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NEW INTEGRAL FUNCTIONS GENERATED BY RISING FACTORIAL POWERS

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We consider new nonelementary functions such as the Fresnel integrals, generated by rising factorial powers. Graphs of such functions are plotted and some of their properties are proved. It is shown, that new integral functions are solutions of second order ordinary differential equations with variable coefficients.

Key words and phrases: factorial power, rising factorial power, Fresnel integrals, power series, Cauchy problem.

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INTRODUCTION

Mathematical models of various natural and industrial processes often lead to problems, such that it is impossible to obtain exact solutions of which by means of well-known classical methods. This is the reason for further development of function theory and numerical analysis. Enlargement of an "library" of nonelementary functions leads to the enlargement of tasks that can be solved in closed form. That's why the introducing of new nonelementary functions and studying of their properties are actual tasks.

In [7], [8] we investigate new nonelementary functions $\mathrm{Sin}(x)$, $\mathrm{Cos}(x)$, constructed by replacing in a power series of classical transcendental functions $\sin x$, $\cos x$ falling factorial powers $n^{\underline{n}}$ (i.e. usual factorials) by corresponding rising factorial powers $n^{\overline{n}}$. Replacing in the Fresnel integrals $\int\limits_0^x \cos t^2 dt$, $\int\limits_0^x \sin t^2 dt$ trigonometric functions by the functions $\mathrm{Cos}(x)$, $\mathrm{Sin}(x)$, we get new real functions

$$\widetilde{C}(x) = \int_0^x \mathrm{Cos}\,(t^2)dt, \quad \widetilde{S}(x) = \int_0^x \mathrm{Sin}\,(t^2)dt.$$

Note, that the Fresnel integrals were originally used in the calculation of a field intensity in an environment related to the bending of light around opaque objects (in diffraction theory). Recently the Fresnel integrals and their various generalizations have been used in vibration theory, in the design of highways and railways, etc (see, for example, [1], [3], [5], [11]–[17], [19] and the references given there).

The aim of this paper is to study the functions $\widetilde{C}(x)$, $\widetilde{S}(x)$.

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1 PRELIMINARIES AND NOTATIONS

For an arbitrary $x \in \mathbb{R}$ and $m \in \mathbb{N}$ the *factorial power* m with the step of $k \in \mathbb{R}$ is the expression

$$x^{m\{k\}} = x(x+k)(x+2k)\cdot\ldots\cdot\big(x+(m-1)k\big).$$

Factorial power $x^{m\{k\}}$ is called *rising* if $k > 0$, and is called *falling* if $k < 0$. By definition, put $x^{0\{k\}} \equiv 1$. If $k = 0$, then we have a simple power, i.e. $x^{m\{0\}} = x^m$.

Rising factorial powers with the step of 1 and falling factorial powers with the step of (-1) we will denote by

$$x^{\overline{m}} = x^{m\{1\}} = x(x+1)\cdot\ldots\cdot(x+m-1), \qquad x^{\underline{m}} = x^{m\{-1\}} = x(x-1)\cdot\ldots\cdot(x-m+1),$$

respectively.

Different notation of factorial powers are used by other authors (see [4], [6], [10], [18]). For example, a rising factorial power m with the step of 1 often denoted by the Pochhammer symbol $(x)_m$, i.e. $(x)_m = x^{\overline{m}}$.

Relation between the factorial function $m!$ and rising (falling) factorials is expressed by the formula

$$m! = 1^{\overline{m}} = m^{\underline{m}}.$$

The main properties of falling factorial powers with the step of (-1) and rising factorial powers with the step of 1 are given by the formulas

$$\Delta x^{\underline{m}} = m x^{\underline{m-1}}, \qquad \overline{\Delta} x^{\overline{m}} = m x^{\overline{m-1}},$$

respectively, where $\Delta f(x) = f(x+1) - f(x)$ is a forward difference of a function $f(x)$ and $\overline{\Delta} f(x) = f(x) - f(x-1)$ is a backward difference of a function $f(x)$.

2 FUNCTIONS Cos (x), Sin (x), DEFINED BY THE RISING FACTORIAL POWERS

The known power series

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)^{\underline{2n}}}\, x^{2n}, \tag{1}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{\underline{2n+1}}}\, x^{2n+1} \tag{2}$$

can be treated as the series constructed with the help of falling factorial powers.

