A note on some identities involving Mersenne numbers

T. Goy, R. Zatorsky
Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine
tarasgoy@yahoo.com, romazatorsky@gmail.com

In this paper, we derive new families of the Mersenne identities with binomial coefficients. These identities can be used to develop new identities of Mersenne numbers.
Keywords: Mersenne sequence, Mersenne number, binomial coefficient.

1. Mersenne numbers. A *Mersenne number*, denoted by M_n, is a number of the form

$$M_n = 2^n - 1, \tag{1}$$

where n is a nonnegative number. The *Mersenne sequence* $\{M_n\}_{n\geq 0}$ can be defined recursively as follows

$$M_0 = 0, \quad M_1 = 1, \quad M_n = 3M_{n-1} - 2M_{n-2}, \tag{2}$$

for $n \geq 2$ (sequence A000930 in On-Line Encyclopedia of Integer Sequences; (Sloane, 2018)). The first fifteen terms of the Mersenne sequence are

$$0,\ 1,\ 3,\ 7,\ 15,\ 31,\ 63,\ 127,\ 255,\ 511,\ 1023,\ 2047,\ 4095,\ 8191,\ 16383,\ \ldots$$

Note that there are Mersenne numbers prime and not prime and the search for Mersenne primes is an active field in number theory and computer science. It is now known that for M_n to be prime, n must be a prime p, though not all M_p are prime.

Many authors studied the Mersenne sequence and its generalizations; see, for example, Catarino, Campos and Vasco (2016), Daşdemir (2019), Goy (2018), Ochalik and Włoch (2018), Zatorsky and Goy (2016) and the references given therein.

For instance, Catarino et al. (2016) established some identities for the common factors of Mersenne numbers and Jacobsthal and Jacobsthal-Lucas numbers, and presented some results with matrices involving Mersenne numbers such as the generating matrix, tridiagonal matrices and circulant matrices. Goy (2018) obtained new identities for Mersenne numbers with binomial coefficients. Ochalik and Włoch (2018) studied generalized Mersenne numbers, their properties, matrix generators and some combinatorial interpretations. Daşdemir (2019) extended the Mersenne numbers to their terms with negative subscripts and derived many identities for new forms of these numbers, including Gelin-Cesaro identity, d'Ocagne's identity, and some sum formulas.

For more information on classical and alternative approaches to the Mersenne numbers see (Jaroma, 2007).

2. Binomial Mersenne identities. In this section, we derived some identities involving Mersenne numbers and binomial coefficients.

Proposition 1. *For all integers* $n \geq 2$ *and* $k \geq 1$, *the following formulas hold*:

$$M_{kn-1} = (-2)^{k-1} M_{n-1}^{k} + \sum_{i=0}^{k-2} \binom{k}{i} (-2)^{i} M_{k-i-1} M_{n}^{k-i} M_{n-1}^{i}, \tag{3}$$

$$M_{kn} = \sum_{i=0}^{k-1} \binom{k}{i} (-2)^{i} M_{k-i} M_{n}^{k-i} M_{n-1}^{i}, \tag{4}$$

where $\binom{k}{i}$ *is the binomial coefficient.*

Proof. We will prove (3) and (4) using induction on k. Clearly, these formulas work when $k = 1$. Now suppose they are true for $k = s$, we show that they are true for $k = s + 1$. Let us first prove (3). Using formula (Ochalik & Włoch, 2018)

$$M_{p+q} = M_{p+1} M_{q} - 2 M_{p} M_{q-1} \tag{5}$$

and well-known binomial identity

$$\binom{s}{i} + \binom{s}{i-1} = \binom{s+1}{i}, \tag{6}$$

we obtain

$$\begin{aligned}
M_{(s+1)n-1} &= M_{(sn-1)+n} = M_{sn} M_{n} - 2 M_{sn-1} M_{n-1} \\
&= \left(\sum_{i=0}^{s-1} \binom{s}{i} (-2)^{i} M_{s-i} M_{n}^{s-i} M_{n-1}^{i} \right) \cdot M_{n} \\
&\quad -2 \left((-2)^{s-1} M_{n-1}^{s} + \sum_{i=0}^{s-2} \binom{s}{i} (-2)^{i} M_{s-i-1} M_{n}^{s-i} M_{n-1}^{i} \right) \cdot M_{n-1} \\
&= \sum_{i=0}^{s-1} \binom{s}{i} (-2)^{i} M_{s-i} M_{n}^{s-i+1} M_{n-1}^{i} \\
&\quad + (-2)^{s} M_{n-1}^{s+1} - 2 \sum_{i=0}^{s-2} \binom{s}{i} (-2)^{i} M_{s-i-1} M_{n}^{s-i} M_{n-1}^{i+1} \\
&= M_{s} M_{n}^{s+1} + \sum_{i=1}^{s-1} \binom{s}{i} (-2)^{i} M_{s-i} M_{n}^{s-i+1} M_{n-1}^{i} \\
&\quad + (-2)^{s} M_{n-1}^{s+1} - 2 \sum_{i=1}^{s-1} \binom{s}{i-1} (-2)^{i-1} M_{s-i} M_{n}^{s-i+1} M_{n-1}^{i} \\
&= (-2)^{s} M_{n-1}^{s+1} + M_{s} M_{n}^{s+1} + \sum_{i=1}^{s-1} \left(\binom{s}{i} + \binom{s}{i-1} \right) (-2)^{i} M_{s-i} M_{n}^{s-i+1} M_{n-1}^{i}
\end{aligned}$$

$$= (-2)^s M_{n-1}^{s+1} + \sum_{i=0}^{s-1} \binom{s+1}{i} (-2)^i M_{s-i} M_n^{s-i+1} M_{n-1}^i.$$

