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Generalized reversed Jensen-Steffensen and related inequalities

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We compare two linear functionals that are negative on convex functions. Further, using Green's functions we give some new conditions for reversed Jensen-Steffensen and related inequalities to hold. Using Green's function we also give refinement of Levinson type generalization of reversed Jensen-Steffensen and related inequalities. The acquired results are then used for constructing mean-value theorems.

Key words and phrases: Jensen-Mercer inequality, Jensen-Steffensen inequality.

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1 Introduction and preliminaries

Fundamental articles regarding convex functions originate by the turn of the 19th century, however, the concrete definition of the convex functions has been progressed by the efforts of the Danish mathematician and engineer J.L.W.V. Jensen [7, 8] from 1905, and his well known inequality. Many years later, the Jensen inequality was contemplated on the weakened conditions, additionally refined, improved, reversed, generalized, etc., and it is yet a consent of stimulus for further inquiry. By weakening the conditions of the Jensen inequality, the reversed Jensen, Jensen-Stefensen and reversed Jensen-Stefensen inequalities were obtained. H.D. Brunk generalized it to a great extent in [4] and his result is known as the Jensen-Brunk inequality also reverse of Jensen-Brunk inequality was given by J.E. Pečarić et. al. (see [11, p. 85]). One more generalization is the Jensen-Boas inequality and the reverse Jensen-Boas inequality (see [3] and [11, p. 86]). All these inequalities and many other results can be found in [11], an outstanding book regarding convex functions. Moreover, many other celebrated inequalities are obtained by making use of the Jensen inequality, like the Hölder inequality, the Cauchy inequality, inequalities between means, and the Young inequality to mention but a few. The power and significance of the Jensen inequality can not be measured because the applications of the aforementioned inequalities are broadly spread in all domains. An interested scholar can view various articles, which apply this inequality (see, for example, [6–18]).

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Here, and in the rest of the article, $[\kappa, \lambda]$ and $[\gamma, \delta]$ are intervals in $\mathbb{R}$, and $\kappa < \lambda$, $\gamma < \delta$. In order to simplify the notation, throughout this paper we use the following notation

$$\overline{\zeta} = \frac{1}{P_n} \sum_{i=1}^{n} p_i \zeta_i \quad \text{and} \quad \overline{\mu} = \frac{1}{\int_\kappa^\lambda d\psi(t)} \int_\kappa^\lambda \mu(t)\, d\psi(t).$$

To begin with, we give a discrete form of Jensen inequality [11, p. 43] (see also [9, p. 6]).

Proposition 1. *Let $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n)$ be n-tuple in $[\gamma, \delta]^n$ and $\boldsymbol{p} = (p_1, \ldots, p_n)$ be nonnegative n-tuple such that $P_n := p_1 + \cdots + p_n > 0$. Then for any convex function $\Upsilon : [\gamma, \delta] \to \mathbb{R}$ we get*

$$\Upsilon\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i \zeta_i\right) \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i \Upsilon(\zeta_i), \tag{1}$$

where

$$P_j = \sum_{i=1}^{j} p_i, \quad j \in \{1, \ldots, n\}. \tag{2}$$

If Υ is strictly convex, then inequality in (1) *is strict, except when $\zeta_1 = \cdots = \zeta_n$.*

Jensen-Steffensen inequality is proved by J.F. Steffensen in [17] (see also [11, p. 57]).

Proposition 2. *Let $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n)$ be a monotonic n-tuple in $[\gamma, \delta]^n$ and $\boldsymbol{p} = (p_1, \ldots, p_n)$ be a real n-tuple such that $\overline{\zeta} \in [\gamma, \delta]$ and the inequalities*

$$0 \leq P_j \leq P_n, \;\; 1 \leq j \leq n, \quad P_n > 0,$$

are satisfied, where P_j are as in (2). *Then for any convex function $\Upsilon : [\gamma, \delta] \to \mathbb{R}$ the inequality* (1) *still holds.*

For later use, let us introduse the Jensen-Steffensen functional $A_{\boldsymbol{\zeta},\boldsymbol{p}}$ with

$$A_{\boldsymbol{\zeta},\boldsymbol{p}}(\Upsilon) = \frac{1}{P_n} \sum_{i=1}^{n} p_i \Upsilon(\zeta_i) - \Upsilon\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i \zeta_i\right).$$

Obviously, (1) can be expressed by $A_{\boldsymbol{\zeta},\boldsymbol{p}}(\Upsilon) \geq 0$.

The integral version of Jensen-Steffensen inequality was established by J.F. Steffensen in [17], however here we acknowledged a variant established by R.P. Boas [3] (see also [11, p. 58]).

Proposition 3. *Let $\mu : [\kappa, \lambda] \longrightarrow [\gamma, \delta]$ be a continuous and monotonic function. Let $\psi : [\kappa, \lambda] \longrightarrow \mathbb{R}$ be either continuous or of bounded variation function such that $\overline{\mu} \in [\gamma, \delta]$ and satisfying*

$$\psi(\kappa) \leq \psi(t) \leq \psi(\lambda) \qquad \text{for all } t \in [\kappa, \lambda], \qquad \psi(\lambda) - \psi(\kappa) > 0.$$

Then for any continuous convex function $\Upsilon : [\gamma, \delta] \longrightarrow \mathbb{R}$ the inequality

$$\Upsilon\left(\frac{1}{\int_\kappa^\lambda d\psi(t)} \int_\kappa^\lambda \mu(t) d\psi(t)\right) \leq \frac{1}{\int_\kappa^\lambda d\psi(t)} \int_\kappa^\lambda \Upsilon(\mu(t)) d\psi(t) \tag{3}$$

holds.

Here equivalently we represent the Jensen-Boas functional $B^{\mu,\psi}_{\kappa,\lambda}$ with

$$B^{\mu,\psi}_{\kappa,\lambda}(\Upsilon) = \frac{1}{\int_\kappa^\lambda d\psi(t)} \int_\kappa^\lambda \Upsilon(\mu(t))d\psi(t) - \Upsilon\left(\frac{1}{\int_\kappa^\lambda d\psi(t)} \int_\kappa^\lambda \mu(t)d\psi(t)\right).$$

R.P. Boas [3] also proved a generalization of Proposition 3, so called Jensen-Boas inequality (see also [13] and [11, p. 59]).

