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NONLOCAL BOUNDARY-VALUE PROBLEMS FOR SYSTEMS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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We study the classical well-posedness of problems with nonlocal two-point conditions for typeless systems of linear partial differential equations with variable coefficients in a cylindrical domain. We prove metric theorems on lower bounds for small denominators that appear in the construction of solutions of such problems.

1. Statement of the Problem

1. Boundary-value problems with nonlocal conditions for hyperbolic, parabolic, and typeless systems of partial differential equations with constant and variable coefficients have been studied by many authors (see, e.g., [1–19] and the bibliography therein). In general, such problems are conditionally well-posed and their solvability is connected with the problem of small denominators.

The present paper is closely related to [13–18] and is devoted to the investigation of the classical well-posedness of problems with nonlocal conditions in a separated variable t and conditions of the Dirichlet type in the variables $x_1, \ldots, x_p$ for typeless systems of linear partial differential equations with variable coefficients dependent on x. We establish conditions for the existence and uniqueness of classical solutions of problems, which are of number-theoretical character. To solve the problem of small denominators appearing in the construction of solutions of the problems considered, we use the metric approach.

We use the following notation: $x = (x_1, \ldots, x_p) \in \mathbf{R}^p$, $s = (s_0, s_1) \in \mathbf{Z}_+^2$, $|s|^* = s_0 + 2s_1$, $[a]$ is the integer part of a number a, $G \subset \mathbf{R}^p$ is a bounded domain with smooth boundary G, $Q = \{(t, x): t \in (0, T), x \in G\}$; $C^{(j,\nu)}$ is the class of functions defined in the domain $\overline{G}$ whose jth derivatives satisfy in $\overline{G}$ the Hölder condition with exponent ν, $0 < \nu < 1$, $A^{(j,\nu)}$ is the class of closed domains such that the functions that determine the equations of boundary surfaces of these domains in local coordinates belong to $C^{(j,\nu)}$, and $\overline{C}^r(\overline{Q})$ is the Banach space of vector functions $v(t, x) = (v_1(t, x), \ldots, v_m(t, x))$ continuous together with all their derivatives up to the rth order inclusive in the domain $\overline{Q}$ with the norm

$$\|v(t, x)\|_{\overline{C}^r(\overline{Q})} = \sum_{j=1}^{m} \sum_{|q| \le r} \max_{(t,x)\in\overline{Q}} \left| \frac{\partial^{|q|} v_j(t, x)}{\partial t^{q_0} \partial t_1^{q_1} \ldots \partial x_p^{q_p}} \right|.$$

2. In the domain Q, consider the problem

$$Pu(t, x) = \sum_{|s|^* \le n} A_s \left(\frac{\partial}{\partial t}\right)^{s_0} (-L)^{s_1} u(t, x) = f(t, x), \tag{1}$$

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$$\sum_{\substack{|s|^* \le n \\ s_0 < n}} B_s(-L)^{s_1} \left(\frac{\partial^{s_0} u(t,x)}{\partial t^{s_0}} \bigg|_{t=0} - \mu \frac{\partial^{s_0} u(t,x)}{\partial t^{s_0}} \bigg|_{t=T} \right) = \varphi(x), \quad \mu \in \mathbf{C} \setminus \{0\}, \tag{2}$$

$$L^j u(t,x) \big|_{\partial G} = 0, \quad j = 0, 1, \ldots, [n/2] - 1, \tag{3}$$

where $u(t,x) = \operatorname{col}(u_1(t,x), \ldots, u_m(t,x))$, $f(t,x) = \operatorname{col}(f_1(t,x), \ldots, f_m(t,x))$, $\varphi(x) = \operatorname{col}(\varphi_1(x), \ldots, \varphi_{nm}(x))$, $A_s = \| a_s^{ij} \|_1^m$ and $B_s = \| b_s^{ij} \|_{nm,m}$ are matrices of sizes $(m \times m)$ and $(nm \times m)$, respectively, with constant complex elements, $\det A_{(n,0)} \ne 0$, and the operator

$$L \equiv \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i} \left(p_{ij}(x) \frac{\partial}{\partial x_j} \right) - q(x), \quad p_{ij}(x) \ge p_0 > 0, \quad q(x) \ge 0,$$

is elliptic in $\overline{G}$. Assume that $\overline{G} \in A^{(2[n/2],\nu)}$, $p_{ij}(x) \in C^{(2[n/2]-1,\nu)}$, $i, j = 1, \ldots, p$, and $q(x) \in C^{(2[n/2]-2,\nu)}$. We do not impose any restrictions on the type of the operator P.

