SOME TRIBONACCI IDENTITIES USING TOEPLITZ–HESSENBERG DETERMINANTS

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We investigate some families of Toeplitz–Hessenberg determinants the entries of which are Tribonacci numbers with successive, even, and odd subscripts.

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Among the several generalizations of Fibonacci numbers, one of the best known is the *Tribonacci sequence* $\{T_n\}_{n\geq 0}$ (Koshy, 2001). This is defined by the recurrence

$$T_{n+1} = T_n + T_{n-1} + T_{n-2},$$

with initial values $T_0 = 0$ and $T_1 = T_2 = 1$.

Many authors studied the Tribonacci sequence and its various properties (Choi & Jo, 2015; Feng, 2011; Tan & Wen, 2007; Zatorsky & Goy, 2016).

A Toeplitz–Hessenberg matrix is an $n \times n$ matrix of the form

$$M_n(a_0, a_1, \dots, a_n) = \begin{pmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{pmatrix},$$

where $a_0 \neq 0$ and $a_k \neq 0$ for at least one $a_k \neq 0$.

Lemma (Merca, 2013). *Let n be a positive integer. Then*

$$\det(M_n) = \sum_{s_1+2s_2+\cdots+ns_n=n} (-a_0)^{n-(s_1+s_2+\cdots+s_n)} p(s) a_1^{s_1} a_2^{s_2} \cdots a_n^{s_n}, \tag{1}$$

where the summation is over nonnegative integers satisfying $s_1 + 2s_2 + \cdots + ns_n = n$, and

$$p(s) = \frac{(s_1 + s_2 + \cdots + s_n)!}{s_1! s_2! \cdots s_n!}$$

is the multinomial coefficient.

Proposition 1. *The following formulas hold*:

$$\det(1, T_0, T_1, \dots, T_{n-1}) = (-1)^{n-1} F_{n-2}, \qquad n \geq 2,$$

$$\det(1, T_1, T_2, \dots, T_n) = (-1)^{n-1} \sum_{i=0}^{\lfloor (n-2)/3 \rfloor} \binom{n-2i-2}{i}, \qquad n \geq 2,$$

$$\det(1, T_1, T_3, \dots, T_{2n-1}) = (-1)^{n-1} \left\lfloor 4 \cdot 3^{n-3} \right\rfloor, \qquad n \geq 1,$$

$$\det(1,T_3,T_4,\dots,T_{n+2}) = (-1)^{n-1}\left\lfloor 3/(n+1)\right\rfloor, \qquad n \geq 2,$$

$$\det(1,T_3,T_5,\dots,T_{2n+1}) = (-2)^{n-1}\sum_{i=0}^{n-1} 2^{-i-\lfloor i/2\rfloor}\begin{pmatrix} n-i-1 \\ \lfloor i/2\rfloor\end{pmatrix}, \qquad n \geq 1,$$

$$\det(1,T_4,T_6,\dots,T_{2n+2}) = (-1)^{n-1}\left(4-\lfloor 2/n\rfloor\right), \qquad n \geq 1,$$

$$\det(1,T_4,T_5,\dots,T_{n+3}) = \sum_{i=0}^{\lfloor n/2\rfloor+1}(-1)^i\begin{pmatrix} n-i \\ i\end{pmatrix}, \qquad n \geq 2,$$

where

$$\begin{pmatrix} n \\ m\end{pmatrix} = \frac{n!}{m!(n-m)!}$$

is the binomial coefficient, F_n *is the* n^{th} *Fibonacci number,* $\lfloor\cdot\rfloor$ *is the floor function.*

Using Formula (1) for determinants in Proposition 1, we obtain the following Tribonacci identities.

Proposition 2. *The following formulas hold*:

$$\sum_{s_1+2s_n+\cdots+ns_n=n}(-1)^{s_1+s_2+\cdots+s_n+1}p(s)T_0^{s_1}T_1^{s_2}\cdots T_{n-1}^{s_n} = F_{n-2}, \qquad n \geq 2,$$

$$\sum_{s_1+2s_n+\cdots+ns_n=n}(-1)^{s_1+s_2+\cdots+s_n+1}p(s)T_1^{s_1}T_2^{s_2}\cdots T_n^{s_n} = \sum_{i=0}^{\lfloor (n-2)/3\rfloor}\begin{pmatrix} n-2i-2 \\ i\end{pmatrix}, \quad n \geq 2,$$

$$\sum_{s_1+2s_n+\cdots+ns_n=n}(-1)^{s_1+s_2+\cdots+s_n+1}p(s)T_1^{s_1}T_3^{s_2}\cdots T_{2n-1}^{s_n} = \left\lfloor 4\cdot 3^{n-3}\right\rfloor, \qquad n \geq 1,$$

$$\sum_{s_1+2s_n+\cdots+ns_n=n}p(s)(-1)^{s_1+s_2+\cdots+s_n+1}T_3^{s_1}T_4^{s_2}\cdots T_{n+2}^{s_n} = \left\lfloor \frac{3}{n+1}\right\rfloor, \qquad n \geq 2,$$

$$\sum_{s_1+2s_n+\cdots+ns_n=n}(-1)^{s_1+s_2+\cdots+s_n+1}p(s)T_3^{s_1}T_5^{s_2}\cdots T_{2n+1}^{s_n} =$$

$$= 2^{n-1}\sum_{i=0}^{n-1}2^{-i-\lfloor i/2\rfloor}\begin{pmatrix} n-i-1 \\ \lfloor i/2\rfloor\end{pmatrix}, \quad n \geq 1,$$

$$\sum_{s_1+2s_n+\cdots+ns_n=n}p(s)(-1)^{s_1+s_2+\cdots+s_n+1}T_4^{s_1}T_6^{s_2}\cdots T_{2n+2}^{s_n} = 4-\lfloor 2/n\rfloor, \qquad n \geq 1,$$

$$\sum_{s_1+2s_n+\cdots+ns_n=n}p(s)(-1)^{s_1+s_2+\cdots+s_n}T_4^{s_1}T_5^{s_2}\cdots T_{n+3}^{s_n} = \sum_{i=0}^{\lfloor n/2\rfloor+1}(-1)^{n+i}\begin{pmatrix} n-i \\ i\end{pmatrix}, n \geq 2,$$

where the summation is over nonnegative integers satisfying

$$s_1 + 2s_2 + \cdots + ns_n = n.$$

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