Proposition 4. *Let $\psi : [\kappa, \lambda] \longrightarrow \mathbb{R}$ be continuous or of bounded variation function satisfying*

$$\psi(\kappa) \leq \psi(t_1) \leq \psi(s_1) \leq \psi(t_2) \leq \cdots \leq \psi(s_{n-1}) \leq \psi(t_n) \leq \psi(\lambda)$$

for all $t_k \in (s_{k-1}, s_k)$, where $s_0 = \kappa$, $s_n = \lambda$ and $\psi(\kappa) < \psi(\lambda)$. If a function μ is continuous and monotonic (either increasing or decreasing) in each of the $n-1$ intervals (s_{k-1}, s_k) and $\overline{\mu} \in [\gamma, \delta]$, then for every continuous convex function $\Upsilon : [\gamma, \delta] \longrightarrow \mathbb{R}$ inequality (3) *holds.*

The following Jensen-Brunk inequality is also the generalization of Jensen-Steffensen inequality (see [4]).

Proposition 5. *Let $\mu : [\kappa, \lambda] \longrightarrow [\gamma, \delta]$ be a continuous and increasing function. Let $\psi : [\kappa, \lambda] \longrightarrow \mathbb{R}$ be continuous or of bounded variation function, such that $\overline{\mu} \in [\gamma, \delta]$ and satisfying $\psi(\kappa) < \psi(\lambda)$. Then for every continuous convex function $\Upsilon : [\gamma, \delta] \longrightarrow \mathbb{R}$ inequality* (3) *holds if and only if*

$$\int_\kappa^s (\mu(s) - \mu(t))\, d\psi(t) \geq 0 \quad \text{and} \quad \int_s^\lambda (\mu(s) - \mu(t))\, d\psi(t) \leq 0$$

for all $s \in [\kappa, \lambda]$.

J.E. Pečarić in his article [12] used the Fuchs generalization of majorization theorem [5] to establish the necessary and sufficient condition for the validity of following reversed Jensen-Steffensen inequality (see also [11, p. 83] and [9, p. 6]).

Proposition 6. *Let $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n)$ be monotonic n-tuple in $[\gamma, \delta]^n$ and $\boldsymbol{p} = (p_1, \ldots, p_n)$ be a real n-tuple such that $P_n > 0$, $\overline{\zeta} \in [\gamma, \delta]$, and there exists $l \in \{1, \ldots, n\}$ such that*

$$P_j \leq 0 \quad \text{for } j < l \quad \text{and} \quad \overline{P}_j \leq 0 \quad \text{for } j > l,$$

where P_j are as in (2) *and*

$$\overline{P}_j = \sum_{i=j}^n p_i, \quad j \in \{1, \ldots, n\}.$$

Then for any convex function $\Upsilon : [\gamma, \delta] \to \mathbb{R}$ the inequality

$$\Upsilon\left(\frac{1}{P_n}\sum_{i=1}^n p_i\zeta_i\right) \geq \frac{1}{P_n}\sum_{i=1}^n p_i\Upsilon(\zeta_i) \tag{4}$$

holds.

J.E. Pečarić also established the following integral version of reversed Jensen-Steffensen inequality in [14], which is a generalization of an inequality of R.E. Barlow et. al. in [2].

Proposition 7. *Let $\mu : [\kappa, \lambda] \longrightarrow [\gamma, \delta]$ be a continuous and monotonic function, and let $\psi : [\kappa, \lambda] \longrightarrow \mathbb{R}$ be a function of bounded variation such that $\overline{\mu} \in [\gamma, \delta]$ and $\psi(\lambda) > \psi(\kappa)$. If there exists $\vartheta \in [\kappa, \lambda]$ such that*

$$\psi(t) \le \psi(\kappa) \quad \text{for} \quad \kappa \le t \le \vartheta,$$
$$\psi(t) \ge \psi(\lambda) \quad \text{for} \quad \vartheta < t \le \lambda,$$

then for any continuous convex function $\Upsilon : [\gamma, \delta] \longrightarrow \mathbb{R}$ the inequality

$$\Upsilon\left(\frac{1}{\int_\kappa^\lambda d\psi(t)} \int_\kappa^\lambda \mu(t)\, d\psi(t)\right) \ge \frac{1}{\int_\kappa^\lambda d\psi(t)} \int_\kappa^\lambda \Upsilon(\mu(t))\, d\psi(t) \tag{5}$$

holds.

Remark 1. *In Proposition 7, the interval $[\kappa, \vartheta]$ and $(\vartheta, \lambda]$ can be recovered by $[\kappa, \vartheta)$ and $[\vartheta, \lambda]$, respectively.*

A reverse of Jensen-Boas inequality is given in [14] (see also [11, p. 86]).

Proposition 8. *Let $\mu : [\kappa, \lambda] \longrightarrow [\gamma, \delta]$ be a continuous function. Let a function $\psi : [\kappa, \lambda] \longrightarrow \mathbb{R}$ is of bounded variation such that $\overline{\mu} \in [\gamma, \delta]$ with $\psi(\lambda) > \psi(\kappa)$, and satisfies*

$$\begin{aligned} \psi(s_{m-1}) \le \psi(t_{m-1}) \le \psi(s_{m-2}) \le \cdots \le \psi(t_1) \le \psi(\kappa) < \psi(\lambda) \\ \le \psi(t_n) \le \cdots \le \psi(t_{m+1}) \le \psi(s_m) \end{aligned} \tag{6}$$

for all $t_k \in (s_{k-1}, s_k)$, where $k \ne m$, $m \in \{1, \dots, n\}$, $s_0 = \kappa$, $s_n = \lambda$, and there exists a $\vartheta \in [s_{m-1}, s_m]$ such that $\psi(t_m) \le \psi(s_{m-1})$ for every $t_m \in [s_{m-1}, \vartheta]$ and $\psi(t_m) \ge \psi(s_m)$ for every $t_m \in (\vartheta, s_m]$, provided that μ is continuous and monotonic (in either direction) in each of the $n-1$ intervals (s_{k-1}, s_k). Then for every continuous convex function $\Upsilon : [\gamma, \delta] \longrightarrow \mathbb{R}$ the inequality (5) *holds.*

The following generalization of reversed Jensen-Steffensen inequality is known as the reversed Jensen-Brunk inequality (see [11, p. 85]).