In analogy to these series in [7], [8] we investigate new nonelementary functions $\mathrm{Cos}(x)$, $\mathrm{Sin}(x)$, constructed with the help of rising factorial powers

$$\mathrm{Cos}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)^{\overline{2n}}}\, x^{2n} = 1 - \frac{x^2}{2\cdot 3} + \frac{x^4}{4\cdot 5\cdot 6\cdot 7} - \ldots + \frac{(-1)^n x^{2n}}{2n\cdot(2n+1)\cdot\ldots\cdot(4n-1)} + \ldots,$$

$$\begin{aligned}\mathrm{Sin}(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{\overline{2n+1}}}\, x^{2n+1} \\ &= \frac{x}{1} - \frac{x^3}{3\cdot 4\cdot 5} + \frac{x^5}{5\cdot 6\cdot 7\cdot 8\cdot 9} - \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)\cdot(2n+2)\cdot\ldots\cdot(4n+1)} + \ldots.\end{aligned}$$

It is clear that

$$\mathrm{Cos}(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2n-1)!}{(4n-1)!} x^{2n}, \tag{3}$$

$$\mathrm{Sin}(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (2n-2)!}{(4n-3)!} x^{2n-1}. \tag{4}$$

Absolutely convergence on the real axis of the series (3) and (4) can easily be shown.

In [8] it is also proved that

$$\mathrm{Cos}(x) = 1 + 2\sqrt{x}\left(\cos\frac{x}{4}\, S\left(\frac{\sqrt{x}}{2}\right) - \sin\frac{x}{4}\, C\left(\frac{\sqrt{x}}{2}\right)\right), \tag{5}$$

$$\mathrm{Sin}(x) = 2\sqrt{x}\left(\cos\frac{x}{4}\, C\left(\frac{\sqrt{x}}{2}\right) + \sin\frac{x}{4}\, S\left(\frac{\sqrt{x}}{2}\right)\right), \tag{6}$$

where $C(p)$ and $S(p)$ are the real Fresnel integrals (cosine-integral and sine-integral) which defined by formulas (see, for example, [2], [18])

$$C(p) = \int_0^p \cos t^2\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)(2n)!} p^{4n+1}, \tag{7}$$

$$S(p) = \int_0^p \sin t^2\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)(2n+1)!} p^{4n+3}. \tag{8}$$

Given (7), (8), the formulas (5), (6) can be rewritten as

$$\mathrm{Cos}\,(x) = 1 + 2\sqrt{x} \int_0^{\frac{\sqrt{x}}{2}} \sin\left(t^2 + \frac{x}{4}\right) dt,$$

$$\mathrm{Sin}(x) = 2\sqrt{x} \int_0^{\frac{\sqrt{x}}{2}} \cos\left(t^2 - \frac{x}{4}\right) dt.$$

Some authors define the Fresnel integrals as

$$C^*(p) = \int_0^p \cos\frac{\pi t^2}{2}\, dt, \qquad S^*(p) = \int_0^p \sin\frac{\pi t^2}{2}\, dt.$$

Then the functions $C^*(p)$, $S^*(p)$ can be represented in the following form

$$\mathrm{Cos}(x) = 1 + \sqrt{2\pi}\sqrt{x}\left(\cos\frac{x}{4}\, S^*\left(\frac{\sqrt{x}}{\sqrt{2\pi}}\right) - \sin\frac{x}{4}\, C^*\left(\frac{\sqrt{x}}{\sqrt{2\pi}}\right)\right),$$

$$\mathrm{Sin}(x) = \sqrt{2\pi}\sqrt{x}\left(\cos\frac{x}{4}\, C^*\left(\frac{\sqrt{x}}{\sqrt{2\pi}}\right) + \sin\frac{x}{4}\, S^*\left(\frac{\sqrt{x}}{\sqrt{2\pi}}\right)\right).$$

Figures 1, 2 show the graphs of functions $y = \mathrm{Cos}(x)$ and $y = \mathrm{Sin}(x)$.

