



Determinant formulas of some Toeplitz–Hessenberg matrices with Catalan entries

TARAS GOY¹ and MARK SHATTUCK^{2,*}

¹Faculty of Mathematics and Computer Science, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk 76018, Ukraine

²Institute for Computational Science and Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam

*Corresponding author.

E-mail: tarasgoy@yahoo.com; mark.shattuck@tdtu.edu.vn

MS received 28 August 2018; accepted 16 March 2019

Abstract. In this paper, we consider determinants of some families of Toeplitz–Hessenberg matrices having various translates of the Catalan numbers for the nonzero entries. These determinant formulas may also be rewritten as identities involving sums of products of Catalan numbers and multinomial coefficients. Combinatorial proofs may be given for several of the identities that are obtained.

Keywords. Toeplitz–Hessenberg matrix; Catalan number; Trudi formula; generating function.

2010 Mathematics Subject Classification. 15A15, 15B05, 05A15.

1. Introduction

The Catalan numbers C_n can be expressed directly in terms of the central binomial coefficients as

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}, \quad n \geq 0,$$

or recursively as follows:

$$C_0 = 1, \quad C_n = \frac{4n-2}{n+1} C_{n-1}, \quad n \geq 1.$$

They may also be computed using the recurrence

$$C_0 = 1, \quad C_{n+1} = \sum_{i=0}^n C_i C_{n-i}, \quad n \geq 0.$$

The first several terms of the Catalan sequence (see entry A000108 in [11]) are

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4869, 16796, 58786, \dots$$

The generating function (g.f.) $C(x)$ for the sequence $\{C_n\}_{n \geq 0}$ is given by

$$C(x) = \sum_{n \geq 0} C_n x^n = \frac{1 - \sqrt{1-4x}}{2x} = \frac{2}{1 + \sqrt{1-4x}}, \quad |x| \leq 1/4,$$

and satisfies the functional equation $C(x) = 1 + xC(x)^2$.

The Catalan numbers have a rich history and many unique properties. They count many types of combinatorial objects, among them, certain lattice paths, permutations, binary trees, polygon triangulations and finite set partitions, and occur as entries in a variety of continued fraction expansions and special matrices (see, among others, [5, 10, 12] and related references therein).

The purpose of the present paper is to investigate the determinants of some families of Toeplitz–Hessenberg matrices whose entries are Catalan numbers with successive, odd or even subscripts. As a consequence, we obtain for these numbers new identities involving multinomial coefficients. Some of the results of this paper were announced without proof in [2]. To establish our main results, we primarily make use of generating functions and combinatorial proofs are provided for some of the special cases.

Recall that a lower *Toeplitz–Hessenberg matrix* is an $n \times n$ matrix of the form

$$M_n(a_0; a_1, \dots, a_n) = \begin{bmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{bmatrix}, \quad (1.1)$$

where $a_0 \neq 0$ and $a_k \neq 0$ for at least one $k > 0$.

The Toeplitz–Hessenberg matrices have been encountered in various scientific and engineering applications (see, for example, [7] and the references therein). It is known that the Toeplitz–Hessenberg matrix determinant $\det(M_n)$ can be evaluated using the *Trudi formula* [9] as follows:

$$\det(M_n) = \sum_{s_1+2s_2+\cdots+ns_n=n} (-a_0)^{n-(s_1+\cdots+s_n)} p_n(s) a_1^{s_1} a_2^{s_2} \cdots a_n^{s_n}, \quad (1.2)$$

where the summation is over integers $s_i \geq 0$ satisfying $s_1 + 2s_2 + \cdots + ns_n = n$ and $p_n(s) = \frac{(s_1+\cdots+s_n)!}{s_1! \cdots s_n!}$ is the multinomial coefficient.

For the sake of brevity, throughout, we will use the notation

$$\det(a_0; a_1, \dots, a_n) = \det(M_n(a_0; a_1, \dots, a_n)).$$

2. Catalan determinants

There is the following formula for the g.f. of the translated Catalan number in terms of $C(x)$.

Lemma 2.1. *Let $i \geq 1$ be a fixed integer. Then we have*

$$C_i(x) := \sum_{n \geq 0} C_{n+i} x^n = C(x)^{i+1} + \sum_{r=1}^{i-1} \sum_{j=0}^{r-1} \frac{(r-j)(2i-r-j-3)!}{(i-r-1)!(i-j-1)!} \binom{r}{j} C(x)^{j+2}. \quad (2.1)$$

Proof. Multiplying both sides of the recurrence

$$C_{n+i} = \sum_{j=0}^{n+i-1} C_j C_{n+i-1-j}, \quad i \geq 1,$$

by x^n , and summing over $n \geq 0$, gives

$$\begin{aligned} C_i(x) &= \sum_{n \geq 0} x^n \sum_{j=0}^{n+i-1} C_j C_{n+i-1-j} \\ &= \sum_{j \geq i-1} C_j \sum_{n \geq j-i+1} C_{n+i-1-j} x^n + \sum_{j=0}^{i-2} C_j \sum_{n \geq 0} C_{n+i-1-j} x^n \\ &= \sum_{j \geq i-1} C_j x^{j-i+1} C(x) + \sum_{j=0}^{i-2} C_j C_{i-1-j}(x) \\ &= C(x)C_{i-1}(x) + \sum_{j=1}^{i-1} C_{i-1-j} C_j(x), \quad i \geq 1, \end{aligned} \quad (2.2)$$

with $C_0(x) = C(x)$. To determine $C_i(x)$, we introduce a second variable and consider $C(x, y) := \sum_{i \geq 0} C_i(x)y^i$. Then multiplying both sides of (2.2) by y^i , and summing over $i \geq 1$, implies

$$\begin{aligned} C(x, y) - C(x) &= yC(x)C(x, y) + \sum_{j \geq 1} C_j(x) \sum_{i \geq j+1} C_{i-1-j} y^i \\ &= yC(x)C(x, y) + \sum_{j \geq 1} C_j(x) y^{j+1} C(y) \\ &= yC(x)C(x, y) + yC(y)(C(x, y) - C(x)). \end{aligned}$$

Solving for $C(x, y)$ in the last equation yields

$$C(x, y) = \frac{C(x)(1 - yC(y))}{1 - y(C(x) + C(y))} = C(x) + \frac{yC(x)^2}{1 - y(C(x) + C(y))}.$$

