}(A_2) + a_{42}a_{43}a_{44}\operatorname{pper}(A_1) + a_{41}a_{42}a_{43}a_{44}\operatorname{pper}(A_0),$$

where $\operatorname{pper}(A_1) = a_{11}$, $\operatorname{pper}A_0 \equiv 1$.

R. Zatorsky and I. Lishchynskyy [10, 13] established connection between the paradeterminats and the lower Hessenberg determinants by formula

$$\operatorname{pper}(A_n) = \begin{vmatrix} \{a_{11}\} & 1 & 0 & \ldots & 0 & 0 \\ -\{a_{21}\} & \{a_{22}\} & 1 & \ldots & 0 & 0 \\ -\{a_{31}\} & -\{a_{32}\} & \{a_{33}\} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\{a_{n-1,1}\} & -\{a_{n-1,2}\} & -\{a_{n-1,3}\} & \ldots & \{a_{n-1,n-1}\} & 1 \\ -\{a_{n1}\} & -\{a_{n2}\} & -\{a_{n3}\} & \ldots & -\{a_{n,n-1}\} & \{a_{nn}\} \end{vmatrix}, \qquad (4)$$

where $\{a_{ij}\}$ is factorial product of the element a_{ij}.

2 A connection between the Horadam numbers with even (odd) indices and parapermanents

In [5] A. Horadam considered the sequence

$$h_1 = p,\ h_2 = q, \quad h_n = h_{n-1} + h_{n-2},\ n \geq 3,$$

where p and q are arbitrary integer numbers. This sequence generalized the Fibonacci sequence:

$$F_1 = 1,\ F_2 = 1, \quad F_n = F_{n-1} + F_{n-2},\ n \geq 3,$$

and the Lucas sequence:

$$L_1 = 2,\ L_2 = 1, \quad L_n = L_{n-1} + L_{n-2},\ n \geq 3.$$

Proposition 2.1. *The following formula is valid:*

$$h_{2n-1} = \left[\begin{array}{cccccccc} p & & & & & & & \\ \frac{h_2}{1} & 1 & & & & & & \\ 0 & \frac{h_4}{h_1} & 1 & & & & & \\ 0 & 0 & \frac{h_6}{h_3} & 1 & & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & & \\ 0 & 0 & 0 & 0 & \cdots & \frac{h_{2n-4}}{h_{2n-7}} & 1 & \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{h_{2n-2}}{h_{2n-5}} & 1 \end{array}\right]. \quad (5)$$

Proof. Expanding the parapermanent (5) by elements of the last raw (see (3)), we have

$$h_{2n-1} = 1 \cdot h_{2n-3} + \frac{h_{2n-2}}{h_{2n-5}} \cdot h_{2n-5} = h_{2n-3} + h_{2n-2}.$$

Obtained equality holds by definition of the sequence $\{h_n\}_{n\geq 1}$. □

Proposition 2.2. *The following formula is valid:*

$$h_{2n} = \left[\begin{array}{cccccccc} q & & & & & & & \\ \frac{h_3}{1} & 1 & & & & & & \\ 0 & \frac{h_5}{h_2} & 1 & & & & & \\ 0 & 0 & \frac{h_7}{h_4} & 1 & & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & & \\ 0 & 0 & 0 & 0 & \cdots & \frac{h_{2n-3}}{h_{2n-6}} & 1 & \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{h_{2n-1}}{h_{2n-4}} & 1 \end{array}\right]. \quad (6)$$

Proof. Using (3), we have

$$h_{2n} = 1 \cdot h_{2n-2} + \frac{h_{2n-1}}{h_{2n-4}} \cdot h_{2n-4} = h_{2n-2} + h_{2n-1}.$$

□

3 Main results

In this section we proved two recurrence formulas expressing the Horadam numbers h_n by the determinant of tridiagonal matrix. As a consequence we received the corresponding formulas for the Fibonacci and Lucas numbers.

Proposition 3.1. *The following formulas are valid:*

$$h_{2n-1} = \frac{1}{h_1 h_3 \cdots h_{2n-5}} \begin{vmatrix} p & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -h_2 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -h_4 & h_1 & h_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -h_6 & h_3 & h_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -h_{2n-4} & h_{2n-7} & h_{2n-7} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -h_{2n-2} & h_{2n-5} \end{vmatrix}, \quad (7)$$

$$h_{2n} = \frac{1}{h_2 h_4 \cdots h_{2n-4}} \begin{vmatrix} q & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -h_3 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -h_5 & h_2 & h_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -h_7 & h_4 & h_4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -h_{2n-3} & h_{2n-6} & h_{2n-6} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -h_{2n-1} & h_{2n-4} \end{vmatrix}. \quad (8)$$

Proof. We prove the formula (7). From (5) using (4), we have

$$h_{2n-1} = \begin{vmatrix} p & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{h_2}{1} & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{h_4}{h_1} & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\frac{h_6}{h_3} & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{h_{2n-4}}{h_{2n-7}} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{h_{2n-2}}{h_{2n-5}} & 1 \end{vmatrix}.$$

After obvious simple transformations, we get (7).

Formula (8) can be proved similarly. □

Example 3.2. *Fibonacci numbers with odd indices:*

$$F_{2n-1} = \frac{1}{F_1 F_3 \cdots F_{2n-5}} \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -F_2 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -F_4 & F_1 & F_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -F_6 & F_3 & F_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -F_{2n-4} & F_{2n-7} & F_{2n-7} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -F_{2n-2} & F_{2n-5} \end{vmatrix}.$$

Example 3.3. *The Fibonacci numbers with even indices:*

$$F_{2n} = \frac{1}{F_2 F_4 \cdots F_{2n-4}} \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -F_3 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -F_5 & F_2 & F_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -F_7 & F_4 & F_4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -F_{2n-3} & F_{2n-6} & F_{2n-6} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -F_{2n-1} & F_{2n-4} \end{vmatrix}.$$

Example 3.4. *The Lucas numbers with odd indices:*

$$L_{2n-1} = \frac{1}{L_1 L_3 \cdots L_{2n-5}} \begin{vmatrix} 2 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -L_2 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -L_4 & L_1 & L_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -L_6 & L_3 & L_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -L_{2n-4} & L_{2n-7} & L_{2n-7} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -L_{2n-2} & L_{2n-5} \end{vmatrix}.$$

Example 3.5. *The Lucas numbers with even indices:*

$$L_{2n} = \frac{1}{L_2 L_4 \cdots L_{2n-4}} \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -L_3 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -L_5 & L_2 & L_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -L_7 & L_4 & L_4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -L_{2n-3} & L_{Ln-6} & L_{2n-6} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -L_{2n-1} & L_{2n-4} \end{vmatrix}.$$

Note, that determinants of matrices, elements of which are classical or generalized Fibonacci numbers, in particular, studied in [1, 2, 3, 4, 6, 7, 8].

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