Proposition 9. *Let $\mu : [\kappa, \lambda] \longrightarrow [\gamma, \delta]$ be continuous and increasing function. Let $\psi : [\kappa, \lambda] \longrightarrow \mathbb{R}$ be a function of bounded variation such that $\overline{\mu} \in [\gamma, \delta]$ and satisfying $\psi(\kappa) < \psi(\lambda)$. Then for every continuous convex function $\Upsilon : [\gamma, \delta] \longrightarrow \mathbb{R}$ the inequality* (5) *holds if and only if*

$$\int_\kappa^s (\mu(s) - \mu(t))\, d\psi(t) \le 0 \ \text{ holds for every } s \in [\kappa, \lambda] \text{ such that } \bar{\mu} \ge \mu(s) \tag{7}$$

and

$$\int_s^\lambda (\mu(s) - \mu(t))\, d\psi(t) \ge 0 \ \text{ holds for every } s \in [\kappa, \lambda] \text{ such that } \bar{\mu} \le \mu(s). \tag{8}$$

Note, that if $\bar{\mu} < \mu(\kappa)$, the condition in (7) *is taken to be vacuous, and if $\bar{\mu} > \mu(\lambda)$, the condition in* (8) *is taken to be vacuous.*

This article is organized in the following manner. Next to the introduction, the section accompanying the main results of the article follows. We analyze the aforestated functionals under the class of 3-convex functions at a point that was established in [1] (our Theorems 1, 2 and 3). Further we give a unified analysis of inequalities of the reversed Jensen-Steffensen type, for the Green's function in Theorems 4 – 7. In the last section, these results are then used to establish Cauchy and Lagrange type mean-value theorems.

2 Main results

Throughout this section, for ease, we represent by e_n the functions $e_n(t) = t^n, n \in \mathbb{N}$. The underlaying class of functions is established in [1].

Definition 1. *Let $\Upsilon : [\gamma, \delta] \longrightarrow \mathbb{R}$ and $c \in (\gamma, \delta)$, where (γ, δ) is the interior of $[\gamma, \delta]$. We say that $\Upsilon \in \mathcal{K}_1^c([\gamma, \delta])$ (resp. $\Upsilon \in \mathcal{K}_2^c([\gamma, \delta])$) if there exists a constant K_Υ such that the function $\Psi(\zeta) = \Upsilon(\zeta) - \frac{K_\Upsilon}{2} e_2(\zeta)$ is concave (resp. convex) on $[\gamma, \delta] \cap (-\infty, c]$ and convex (resp. concave) on $[\gamma, \delta] \cap [c, \infty)$.*

Remark 2. *A function $\Upsilon \in \mathcal{K}_1^c([\gamma, \delta])$ is said to be 3-convex at point c. It was shown in [1] that a function Υ is 3-convex on an interval if and only if it is 3-convex at every point of interval. It was also shown in [1] that if $\Upsilon''(c)$ exists, then $K_\Upsilon = \Upsilon''(c)$.*

In the following theorem we analyse the reversed Jensen-Steffensen functionals on the reversed Jensen-Boas conditions.

Theorem 1. *Let $c \in (\gamma, \delta)$ and let $\mu : [\kappa_1, \lambda_1] \to [\gamma, \delta]$ and $\nu : [\kappa_2, \lambda_2] \to [\gamma, \delta]$ be continuous functions such that $\mu([\kappa_1, \lambda_1]) \subset [\gamma, c]$ and $\nu([\kappa_2, \lambda_2]) \subset [c, \delta]$. Let $\psi : [\kappa_1, \lambda_1] \to \mathbb{R}$ be a function of bounded variation such that $\bar{\mu} \in [\gamma, c]$ with $\psi(\kappa_1) < \psi(\lambda_1)$, and satisfying*

$$\begin{aligned}\psi(s_{l-1}) \leq \psi(t_{l-1}) \leq \psi(s_{l-2}) \leq \cdots \leq \psi(t_1) \leq \psi(\kappa_1) < \psi(\lambda_1) \\ \leq \psi(t_n) \leq \cdots \leq \psi(t_{m+1}) \leq \psi(s_m)\end{aligned}$$

for all $t_k \in (s_{k-1}, s_k)$, where $k \neq l$, $l \in \{1, \ldots, n\}$, $s_0 = \kappa_1$, $s_n = \lambda_1$, and there exists a $\vartheta \in [s_{l-1}, s_l]$ such that $\psi(t_l) \leq \psi(s_{l-1})$ for every $t_l \in [s_{l-1}, \vartheta]$ and $\psi(t_l) \geq \psi(s_l)$ for every $t_l \in (\vartheta, s_l]$, provided that μ is continuous and monotonic (in either direction) in each of the $n-1$ intervals (s_{k-1}, s_k). Let $\sigma : [\kappa_2, \lambda_2] \to \mathbb{R}$ be a function of bounded variation such that $\bar{\nu} \in [c, \delta]$ with $\sigma(\kappa_2) < \sigma(\lambda_2)$, and satisfying

$$\begin{aligned}\sigma(u_{l-1}) \leq \sigma(v_{l-1}) \leq \sigma(u_{l-2}) \leq \cdots \leq \sigma(v_1) \leq \sigma(\kappa_2) < \sigma(\lambda_2) \\ \leq \sigma(v_n) \leq \cdots \leq \sigma(v_{l+1}) \leq \sigma(u_l)\end{aligned}$$

for all $v_k \in (u_{k-1}, u_k)$, where $k \neq l$, $l \in \{1, \ldots, n\}$, $u_0 = \kappa_2$, $u_n = \lambda_2$, and there exists a $\vartheta \in [u_{l-1}, u_l]$ such that $\sigma(v_l) \leq \sigma(u_{l-1})$ for every $v_l \in [u_{l-1}, \vartheta]$ and $\sigma(v_l) \geq \sigma(u_l)$ for every $v_l \in (\vartheta, u_l]$, provided that ν is continuous and monotonic (in either direction) in each of the $n-1$ intervals (u_{k-1}, u_k).