Under conditions imposed above on the domain G and the coefficients of the operator L, the eigenvalue problem

$$LX(x) = -\lambda X(x), \quad X(x)\big|_{\partial G} = 0 \tag{4}$$

has a complete system of classical eigenfunctions $\{X_k(x), k \in \mathbf{N}\}$ orthonormal in $L_2(G)$, and all eigenvalues λ_k, $k \in \mathbf{N}$, are positive. Denote the set of these eigenvalues by Λ. Furthermore, $X_k(x) \in C^{2[n/2]}(\overline{G})$, $k \in \mathbf{N}$, and the following estimates are true [20, 21]:

$$(\forall \lambda_k > K_1) \quad c_0 k^{2/p} \le \lambda_k \le c_1 k^{2/p}, \quad 0 < c_0 \le c_1, \tag{5}$$

$$\max_{x \in \overline{G}} \left| X_k^{(j)}(x) \right| \le c_2 \lambda_k^{p/4 + j/2}, \quad c_2 = c_2(j), \quad j = 0, 1, \ldots, 2[n/2]. \tag{6}$$

Let $f(t,x) \in \overline{C}([0,T], \overline{L}_2(G))$ and $\varphi(x) \in \overline{L}_2(G)$. Then the following expansions are true:

$$f(t,x) = \sum_{k=1}^{\infty} f_k(t) X_k(x), \quad f_k(t) = \operatorname{col}(f_{k1}(t), \ldots, f_{km}(t)),$$

$$\varphi(x) = \sum_{k=1}^{\infty} \varphi_k X_k(x), \quad \varphi_k = \operatorname{col}\left(\varphi_{k1}(t), \ldots, \varphi_{k,nm}\right),$$

where

$$f_{ki}(t) = \int_G f_i(t,x) X_k(x) dx, \quad i = 1, \ldots, m,$$

$$\varphi_{kj} = \int_G \varphi_j(x) X_k(x) dx, \quad j = 1, \ldots, nm.$$

3. We seek a solution of problem (1)–(3) in the form of a vector series

$$u(t,x) = \sum_{k=1}^{\infty} u_k(t) X_k(x), \qquad u_k(t) = \operatorname{col}(u_{k1}(t), \dots, u_{km}(t)). \tag{7}$$

If series (7) and series obtained from it by termwise differentiation with respect to the variables $x_1, \dots, x_p$ up to the order $2[n/2]$ are uniformly convergent in the domain $\overline{Q}$, then the vector function $u(t,x)$ defined by (7) satisfies the boundary conditions (3). Each vector function $u_k(t)$, $k \in \mathbf{N}$, is a solution of the following problem for a system of ordinary differential equations with nonlocal conditions:

$$\sum_{|s|^* \le n} A_s \lambda_k^{s_1} u_k^{(s_0)}(t) = f_k(t), \tag{8}$$

$$\sum_{\substack{|s|^* \le n \\ s_0 < n}} B_s \lambda_k^{s_1} \left(u_k^{(s_0)}(0) - \mu u_k^{(s_0)}(T) \right) = \varphi_k. \tag{9}$$

Consider the homogeneous problem that corresponds to problem (8), (9):

$$\sum_{|s|^* \le n} A_s \lambda_k^{s_1} u_k^{(s_0)}(t) = 0, \tag{10}$$

$$\sum_{\substack{|s|^* \le n \\ s_0 < n}} B_s \lambda_k^{s_1} \left(u_k^{(s_0)}(0) - \mu u_k^{(s_0)}(T) \right) = 0. \tag{11}$$

Assume that, for all $\lambda_k \in \Lambda$, the roots $\eta_j \equiv \eta_j(\lambda_k)$, $j = 1, \dots, nm$, of the characteristic equation

$$M(\eta, \lambda_k) \equiv \det \left\| \sum_{|s|^* \le n} A_s \lambda_k^{s_1} \eta^{s_0} \right\| = 0 \tag{12}$$

are simple and are not equal to zero.[3] Then, for every η_j,

$$\operatorname{rang} \left\| \sum_{|s|^* \le n} A_s \lambda_k^{s_1} \eta_j^{s_0} \right\| = m - 1, \qquad j = 1, \dots, nm,$$

and, therefore, at least one minor of the $(m-1)$th order of the determinant $M(\eta_j, \lambda_k)$ is not equal to zero (let it be the minor of an element of the row with the number $l = l(j)$). The homogeneous system of differential equations (10) has the following fundamental system of solutions:

$$Y_{kj}(t) = \operatorname{col}(h_{l1}(\eta_j), \dots, h_{lm}(\eta_j)) \exp(\eta_j t), \qquad j = 1, \dots, nm. \tag{13}$$

Here, $h_{lr}(\eta_j)$, $r = 1, \dots, m$, are the minors of the elements of the row with the number $l = l(j)$ of the determinant $M(\eta_j, \lambda_k)$, which are calculated according to the formulas

[3] The results of the present paper with insignificant modifications can be extended to the case where Eq. (12) has multiple roots.

$$h_{lr}(\eta_j(\lambda_k)) = \sum_{\substack{|q|^* \le n(m-1) \\ q_1 \le [n/2](m-1)}} \xi_q^{lr} \lambda_k^{q_1} \eta_j^{q_0}, \quad r = 1, \dots, m, \quad j = 1, \dots, nm, \tag{14}$$

$$\xi_q^{lr} \equiv \xi_{q_0, q_1}^{lr} = \sum_{\substack{\sum_{\beta=1}^{m} \omega_i(\beta) = q_i;\ i=0;1 \\ \beta \ne r}} \det \left\| a_{\omega_0(\beta), \omega_1(\beta)}^{\gamma\beta} \right\|_{\substack{\gamma, \beta = 1, \dots, m, \\ \gamma \ne l,\ \beta \ne r}}, \quad r = 1, \dots, m, \tag{15}$$

where $a_{\omega_0(\beta), \omega_1(\beta)}^{\gamma\beta}$, $\gamma = 1, \dots, m$, are the elements of the βth row of the matrix A_s, $s = (\omega_0(\beta), \omega_1(\beta))$.

