ON RECURRENT FORMULAS FOR THIRD-ORDER HORADAM NUMBERS

T.P. Goy

Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine
E-mail: tarasgoy@yahoo.com

The generalized Fibonacci sequence $h_n = h_n(a, b; p, q)$ is defined as follows

$$h_0 = a, \quad h_1 = b, \quad h_n = ph_{n-1} + qh_{n-2}, \qquad n \geqslant 2, \tag{1}$$

where a, b, p and q are arbitrary complex numbers with $q \neq 0$. Since the numbers h_n were first studied by Alwyn Horadam (see [4, 5]), they are called *Horadam numbers.*

In [6] the Horadam recurrence relation (1) is extended to higher order recurrence relations. In fact, third-order Horadam numbers $H_n = H_n(a, b, c; p, q, r)$ are defined by

$$H_n = pH_{n-1} + qH_{n-2} + rH_{n-3}, \qquad n \geqslant 3, \tag{2}$$

with initial values $H_0 = a$, $H_1 = b$ and $H_2 = c$.

In equation (2), for special choices of a, b, c, p, q and r, the following recurrence relations can be obtained:

- for $a = b = c = p = q = r = 1$, it is obtained *tribonacci numbers* (sequence A000073 from [7]);
- for $a = 3$, $b = 1$, $c = 3$, $p = q = r = 1$, it is obtained *tribonacci-Lucas numbers* (sequence A001644);
- for $a = 1$, $b = c = 0$, $p = 0$, $q = r = 1$, it is obtained *Padovan numbers* (sequence A000931);
- for $a = 3$, $b = 0$, $c = 2$, $p = 0$, $q = r = 1$, it is obtained *Perrin numbers* (sequence A001608);
- for $a = 0$, $b = 1$, $c = 1$, $p = 1$, $q = 0$, $r = 1$, it is obtained *Fibonacci-Narayana numbers* (sequence A000930).

Applying the apparatus of triangular matrices (see, for example, [8] and the bibliography given there), we proved new recurrent formulas expressing third-order Horadam numbers H_n with even and odd subscripts via recurrent determinants of four-diagonal matrix of order n.

Our approach is similar to spirit in [1, 2, 3].

Let P_n and Q_n denote the $n \times n$ four-diagonal matrices

$$P_n = \begin{pmatrix} H_1 & H_1 & 0 & & & & & \\ 0 & H_3 & H_3 & & & & 0 & \\ 0 & 0 & H_5 & H_5 & & & & \\ 0 & pH_6 & -rH_4 & qH_5 & H_7 & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & & pH_{2n-6} & -rH_{2n-8} & qH_{2n-7} & H_{2n-5} & 0 \\ & 0 & & & pH_{2n-4} & -rH_{2n-6} & qH_{2n-5} & H_{2n-3} \\ & & & & 0 & pH_{2n-2} & -rH_{2n-4} & qH_{2n-3} \end{pmatrix},$$

and

$$Q_n = \begin{pmatrix} H_0 & H_0 & 0 & & & & & \\ 0 & H_2 & H_2 & & & & & \\ 0 & 0 & H_4 & H_4 & & & 0 & \\ 0 & pH_5 & -rH_3 & qH_4 & H_6 & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & & pH_{2n-7} & -rH_{2n-9} & qH_{2n-8} & H_{2n-6} & 0 \\ & 0 & & & pH_{2n-5} & -rH_{2n-7} & qH_{2n-6} & H_{2n-4} \\ & & & & 0 & pH_{2n-3} & -rH_{2n-5} & qH_{2n-4} \end{pmatrix}.$$

Theorem. *For all $n \geqslant 1$, the following formulas hold:*

$$H_{2n-1} = \frac{\det P_n}{\prod\limits_{i=1}^{n-1} H_{2i-1}}, \qquad H_{2n-2} = \frac{\det Q_n}{\prod\limits_{i=1}^{n-1} H_{2i-2}},$$

By choosing other suitable values on a, b, c, p, q and r, we can also obtain the tribonacci, tribonacci-Lucas, Padovan, Perrin and Fibonacci-Narayana numbers in term of recurrent determinants of four-diagonal matrix.

Reference

[1] T. Goy, *Fibonacci and Lucas numbers via the determinants of tridiagonal matrix.* Notes on Number Theory and Discrete Mathematics, 24 (2018), no. 1, 27-43.

[2] T. Goy, *Horadam sequence through recurrent determinants of tridiagonal matrix.* Kragujevac Journal of Mathematic, 42 (2018), no. 4, 527-532.

[3] T. Goy, *On identities with multinomial coefficients for Fibonacci-Narayana sequence.* Annales Mathematicae et Informaticae, 49 (2018), 75-84.

[4] A.F. Horadam, *Basic properties of a certain generalized sequence of numbers.* Fibonacci Quarterly, 3 (1965), no. 3, 161-176.

[5] A.F. Horadam, *Special properties of the sequence $W_n(a, b; p, q)$.* Fibonacci Quarterly, 5 (1967), no. 5, 424-434.

[6] A.G. Shannon and A.F. Horadam, *Some properties of third-order recurrence relations.* Fibonacci Quarterly, 10 (1972), no. 2, 135-146.

[7] N.J.A. Sloane (ed.), The On-Line Encyclopedia of Integer Sequences. Available at http://oeis.org.

[8] R. Zatorsky and T. Goy, *Parapermanents of triangular matrices and some general theorems on number sequences.* Journal of Integer Sequences, 19 (2016), no. 2, Article 16.2.2.