If $\Upsilon \in \mathcal{K}_1^c([\gamma, \delta])$ is continuous and $B_{\kappa_1, \lambda_1}^{\mu, \psi}(e_2) = B_{\kappa_2, \lambda_2}^{\nu, \sigma}(e_2)$, i.e.

$$\frac{\int_{\kappa_1}^{\lambda_1} d\psi(t) \int_{\kappa_1}^{\lambda_1} \mu^2(t) - \left(\int_{\kappa_1}^{\lambda_1} \mu(t) d\psi(t)\right)^2}{\left(\int_{\kappa_1}^{\lambda_1} d\psi(t)\right)^2} = \frac{\int_{\kappa_2}^{\lambda_2} d\sigma(t) \int_{\kappa_2}^{\lambda_2} \nu^2(t) - \left(\int_{\kappa_2}^{\lambda_2} \nu(t) d\sigma(t)\right)^2}{\left(\int_{\kappa_2}^{\lambda_2} d\sigma(t)\right)^2}, \tag{9}$$

then

$$B_{\kappa_1, \lambda_1}^{\mu, \psi}(\Upsilon) \geq B_{\kappa_2, \lambda_2}^{\nu, \sigma}(\Upsilon). \tag{10}$$

Proof. Let $\Psi(\zeta) = \Upsilon(\zeta) - \frac{K_\Upsilon}{2}e_2(\zeta)$, where K_Υ is the constant from Definition 1. Since $\Psi : [\gamma,\delta] \cap (-\infty, c] \to \mathbb{R}$ is concave, reversed Jensen-Steffensen inequality (5) implies

$$0 \geq B^{\mu,\psi}_{\kappa_1,\lambda_1}(-\Psi) = -B^{\mu,\psi}_{\kappa_1,\lambda_1}(\Upsilon) + \frac{K_\Upsilon}{2}B^{\mu,\psi}_{\kappa_1,\lambda_1}(e_2). \tag{11}$$

Similarly, $\Psi : [\gamma,\delta] \cap [c,\infty) \to \mathbb{R}$ is convex, so,

$$0 \geq B^{\nu,\sigma}_{\kappa_2,\lambda_2}(\Psi) = B^{\nu,\sigma}_{\kappa_2,\lambda_2}(\Upsilon) - \frac{K_\Upsilon}{2}B^{\nu,\sigma}_{\kappa_2,\lambda_2}(e_2). \tag{12}$$

Adding up (11) and (12) we obtain

$$0 = \frac{K_\Upsilon}{2}\left(B^{\nu,\sigma}_{\kappa_2,\lambda_2}(e_2) - B^{\mu,\psi}_{\kappa_1,\lambda_1}(e_2)\right) \geq B^{\nu,\sigma}_{\kappa_2,\lambda_2}(\Upsilon) - B^{\mu,\psi}_{\kappa_1,\lambda_1}(\Upsilon),$$

which completes the proof. □

By the similar reasoning we can analyse the discrete reversed Jensen-Steffensen functionals on the reversed Jensen-Steffensen conditions.

Theorem 2. *Let $\boldsymbol{\zeta} = (\zeta_1,\ldots,\zeta_n)$ and $\boldsymbol{\rho} = (\rho_1,\ldots,\rho_m)$ be two real monotonic tuples from $[\gamma,\delta]$ such that $\zeta_i \in [\gamma,c]$ for all $i \in \{1,\ldots,n\}$ and $\rho_j \in [c,\delta]$ for all $j \in \{1,\ldots,m\}$. Let $\boldsymbol{p} = (p_1,\ldots,p_n)$ and $\boldsymbol{q} = (q_1,\ldots,q_m)$ are real tuples such that $\overline{\zeta} \in [\gamma,c]$ with $P_n > 0$, and $\overline{\rho} \in [c,\delta]$ with $Q_m > 0$. Moreover, there exist $l \in \{1,\ldots,n\}$ and $k \in \{1,\ldots,m\}$ such that*

$$P_i \leq 0 \quad \text{for } i < l, \quad \text{and} \quad \overline{P}_i \leq 0 \quad \text{for } i > l; \tag{13}$$

$$Q_j \leq 0 \quad \text{for } j < k, \quad \text{and} \quad \overline{Q}_j \leq 0 \quad \text{for } j > k. \tag{14}$$

If $A_{\boldsymbol{\zeta},\boldsymbol{p}}(e_2) = A_{\boldsymbol{\rho},\boldsymbol{q}}(e_2)$, i.e.

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i\zeta_i^2 - \overline{\zeta}^2 = \frac{1}{Q_m}\sum_{i=1}^{m} q_i\rho_i^2 - \overline{\rho}^2, \tag{15}$$

then for every $\Upsilon \in \mathcal{K}_1^c([\gamma,\delta])$ we have

$$A_{\boldsymbol{\zeta},\boldsymbol{p}}(\Upsilon) \geq A_{\boldsymbol{\rho},\boldsymbol{q}}(\Upsilon).$$

Also by the similar reasoning we can analyse the reversed Jensen-Steffensen functionals on the reversed Jensen-Brunk conditions.

Theorem 3. *Let $c \in (\gamma,\delta)$ and let $\mu : [\kappa_1,\lambda_1] \to [\gamma,\delta]$ and $\nu : [\kappa_2,\lambda_2] \to [\gamma,\delta]$ be continuous and increasing functions such that $\mu([\kappa_1,\lambda_1]) \subset [\gamma,c]$ and $\nu([\kappa_2,\lambda_2]) \subset [c,\delta]$. Let $\psi : [\kappa_1,\lambda_1] \to \mathbb{R}$ and $\sigma : [\kappa_2,\lambda_2] \to \mathbb{R}$ be functions of bounded variation such that $\bar{\mu} \in [\gamma,c]$, $\bar{\nu} \in [c,\delta]$, and satisfying $\psi(\kappa_1) < \psi(\lambda_1)$, $\sigma(\kappa_2) < \sigma(\lambda_2)$. Let*