Problem (10), (11) has nontrivial solutions if and only if its characteristic determinant $\Delta(\lambda_k)$ is equal to zero [22]. The determinant $\Delta(\lambda_k)$ is calculated according to the formula

$$\Delta(\lambda_k) = D(\lambda_k) E(\lambda_k) \prod_{j=1}^{nm} \left(1 - \mu \exp(\eta_j(\lambda_k) T)\right) \prod_{1 \le i < j \le nm} \left(\eta_j(\lambda_k) - \eta_i(\lambda_k)\right), \tag{16}$$

where

$$D(\lambda_k) = \det \left\| \sum_{s_1 \le (n - s_0)/2} b_s^{jr} \lambda_k^{s_1} \right\|_{\substack{j = 1, \dots, nm \\ r = 1, \dots, m;\ s_0 = 0, 1, \dots, n-1}}, \tag{17}$$

$$E(\lambda_k) = \begin{vmatrix} E_0^1 & E_1^1 & \dots & E_{n-2}^1 & E_{n-1}^1 & \dots & E_{n(m-1)}^1 & 0 & \dots & 0 \\ 0 & E_0^1 & \dots & E_{n-3}^1 & E_{n-2}^1 & \dots & E_{n(m-1)-1}^1 & E_{n(m-1)}^1 & \dots & 0 \\ \hdotsfor{10} \\ 0 & 0 & \dots & 0 & E_0^1 & \dots & E_{n(m-1)-n+1}^1 & E_{n(m-1)-n+2}^1 & \dots & E_{n(m-1)}^1 \\ \hdotsfor{10} \\ E_0^m & E_1^m & \dots & E_{n-2}^m & E_{n-1}^m & \dots & E_{n(m-1)}^m & 0 & \dots & 0 \\ 0 & E_0^m & \dots & E_{n-3}^m & E_{n-2}^m & \dots & E_{n(m-1)-1}^m & E_{n(m-1)}^m & \dots & 0 \\ \hdotsfor{10} \\ 0 & 0 & \dots & 0 & E_0^m & \dots & E_{n(m-1)-n+1}^m & E_{n(m-1)-n+2}^m & \dots & E_{n(m-1)}^m \end{vmatrix}, \tag{18}$$

$$E_j^r = E_j^r(\lambda_k) = \sum_{d=0}^{[(n(m-1)-j)/2]} \xi_{j,d}^{lr} \lambda_k^d, \quad j = 0, 1, \dots, n(m-1), \quad r = 1, \dots, m,$$

and $\xi_{j,d}^{lr}$ are defined by (15).

Remark 1. The determinant $E(\lambda_k)$ is not equal to zero for all $\lambda_k \in \Lambda$ because it is a factor in the expression for the determinant

$$W(\lambda_k) \equiv \det \left\| Y_{kj}^{(q)}(t) \right\|_{\substack{j = 1, \dots, nm, \\ q = 0, 1, \dots, n-1}},$$

which, as is known [22], is not equal to zero, and

$$W(\lambda_k) = E(\lambda_k) \prod_{j=1}^{nm} \exp(\eta_j(\lambda_k) t) \prod_{1 \le i < j \le nm} \left(\eta_j(\lambda_k) - \eta_i(\lambda_k)\right).$$

Theorem 1. *In order that problem (1)–(3) be uniquely solvable in the space* $\overline{C}^n(\overline{Q})$, *it is necessary and sufficient that the following conditions be satisfied:*

$$(\forall \lambda_k \in \Lambda) \quad 1-\mu \exp(\eta_j(\lambda_k)T) \neq 0, \quad j = 1, \ldots, nm; \quad D(\lambda_k) \neq 0. \tag{19}$$

Proof. The proof is similar to the proof of Theorem 5.3 in [16, Sec. 2] and follows from (16), (17), Remark 1, and the theorem on the uniqueness of the Fourier expansion of a function from the space $L_2(G)$ in a complete system of orthogonal functions.

