SOME IDENTITIES INVOLVING PELL NUMBERS

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The Pell numbers are an integer sequence defined by recurrence relation: $P_0 = 0$, $P_1 = 1$ and

$$P_n = 2P_{n-1} + P_{n-2}, \qquad n \geq 2.$$

The Pell sequence has a rich history and many remarkable properties. As well as being used to approximate the square root of 2, the Pell numbers can be used to find square triangular numbers, to construct integer approximations to the right isosceles triangle, and to solve certain combinatorial enumeration problems (see entry A000129 in [5], for more details).

We investigate some families of Toeplitz-Hessenberg determinants the entries of which are Pell numbers. As result, we obtained new identities with multinomial coefficients for these numbers.

Our approach is similar in spirit to [1–4].

Proposition. *Let $n \geq 1$, except when noted otherwise. Then*

$$\sum_{\tau_n=n} (-1)^{|t|} m_n(t) \left(\frac{P_0}{2}\right)^{t_1} \left(\frac{P_1}{2}\right)^{t_2} \cdots \left(\frac{P_{n-1}}{2}\right)^{t_n} = \frac{(2-\sqrt{6})^{n-1} - (2+\sqrt{6})^{n-1}}{\sqrt{6}\cdot 2^n},$$

$$\sum_{\tau_n=n} m_n(t) \left(\frac{P_0}{2}\right)^{t_1} \left(\frac{P_1}{2}\right)^{t_2} \cdots \left(\frac{P_{n-1}}{2}\right)^{t_n} = \frac{(2+\sqrt{10})^{n-1} - (2-\sqrt{10})^{n-1}}{\sqrt{10}\cdot 2^n},$$

$$\sum_{\tau_n=n} (-1)^{|t|} m_n(t) \left(\frac{P_1}{2}\right)^{t_1} \left(\frac{P_2}{2}\right)^{t_2} \cdots \left(\frac{P_{n}}{2}\right)^{t_n} = \frac{1-(-4)^n}{5(-2)^n},$$

$$\sum_{\tau_n=n} m_n(t) \left(\frac{P_1}{2}\right)^{t_1} \left(\frac{P_2}{2}\right)^{t_2} \cdots \left(\frac{P_{n}}{2}\right)^{t_n} = \frac{(5+\sqrt{41})^{n} - (5-\sqrt{41})^{n}}{\sqrt{41}\cdot 4^n},$$

$$\sum_{\tau_n=n} (-1)^{|t|} m_n(t) \left(\frac{P_2}{2}\right)^{t_1} \left(\frac{P_3}{2}\right)^{t_2} \cdots \left(\frac{P_{n+1}}{2}\right)^{t_n} = \frac{(1-\sqrt{3})^{n+1} - (1+\sqrt{3})^{n+1}}{\sqrt{3}\cdot 2^{n+1}},$$

$$\sum_{\tau_n=n} (-1)^{|t|} m_n(t) \left(\frac{P_3}{2}\right)^{t_1} \left(\frac{P_4}{2}\right)^{t_2} \cdots \left(\frac{P_{n+2}}{2}\right)^{t_n} = \frac{1}{(-2)^n}, \qquad n \geq 2,$$

$$\sum_{\tau_n=n} m_n(t) \left(\frac{P_0}{2}\right)^{t_1} \left(\frac{P_2}{2}\right)^{t_2} \cdots \left(\frac{P_{2n-2}}{2}\right)^{t_n} = 6^{n-2}, \qquad n \geq 2,$$

$$\sum_{\tau_n=n} (-1)^{|t|} m_n(t) \left(\frac{P_2}{2}\right)^{t_1} \left(\frac{P_4}{2}\right)^{t_2} \cdots \left(\frac{P_{2n}}{2}\right)^{t_n} = \frac{(5+\sqrt{21})^{n} - (5-\sqrt{21})^{n}}{\sqrt{21}\cdot 2^n},$$

$$\sum_{\tau_n=n} (-1)^{|t|} m_n(t) \left(\frac{P_4}{2}\right)^{t_1} \left(\frac{P_6}{2}\right)^{t_2} \cdots \left(\frac{P_{2n+2}}{2}\right)^{t_n} = 0, \qquad n \geq 3,$$

where $\tau_n = t_1 + 2t_2 + \cdots + nt_n$, $|t| = t_1 + \cdots + t_n$, $m_n(t) = \frac{|t|!}{t_1! \cdots t_n!}$ is the multinomial coefficient, and the summation is over integers $t_i \geq 0$ satisfying $\tau_n = n$.

References

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