$$\int_{\kappa_1}^{s} (\mu(s) - \mu(t))\, d\psi(t) \leq 0 \quad \text{holds for every } s \in [\kappa_1,\lambda_1] \text{ such that } \bar{\mu} \geq \mu(s) \tag{16}$$

and

$$\int_{s}^{\lambda_1} (\mu(s) - \mu(t))\, d\psi(t) \geq 0 \quad \text{holds for every } s \in [\kappa_1,\lambda_1] \text{ such that } \bar{\mu} \leq \mu(s), \tag{17}$$

and let

$$\int_{\kappa_2}^{s} (\nu(s) - \nu(t))\, d\sigma(t) \leq 0 \quad \textit{holds for every } s \in [\kappa_2, \lambda_2] \textit{ such that } \bar{\nu} \geq \nu(s) \tag{18}$$

and

$$\int_{s}^{\lambda_2} (\nu(s) - \nu(t))\, d\sigma(t) \geq 0 \quad \textit{holds for every } s \in [\kappa_2, \lambda_2] \textit{ such that } \bar{\nu} \leq \nu(s). \tag{19}$$

Note, that if $\bar{\mu} < \mu(\kappa_1)$*, the condition in* (16) *is taken to be vacuous, and if* $\bar{\mu} > \mu(\lambda_1)$*, the condition in* (17) *is taken to be vacuous. Moreover, if* $\bar{\nu} < \nu(\kappa_2)$*, the condition in* (18) *is taken to be vacuous, and if* $\bar{\nu} > \nu(\lambda_2)$*, the condition in* (19) *is taken to be vacuous.*

If (9) *is satisfied, then for any continuous* $\Upsilon \in \mathcal{K}_1^c([\gamma, \delta])$ *inequality* (10) *holds.*

In [10], J.E. Pečarić et. al. generalizes the Jensen inequality through Green's function for real Stieltjes measure which is not necessarily positive, nor increasing. Motivated by these results we established analogous results for reversed Jensen-Steffensen inequality in this section. These established results gave us generalized refinement of Theorem 2. The Green function is defined on $[\gamma, \delta] \times [\gamma, \delta]$ by

$$G(\theta, \eta) = \begin{cases} \frac{(\theta-\delta)(\eta-\gamma)}{(\delta-\gamma)}, & \text{for } \gamma \leq \eta \leq \theta, \\ \frac{(\eta-\delta)(\theta-\gamma)}{(\delta-\gamma)}, & \text{for } \theta \leq \eta \leq \delta, \end{cases} \tag{20}$$

which is continuous and convex regarding both η and θ. Under both η and θ the function G is continuous, and, using integrating by parts, we can easily show that any function $\Upsilon : [\gamma, \delta] \to \mathbb{R}$, $\Upsilon \in \mathcal{C}^2[\gamma, \delta]$, can be represented by

$$\Upsilon(\zeta) = \frac{\delta - \zeta}{\delta - \gamma}\Upsilon(\gamma) + \frac{\zeta - \gamma}{\delta - \gamma}\Upsilon(\delta) + \int_{\gamma}^{\delta} G(\zeta, \eta)\Upsilon''(\eta)d\eta. \tag{21}$$

To establish the generalized refinement of Theorem 2, first we have to prove using Green's functions some new conditions for reversed Jensen-Steffensen inequality under reversed Jensen-Boas conditions to hold.

Theorem 4. *Let* $\mu : [\kappa, \lambda] \longrightarrow [\gamma, \delta]$ *be a continuous function. Let* $\psi : [\kappa, \lambda] \longrightarrow \mathbb{R}$ *is a function of bounded variation such that* $\overline{\mu} \in [\gamma, \delta]$ *with* $\psi(\lambda) > \psi(\kappa)$*, and satisfies* (6)*.*

Then the following statements are equivalent.

(1) *For every continuous convex function* $\Upsilon : [\gamma, \delta] \to \mathbb{R}$ *the inequality* (5) *holds.*

(2) *For all* $\eta \in [\gamma, \delta]$ *the inequality*

$$G\left(\frac{1}{\int_{\kappa}^{\lambda} d\psi(t)} \int_{\kappa}^{\lambda} \mu(t)d\psi(t), \eta\right) \geq \frac{1}{\int_{\kappa}^{\lambda} d\psi(t)} \int_{\kappa}^{\lambda} G(\mu(t), \eta)d\psi(t) \tag{22}$$

holds, where the function $G : [\gamma, \delta] \times [\gamma, \delta] \to \mathbb{R}$ *is defined in* (20)*.*

Proof. $(1) \Rightarrow (2)$. Let (1) holds. While the function $G(.,\eta)$, $\eta \in [\gamma,\delta]$, is also convex and continuous accordingly also (5) holds for the function $G(.,\eta)$, i.e. (22) holds.

$(2) \Rightarrow (1)$. Let (2) holds. We notice that we can easily shown every function $\Upsilon : [\gamma,\delta] \to \mathbb{R}$, $\Upsilon \in \mathcal{C}^2([\gamma,\delta])$, in the structure of (21), where the function G is defined in (20). With simple calculations, using (21), we can obtain with ease that

$$\Upsilon\left(\frac{1}{\int_\kappa^\lambda d\psi(t)}\int_\kappa^\lambda \mu(t)d\psi(t)\right) - \frac{1}{\int_\kappa^\lambda d\psi(t)}\int_\kappa^\lambda \Upsilon(\mu(t))d\psi(t)$$
$$= \int_\gamma^\delta \left[G\left(\frac{1}{\int_\kappa^\lambda d\psi(t)}\int_\kappa^\lambda \mu(t)d\psi(t),\eta\right) - \frac{1}{\int_\kappa^\lambda d\psi(t)}\int_\kappa^\lambda G(\mu(t),\eta)d\psi(t)\right]\Upsilon''(\eta)d\eta.$$

If the Υ is also convex, then $\Upsilon''(\eta) \geq 0$ for all $\eta \in [\gamma,\delta]$. Therefore, if (22) holds for every $\eta \in [\gamma,\delta]$, then for the convexity of $\Upsilon : [\gamma,\delta] \to \mathbb{R}$ with $\Upsilon \in \mathcal{C}^2([\gamma,\delta])$, it follows that inequality (5) holds.