4. Consider the problem of the existence of a solution of problem (1)–(3). Let conditions (19) be satisfied. Then, for every $\lambda_k \in \Lambda$, there exists a unique solution of problem (8), (9), which can be represented in the form of sum

$$u_k(t) = U_k(t) + V_k(t),$$

where $U_k(t) = \mathrm{col}\,(U_{k1}(t), \ldots, U_{km}(t))$ and $V_k(t) = \mathrm{col}\,(V_{k1}(t), \ldots, V_{km}(t))$ are solutions of problems (9), (10) and (8), (11), respectively. The components of the vector functions $U_k(t)$ and $V_k(t)$ are determined by the relations

$$U_{kj}(t) = \sum_{q=1}^{nm}\sum_{l=1}^{nm}\sum_{\alpha=1}^{nm}\sum_{p=1}^{nm}(-1)^{q-1}h_j(\eta_q(\lambda_k))D_{lp}(\lambda_k)E_{p\alpha}(\lambda_k)S^q_{nm-\alpha}$$

$$\times\left(E(\lambda_k)D(\lambda_k)(1-\mu\exp(\eta_q(\lambda_k)T))\prod_{i=1,\, i\neq q}^{nm}(\eta_q(\lambda_k)-\eta_i(\lambda_k))\right)^{-1}\varphi_{kl}\exp(\eta_q(\lambda_k)t), \quad j = 1, \ldots, m, \tag{20}$$

$$V_{kj}(t) = \int_0^T \sum_{r=1}^{m} G_{k,j,r}(t,\tau)f_{kr}(\tau)d\tau, \quad j = 1, \ldots, m, \tag{21}$$

where $D_{ij}(\lambda_k)$ and $E_{ij}(\lambda_k)$ are the determinants obtained from $D(\lambda_k)$ and $E(\lambda_k)$, respectively, by deleting the ith row and jth column, S^q_γ is the sum of all possible products of γ factors $\eta_j(\lambda_k)$, $j = 1, \ldots, nm$, $i \neq q$ $(S^q_0 \equiv 1)$, and $G_{k,j,r}(t,\tau)$, $j, r = 1, \ldots, m$, are the elements of the Green matrix of problem (8), (9), which are determined by the following relations in the square $K_T = \{(t,\tau) \in \mathbf{R}^2_+ : 0 \le t, \tau \le T\}$ except for the sides $\tau = 0$ and $\tau = T$:

$$G_{k,j,r}(t,\tau) = (2D(\lambda_k))^{-1}\sum_{q=1}^{nm}\sum_{\alpha=1}^{nm}D_{r\alpha}(\lambda_k)S^q_{nm-\alpha}\prod_{\substack{\beta=1\\ \beta\neq q}}^{nm}(\eta_\beta(\lambda_k)-\eta_q(\lambda_k))^{-1}$$

$$\times\Bigg((-1)^{n(q-1)+1}\operatorname{sgn}(t-\tau)h_j(\eta_q(\lambda_k))\exp(\eta_q(\lambda_k)(t-\tau))$$

$$+\sum_{l=1}^{nm}\sum_{i=1}^{nm}\sum_{p=1}^{nm}\sum_{|s|^*\le n}(-1)^{(n-1)p+1}\,b_s^{ip}\lambda_k^{s_1}\eta_q^{s_0}(\lambda_k)h_p(\eta_q(\lambda_k))h_j(\eta_l(\lambda_k))$$

$$\times \exp\big(\eta_q(\lambda_k)t\big)D_{lp}(\lambda_k)E_{pi}(\lambda_k)S^q_{nm-l}(D(\lambda_k)E(\lambda_k))^{-1}$$

$$\times \prod_{\substack{\alpha=1\\ \alpha\neq l}}^{nm}(\eta_\alpha(\lambda_k)-\eta_l(\lambda_k))^{-1}\frac{1+\mu\exp(\eta_l(\lambda_k)T)}{1-\mu\exp(\eta_l(\lambda_k)T)}\Bigg),\qquad j,r=1,\ldots,m, \tag{22}$$

where S^q_γ are the same as in relations (20). On the side $\tau = 0$ $(\tau = T)$ of the square K_T, each function $G_{k,j,r}(t,\tau)$, $j, r = 1,\ldots,m$, is defined by continuity from the right (left).

A solution of problem (1)–(3) can be formally represented by the series

$$u(t,x) = \sum_{k=1}^{\infty}(U_k(t)+V_k(t))X_k(x), \tag{23}$$

where the components of the vector functions $U_k(t)$ and $V_k(t)$ are defined by relations (20)–(22). In general, the problem of convergence of series (23) is connected with the problem of small denominators because the expressions

$$1-\mu\exp\big(\eta_j(\lambda_k)T\big),\qquad \prod_{\substack{q=1\\ q\neq j}}^{nm}\big(\eta_j(\lambda_k)-\eta_q(\lambda_k)\big),\quad j=1,\ldots,nm,\qquad D(\lambda_k),\qquad E(\lambda_k),$$

which appear as denominators in relations (20) and (22), are not equal to zero and can be arbitrarily small in modulus for an infinite set of $\lambda_k \in \Lambda$.