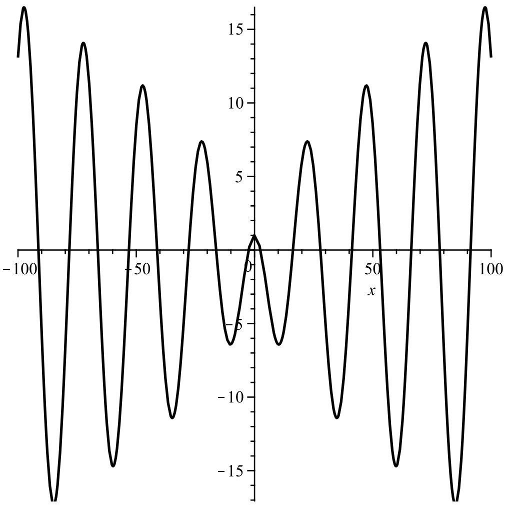

Fig. 1. Graph of the function $y = \mathrm{Cos}(x)$

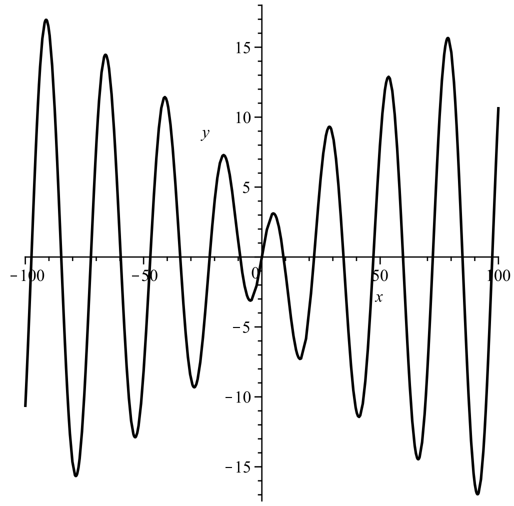

Fig. 2. Graph of the function $y = \mathrm{Sin}(x)$

3 INTEGRAL FUNCTIONS $\widehat{C}(x)$, $\widehat{S}(x)$ AND THEIR PROPERTIES

We will denote by $\widehat{C}(x)$ the function defined by formula

$$\widehat{C}(x) = \int_0^x \mathrm{Cos}\,(t^2)\,dt. \tag{9}$$

By (3), (7) we obtain the following series expansion of function $\widehat{C}(x)$:

$$\widehat{C}(x) = x + \sum_{n=1}^{\infty} \frac{(-1)^n\,(2\,n-1)!}{(4\,n-1)!\,\,(4\,n+1)}\,x^{4\,n+1}.$$

Then, since

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)!}{(4n-1)!\,(4n+1)}\,x^{4\,n+1} &= -\,2x + 2\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+1}}{16^n}\sum_{s=0}^{n}\frac{1}{(2s)!(2n-2s)!(4n-4s+1)} \\ &\quad +\,2\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+5}}{16^{n+1}}\sum_{s=0}^{n}\frac{1}{(2s+1)!(2n-2s+1)!(4n-4s+3)} \\ &= -\,2x + 4\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!\,4^{2\,n}}\,x^{4n}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!\,(4\,n+1)2^{4\,n+1}}\,x^{4n+1} \\ &\quad +4\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!\,4^{2\,n+1}}\,x^{4n+2}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!\,(4n+3)2^{4n+3}}\,x^{4n+3},\end{aligned}$$

from (3), using (1), (2), we obtain

$$\widehat{C}(x) = -\,x + 4\left(\cos\frac{x^2}{4}\,C\left(\frac{x}{2}\right) + \sin\frac{x^2}{4}\,S\left(\frac{x}{2}\right)\right). \tag{10}$$

The graph of $y = \widehat{C}(x)$ is plotted in the Figure 3 (the dashed line is graph of the function $y = -x$).

