which gives

$$\sum_{n=0}^{\infty}[P_n(x)-F_n(x)]\,t^n = x\left(\sum_{i=0}^{\infty}P_i(x)t^i\right)\left(\sum_{j=0}^{\infty}F_j(x)t^j\right).$$

After comparing coefficients of t^n, we obtain

$$P_n(x)=F_n(x)+x\sum_{s=0}^{n}P_{n-s}(x)\,F_s(x).$$

The second relation follows from

$$\frac{2-xt}{1-2xt-t^2}-\frac{2-xt}{1-xt-t^2}=x\cdot\frac{t}{1-2xt-t^2}\cdot\frac{2-xt}{1-xt-t^2},$$

which leads to

$$2P_{n+1}(x)-xP_n(x)=L_n(x)+x\sum_{s=0}^{n}P_{n-s}(x)\,L_s(x).$$

When $x=1$ these relations reduce to

$$P_n=F_n+\sum_{s=0}^{n}P_{n-s}F_s,\qquad\text{and}\qquad 2P_{n+1}-P_n=L_n+\sum_{s=0}^{n}P_{n-s}L_s.$$

Solution 2 by T. Goy, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

Define $D_0=1$, and let

$$D_n=\operatorname{perm}\begin{pmatrix} F_1 & 1 & 0 & \cdots & 0 & 0\\ F_2 & F_1 & 1 & \cdots & 0 & 0\\ F_3 & F_2 & F_1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ F_{n-1} & F_{n-2} & F_{n-3} & \cdots & F_1 & 1\\ F_n & F_{n-1} & F_{n-2} & \cdots & F_2 & F_1 \end{pmatrix}.$$

It is easy to verify that $D_1=P_1$ and $D_2=P_2$, and $D_n=\sum_{i=1}^{n}F_iD_{n-i}$. It follows that

$$\begin{aligned} D_n &= F_1D_{n-1}+\sum_{i=2}^{n}\big(F_{i-1}+F_{i-2}\big)D_{n-i}\\ &= D_{n-1}+\sum_{i=1}^{n-1}F_iD_{n-1-i}+\sum_{i=0}^{n-2}F_iD_{n-2-i}\\ &= 2D_{n-1}+D_{n-2}. \end{aligned}$$

Since D_n satisfies the Pell recurrence relation with $D_1=P_1$ and $D_2=P_2$, we determine that $D_n=P_n$ for all integers $n\ge 1$. Since

$$\operatorname{perm}\begin{pmatrix} a_1 & a_0 & \cdots & 0 & 0\\ a_2 & a_1 & \ddots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ a_{n-1} & a_{n-2} & \cdots & a_1 & a_0\\ a_n & a_{n-1} & \cdots & a_2 & a_1 \end{pmatrix} = \sum_{\substack{t_1,t_2,\ldots,t_n\ge 0\\ t_1+2t_2+\cdots+nt_n=n}} a_0^{\,n-t_1-\cdots-t_n}\,\frac{(t_1+\cdots+t_n)!}{t_1!\cdots t_n!}\,a_1^{t_1}a_2^{t_2}\cdots a_n^{t_n},$$

we obtain

$$P_n = \sum_{\substack{t_1,t_2,\dots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} \frac{(t_1+\cdots+t_n)!}{t_1!\cdots t_n!} F_1^{t_1}F_2^{t_2}\cdots F_n^{t_n}.$$

Editor's Notes: Frontczak derived a similar result for the generalized Pell sequence with arbitrary initial values, thereby obtaining the same result in Solution 1. Using a different method, Edgar derived a formula in terms of compositions of n that can be expressed as

$$P_n = \sum_{k=1}^{n} \sum_{\substack{i_1,i_2,\dots,i_k\geq 1\\ i_1+i_2+\cdots+i_k=n}} F_{i_1}F_{i_2}\cdots F_{i_k},$$

which is equivalent to Solution 2. Fedak and the proposer compared the Binet's formulas of P_n to those of F_n and L_n. Fedak found that

$$P_n = \frac{\sqrt{5}\,(\gamma^n+\delta^n)F_n + (\gamma^n-\delta^n)L_n}{4\sqrt{2}},$$

where $\gamma = \frac{1+\sqrt{2}}{\alpha}$ and $\delta = \frac{1-\sqrt{2}}{\beta}$, and the proposer proved that $P_n = \text{round}\left(\frac{r^n(L_n+F_n\sqrt{5})}{4\sqrt{2}}\right)$, where $r = \frac{2(1+\sqrt{2})}{1+\sqrt{5}}$.

Also solved by Tom Edgar, I. V. Fedak, Robert Frontczak, Raphael Schumacher, and the proposer.

A Not-So-Obvious Application of Cauchy-Schwarz Inequality

B-1241 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine. (Vol. 57.1, February 2019)

For all positive integers n, prove that

$$\frac{F_{n+2}}{L_{n+2}} + \frac{F_{n+1}}{L_{n+1}} + \frac{F_n}{L_{n+1}+F_{n+2}} > 1.$$

Solution by Hideyuki Ohtsuka, Saitama, Japan.

By Cauchy-Schwarz inequality, for $a, b, c > 0$, we have

$$[a(a+2b)+b(b+2c)+c(c+2a)]\left(\frac{a}{a+2b}+\frac{b}{b+2c}+\frac{c}{c+2a}\right) \geq (a+b+c)^2.$$

From the above inequality and the identity

$$a(a+2b)+b(b+2c)+c(c+2a) = a^2+b^2+c^2+2ab+2bc+2ca = (a+b+c)^2,$$

we obtain

$$\frac{a}{a+2b}+\frac{b}{b+2c}+\frac{c}{c+2a} \geq 1,$$

where equality occurs if and only if $a=b=c$. Using the identity $L_m = F_{m+1}+F_{m-1}$, we find

$$\begin{aligned}\frac{F_{n+2}}{L_{n+2}} + \frac{F_{n+1}}{L_{n+1}} + \frac{F_n}{L_{n+1}+F_{n+2}} &= \frac{F_{n+2}}{F_{n+3}+F_{n+1}} + \frac{F_{n+1}}{F_{n+2}+F_n} + \frac{F_n}{F_n+2F_{n+2}}\\ &= \frac{F_{n+2}}{F_{n+2}+2F_{n+1}} + \frac{F_{n+1}}{F_{n+1}+2F_n} + \frac{F_n}{F_n+2F_{n+2}} > 1.\end{aligned}$$