Let us introduce functional spaces that will be used for the investigation of the solvability of problem (1)–(3). Denote

$$B^\omega_q = \left\{v(x)\in L_2(G):\ v(x) = \sum_{k=1}^{\infty}v_kX_k(x),\quad \|v(x)\|_{q,\omega} \equiv \sum_{k=1}^{\infty}|v_k|\exp\big(q\lambda_k^\omega\big)<\infty\right\},\qquad q>0,\quad \omega>0,$$

and let $C\big([0,T],\ B^\omega_q\big)$ be the space of functions $w(t,x)$ that are defined and continuous in the domain Q and, for every $t\in[0,T]$, belong to B^ω_q with the norm

$$\|w(t,x)\|_{C([0,T],\ B^\omega_q)} = \sum_{k=1}^{\infty}\max_{t\in[0,T]}|w_k(t)|\exp\big(q\lambda_k^\omega\big),$$

where

$$w_k(t) = \int_G w(t,x)X_k(x)dx,\qquad k\in\mathbf{N}.$$

Note that Eq. (12) implies the following estimates:

$$(\forall\lambda_k>K_2)\quad |\eta_j(\lambda_k)| \le \alpha\lambda_k^{1/2},\qquad j=1,\ldots,nm,\quad \alpha>0. \tag{24}$$

Theorem 2. *Suppose that there exist positive constants m_j, γ_j, $j = 1,\ldots,4$, such that the following inequalities hold for all $\lambda_k\in\Lambda$, $\lambda_k>K_3$:*

$$|1-\mu\exp(\eta_j(\lambda_k)T)| \geq m_1\lambda_k^{-\gamma_1}\exp(-|\operatorname{Re}\eta_j(\lambda_k)|T), \quad j=1,\ldots,nm, \tag{25}$$

$$\prod_{\substack{q=1\\ q\neq j}}^{nm}|\eta_j(\lambda_k)-\eta_q(\lambda_k)| \geq m_2\lambda_k^{-\gamma_2}, \quad j=1,\ldots,nm, \tag{26}$$

$$|D(\lambda_k)| \geq m_3\lambda_k^{-\gamma_3}, \tag{27}$$

$$|E(\lambda_k)| \geq m_4\lambda_k^{-\gamma_4}. \tag{28}$$

If $f_i(t,x)\in C([0,T], B_q^{1/2})$, $i=1,\ldots,m$, $\varphi_j(x)\in B_q^{1/2}$, $j=1,\ldots,nm$, *where* $q>\alpha T$, *then there exists the unique solution of problem (1)–(3) in the space* $\overline{C}^n(\overline{D})$, *which continuously depends on the vector functions* $f(t,x)$ *and* $\varphi(x)$.

Proof. It follows from relations (20)–(23) and estimates (6) and (24)–(28) that

$$\|u(t,x)\|_{\overline{C}^n(\overline{D})} \leq c_3\sum_{k\leq K}\left(\sum_{j=1}^{nm}|\varphi_{kj}|+\sum_{i=1}^{m}\max_{t\in[0,T]}|f_{ki}(t)|\right)$$

$$+\,c_4\sum_{k>K}\left(\sum_{j=1}^{nm}|\varphi_{kj}|\lambda_k^{\sigma_1}\exp(\alpha T\sqrt{\lambda_k})+\sum_{i=1}^{m}\max_{t\in[0,T]}|f_{ki}(t)|\lambda_k^{\sigma_2}\exp(\alpha T\sqrt{\lambda_k})\right), \tag{29}$$

where

$$K = \max(K_1,K_2,K_3), \quad \sigma_1 = \chi-(n+3)/2, \quad \sigma_2 = \chi+p/4,$$

$$\chi = nm+m[n/2][(n+1)/2]-[n/2]-[n(m-1)/2]+n\sum_{j=1}^{m-1}[n(m-j)/2]+\sum_{j=1}^{4}\gamma_j.$$

By using the elementary inequality

$$q^\delta \leq c_5\exp(\rho q), \quad c_5 = c_5(\delta), \quad q>0,$$

which holds for arbitrary $\delta\geq 0$ and $\rho>0$, we obtain from estimates (5) and (29) that

$$\|u(t,x)\|_{\overline{C}^n(\overline{D})} \leq c_6\sum_{k=1}^{\infty}\left(\sum_{j=1}^{nm}|\varphi_{kj}|\exp(q\sqrt{\lambda_k})+\sum_{i=1}^{m}\max_{t\in[0,T]}|f_{ki}(t)|\exp(q\sqrt{\lambda_k})\right)$$

$$= c_6\left(\sum_{j=1}^{nm}\|\varphi_j(x)\|_{q,1/2}+\sum_{i=1}^{m}\|f_i(t,x)\|_{C([0,T],\,B_q^{1/2})}\right) < \infty.$$

The theorem is proved.

5. Let us investigate under what conditions estimates (25)–(28) can be satisfied.

Lemma 1. *Let* $\Phi(\lambda_k)$ *be a bounded sequence of real numbers. Then for almost all (with respect to the Lebesgue measure in* $\mathbf{R}$*) numbers* $T > 0$, *the following inequality holds for all (except finitely many) pairs of numbers* $(\lambda_k, d), \lambda_k \in \Lambda, d \in \mathbf{Z}$:

$$\left|\Phi(\lambda_k) - Td/\sqrt{\lambda_k}\right| \geq \lambda_k^{-(p+1)/2-\delta}, \quad 0 < \delta < 1.$$

Proof. The proof can be carried out according to the scheme of the proof of Lemma 2.4 in Sec. 1 of [16] with regard for estimates (5).

