CONSTRUCTION OF CHEBYSHEV POLYNOMIALS OF THE FIRST AND SECOND KINDS IN TERMS OF THE DETERMINANT OF TRIDIAGONAL MATRICES

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Chebyshev polynomials crop up in virtually every area of numerical analysis, and they hold particular importance in recent advances in subjects such as orthogonal polynomials, polynomial approximation, numerical integration, combinatorics, statistics, and spectral methods. There are many interesting and unique properties of these polynomials, which can be found in several textbooks and articles, for example [1–4].

The Chebyshev polynomials $T_n(x)$ of the first kind are defined by the two-order recurrence relation

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

while the Chebyshev polynomials $U_n(x)$ of the second kind are defined by the recurrence relation

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x).$$

Using the apparatus of triangular matrices [5, 6], we obtain the recurrent formulas for Chebyshev polynomials of the first and second kinds with even (odd) indices via determinant of tridiagonal matrices.

Theorem 1. *For $n \geqslant 1$ the following recurrent formulas are hold:*

$$T_{2n-2}(x) = (-1)^n \begin{vmatrix} -1 & -1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 2xT_1(x) & 1 & -1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 2x\frac{T_3(x)}{T_0(x)} & 1 & -1 & \ldots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 2x\frac{T_{2n-5}(x)}{T_{2n-8}(x)} & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2x\frac{T_{2n-3}(x)}{T_{2n-6}(x)} & 1 \end{vmatrix}$$

and

$$T_{2n-1}(x) = (-1)^n \begin{vmatrix} -x & -1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 2xT_2(x) & 1 & -1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 2x\frac{T_4(x)}{T_1(x)} & 1 & -1 & \ldots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \ldots & 2x\frac{T_{2n-4}(x)}{T_{2n-7}(x)} & 1 & -1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 2x\frac{T_{2n-2}(x)}{T_{2n-5}(x)} & 1 \end{vmatrix}.$$

Theorem 2. *For $n \geqslant 1$ the following recurrent formulas are hold:*

$$U_{2n-2}(x) = (-1)^n \begin{vmatrix} -1 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 2xU_1(x) & 1 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 2x\frac{U_3(x)}{U_0(x)} & 1 & -1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2x\frac{U_{2n-5}(x)}{U_{2n-8}(x)} & 1 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 2x\frac{U_{2n-3}(x)}{U_{2n-6}(x)} & 1 \end{vmatrix}$$

and

$$U_{2n-1}(x) = (-1)^n \begin{vmatrix} -2x & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 2xU_2(x) & 1 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 2x\frac{U_4(x)}{U_1(x)} & 1 & -1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2x\frac{U_{2n-4}(x)}{U_{2n-7}(x)} & 1 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 2x\frac{U_{2n-2}(x)}{U_{2n-5}(x)} & 1 \end{vmatrix}.$$

A similar formulas can be obtained for Chebyshev polynomials of the third and fourth kinds. Recall that the n-th Chebyshev polynomials of the third and fourth kinds are defined to be the polynomials $V_n(x)$ and $W_n(x)$ satisfying the same recurrent relation

$$X_{n+1}(x) = 2xX_n(x) - X_{n-1}(x),$$

where $n \geqslant 1$, with initial conditions $V_0(x) = 0$, $V_1(x) = 2x - 1$ and $W_0(x) = 0$, $W_1(x) = 2x + 1$, respectively.

References

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