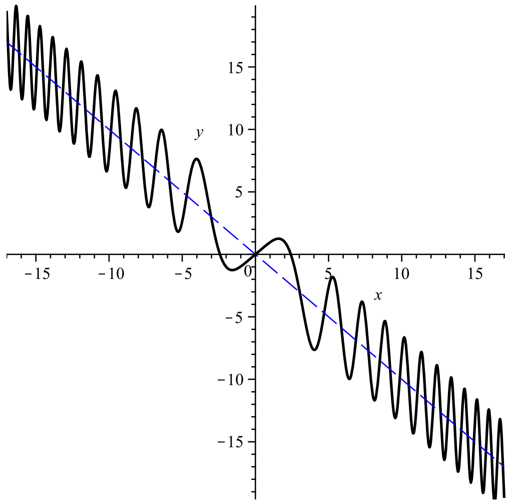

Fig. 3. Graph of the function $y = \widehat{C}(x)$

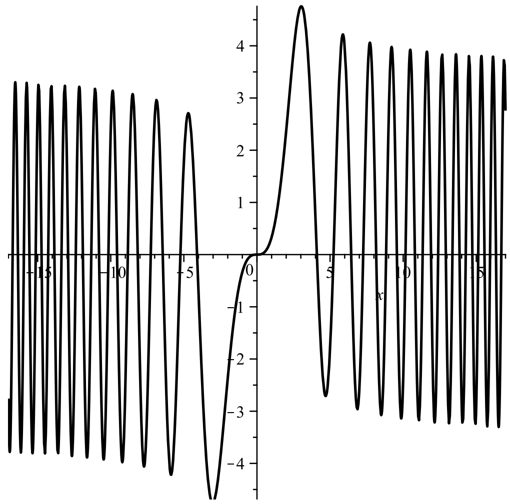

Fig. 4. The graph of the function $y = \widehat{S}(x)$

Define the following function

$$\widehat{S}(x) = \int_0^x \mathrm{Sin}(t^2)\, dt. \tag{11}$$

From (11) and (4) we obtain presentation of function $\widehat{S}(x)$ in the form

$$\widehat{S}(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2\,n-2)!}{(4n-3)!\,(4\,n-1)}\, x^{4\,n-1}. \tag{12}$$

Since

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2\,n-2)!}{(4n-3)!\,(4\,n-1)}\, x^{4\,n-1} =& \frac{x^3}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{16^n} \sum_{s=0}^{n} \frac{1}{(2s+1)!(2n-2s)!(4n-4s+1)} \\ &+ \frac{x^3}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{16^n} \sum_{s=0}^{n} \frac{1}{(2s)!(2n-2s+1)!(4n-4s+3)} \\ =& 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,4^{2\,n+1}}\, x^{4n+1} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!\,(4\,n+1)2^{4\,n+1}}\, x^{4n+1} \\ &- 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!\,4^{2\,n}}\, x^{4n+2} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,(4n+3)2^{4n+3}}\, x^{4n+3},\end{aligned}$$

from (12), using (1), (2), (7), (8), we get

$$\widehat{S}(x) = 4\left(\sin\frac{x^2}{4}\, C\left(\frac{x}{2}\right) - \cos\frac{x^2}{4}\, S\left(\frac{x}{2}\right)\right). \tag{13}$$

Graph of the function $y = \widehat{S}(x)$ is plotted in the Figure 4.

The following proposition establishes a relation between the new functions $\widehat{C}(x)$, $\widehat{S}(x)$ and classical Fresnel integrals.

Proposition 3.1. *For all* $x \in \mathbb{R}$

$$\left(\widehat{C}(x)+x\right)^2+\widehat{S}^{\,2}(x)=16\Big(C^2\Big(\frac{x}{2}\Big)+S^2\Big(\frac{x}{2}\Big)\Big). \tag{14}$$

Proof. Squaring and adding the formulas

$$\cos\frac{x^2}{4}\,C\Big(\frac{x}{2}\Big)+\sin\frac{x^2}{4}\,S\Big(\frac{x}{2}\Big)=\frac{\widehat{C}(x)+x}{4},$$

$$\sin\frac{x^2}{4}\,C\Big(\frac{x}{2}\Big)-\cos\frac{x^2}{4}\,S\Big(\frac{x}{2}\Big)=\frac{\widehat{S}(x)}{4},$$

which may be derived from (10), (13) respectively, we obtain formula (14). □

The following proposition establishes a relation between the functions $\widehat{C}(x)$, $\widehat{S}(x)$ and $\mathrm{Cos}(x)$, $\mathrm{Sin}(x)$.