Theorem 3. *For almost all (with respect to the Lebesgue measure in* $\mathbf{R}$*)* $T > 0$ *and arbitrary fixed* μ *and* a_s^{ij}, $|s|^* \leq n$, $i, j = 1, \ldots, m$, *inequalities (25) hold for* $\gamma_1 > p/2$ *for all (except finitely many) values* $\lambda_k \in \Lambda$.

Proof. By using the inequality $\sin x \geq 2x/\pi$, which is true for all $x \in [0, \pi/2]$, we get

$$\left|1 - \mu \exp(\eta_j(\lambda_k)T)\right| \geq |\mu| \exp(\operatorname{Re}\eta_j(\lambda_k)T) \left|\sin\left(\psi + \operatorname{Im}\eta_j(\lambda_k)T\right)\right|$$

$$> |\mu| \exp\left(-\left|\operatorname{Re}\eta_j(\lambda_k)\right|T\right) \left|\frac{\psi + \operatorname{Im}\eta_j(\lambda_k)T}{\pi} - d_j(\lambda_k)\right|, \quad j = 1, \ldots, nm, \tag{30}$$

where $\psi = \arg\mu$ and $d_j(\lambda_k) \in \mathbf{Z}$ is such that

$$\left|\frac{\psi + \operatorname{Im}\eta_j(\lambda_k)T}{\pi} - d_j(\lambda_k)\right| \leq \frac{1}{2}.$$

By using Lemma 1 and estimates (24) and (30), we establish that, for all (except finitely many) $\lambda_k \in \Lambda$ and almost all numbers $T > 0$, the following estimates are true:

$$\left|1 - \mu \exp(\eta_j(\lambda_k)T)\right| \geq |\mu| T^{-1}\sqrt{\lambda_k} \exp(\operatorname{Re}\eta_j(\lambda_k)T) \left|\frac{\psi T/\pi + \operatorname{Im}\eta_j(\lambda_k)T^2/\pi}{\sqrt{\lambda_k}} - \frac{T d_j(\lambda_k)}{\sqrt{\lambda_k}}\right|$$

$$\geq |\mu| T^{-1} \lambda_k^{-p/2-\delta} \exp\left(-\left|\operatorname{Re}\eta_j(\lambda_k)\right|T\right), \quad j = 1, \ldots, nm,$$

where δ is an arbitrary positive number. The theorem is proved.

By $Y \in \mathbf{R}^\sigma$, we denote the vector composed of the real and imaginary parts of the numbers a_s^{ij}, $|s|^* \leq n$, $i, j = 1, \ldots, m$, of system (1), where

$$\sigma = 2m^2\left(n + 1 + \sum_{j=1}^{n}[j/2]\right).$$

Theorem 4. *For almost all (with respect to the Lebesgue measure in the space* $\mathbf{R}^\sigma$*) vectors* Y, *inequalities (26) hold for all (except finitely many)* $\lambda_k \in \Lambda$ *for* $\gamma_2 \geq (nm-1)(p-2-2m[n/2]+nm)/4$.

Proof. Let us represent the polynomial $M(\eta, \lambda_k)$ in the form

$$M(\eta, \lambda_k) = \sum_{j=0}^{nm} H_j(\lambda_k)\eta^j,$$

where $H_j(\lambda_k) \equiv H_j$ is a polynomial in λ_k whose degree does not exceed $[(nm-j)/2]$ and whose coefficients are expressed via the elements of the matrices A_s, $j = 0, 1, \ldots, nm$, $H_{nm} = \det A_{(n,0)}$. For the discriminant $W(M)$ of the polynomial $M(\eta, \lambda_k)$, the following representations are true:

$$W(M) = H_{nm}^{2(nm-1)} \prod_{1 \le i < j \le nm} \big(\eta_j(\lambda_k) - \eta_i(\lambda_k)\big)^2, \tag{31}$$

$$W(M) = \frac{(-1)^{nm(nm-1)/2}}{H_{nm}} \times \begin{vmatrix} H_{nm} & H_{nm-1} & \ldots & H_0 & 0 & \ldots & 0 \\ 0 & H_{nm} & \ldots & H_1 & H_0 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & H_{nm-2} & H_{nm-3} & \ldots & H_0 \\ nmH_{nm} & (nm-1)H_{nm-1} & \ldots & 0 & 0 & \ldots & 0 \\ 0 & nmH_{nm} & \ldots & H_1 & 0 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & (nm-2)H_{nm-2} & (nm-3)H_{nm-3} & \ldots & H_1 \end{vmatrix}. \tag{32}$$

By using the scheme of the proof of Theorem 6 in [23], we establish that, for almost all (with respect to the Lebesgue measure in $\mathbf{R}^{\sigma}$) vectors Y and all (except finitely many) $\lambda_k \in \Lambda$, the following inequality is true:

$$|\operatorname{Re} W(M)| \ge \lambda_k^{-\nu-\varepsilon}, \quad \nu = (nm-1)(p/2 - m[n/2]), \quad \varepsilon > 0. \tag{33}$$