Extracting the coefficient of y gives $C_1(x) = [y^1]C(x, y) = C(x)^2$, which is in accordance with the recurrence for C_{n+1} . If $i \geq 2$, then we have

$$\begin{aligned} C_i(x) &= [y^i]C(x, y) = C(x)^{i+1} + C(x)^2 \sum_{r=1}^{i-1} \sum_{j=0}^{r-1} \binom{r}{j} C(x)^j [y^{i-r-1}] (C(y))^{r-j} \\ &= C(x)^{i+1} + C(x)^2 \sum_{r=1}^{i-1} \sum_{j=0}^{r-1} \binom{r}{j} C(x)^j \frac{(r-j)(2i-r-j-3)!}{(i-r-1)!(i-j-1)!}, \end{aligned}$$

which implies (2.1), where we have used in the last equality the formula

$$C(x)^k = \left(\frac{1 - \sqrt{1 - 4x}}{2x} \right)^k = \sum_{n \geq 0} \frac{k(2n + k - 1)!}{n!(n + k)!} x^n, \quad k \geq 1,$$

from [13, equation (2.5.16)]. \square

Let $f_i(x) = \sum_{n \geq 0} \det(a; C_{1+i}, \dots, C_{n+i}) x^n$, where $i \geq -1$ is fixed and $a \neq 0$ is arbitrary.

Theorem 2.2. *We have*

$$f_i(x) = \frac{a}{a - C_i + C_i(-ax)}, \quad i \geq 0, \quad (2.3)$$

where $C_i(x)$ is given by (2.1). In particular,

$$f_0(x) = \frac{2a(a - 1)x - 1 - \sqrt{1 + 4ax}}{2(a - 1)^2x - 2} \quad (2.4)$$

and

$$f_1(x) = \frac{2a^2(a - 1)x^2 + 2ax + 1 + \sqrt{1 + 4ax}}{2a(a - 1)^2x^2 + 4(a - 1)x + 2}. \quad (2.5)$$

Proof. Let M_n denote an $n \times n$ Toeplitz–Hessenberg matrix with $a_0 = a$ and $a_k = C_{k+i}$. Expanding the determinant repeatedly along the first row gives

$$\det(M_n) = \sum_{k=1}^n (-a)^{k-1} C_{k+i} \det(M_{n-k}), \quad n \geq 1,$$

with $\det(M_0) = 1$. Multiplying both sides of this recurrence by x^n , and summing over $n \geq 1$, implies

$$\begin{aligned} f_i(x) - 1 &= \sum_{n \geq 1} \det(M_n) x^n = \sum_{n \geq 1} x^n \sum_{k=1}^n (-a)^{k-1} C_{k+i} \det(M_{n-k}) \\ &= \sum_{k \geq 1} (-a)^{k-1} C_{k+i} \sum_{n \geq k} \det(M_{n-k}) x^n = \sum_{k \geq 1} (-a)^{k-1} C_{k+i} x^k f_i(x) \\ &= -\frac{f_i(x)}{a} \sum_{k \geq 1} C_{k+i} \cdot (-ax)^k = -\frac{f_i(x)}{a} (C_i(-ax) - C_i). \end{aligned}$$

Solving for $f_i(x)$ in the last equation gives (2.3). Taking $i = 0$ and $i = 1$ in (2.3), and noting $C_0(z) = C(z)$ and $C_1(z) = C(z)^2$, yields the formulas stated for $f_0(x)$ and $f_1(x)$, respectively, after simplification. \square

We observe the following special cases of the prior result.

COROLLARY 2.3

We have

$$\det(1; C_1, \dots, C_n) = (-1)^{n-1} C_{n-1}, \quad n \geq 1, \quad (2.6)$$

$$\det(-1; C_1, \dots, C_n) = (2n-1)C_{n-1}, \quad n \geq 1, \quad (2.7)$$

$$\det(2; C_1, \dots, C_n) = 1 + \sum_{i=1}^{n-1} (-1)^i 2^{i+1} C_i, \quad n \geq 1, \quad (2.8)$$

$$\det(1; C_2, \dots, C_{n+1}) = (-1)^{n-1} C_{n-1}, \quad n \geq 2. \quad (2.9)$$

Proof. Letting $a = 1$ in (2.4) gives

$$\begin{aligned} \frac{1 + \sqrt{1+4x}}{2} &= 1 + x \left(\frac{1 - \sqrt{1+4x}}{-2x} \right) \\ &= 1 + x \sum_{n \geq 0} C_n \cdot (-x)^n = 1 + \sum_{n \geq 1} (-1)^{n-1} C_{n-1} x^n, \end{aligned}$$

which implies the first formula. Letting $a = -1$ in (2.4) gives

$$\frac{4x - 1 - \sqrt{1-4x}}{8x - 2} = \frac{1}{2} - \frac{\sqrt{1-4x}}{8x - 2} = \frac{1}{2} + \frac{1}{2\sqrt{1-4x}} = 1 + \sum_{n \geq 1} \frac{1}{2} \binom{2n}{n} x^n,$$

from which (2.7) follows from the fact $\frac{1}{2} \binom{2n}{n} = (2n-1)C_{n-1}$. Letting $a = 2$ in (2.4) gives

$$\begin{aligned} &\frac{4x - 1 - \sqrt{1+8x}}{2x - 2} \\ &= 2 - \frac{3}{2(1-x)} + \frac{1}{2(1-x)} \sum_{n \geq 0} \binom{1/2}{n} (8x)^n \\ &= 2 - \frac{3}{2(1-x)} + \frac{1}{2(1-x)} \left(1 + 4x + \sum_{n \geq 2} \frac{(-1)^{n-1} (2n-3)!! (4x)^n}{n!} \right) \\ &= 2 + \frac{2x-1}{1-x} + \frac{1}{1-x} \sum_{n \geq 2} (-1)^{n-1} \frac{1}{n} \binom{2n-2}{n-1} (2x)^n \\ &= \frac{1}{1-x} + \frac{1}{1-x} \sum_{n \geq 2} (-1)^{n-1} 2^n C_{n-1} x^n. \end{aligned}$$

Extracting the coefficient of x^n in the last expression yields (2.8) for $n \geq 2$, which is also seen to hold for $n = 1$. Finally, letting $a = 1$ in (2.5) gives

$$\frac{2x + 1 + \sqrt{1+4x}}{2} = 1 + x + x \left(\frac{1 - \sqrt{1+4x}}{-2x} \right)$$

$$= 1 + x + \sum_{n \geq 1} (-1)^{n-1} C_{n-1} x^n,$$

which implies (2.9). \square

Remark 2.4. Equation (2.6) is previously known (see [8, Corollary 8]), where an algebraic proof is given. In the last section, we provide combinatorial proofs of this formula and several others.

