Some relationship between Chebyshev and Fibonacci polynomials

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The Chebyshev polynomials $T_n(x)$ of the first kind, the Chebyshev polynomials $U_n(x)$ of the second kind, and the Fibonacci polynomials $F_n(x)$ are respectively defined by the recurrence relations as follows [3,4]: for $n \geq 2$,

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x),$$

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x),$$

$$F_0(x) = 0, \quad F_1(x) = 1, \quad F_n(x) = xF_{n-1}(x) + F_{n-2}(x).$$

We establish new connection formulas between Fibonacci polynomials and Chebyshev polynomials of the first and second kinds (see Theorems 1 and 2, respectively). This is achieved by relating the respective generating functions to each other; see [1] and [2] for more details of this method.

Theorem 1. *For $n \geq 1$, the following identities hold:*

$$F_n(x) = T_{n-1}(x) - \sum_{k=1}^{n-2} \big(xT_{n-k-1}(x) - 2T_{n-k-2}(x)\big) F_k(x);$$

$$x^2 F_{2n}(x) = T_{2n+1}(x) - T_{2n-1}(x) - (3x^2 - 4) \sum_{k=0}^{n-1} F_{2k+1}(x) T_{2(n-k)-1}(x);$$

$$F_{2n}(x) - (2x^2 - 1)F_{2n-2}(x) = xT_{2n-2}(x) - (3x^2 - 4) \sum_{k=1}^{n-1} F_{2k}(x) T_{2(n-k-1)}(x);$$

$$\begin{gathered} xF_n(x) + (4x^3 - x^2 - 3x)F_{n-1}(x) \\ = T_{2n-1}(x) - \sum_{k=1}^{n-2} \big((4x^2 - x - 2)T_{2(n-k)-1}(x) - 2T_{2(n-k)-3}(x)\big) F_k(x); \end{gathered}$$

$$\begin{gathered} F_n(x) + (2x^2 - x - 1)F_{n-1}(x) \\ = T_{2n-2}(x) - \sum_{k=1}^{n-2} \big((4x^2 - x - 2)T_{2(n-k-1)}(x) - 2T_{2(n-k-2)}(x)\big) F_k(x); \end{gathered}$$

$$F_{2n+1}(x) - (2x^2 - 1)F_{2n-1}(x)$$
$$= T_{2n}(x) - T_{2n-2}(x) - (3x^2 - 4)\sum_{k=0}^{n-1} T_{2(n-k-1)}(x)F_{2k+1}(x);$$

$$x^2F_{2n-1}(x) = xT_{2n-1}(x) - (3x^2 - 4)\sum_{k=1}^{n-1} T_{2(n-k)-1}(x)F_{2k}(x).$$

Theorem 2. *For $n \geq 1$, the following identities hold:*

$$F_n(x) + xF_{n-1}(x) = U_{n-1}(x) - \sum_{k=1}^{n-2}\big(xU_{n-k-1}(x) - 2U_{n-k-2}(x)\big)F_k(x);$$

$$2xF_{2n+1}(x) = U_{2n+1}(x) - U_{2n-1}(x) - (3x^2 - 4)\sum_{k=0}^{n-1} F_{2k+1}(x)U_{2(n-k)-1}(x);$$

$$F_{2n}(x) + F_{2n-2}(x) = xU_{2n-2}(x) - (3x^2 - 4)\sum_{k=1}^{n-1} F_{2k}(x)U_{2(n-k-1)}(x);$$

$$2xF_n(x) + (8x^3 - 2x^2 - 4x)F_{n-1}(x)$$
$$= U_{2n-1}(x) - \sum_{k=1}^{n-2}\big((4x^2 - x - 2)U_{2(n-k)-1}(x) - 2U_{2(n-k)-3}(x)\big)F_k(x);$$

$$F_n(x) + (4x^2 - x - 1)F_{n-1}(x)$$
$$= U_{2n-2}(x) - \sum_{k=1}^{n-2}\big((4x^2 - x - 2)U_{2(n-k-1)}(x) - 2U_{2(n-k-2)}(x)\big)F_k(x);$$

$$F_{2n+1}(x) + F_{2n-1}(x)$$
$$= U_{2n}(x) - U_{2n-2}(x) - (3x^2 - 4)\sum_{k=0}^{n-1} U_{2(n-k-1)}(x)F_{2k+1}(x);$$

$$2xF_{2n}(x) = xU_{2n-1}(x) - (3x^2 - 4)\sum_{k=1}^{n-1} U_{2(n-k)-1}(x)F_{2k}(x).$$

[1] R. Frontczak, *Some Fibonacci-Lucas-Tribonacci-Lucas identities*, Fibonacci Quart., **56** (3) (2018), 263–274.

[2] R. Frontczak, *Relations for generalized Fibonacci and Tribonacci sequences*, Notes Number Theory Discrete Math., **25** (1) (2019), 178–192.

[3] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, 2 ed., Wiley, New York, 2017.

[4] J.C. Mason, D.C. Handscomb, *Chebyshev Polynomials*, Chapman and Hall/CRC, Boca Raton, 2002.