In the end, observe that it is not essential to claim the existence of the double derivative of the function Υ (see [11, p. 172]). The condition of the differentiability can be directly removed by applying the fact that it is feasible to approximate uniformly a continuous convex function by convex polynomials.

The last part of our theorem can be proved analogously. □

Observe that for each continuous concave function $\Upsilon : [\gamma,\delta] \to \mathbb{R}$ inequality (5) is reversed, i.e. the underlaying corollary holds.

Corollary 1. *Under the conditions of Theorem* 4*, following two statements are equivalent.*

$(1')$ *For every continuous concave function* $\Upsilon : [\gamma,\delta] \to \mathbb{R}$ *the reverse inequality in* (5) *holds.*

$(2')$ *For all* $\eta \in [\gamma,\delta]$ *inequality* (22) *holds, where the function* G *is defined in* (20).

Further, the statements $(1')$ *and* $(2')$ *are also equivalent in case that we alter inequality sign in both* $(1')$ *and* $(2')$.

Now we give a Levinson type generalization of the result from Theorem 4, which will also refined the result of Theorem 1.

Theorem 5. *Let the assumptions of Theorem* 1 *are fulfilled.*

If condition (9) *holds, and for all* $\eta_1 \in [\gamma, c]$ *and for all* $\eta_2 \in [c,\delta]$ *we have*

$$G(\bar{\mu},\eta_1) \geq \frac{\int_{\kappa_1}^{\lambda_1} G(\mu(\zeta),\eta_1)d\psi(\zeta)}{\int_{\kappa_1}^{\lambda_1} d\psi(\zeta)} \quad \text{and} \quad G(\bar{\nu},\eta_2) \geq \frac{\int_{\kappa_2}^{\lambda_2} G(\nu(\zeta),\eta_2)d\sigma(\zeta)}{\int_{\kappa_2}^{\lambda_2} d\sigma(\zeta)}, \tag{23}$$

where the function G *is defined in* (20)*, then for every continuous function* $\Upsilon \in \mathcal{K}_1^c([\gamma,\delta])$ *we have*

$$B_{\kappa_1,\lambda_1}^{\mu,\psi}(\Upsilon) \geq \frac{K_\Upsilon}{2}D \geq B_{\kappa_2,\lambda_2}^{\nu,\sigma}(\Upsilon). \tag{24}$$

The statement also holds if we reverse all signs of inequalities in (23) *and* (24).

Proof. Let $\Upsilon \in \mathcal{K}_1^c([\gamma,\delta])$ be continuous function on $[\gamma,\delta]$ and let $\Psi(\zeta) = \Upsilon(\zeta) - \frac{K_\Upsilon}{2}e_2(\zeta)$, where K_Υ is constant from Definition 1.

As the function Ψ is concave and continuous on $[\gamma,c]$ and (23) hold for every $\eta_1 \in [\gamma,c]$, it follows from Corollary 1 that

$$\Psi(\bar{\mu}) \leq \frac{\int_{\kappa_1}^{\lambda_1} \Psi(\mu(\zeta))d\psi(\zeta)}{\int_{\kappa_1}^{\lambda_1} d\psi(\zeta)}.$$

When we reorganize the former inequality, we obtain

$$\frac{\int_{\kappa_1}^{\lambda_1} \Upsilon(\mu(\zeta))d\psi(\zeta)}{\int_{\kappa_1}^{\lambda_1} d\psi(\zeta)} - \Upsilon(\bar{\mu}) \geq \frac{K_\Upsilon}{2}\left[\frac{\int_{\kappa_1}^{\lambda_1} \mu^2(\zeta)d\psi(\zeta)}{\int_{\kappa_1}^{\lambda_1} d\psi(\zeta)} - \bar{\mu}^2\right]. \tag{25}$$

As the function Ψ is convex and continuous on $[c,\delta]$ and for all $\eta_2 \in [c,\delta]$ the inequalities (23) hold, it follows from Theorem 4 that

$$\Psi(\bar{\nu}) \geq \frac{\int_{\kappa_2}^{\lambda_2} \Psi(\nu(\zeta))d\sigma(\zeta)}{\int_{\kappa_2}^{\lambda_2} d\sigma(\zeta)}.$$

When we rearrange the previous inequality, we get

$$\frac{K_\Upsilon}{2}\left[\frac{\int_{\kappa_2}^{\lambda_2} \nu^2(\zeta)d\sigma(\zeta)}{\int_{\kappa_2}^{\lambda_2} d\sigma(\zeta)} - \bar{\nu}^2\right] \geq \frac{\int_{\kappa_2}^{\lambda_2} \Upsilon(\nu(\zeta))d\sigma(\zeta)}{\int_{\kappa_2}^{\lambda_2} d\sigma(\zeta)} - \Upsilon(\bar{\nu}). \tag{26}$$

Inequality (24) follows directly by combining (25) and (26), and taking into consideration the condition (9). □

Corollary 2. *Under the conditions of Theorem* 5 *we have the following statements.*

$(1'')$ *If for all* $\eta_1 \in [\gamma,c]$ *and for all* $\eta_2 \in [c,\delta]$ *inequalities* (23) *hold, where the function* G *is defined in* (20)*, then for every continuous function* $\Upsilon \in \mathcal{K}_2^c([\gamma,\delta])$ *the reverse inequalities hold in* (24)*.*

$(2'')$ *If for all* $\eta_1 \in [\gamma,c]$ *and for all* $\eta_2 \in [c,\delta]$ *the reverse inequalities* (23) *hold, then for every continuous function* $\Upsilon \in \mathcal{K}_2^c([\gamma,\delta])$ *the inequalities* (24) *hold.*

Following theorem gave some new conditions for reversed Jensen-Steffensen inequality to hold using Green's functions.

Theorem 6. *Let* $\boldsymbol{\zeta} = (\zeta_1,\ldots,\zeta_n)$ *be real monotonic* n*-tuple such that* $\zeta_i \in [\gamma,\delta]$ *for all* $i \in \{1,\ldots,n\}$*. Let* $\boldsymbol{p} = (p_1,\ldots,p_n)$ *be real* n*-tuple such that* $\frac{1}{P_n}\sum_{i=1}^{n} p_i\zeta_i \in [\gamma,\delta]$ *with* $P_n > 0$*. Moreover, there exists* $l \in \{1,\ldots,n\}$*, satisfying* (13)*.*

Then the following statements are equivalent.