Let

$$e(x) = \sum_{n \geq 0} \det(a; C_2, \dots, C_{2n}) x^n$$

and

$$o(x) = \sum_{n \geq 0} \det(a; C_1, \dots, C_{2n-1}) x^n.$$

A similar argument to the proof of Theorem 2.2 yields the following result.

Theorem 2.5. *We have*

$$f_{-1}(x) = \frac{2a + 1 + \sqrt{1 + 4ax}}{2(a + 1 - x)}, \quad (2.10)$$

$$e(x) = \frac{a(\sqrt{1 + 4y} + \sqrt{1 - 4y})}{(a - 1)(\sqrt{1 + 4y} + \sqrt{1 - 4y}) + 2}, \quad (2.11)$$

$$o(x) = \frac{4a}{4a + 2 - \sqrt{1 + 4y} - \sqrt{1 - 4y}}, \quad (2.12)$$

where $y = \sqrt{-ax}$.

We note the following special cases of the prior formulas.

COROLLARY 2.6

If $n \geq 1$, then

$$\det(-1; C_0, \dots, C_{n-1}) = C_n, \quad (2.13)$$

$$\det(2; C_0, \dots, C_{n-1}) = \frac{1}{3^n} \left(1 + 2 \sum_{i=0}^{n-1} (-6)^i C_i \right), \quad (2.14)$$

$$\det(1; C_2, \dots, C_{2n}) = 2(-1)^{n-1} C_{2n-1}. \quad (2.15)$$

Proof. Taking $a = -1$ in (2.10) yields (2.13). A direct calculation shows that the g.f. of the right side of (2.14) over $n \geq 1$ is given by $\frac{2x-1+\sqrt{1+8x}}{2(3-x)}$. Adding 1 to this expression (to account for $n = 0$) yields $\frac{5+\sqrt{1+8x}}{2(3-x)}$, which corresponds to the $a = 2$ case of (2.10).

Finally, for (2.15), note that taking $a = 1$ in (2.11) gives

$$\begin{aligned} & \frac{\sqrt{1 + 4\sqrt{-x}} + \sqrt{1 - 4\sqrt{-x}}}{2} \\ &= 1 + 2x \left(\frac{1}{-2x} - \frac{\sqrt{1 + 4\sqrt{-x}} + \sqrt{1 - 4\sqrt{-x}}}{-4x} \right) \\ &= 1 + 2x \sum_{n \geq 0} C_{2n+1} \cdot (-x)^n = 1 + 2 \sum_{n \geq 1} (-1)^{n-1} C_{2n-1} x^n, \end{aligned}$$

where the second equality is obtained by replacing x by \sqrt{x} (and then x by $-x$) in the odd part of $C(x)$ which is given by

$$\sum_{n \geq 0} C_{2n+1} x^{2n+1} = \frac{1}{2x} - \frac{\sqrt{1 + 4x} + \sqrt{1 - 4x}}{4x}.$$

□

Remark 2.7. Formula (2.13) occurs as a special case of Proposition 6 in [4].

We have the following general formula for all a for the coefficients of $f_0(x)$ given by (2.4).

Theorem 2.8. *If $n \geq 1$, then*

$$\det(-a; C_1, \dots, C_n) = \frac{1}{n} \sum_{i=0}^{n-1} (n-i) a^i \binom{2n}{i}. \tag{2.16}$$

Proof. We compute the g.f. of the right side of (2.16) for $n \geq 1$. To do so, first note

$$\frac{n-i}{n} \binom{2n}{i} = \binom{2n}{i} - 2 \binom{2n-1}{i-1}, \quad i \geq 1,$$

so that

$$\begin{aligned} \frac{1}{n} \sum_{i=0}^{n-1} (n-i) a^i \binom{2n}{i} &= \sum_{i=0}^{n-1} a^i \binom{2n}{i} - 2 \sum_{i=1}^{n-1} a^i \binom{2n-1}{i-1} \\ &= \sum_{i=0}^{n-1} a^i \binom{2n}{i} - 2a \sum_{i=0}^{n-2} a^i \binom{2n-1}{i}. \end{aligned}$$

Let $A_n = \sum_{i=0}^{n-1} a^i \binom{2n}{i}$ and $B_n = \sum_{i=0}^{n-2} a^i \binom{2n-1}{i}$ for $n \geq 1$. By the recurrence for the binomial coefficients, we have

$$A_{n+1} = \sum_{i=0}^n a^i \binom{2n+2}{i} = 1 + \sum_{i=1}^n a^i \binom{2n+1}{i} + \sum_{i=1}^n a^i \binom{2n+1}{i-1}$$

$$\begin{aligned}
&= 1 + \sum_{i=1}^n a^i \binom{2n}{i} + \sum_{i=1}^n a^i \binom{2n}{i-1} + a + \sum_{i=1}^{n-1} a^{i+1} \binom{2n+1}{i} \\
&= a + 1 + \sum_{i=1}^n a^i \binom{2n}{i} + \sum_{i=0}^{n-1} a^{i+1} \binom{2n}{i} \\
&\quad + \sum_{i=1}^{n-1} a^{i+1} \binom{2n}{i} + \sum_{i=0}^{n-2} a^{i+2} \binom{2n}{i} \\
&= \sum_{i=0}^{n-1} a^i \binom{2n}{i} + a^n \binom{2n}{n} \\
&\quad + 2a \sum_{i=0}^{n-1} a^i \binom{2n}{i} + a^2 \sum_{i=0}^{n-1} a^i \binom{2n}{i} - a^{n+1} \binom{2n}{n-1} \\
&= (a+1)^2 A_n + a^n \binom{2n}{n} - a^{n+1} \binom{2n}{n-1} \\
&= (a+1)^2 A_n + a^n (1-a) \binom{2n}{n} + a^{n+1} C_n, \quad n \geq 1, \tag{2.17}
\end{aligned}$$