Since $|W(M)| \ge |\operatorname{Re} W(M)|$, it follows from (31) that the following estimate holds for almost all vectors $Y \in \mathbf{R}^{\sigma}$:

$$\prod_{1 \le i < j \le nm} |\eta_j(\lambda_k) - \eta_i(\lambda_k)| \ge \lambda_k^{-(nm-1)(p-2m[n/2])/4 - \varepsilon/2}. \tag{34}$$

The equality

$$\prod_{\substack{q=1 \\ q \ne j}}^{nm} (\eta_q(\lambda_k) - \eta_j(\lambda_k)) = \prod_{1 \le i < j \le nm} |\eta_j(\lambda_k) - \eta_i(\lambda_k)| \prod_{\substack{1 \le \alpha < \beta \le nm \\ \alpha \ne j, \beta \ne j}} |\eta_\alpha(\lambda_k) - \eta_\beta(\lambda_k)|^{-1}$$

and estimates (24) and (34) imply that, for almost all vectors $Y \in \mathbf{R}^{\sigma}$,

$$\prod_{\substack{q=1 \\ q \ne j}}^{nm} (\eta_q(\lambda_k) - \eta_j(\lambda_k)) \ge c_7 \lambda_k^{-(nm-1)(p-2+nm-2m[n/2])/4 - \varepsilon/2}, \quad j = 1, \ldots, nm, \ c_7 > 0.$$

The theorem is proved.

Now consider inequalities (27) and (28). The determinants $E(\lambda_k)$ and $D(\lambda_k)$ are polynomials of the form

$$E(\lambda_k) = \sum_{r=0}^{M_1} E_r \lambda_k^r,$$

$$D(\lambda_k) = \sum_{j=0}^{M_2} D_j \lambda_k^j,$$

where

$$M_1 = n \sum_{j=1}^{m-1} [n(m-j)/2],$$

$$M_2 = m[n/2][(n+1)/2],$$

and the coefficients E_r and D_j are expressed via the elements of the matrices A_s and B_s, respectively. Let $\beta_1 = \left(\beta_1^{(1)}, \ldots, \beta_\xi^{(1)}\right)$ and $\beta_2 = \left(\beta_1^{(2)}, \ldots, \beta_\xi^{(2)}\right)$ be vectors composed, respectively, of the real and imaginary parts of the numbers b_s^{ij}; here,

$$\xi = m^2 n \left(n + \sum_{j=1}^{n} [j/2] \right).$$

The following statements are true:

Theorem 5. *For almost all (with respect to the Lebesgue measure in* $\mathbf{R}^\xi$*) vectors* β_1 *and arbitrary fixed* β_2 *(or for almost all* β_2 *and arbitrary fixed* β_1*), inequality (27) holds for* $\gamma_3 > p/2$ *for all (except finitely many) values* $\lambda_k \in \Lambda$.

Theorem 6. *For almost all (with respect to the Lebesgue measure in* $\mathbf{R}^\sigma$*) vectors* Y*, inequality (28) holds for* $\gamma_4 > p/2$ *for all (except finitely many) values* $\lambda_k \in \Lambda$.

If the free terms of the polynomials $D(\lambda_k)$ and $E(\lambda_k)$ are not equal to zero, then Theorems 5 and 6 can be proved according to the scheme of the proof of Theorem 4 in [23]. If the polynomials $D(\lambda_k)$ and $E(\lambda_k)$ do not contain free terms, then the proof of these theorems is similar to that of Theorem 6 in [24].

The results are generalized to the case of the following problem for a typeless system of linear partial differential equations perturbed by a nonlinear integro-differential operator:

$$\frac{\partial^n u_j(t,x)}{\partial t^{n_j}} - \sum_{r=1}^{m} P_{jr}\left(\frac{\partial}{\partial t}, L\right) u_r(t,x) = f_j(t,x) + \varepsilon \int_G \sum_{q=1}^{m} K_{jq}(t,x,y) F_q(t,y,\bar{u}(t,y))\,dy, \qquad j = 1, \ldots, m,$$

$$\sum_{j=1}^{m} \sum_{\substack{s \le n_j \\ l \le H}} b_{sl}^{qj} (-L)^l \left(\left.\frac{\partial^s u_j(t,x)}{\partial t^s}\right|_{t=0} - \mu \left.\frac{\partial^s u_j(t,x)}{\partial t^s}\right|_{t=T} \right) = \varphi_q(x), \qquad q = 1, \ldots, n,$$

$$L^q u_j(t,x)\big|_{\partial G} = 0, \quad j = 1, \ldots, m, \quad q = 0, 1, \ldots, H-1,$$

where

$$n_1 + \ldots + n_m = n, \qquad P_{jr}\left(\frac{\partial}{\partial t}, L\right) \equiv \sum_{\substack{s<n_j \\ l\le H}} p_{jr}^{sl}\left(\frac{\partial}{\partial t}\right)^s (-L)^l,$$

$$j, r = 1, \ldots, m; \quad p_{jr}^{sl},\ b_{sl}^{qj} \in \mathbf{C}, \quad \varepsilon,\ \mu \in \mathbf{C}\setminus\{0\},$$

$$\bar{u}(t,y) = \left\{\frac{\partial^{s_0+|s|}u_q(t,y)}{\partial t^{s_0}\,\partial y_1^{s_1}\ldots\partial y_p^{s_p}};\ s_0 \le n_q,\ |s| \le 2H,\ q = 1,\ldots,m\right\}.$$