Proposition 3.2. *For all* $x \in \mathbb{R}$

$$\left(\widehat{C}(x)+x\right)^2+\widehat{S}^{\,2}(x)=\frac{8}{x^2}\left(\left(1-\mathrm{Cos}\,(x^2)\right)^2+\mathrm{Sin}^2(x^2)\right). \tag{15}$$

Proof. From (5), (6) it follows that

$$(1-\mathrm{Cos}\,(x))^2+\mathrm{Sin}^2(x)=2x\left(C^2\left(\frac{\sqrt{x}}{2}\right)+S^2\left(\frac{\sqrt{x}}{2}\right)\right).$$

Hence, using (14), we get (15). □

4 Differential equations of functions $\widehat{S}(x)$, $\widehat{C}(x)$

In this section it is shown that both functions $\widehat{C}(x)$, $\widehat{S}(x)$ are solutions of the Cauchy problem for the inhomogeneous linear ordinary differential equation of second order with continuous coefficients.

Proposition 4.1. *The functions* $\widehat{C}(x)$, $\widehat{S}(x)$ *are solutions of the Cauchy problems*

$$4xy''-4y'+x^3y=-x^4-4,\quad y(0)=0,\ y'(0)=1; \tag{16}$$

$$4xy''-4y'+x^3y=4x^2,\quad y(0)=0,\ y'(0)=0. \tag{17}$$

respectively.

Proof. Using (9), (11) we obtain that the functions $\widehat{C}(x)$, $\widehat{S}(x)$ satisfy the corresponding initial conditions. It remains to check that these functions are the solutions of differential equations from (16), (17).

First of all, using (10), we find the first and second derivatives of the function $\widehat{C}(x)$

$$\widehat{C}'(x)=1-2\,x\left(\sin\frac{x^2}{4}\,C\Big(\frac{x}{2}\Big)-\cos\frac{x^2}{4}\,S\Big(\frac{x}{2}\Big)\right), \tag{18}$$

$$\widehat{C}''(x)=-2\left(\sin\frac{x^2}{4}\,C\Big(\frac{x}{2}\Big)-\cos\frac{x^2}{4}\,S\Big(\frac{x}{2}\Big)\right)-x^2\left(\cos\frac{x^2}{4}\,C\Big(\frac{x}{2}\Big)+\sin\frac{x^2}{4}\,S\Big(\frac{x}{2}\Big)\right). \tag{19}$$

We obtain the differential equation (16) from (3), (18), (19) by elimination the expressions

$$\cos\frac{x^2}{4}\,C\Big(\frac{x}{2}\Big)+\sin\frac{x^2}{4}\,S\Big(\frac{x}{2}\Big),\quad \cos\frac{x^2}{4}\,S\Big(\frac{x}{2}\Big)-\sin\frac{x^2}{4}\,C\Big(\frac{x}{2}\Big). \tag{20}$$

The proof for $\widehat{S}(x)$ is similar. We obtain the differential equation (17) by elimination the expressions (20) from (4) and formulas

$$\widehat{S}'(x)=2x\bigg(\cos\frac{x^2}{4}\,C\Big(\frac{x}{2}\Big)+\sin\frac{x^2}{4}\,S\Big(\frac{x}{2}\Big)\bigg),$$

$$\widehat{S}''(x)=x-x^2\bigg(\sin\frac{x^2}{4}\,C\Big(\frac{x}{2}\Big)-\cos\frac{x^2}{4}\,S\Big(\frac{x}{2}\Big)\bigg)+2\bigg(\cos\frac{x^2}{4}\,C\Big(\frac{x}{2}\Big)+\sin\frac{x^2}{4}\,S\Big(\frac{x}{2}\Big)\bigg).$$

□

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Гой Т.П., Заторський Р.А. *Нові інтегральні функції, породжені зростаючими факторіальними степенями* // Карпатські математичні публікації. — 2013. — Т.5, №2. — С. 217–224.

Запропоновані нові неелементарні функції типу інтегралів Френеля, побудовані при допомозі зростаючих факторіальних степенів. Встановлені деякі властивості цих інтегральних функцій, побудовані їх графіки. Виведені звичайні диференціальні рівняння, розв'язками яких є нові інтегральні функції.

Ключові слова і фрази: факторіальний степінь, зростаючий факторіальний степінь, інтеграли Френеля, степеневі ряди, задача Коші.

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Рассматриваются новые неэлементарные функции типа интегралов Френеля, порожденные возрастающими факториальными степенями. Установлены некоторые свойства этих интегральных функций, построены их графики. Выведены обыкновенные дифференциальные уравнения, решениями которых есть новые интегральные функции.

Ключевые слова и фразы: факториальная степень, возростающая факториальная степень, интегралы Френеля, степенные ряды, задача Коши.