where we have used the fact $\binom{2n}{n-1} = \binom{2n}{n} - C_n$ in the last equality. Also, we have

$$\begin{aligned}
B_{n+1} &= \sum_{i=0}^{n-1} a^i \binom{2n+1}{i} = \sum_{i=0}^{n-1} a^i \binom{2n}{i} + \sum_{i=1}^{n-1} a^i \binom{2n}{i-1} \\
&= \sum_{i=0}^{n-1} a^i \binom{2n}{i} + \sum_{i=0}^{n-1} a^{i+1} \binom{2n}{i} - a^n \binom{2n}{n-1} \\
&= (a+1)A_n - a^n \binom{2n}{n-1}, \quad n \geq 1. \tag{2.18}
\end{aligned}$$

Let $A(x) = \sum_{n \geq 1} A_n x^n$ and $B(x) = \sum_{n \geq 1} B_n x^n$. We seek the difference $A(x) - 2aB(x)$. From (2.17), we get

$$\begin{aligned}
A(x) - x &= (a+1)^2 x A(x) + (1-a)x \left(\frac{1}{\sqrt{1-4ax}} - 1 \right) \\
&\quad + ax \left(\frac{1 - \sqrt{1-4ax}}{2ax} - 1 \right),
\end{aligned}$$

which implies

$$\begin{aligned}
(1 - (a+1)^2 x) A(x) &= \frac{(1-a)x}{\sqrt{1-4ax}} + \frac{1 - \sqrt{1-4ax}}{2} \\
&= \frac{(2a+2)x - 1 + \sqrt{1-4ax}}{2\sqrt{1-4ax}},
\end{aligned}$$

or

$$A(x) = \frac{(2a+2)x - 1 + \sqrt{1-4ax}}{2(1-(a+1)^2x)\sqrt{1-4ax}}.$$

From (2.18), we get

$$\begin{aligned} B(x) &= (a+1)x A(x) - x \sum_{n \geq 1} \left(\binom{2n}{n} - C_n \right) (ax)^n \\ &= (a+1)x A(x) - x \left(\frac{1}{\sqrt{1-4ax}} - \frac{1 - \sqrt{1-4ax}}{2ax} \right) \\ &= (a+1)x A(x) - \frac{1 - 2ax - \sqrt{1-4ax}}{2a\sqrt{1-4ax}}. \end{aligned}$$

Thus,

$$\begin{aligned} A(x) - 2aB(x) &= (1 - 2a(a+1)x)A(x) + \frac{1 - 2ax - \sqrt{1-4ax}}{\sqrt{1-4ax}} \\ &= \frac{(1-2a(a+1)x)(2(a+1)x - 1 + \sqrt{1-4ax}) + 2(1-(a+1)^2x)(1 - 2ax - \sqrt{1-4ax})}{2(1-(a+1)^2x)\sqrt{1-4ax}} \\ &= \frac{1 - 4ax - (1 - 2(a+1)x)\sqrt{1-4ax}}{2(1-(a+1)^2x)\sqrt{1-4ax}} = \frac{\sqrt{1-4ax} + 2(a+1)x - 1}{2(1-(a+1)^2x)}. \end{aligned}$$

Therefore, we have $A(x) - 2aB(x) + 1 = \frac{1-2a(a+1)x + \sqrt{1-4ax}}{2(1-(a+1)^2x)}$, which is the same expression as that given for $f_0(x)$ in (2.4) with a replaced by $-a$. Identity (2.16) now follows from the various definitions. \square

Remark 2.9. Comparing the $a = -1$ and $a = 1$ cases of (2.16) with (2.6) and (2.7), respectively, yields the following pair of Catalan number formulas:

$$C_n = \frac{(-1)^n}{n+1} \sum_{i=0}^n (-1)^i (n+1-i) \binom{2n+2}{i}, \quad n \geq 0, \quad (2.19)$$

$$C_n = \frac{1}{(n+1)(2n+1)} \sum_{i=0}^n (n+1-i) \binom{2n+2}{i}, \quad n \geq 0. \quad (2.20)$$

Note that (2.19) may be rewritten as

$$(-1)^n \binom{2n}{n} = \sum_{i=0}^n (-1)^i \binom{n+1-i}{1} \binom{2n+2}{i},$$

and thus corresponds to a particular case of [3, equation (5.25)]. Further identities may be obtained by comparing other expressions from Corollaries 2.3 and 2.6 above with specific cases of (2.16) (and of (2.21) below).

Analogous to Theorem 2.8, we have the following expression for the coefficients of $f_j(x)$ when $j = -1$.

Theorem 2.10. *If $n \geq 1$, then*

$$\det(-a; C_0, \dots, C_{n-1}) = \frac{1}{n} \sum_{i=0}^{n-1} (n-i)a^i \binom{n-1+i}{i}. \quad (2.21)$$

Proof. We compute the g.f. of the right side of (2.21) for $n \geq 1$. First, note

$$\frac{n-i}{n} \binom{n-1+i}{i} = \binom{n+i}{n} - 2 \binom{n-1+i}{n}$$

so that

$$\begin{aligned} \frac{1}{n} \sum_{i=0}^{n-1} (n-i)a^i \binom{n-1+i}{i} &= \sum_{i=0}^{n-1} a^i \binom{n+i}{n} - 2 \sum_{i=1}^{n-1} a^i \binom{n-1+i}{n} \\ &= \sum_{i=0}^{n-1} a^i \binom{n+i}{n} - 2 \sum_{i=0}^{n-2} a^{i+1} \binom{n+i}{n} \\ &= a^{n-1} \binom{2n-1}{n-1} - (2a-1)A_n, \end{aligned}$$

where $A_n = \sum_{i=0}^{n-2} a^i \binom{n+i}{n}$ for $n \geq 2$, with $A_1 = 0$. Now observe