(1) *For every continuous convex function* $\Upsilon : [\gamma,\delta] \to \mathbb{R}$ *the inequality* (4) *holds.*

(2) *For all* $\eta \in [\gamma,\delta]$ *the inequality*

$$G\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i\zeta_i, \eta\right) \geq \frac{1}{P_n}\sum_{i=1}^{n} p_i G(\zeta_i,\eta)$$

holds, where the function $G : [\gamma,\delta] \times [\gamma,\delta] \to \mathbb{R}$ *is defined in* (20)*.*

Now we give a Levinson type generalization of the result from Theorem 6, which will also refined the result of Theorem 2.

Theorem 7. *Under all the assumptions of Theorem* 2 *we have the following.*
If condition (15) *holds, and for all* $\eta_1 \in [\gamma, c]$ *and for all* $\eta_2 \in [c, \delta]$ *we have*

$$G(\bar{\zeta}, \eta_1) \geq \frac{1}{P_n} \sum_{i=1}^{n} p_i G(\zeta_i, \eta_1) \quad \text{and} \quad G(\bar{\rho}, \eta_2) \geq \frac{1}{Q_m} \sum_{j=1}^{m} q_j G(\rho_j, \eta_2),$$

where the function G is defined in (20)*, then for every continuous function* $\Upsilon \in \mathcal{K}_1^c([\gamma, \delta])$ *we have*

$$A_{\boldsymbol{\zeta},\boldsymbol{p}}(\Upsilon) \geq \frac{K_\Upsilon}{2} C \geq A_{\boldsymbol{\rho},\boldsymbol{q}}(\Upsilon).$$

3 Mean value results

In this section, we establish linear functionals as certain differences of the linear functionals from the previous section. We utilize the linearity of these functionals to obtain two mean value theorems. For tuples $\boldsymbol{\zeta}, \boldsymbol{p} \in \mathbb{R}^n$ and $\boldsymbol{\rho}, \boldsymbol{q} \in \mathbb{R}^m$ that fulfilling the conditions of Theorem 2, i.e. such that $\boldsymbol{\zeta} \in [\gamma, \delta]^n$ and $\boldsymbol{\rho} \in [\gamma, \delta]^m$ are monotonic and (13), (14) and (15) hold, we represent the linear functional

$$\wedge_1(\Upsilon) = \wedge_1(\Upsilon; \boldsymbol{\zeta}, \boldsymbol{p}, \boldsymbol{\rho}, \boldsymbol{q}) = A_{\boldsymbol{\rho},\boldsymbol{q}}(\Upsilon) - A_{\boldsymbol{\zeta},\boldsymbol{p}}(\Upsilon). \tag{27}$$

The linear functional $\wedge_1$ depends on the selection of $\boldsymbol{\zeta}, \boldsymbol{p}, \boldsymbol{\rho}$ and $\boldsymbol{q}$, but we will exclude them from the representation, when they are clear from the text. By Theorem 2, for every continuous $\Upsilon \in \mathcal{K}_1^c([\gamma, \delta])$ we have $\wedge_1(\Upsilon) \geq 0$.

Similarly under the assumptions of Theorem 1, we define the linear operator

$$\wedge_2(\Upsilon) = \wedge_2(\Upsilon; \mu, \psi, \kappa_1, \lambda_1, \nu, \sigma, \kappa_2, \lambda_2) = B_{\kappa_2,\lambda_2}^{\nu,\sigma}(\Upsilon) - B_{\kappa_1,\lambda_1}^{\mu,\psi}(\Upsilon).$$

We establish the mean value results for the linear functional $\wedge_1$, whereas related results are also valid for linear functional $\wedge_2$. The following theorem is a mean value result, which is of Lagrange type.

Theorem 8. *Let* $\boldsymbol{\zeta}, \boldsymbol{p}, \boldsymbol{\rho}$ *and* $\boldsymbol{q}$ *be as in Theorem* 2 *and let* $\wedge_1$ *be given by* (27)*. Then for* $\Upsilon \in \mathcal{C}^3([\gamma, \delta])$ *there exists* $\xi \in [\gamma, \delta]$ *such that*

$$\wedge_1(\Upsilon) = \frac{\Upsilon'''(\xi)}{6} = \left[\frac{1}{Q_m} \sum_{i=1}^{m} q_i \rho_i^3 - \left(\frac{1}{Q_n} \sum_{i=1}^{m} q_i \rho_i\right)^3 - \frac{1}{P_n} \sum_{i=1}^{n} p_i \zeta_i^3 + \left(\frac{1}{P_n} \sum_{i=1}^{n} p_i \zeta_i\right)^3\right]. \tag{28}$$

Proof. Since $\Upsilon \in \mathcal{C}^3([\gamma, \delta])$, there exist $m = \min_{\zeta \in [\gamma,\delta]} \Upsilon'''(\zeta)$ and $M = \max_{\zeta \in [\gamma,\delta]} \Upsilon'''(\zeta)$.

Define

$$\Upsilon_1(\zeta) = \Upsilon(\zeta) - \tfrac{m}{6} e_3(\zeta),$$
$$\Upsilon_2(\zeta) = \tfrac{M}{6} e_3(\zeta) - \Upsilon(\zeta).$$

These functions satisfy $\Upsilon_i'''(\zeta) \geq 0$, $i = 1, 2$, so they are three times differentiable 3-convex functions.