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REFERENCES

1. A. A. Makarov, "On necessary and sufficient conditions of solvability of a boundary-value problem in a layer for systems of partial differential equations," *Differents. Uravn.*, **17**, No. 2, 320–324 (1981).
2. G. B. Savchenko, "On the well-posedness of one nonlocal boundary-value problem," *Differents. Uravn.*, **21**, No. 8, 1450–1453 (1985).
3. V. K. Romanko, "Mixed boundary-value problems for one system of equations," *Dokl. Akad. Nauk SSSR*, **286**, No. 1, 47–50 (1986).
4. V. K. Romanko, "On general boundary-value problems for nonlinear systems of equations with constant coefficients," *Mat. Met. Fiz.-Mekh. Polya*, **23**, 3–7 (1986).
5. E. Sinestrari and G. F. Webb, "Nonlinear hyperbolic systems with nonlocal boundary conditions," *J. Math. Anal. Appl.*, **121**, No. 2 449–464 (1987).
6. I. Ya. Kmit', "On one problem with nonlocal time conditions for systems of hyperbolic type," *Mat. Met. Fiz.-Mekh. Polya*, **37**, 21–25 (1994).
7. V. V. Marinets, "On some problems for systems of nonlinear partial differential equations," *Differents. Uravn.*, **24**, No. 8, 1393–1397 (1988).
8. Yu. A. Mamedov, "On correct solvability of general mixed problems," *Differents. Uravn.*, **26**, No. 3, 534–537 (1990).
9. V. A. Malovichko, "On the solvability of nonlocal boundary-value problems for pseudoparabolic systems and systems of composite type," *Mat. Zametki*, **47**, No. 5, 151–153 (1990).
10. Yu. A. Mitropol'skii and L. B. Urmancheva, "On a two-point problem for systems of hyperbolic equations," *Ukr. Mat. Zh.*, **42**, No. 2 1657–1663 (1990).
11. T. I. Kiguradze, "On one boundary-value problem for hyperbolic systems," *Dokl. Ros. Akad. Nauk, Mat.*, **328**, No. 2, 135–138 (1993).
12. T. I. Kiguradze, "On solutions of quasilinear hyperbolic systems bounded and periodic in a strip," *Differents. Uravn.*, **30**, No. 10, 1760–1773 (1994).
13. Yu. M. Berezanskii, *Expansion in Eigenfunctions of Self-Adjoint Operators* [in Russian], Naukova Dumka, Kiev (1965).
14. V. I. Polishchyuk and B. I. Ptashnyk, "Periodic solutions of a system of partial differential equations with constant coefficients," *Ukr. Mat. Zh.*, **32**, No. 2, 239–243 (1980).
15. V. S. Il'kiv and B. I. Ptashnyk, "On a problem with nonlocal conditions for systems of partial differential equations with constant coefficients," *Differents. Uravn.*, **20**, No. 6, 1012–1023 (1984).
16. B. I. Ptashnyk, *Ill-Posed Boundary-Value Problems for Partial Differential Equations* [in Russian], Naukova Dumka, Kiev (1984).
17. N. M. Zadorozhna, *Problems with Nonlocal Boundary-Value Conditions for Parabolic Equations and Systems* [in Ukrainian], Author's Abstract of the Candidate-Degree Thesis (Physics and Mathematics), L'viv (1995).
18. V. S. Il'kiv and B. I. Ptashnyk, "Representations and investigation of solutions of a nonlocal boundary-value problem for a system of partial differential equations," *Ukr. Mat. Zh.*, **48**, No. 2, 184–194 (1996).

19. M. I. Matviichuk, "On one nonlocal parabolic boundary-value problem," *Ukr. Mat. Zh.*, **48**, No. 3, 362–367 (1996).

20. V. P. Mikhailov, *Partial Differential Equations* [in Russian], Nauka, Moscow (1983).

21. V. A. Il'in and I. A. Shishmarev, "Estimates of eigenfunctions of an elliptic operator uniform in a closed domain and their derivatives," *Izv. Akad. Nauk SSSR, Ser. Mat.*, **24**, 883–896 (1960).

22. M. A. Naimark, *Linear Differential Operators* [in Russian], Nauka, Moscow (1969).

23. V. I. Bernik, B. I. Ptashnyk, and B. O. Salyga, "Analog of the multipoint problem for a hyperbolic equation with constant coefficients," *Differents. Uravn.*, **13**, No. 4, 637–645 (1977).

24. V. G. Sprindzhuk, *Metric Theory of Diophantine Approximations* [in Russian], Nauka, Moscow (1977).