$$\begin{aligned} A_{n+1} - A_n &= a^{n-1} \binom{2n}{n-1} + \sum_{i=0}^{n-2} a^i \left[\binom{n+1+i}{i} - \binom{n+i}{i} \right] \\ &= a^{n-1} \binom{2n}{n-1} + \sum_{i=1}^{n-2} a^i \binom{n+i}{i-1} \\ &= a^{n-1} \binom{2n}{n-1} + \sum_{i=0}^{n-3} a^{i+1} \binom{n+1+i}{i} \\ &= a^{n-1} \binom{2n}{n-1} + aA_{n+1} - a^n \binom{2n}{n-1} \\ &\quad - a^{n-1} \binom{2n-1}{n-2}, \quad n \geq 2, \end{aligned}$$

which may be rewritten as

$$\begin{aligned} (a-1)A_{n+1} + A_n &= a^{n-1}(a-1) \binom{2n}{n-1} + a^{n-1} \binom{2n-1}{n+1} \\ &= a^{n-1}(a-1) \left(\binom{2n}{n} - C_n \right) + a^{n-1} \left(\frac{1}{2} \binom{2n}{n} - C_n \right) \\ &= \left(a^n - \frac{1}{2} a^{n-1} \right) \binom{2n}{n} - a^n C_n, \quad n \geq 1. \quad (2.22) \end{aligned}$$

Let $A(x) = \sum_{n \geq 1} A_n x^n$. Then (2.22) implies

$$\begin{aligned} \left(\frac{a-1}{x} + 1\right) A(x) &= \left(1 - \frac{1}{2a}\right) \left(\frac{1}{\sqrt{1-4ax}} - 1\right) - \frac{1 - \sqrt{1-4ax}}{2ax} + 1 \\ &= \frac{2a-1}{2a} \left(\frac{1 - \sqrt{1-4ax}}{\sqrt{1-4ax}}\right) + \frac{2ax-1 + \sqrt{1-4ax}}{2ax} \\ &= \frac{1 - (2a+1)x + (x-1)\sqrt{1-4ax}}{2ax\sqrt{1-4ax}} \end{aligned}$$

so that

$$A(x) = \frac{1 - (2a+1)x + (x-1)\sqrt{1-4ax}}{2a(a-1+x)\sqrt{1-4ax}}.$$

Thus, the desired g.f. is given by

$$\begin{aligned} \sum_{n \geq 1} a^{n-1} \binom{2n-1}{n-1} x^n - (2a-1)A(x) &= \frac{1}{2a} \sum_{n \geq 1} \binom{2n}{n} (ax)^n - (4a-1)A(x) \\ &= \frac{1}{2a} \left(\frac{1}{\sqrt{1-4ax}} - 1\right) - (2a-1)A(x) \\ &= \frac{(a-1+x)(1 - \sqrt{1-4ax}) - (2a-1) + (4a^2-1)x - (2a-1)(x-1)\sqrt{1-4ax}}{2a(a-1+x)\sqrt{1-4ax}} \\ &= \frac{4a^2x - a + a(1-2x)\sqrt{1-4ax}}{2a(a-1+x)\sqrt{1-4ax}} = \frac{4ax - 1 + (1-2x)\sqrt{1-4ax}}{2(a-1+x)\sqrt{1-4ax}} \\ &= \frac{1-2x - \sqrt{1-4ax}}{2(a-1+x)}. \end{aligned}$$

Adding 1 to this last expression yields $\frac{2a-1-\sqrt{1-4ax}}{2(a-1+x)}$, which is the same as that given for $f_{-1}(x)$ in (2.10) with a replaced by $-a$, and thus implies (2.21). \square

We obtain the following binomial identity as a consequence of the prior results.

COROLLARY 2.11

We have

$$\begin{aligned} \det(2; C_1, \dots, C_n) &= \frac{1}{n} \sum_{i=0}^{n-1} (n-i)(-2)^i \binom{2n}{i} \\ &= \frac{(-1)^{n-1}}{n} \sum_{i=0}^{n-1} (n-i)2^i \binom{n-1+i}{i}, \quad n \geq 1, \end{aligned} \tag{2.23}$$

$$\det(2; C_3, \dots, C_{n+2}) = \frac{(-1)^{n-1}}{n} \sum_{i=0}^{n-1} (n-i)2^i \binom{n-1+i}{i}, \quad n \geq 2. \quad (2.24)$$

Proof. To show formula (2.23), note that by Theorem 2.10, the g.f. of the quantity $\frac{(-1)^{n-1}}{n} \sum_{i=0}^{n-1} (n-i)2^i \binom{n-1+i}{i}$ for $n \geq 1$ can be obtained by replacing x by $-x$ in the $a = -2$ case of $f_{-1}(x) - 1$ and then multiplying the expression that results by -1 . Doing so gives $\frac{2x+1-\sqrt{1+8x}}{2(x-1)}$, which corresponds to the $a = 2$ case of $f_0(x) - 1$.

By $i = a = 2$ case of Theorem 2.2, we have

$$\sum_{n \geq 0} \det(2; C_3, \dots, C_{n+2})x^n = \frac{2}{C_2(-2x)} = \frac{2}{C(-2x)^2 + C(-2x)^3}.$$

By (2.23) and the formula for $f_0(x)$ when $a = 2$, to establish (2.24), it suffices to show that

$$\frac{2}{C(-2x)^2 + C(-2x)^3} - 1 - 5x = \frac{4x - 1 - \sqrt{1+8x}}{2(x-1)} - 1 - x.$$

By the fact $zC(z)^2 = C(z) - 1$, the last equality is equivalent to

$$\begin{aligned} \frac{-4x}{C(-2x)^2 - 1} &= \frac{-4x}{C(-2x) - 1 + C(-2x)(C(-2x) - 1)} \\ &= \frac{4x - 1 - \sqrt{1+8x}}{2(x-1)} + 4x, \end{aligned}$$

or

$$\frac{8x^2}{2x + C(-2x) - 1} = \frac{8x^2 - 4x - 1 - \sqrt{1+8x}}{2(x-1)}.$$

Substituting $C(-2x) = \frac{1-\sqrt{1+8x}}{-4x}$, one may verify the validity of the last equation, which completes the proof of (2.24). \square

Remark 2.12. Comparing the generating functions, one sees that only when $a = 2$ is there an identity of the form

$$\frac{1}{n} \sum_{i=0}^{n-1} (n-i)(-a)^i \binom{2n}{i} = \frac{(-1)^{n-1}}{n} \sum_{i=0}^{n-1} (n-i)a^i \binom{n-1+i}{i}, \quad n \geq 1.$$

3. Multinomial extension of the Catalan determinants

The Trudi formula (1.2), taken together with Corollaries 2.3, 2.6, 2.11 and Theorems 2.8 and 2.10, yields the following result.