Therefore, $\Upsilon_1, \Upsilon_2 \in \mathcal{K}_1^c([\gamma,\delta])$ (see Remark 2) and by Theorem 2 we have $\wedge_1(\Upsilon_i) \leq 0$, $i = 1,2$, so

$$\frac{m}{6}\wedge_1(e_3) \geq \wedge_1(\Upsilon) \geq \frac{M}{6}\wedge_1(e_3). \tag{29}$$

Since e_3 is 3-convex, by Theorem 2 we have

$$0 \geq \wedge_1(e_3) = \frac{1}{Q_m}\sum_{i=1}^{m} q_i\rho_i^3 - \left(\frac{1}{Q_n}\sum_{i=1}^{m} q_i\rho_i\right)^3 - \frac{1}{P_n}\sum_{i=1}^{n} p_i\zeta_i^3 + \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i\zeta_i\right)^3.$$

If $\wedge_1(e_3) = 0$, then (29) implies $\wedge_1(\Upsilon) = 0$ and (28) holds for every $\xi \in [\gamma,\delta]$. Otherwise, dividing (29) by $\frac{\wedge_1(e_3)}{6} < 0$, we get

$$m \leq \frac{6\wedge_1(\Upsilon)}{\wedge_1(e_3)} \leq M.$$

So, continuity of Υ''' insures existence of $\xi \in [\gamma,\delta]$ satisfying (28). □

The following theorem is a mean value result, which is Cauchy type.

Theorem 9. *Let* $c, [\gamma,\delta], \boldsymbol{\zeta}, \boldsymbol{p}, \boldsymbol{\rho}, \boldsymbol{q}$ *and* $\wedge_1$ *be as in Theorem* 8 *and let* $\Upsilon, \psi \in \mathcal{C}^3([\gamma,\delta])$. *If* $\wedge_1(\psi) \neq 0$*, then there exists* $\xi \in [\gamma,\delta]$ *such that either*

$$\frac{\wedge_1(\Upsilon)}{\wedge_1(\psi)} = \frac{\Upsilon'''(\xi)}{\psi'''(\xi)},$$

or

$$\Upsilon'''(\xi) = \psi'''(\xi) = 0.$$

Proof. Define $\tau \in \mathcal{C}^3([\gamma,\delta])$ by $\tau(\zeta) = \alpha\Upsilon(\zeta) - \beta\psi(\zeta)$, where $\alpha = \wedge_1(\psi)$, $\beta = \wedge_1(\Upsilon)$. Because linearity of $\wedge_1$ we have $\wedge_1(\tau) = 0$. Now, by Theorem 8 there exist $\xi, \xi_1 \in [\gamma,\delta]$ such that

$$0 = \wedge_1(\tau) = \tfrac{\tau'''(\xi)}{6}\wedge_1(e_3),$$
$$0 \neq \wedge_1(\psi) = \tfrac{\psi'''(\xi_1)}{6}\wedge_1(e_3).$$

Therefore, $\wedge_1(e_3) \neq 0$ and

$$0 = \tau'''(\xi) = \alpha\Upsilon'''(\xi) - \beta\psi'''(\xi),$$

that gives the claim of the theorem. □

References

[1] Baloch I.A., Pečarić J.E., Praljak M. *Generalization of Levinson's inequality*. J. Math. Inequal. 2015, **9** (2), 571–586. doi:10.7153/jmi-09-49

[2] Barlow R.E., Marshall A.W., Proschan F. *Some Inequalities for Starshaped and Convex Functions*. Pacif. J. Math. 1969, **29** (1), 19–42.

[3] Boas R.P. *The Jensen-Steffensen Inequality*. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. 1970, **302–319**, 1–8.

[4] Brunck H.D. *On an inequality for convex functions*. Proc. Amer. Math. Soc. 1956, **7** (5), 817–824. doi: 10.2307/2033544

[5] Fuchs L. *A New Proof of an Inequality of Hardy-Littlewood Pólya*. Mat. Tidssker. 1947, **B**, 53–54.

[6] Jakšetić J., Pečarić J.E., Praljak M. *Generalized Jensen-Steffensen and related inequalities*. J. Math. Inequal. 2015, **9** (4), 1287–1302. doi:10.7153/jmi-09-98

[7] Jensen J.L.W.V. *Om konvekse Funktioner og Uligheder imellem Middelværdier*. Nyt. Tidsskr. Math. 1905, **16B**, 49–69. (in Dutch)

[8] Jensen J.L.W.V. *Sur les fonctions convexes et les inégalités entre les valeurs moyennes*. Acta Math. 1906, **30** (1), 175–193. doi:10.1007/BF02418571 (in French)

[9] Mitrinović D.S., Pečarić J.E., Fink A.M. Classical and New Inequalities In Analysis. Kluwer Academic Publ., Dordrecht, 1993.

[10] Pečarić J.E., Perić I., Rodić M.L. Uniform treatment of Jensen type inequalities. Math. Rep. 2014, **16** (2), 183–205.

[11] Pečarić J.E., Proschan F., Tong Y.L. Convex functions, partial orderings and statistical applications. Academic Press, New York, 1992.

[12] Pečarić J.E. *Inverse of Jensen's Steffensen Inequality*. Glassnik Mathemat. 1981, **16** (36), 229–233.

[13] Pečarić J.E. *A short proof of a variant of Jensen's inequality*. J. Math. Anal. Appl. 1982, **87** (1), 278–280. doi: 10.1016/0022-247X(82)90166-4

[14] Pečarić J.E. *On variants of Jensen's inequality*. Glassnik Mathemat. 1983, **18** (38), 281–289.

[15] Mikić R., Pečarić J.E., Rodić M. *Levinson's type generalization of the Jensen inequality and its converse for real Stieltjes measure*. J. Inequal. Appl. 2017, article 4 (2017). doi:10.1186/s13660-016-1274-y

[16] Roberts A.W., Varberg D.E. Convex Functions. Academic Press, New York-London, 1973.

[17] Steffensen J.F. *On Certain Inequalities and Methods of Approximation*. J. Inst. Actuaries 1919, **51** (3), 274–297.

[18] Vasić P.M., Pečarić J.E. *On Jensen Inequality*. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. 1979, **639–677**, 50–54.

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Ми порівнюємо два лінійні функціонали, які є від'ємні на опуклих функціях. Крім того, використовуючи функції Гріна, ми наводимо деякі нові умови для виконання обернених нерівностей Єнсена-Штеффенсена та пов'язаних нерівностей. Використовуючи функцію Гріна, ми також надаємо уточнення узагальнення типу Левінсона обернених нерівностей Єнсена-Штеффенсена та пов'язаних нерівностей. Відтак отримані результати використовуються для побудови теорем про середнє значення.

Ключові слова і фрази: нерівність Єнсена-Мерсера, нерівність Єнсена-Штеффенсена.