Theorem 3.1. Let $n \geq 1$, except when noted otherwise, and $a \neq 0$. Then

$$\begin{aligned} \sum_{\sigma_n=n} (-1)^{|\sigma|} p_n(s) C_1^{s_1} C_2^{s_2} \cdots C_n^{s_n} &= -C_{n-1}, \\ \sum_{\sigma_n=n} p_n(s) C_1^{s_1} C_2^{s_2} \cdots C_n^{s_n} &= (2n-1)C_{n-1}, \\ \sum_{\sigma_n=n} (-1)^{|\sigma|} p_n(s) \left(\frac{C_1}{2}\right)^{s_1} \left(\frac{C_2}{2}\right)^{s_2} \cdots \left(\frac{C_n}{2}\right)^{s_n} \\ &= \frac{1}{(-2)^n} \left(1 + \sum_{i=1}^{n-1} (-1)^i 2^{i+1} C_i\right), \\ \sum_{\sigma_n=n} (-1)^{|\sigma|} p_n(s) C_2^{s_1} C_3^{s_2} \cdots C_{n+1}^{s_n} &= -C_{n-1}, \quad n \geq 2, \\ \sum_{\sigma_n=n} p_n(s) C_0^{s_1} C_1^{s_2} \cdots C_{n-1}^{s_n} &= C_n, \\ \sum_{\sigma_n=n} (-1)^{|\sigma|} p_n(s) \left(\frac{C_0}{2}\right)^{s_1} \left(\frac{C_1}{2}\right)^{s_2} \cdots \left(\frac{C_{n-1}}{2}\right)^{s_n} \\ &= \frac{1}{(-6)^n} \left(1 + 2 \sum_{i=0}^{n-1} (-6)^i C_i\right), \\ \sum_{\sigma_n=n} (-1)^{|\sigma|} p_n(s) C_2^{s_1} C_4^{s_2} \cdots C_{2n}^{s_n} &= -2C_{2n-1}, \\ \sum_{\sigma_n=n} (-1)^{|\sigma|} p_n(s) \left(\frac{C_1}{2}\right)^{s_1} \left(\frac{C_2}{2}\right)^{s_2} \cdots \left(\frac{C_n}{2}\right)^{s_n} &= \frac{1}{n} \sum_{i=0}^{n-1} (n-i)(-2)^{i-n} \binom{2n}{i} \\ &= -\frac{1}{n} \sum_{i=0}^{n-1} (n-i) 2^{i-n} \binom{n-1+i}{i}, \\ \sum_{\sigma_n=n} (-1)^{|\sigma|} p_n(s) \left(\frac{C_3}{2}\right)^{s_1} \left(\frac{C_4}{2}\right)^{s_2} \cdots \left(\frac{C_{n+2}}{2}\right)^{s_n} \\ &= -\frac{1}{n} \sum_{i=0}^{n-1} (n-i) 2^{i-n} \binom{n-1+i}{i}, \quad n \geq 2, \\ \sum_{\sigma_n=n} p_n(s) \left(\frac{C_1}{a}\right)^{s_1} \left(\frac{C_2}{a}\right)^{s_2} \cdots \left(\frac{C_n}{a}\right)^{s_n} &= \frac{1}{n} \sum_{i=0}^{n-1} (n-i) a^{i-n} \binom{2n}{i}, \\ \sum_{\sigma_n=n} p_n(s) \left(\frac{C_0}{a}\right)^{s_1} \left(\frac{C_1}{a}\right)^{s_2} \cdots \left(\frac{C_{n-1}}{a}\right)^{s_n} &= \frac{1}{n} \sum_{i=0}^{n-1} (n-i) a^{i-n} \binom{n-1+i}{i}, \end{aligned}$$

where $|\sigma| = s_1 + \cdots + s_n$, $\sigma_n = s_1 + 2s_2 + \cdots + ns_n$ and $p_n(s) = \frac{|s|!}{s_1! \cdots s_n!}$.

4. Combinatorial proofs

Recall that the determinant of an $n \times n$ matrix $A = (a_{i,j})$ is given by

$$\det(A) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{\text{sgn}(\sigma)} a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)},$$

where $\text{sgn}(\sigma)$ denotes the sign of the permutation σ . Assume that the permutations are expressed in the *standard cycle form*, i.e., within each cycle, the smallest element is first, with cycles arranged from left to right in increasing order of smallest elements. Suppose A is a Toeplitz–Hessenberg matrix of size n . Then the only permutations α that contribute to the determinant sum are those in which each cycle comprises a set of consecutive integers in increasing order. Such α may clearly be regarded as compositions of n , upon identifying the cycle lengths as parts. So if α is such a permutation, then each part of size i in the corresponding composition (which we will also denote by α) is weighted by $a_0^{i-1} a_i$.

Furthermore, observe that the sign of the composition α is the same as that of the associated permutation and is given by $(-1)^{n-\mu(\alpha)}$, where $\mu(\alpha)$ denotes the number of parts of α . Thus, in the case when A is Toeplitz–Hessenberg of size n , the determinant is a signed sum over weighted compositions α of n , where the sign of α is as stated, and the weight of α is the product of the weights of its individual parts with a part of size i having weight $a_0^{i-1} a_i$ for $i \geq 1$. Note that in the case when $a_0 = -1$, the sign of α is the same as the product of the factors a_0^{i-1} taken over all of the parts of α . Thus, $\det(-1; a_1, \dots, a_n)$ is a positive sum of weighted compositions of n where the weight of a composition (x_1, \dots, x_r) is given by $\prod_{i=1}^r a_{x_i}$.

In this section, we provide combinatorial proofs of some of the prior explicit formulas for the determinant using a lattice path interpretation. For combinatorial proofs of other Catalan determinant formulas, see, e.g., [1,6]. Recall that a *Dyck path* of semilength n where $n > 0$ is a lattice path starting at the origin and ending at $(2n, 0)$ that never dips below the x -axis and contains two types of steps – an upstep $U = (1, 1)$ and a downstep $D = (1, -1)$. Let \mathcal{C}_n denote the set of Dyck paths of semilength n . It is well-known that the cardinality of \mathcal{C}_n is given by C_n (see [12] for a complete list of structures enumerated by the Catalan numbers).

Recall that $\rho \in \mathcal{C}_n$ for $n \geq 1$ can be decomposed as $\rho = \rho_1 \rho_2 \cdots \rho_s$ for some $s \geq 1$ where each ρ_i , $1 \leq i \leq s$, is of the form $\rho_i = U \rho'_i D$ with ρ'_i a (possibly empty) Dyck path. The ρ_i are often referred to as the *components* or *units* of ρ . A *return* of ρ refers to a downstep ending at $(2m, 0)$ for some $1 \leq m \leq n$, with an *internal return* one in which $m < n$. Note that a component is a section of ρ lying between two successive returns (counting the starting point as a return).

Combinatorial proofs of (2.6), (2.7) and (2.13). We first show (2.13). To do so, first observe that the expansion of $\det(-1; C_0, \dots, C_n)$ in terms of permutations contains only positive terms. Thus, $\det(-1; C_0, \dots, C_{n-1})$ gives the *cardinality* of the set of sequences of ordered pairs $(a_1, \lambda_1), (a_2, \lambda_2), \dots$ such that $a_1 + a_2 + \cdots = n$, $a_i > 0$ and $\lambda_i \in \mathcal{C}_{a_i-1}$ for all i . We map such a sequence of ordered pairs to $U \lambda_1 D U \lambda_2 D \cdots$, which is seen to be the decomposition of some member of \mathcal{C}_n according to its units. Thus, the mapping is reversible and hence a bijection, which implies (2.13).

To show (2.7), let \mathcal{A}_n consist of sequences of ordered pairs (a_i, λ_i) as in the prior proof except that $\lambda_i \in \mathcal{C}_{a_i}$ for all i . Note that $\det(-1; C_1, \dots, C_n)$ gives the cardinality of \mathcal{A}_n . Let \mathcal{B}_n consist of lattice paths from $(0, 0)$ to $(2n, 0)$ using $(1, 1)$ and $(1, -1)$ steps such that the last step is $(1, -1)$. By symmetry, $|\mathcal{B}_n| = \frac{1}{2} \binom{2n}{n}$. Note that $\frac{1}{2} \binom{2n}{n} = \binom{2n-1}{n} = (2n-1)C_{n-1}$,

so to complete the proof of (2.7), it suffices to define a bijection between \mathcal{A}_n and \mathcal{B}_n . Let $\lambda = (a_1, \lambda_1), \dots, (a_r, \lambda_r) \in \mathcal{A}_n$. For each $1 \leq i \leq r$, let λ'_i be obtained from λ_i by reflecting all of λ_i except its last component in the x -axis. Note that the first step of λ'_i is $(1, 1)$ if and only if λ_i has only one component, in which case $\lambda'_i = \lambda_i$. Let $\lambda' = \lambda'_1 \cup \lambda'_2 \cup \dots \cup \lambda'_r$, where \cup denotes concatenation (of lattice paths). One may verify that $\lambda' \in \mathcal{B}_n$ and that the mapping $\lambda \mapsto \lambda'$ is reversible, upon considering the number of components of λ' lying above the x -axis.

Finally, to show (2.6), let \mathcal{A}_n be as before but now with the sign of $\lambda \in \mathcal{A}_n$ given by $(-1)^{n-r}$, where r denotes the number of pairs. Then $\det(1; C_1, \dots, C_n)$ gives the sum of the signs of all members of \mathcal{A}_n , by definition. We apply the mapping $\lambda \mapsto \lambda'$ from above to members of \mathcal{A}_n . Note that the sign of $\rho \in \mathcal{B}_n$ equals $(-1)^{n-\mu(\rho)}$, where $\mu(\rho)$ denotes the number of components of ρ lying above the x -axis. Define an involution on \mathcal{B}_n by considering the penultimate component of ρ , if it exists, and reflecting it in the x -axis. Note that this operation reverses the sign (since the μ value changes by one) and fails to be defined for those members of \mathcal{B}_n containing a single (positive) component. Such members of \mathcal{B}_n clearly number C_{n-1} with each having sign $(-1)^{n-1}$, which implies (2.6). \square

Proof of (2.9). Let \mathcal{D}_n denote the set of sequences of ordered pairs $\lambda = (a_1, \lambda_1), \dots, (a_r, \lambda_r)$, where $1 \leq r \leq n$ such that $a_1 + \dots + a_r = n$, $a_i > 0$ and $\lambda_i \in \mathcal{C}_{a_i+1}$ for all i . Define the sign of λ to be $(-1)^{n-r}$. Then $\det(1; C_2, \dots, C_{n+1})$ gives the sum of the signs of all members of \mathcal{D}_n . One may regard members of \mathcal{D}_n alternatively as follows. Given $\pi \in \mathcal{C}_n$, we mark r of the returns to the x -axis, including the final return, by a dot (\bullet). Also, put a dot just prior to the first step of π (so that there are $r + 1$ dots altogether). Between any two dots, we place exactly one \times and it either marks one of the internal returns of the subpath or goes at the beginning or the end. If the \times is put at the beginning of the subpath, then it goes to the right of the initial dot, while if the \times is put at the end, then it goes to the left of the terminal dot. Thus, between any two \bullet there is exactly one \times , and hence it is possible for there to be an \times both directly to the left and to the right of the same \bullet marking an internal return of π . Let \mathcal{E}_n denote the set of such marked Dyck paths of semilength n and define the sign as $(-1)^{n-r}$, where r is the number of \times 's.

Note that a member $\alpha \in \mathcal{E}_n$ may be expressed as $\alpha = \alpha_1 \cdots \alpha_r$, where α_i is the subpath of α lying between the i -th and $(i + 1)$ -st \bullet (recall that the initial and final positions of α always correspond to the first and to the last \bullet). The α_i may be decomposed further as $\alpha_i = \beta_i \gamma_i$, where β_i denotes the section of α_i to the left of the return marked by \times and γ_i the remainder of α_i . Note that it is possible for either β_i or γ_i to be empty. Consider replacing α_i with $\beta_i U \gamma_i D$ for each i (that is, we raise the section γ_i vertically by one unit) and then form the sequence of ordered pairs $(|\alpha_1|, \beta_1 U \gamma_1 D), (|\alpha_2|, \beta_2 U \gamma_2 D), \dots$. This operation is seen to define a sign-preserving bijection between \mathcal{E}_n and \mathcal{D}_n which can be reversed by considering the final unit within the lattice path component of each ordered pair. Thus, the sum of the signs of all the members of \mathcal{E}_n is given by $\det(1; C_2, \dots, C_{n+1})$.

To complete the proof, it suffices to identify a sign-changing involution on \mathcal{E}_n , the set of survivors of which has sum of signs equal to $(-1)^{n-1} C_{n-1}$. Let $\mathcal{E}'_n \subseteq \mathcal{E}_n$ comprise those members in which $r = 1$ with no internal returns or $r = 2$ with exactly one internal return, where this return is marked only by \bullet . Note that the sum of the signs of members of \mathcal{E}'_n equals $2(-1)^{n-1} C_{n-1} + (-1)^{n-2} C_{n-1} = (-1)^{n-1} C_{n-1}$ as $C_{n-1} = \sum_{i=0}^{n-2} C_i C_{n-2-i}$ for $n \geq 2$ in the latter case (one can define a bijection between this case and part of the former case, if desired). Within a member of $\mathcal{E}_n - \mathcal{E}'_n$, consider the leftmost internal return R that is *not* marked only by \bullet (note that if $r > 2$, such an internal return always exists). Then

R is either marked by $\bullet \times$, $\times \bullet$, $\times \bullet \times$ or \times , or is unmarked altogether. If R is marked by $\times \bullet \times$, then replace with \times , and vice versa, while if R is marked by $\bullet \times$ or $\times \bullet$, then delete the designation and take R to be unmarked altogether, and vice versa. Note that in the latter case when R is unmarked and one is adding back the designation, the choice of either $\bullet \times$ or $\times \bullet$ is dictated by where an \times is needed. Combining the two operations is then seen to define an involution on $\mathcal{E}_n - \mathcal{E}'_n$. Since this involution always reverses the sign, the proof of (2.9) is complete. \square

Proof of (2.15). Let \mathcal{F}_n denote the set of ‘marked’ Dyck paths of semilength $2n$ wherein some subset of the even returns to the x -axis is marked including the beginning and the end (by an *even/odd* return, we mean a downstep ending at a point $(4i, 0)/(4i - 2, 0)$ for some $1 \leq i \leq n$). Note that if exactly $s - 1$ of the internal even returns are marked within $\rho \in \mathcal{F}_n$ where $1 \leq s \leq n$, then ρ is divided into s sections by the markings and we define the sign of ρ to be $(-1)^{n-s}$. Upon splitting the various ρ into sections (each of positive even semilength) according to the markings, one sees that $\det(1; C_2, \dots, C_{2n})$ gives the sum of the signs of all members of \mathcal{F}_n .

We now define an involution on \mathcal{F}_n as follows. First, let $\mathcal{F}'_n \subseteq \mathcal{F}_n$ comprise those members in which all internal returns are odd (with this condition holding vacuously for paths having only one component). Note that the sign of each member of \mathcal{F}'_n is $(-1)^{n-1}$. Define an involution on $\mathcal{F} - \mathcal{F}'_n$ by considering the leftmost even internal return and either marking it or removing the marking from it. To complete the proof of (2.15), we must show that $|\mathcal{F}'_n| = 2C_{2n-1}$. Since there are clearly C_{2n-1} paths with exactly one component, this amounts to showing that there are C_{2n-1} members $\rho \in \mathcal{F}_n$ in which every internal return is odd with at least one such return. These ρ may be decomposed as $\rho = \alpha_1 \beta_1 \cdots \beta_r \alpha_2$, where α_1 and α_2 are components of ρ of odd semilength, $r \geq 0$ and each β_i is a component of even semilength. Let α'_1 be obtained from α_1 by removing the initial $(1, 1)$ step as well as the final $(1, -1)$ step. Let $\rho' = \alpha'_1 \alpha_2 \beta_1 \cdots \beta_r \in C_{2n-1}$. Then the mapping $\rho \mapsto \rho'$ is seen to define a bijection with C_{2n-1} , as desired, which can be reversed upon considering the rightmost component of odd semilength within a member of C_{2n-1} . \square

References

- [1] Benjamin A T, Cameron N T, Quinn J J and Yerger C R, Catalan determinants – A combinatorial approach, *Congr. Numer.* **200** (2010) 27–34
- [2] Goy T, On new Catalan identities using Toeplitz–Hessenberg matrices, Abstracts of the 11th International Algebraic Conference in Ukraine (2017)
- [3] Graham R L, Knuth D E and Patashnik O, *Concrete Mathematics: A Foundation for Computer Science*, second edition (1994) (Boston: Addison-Wesley)
- [4] Janjić M, Determinants and recurrence sequences, *J. Integer Seq.* **15** (2012) Art. 12.3.5
- [5] Koshy T, *Catalan Numbers with Applications* (2008) (Oxford: Oxford University Press)
- [6] Mays M E and Wojciechowski J, A determinant property of Catalan numbers, *Discrete Math.* **211** (2000) 125–133
- [7] Merca M, A note on the determinant of a Toeplitz–Hessenberg matrix, *Spec. Matrices* **1** (2013) 10–16
- [8] Merca M, A generalization of the symmetry between complete and elementary symmetric functions, *Indian J. Pure. Appl. Math.* **45(1)** (2014) 75–89
- [9] Muir T, *The Theory of Determinants in the Historical Order of Development* (1960) (Mineola: Dover Publications) vol. 3
- [10] Roman S, *An Introduction to Catalan Numbers* (2015) (Basel: Birkhauser)

- [11] Sloane N J A ed., The On-Line Encyclopedia of Integer Sequences, published electronically at <https://oeis.org>
- [12] Stanley R P, Catalan Numbers (2015) (Cambridge, UK: Cambridge University Press)
- [13] Wilf H S, generatingfunctionology (1994) (Cambridge, MA: Academic Press)

COMMUNICATING EDITOR: Parameswaran Sankaran