Therefore, by induction, Formula (3) works for all positive integers s.
Now, we prove Formula (4). Using (2)–(6), we obtain

$$M_{(s+1)n} = M_{(sn-1)+(n+1)} = M_{sn} M_{n+1} - 2M_{sn-1} M_n$$

$$= \sum_{i=0}^{s-1} \binom{s}{i} (-2)^i M_{s-i} M_n^{s-i} M_{n-1}^i \cdot (3M_n - 2M_{n-1})$$

$$-2\left((-2)^{s-1} M_{n-1}^s + \sum_{i=0}^{s-2} \binom{s}{i} (-2)^i M_{s-i-1} M_n^{s-i} M_{n-1}^i \right) \cdot M_n$$

$$= 3\sum_{i=0}^{s-1} \binom{s}{i} (-2)^i M_{s-i} M_n^{s-i+1} M_{n-1}^i - 2\sum_{i=0}^{s-1} \binom{s}{i} (-2)^i M_{s-i} M_n^{s-i} M_{n-1}^{i+1}$$

$$+(-2)^s M_{n-1}^s M_n - 2\sum_{i=0}^{s-2} \binom{s}{i} (-2)^i M_{s-i-1} M_n^{s-i+1} M_{n-1}^i$$

$$= 3\binom{s}{s-1} (-2)^{s-1} M_1 M_n^2 M_{n-1}^{s-1} + 3\sum_{i=0}^{s-2} \binom{s}{i} (-2)^i M_{s-i} M_n^{s-i+1} M_{n-1}^i$$

$$-2 \sum_{i=1}^{s} \binom{s}{i-1} (-2)^{i-1} M_{s-i+1} M_n^{s-i+1} M_{n-1}^{i+1} + (-2)^s M_{n-1}^s M_n$$

$$-2 \sum_{i=0}^{s-2} \binom{s}{i} (-2)^i M_{s-i-1} M_n^{s-i+1} M_{n-1}^i$$

$$= 3s(-2)^{s-1} M_n^2 M_{n-1}^{s-1} + \sum_{i=0}^{s-2} \binom{s}{i} (-2)^i M_{s-i+1} M_n^{s-i+1} M_{n-1}^i$$

$$+\sum_{i=1}^{s} \binom{s}{i-1} (-2)^i M_{s-i+1} M_n^{s-i+1} M_{n-1}^i + (-2)^s M_{n-1}^s M_n$$

$$= M_{s+1} M_n^{s+1} + \sum_{i=1}^{s} \left(\binom{s}{i} + \binom{s}{i-1} \right) (-2)^i M_{s-i+1} M_n^{s-i+1} M_{n-1}^i$$

$$= \sum_{i=0}^{s} \binom{s+1}{i} (-2)^i M_{s-i+1} M_n^{s-i+1} M_{n-1}^i.$$

Therefore, by induction, Formula (4) is true for all positive integers s.

Proposition 3. *For all integers* $n \geq 2$, $k \geq 0$, *and* $0 < r < k$, *the following formula hold*:

$$M_{kn+r} = \sum_{i=0}^{k} \binom{k}{i} (-2)^i M_{k-i+r} M_n^{k-i} M_{n-1}^i. \tag{7}$$

Proof. We use the induction on *r*. By (2)–(4), if $r = 1$, then we have

$$\begin{aligned}
M_{kn+1} &= 2M_{kn} - 3M_{kn-1} \\
&= 2\sum_{i=0}^{k-1} \binom{k}{i} (-2)^i M_{k-i} M_n^{k-i} M_{n-1}^i \\
&\quad -3(-2)^{k-1} M_{n-1}^k - 3\sum_{i=0}^{k-2} \binom{k}{i} (-2)^i M_{k-i-1} M_n^{k-i} M_{n-1}^i \\
&= 2\binom{k}{k-1}(-2)^{k-1} M_1 M_n M_{n-1}^{k-1} + 2\sum_{i=0}^{k-2} \binom{k}{i}(-2)^i M_{k-i} M_n^{k-i} M_{n-1}^i \\
&\quad -3(-2)^{k-1} M_{n-1}^k - 3\sum_{i=0}^{k-2} \binom{k}{i} (-2)^i M_{k-i-1} M_n^{k-i} M_{n-1}^i \\
&= (-2)^k k M_n M_{n-1}^{k-1} - 3(-2)^{k-1} M_{n-1}^k \\
&\quad + \sum_{i=0}^{k-2} \binom{k}{i}(-2)^i (2M_{k-i} - 3M_{k-1-i}) M_n^{k-i} M_{n-1}^i \\
&= \sum_{i=0}^{k} \binom{k}{i} (-2)^i M_{k-i+1} M_n^{k-i} M_{n-1}^i. \qquad (7)
\end{aligned}$$

Suppose that Formula (7) is true for $r = s$ and proved it for $r = s + 1$. Using (5) for $p = kn$, $q = s + 1$, and (7), we then obtain

$$M_{kn+(s+1)} = M_{kn+1} M_{s+1} - 2M_{kn} M_s$$

$$\begin{aligned}
&= \sum_{i=0}^{k} \binom{k}{i}(-2)^i M_{k-i+1} M_n^{k-i} M_{n-1}^i \cdot M_{s+1} - 2\sum_{i=0}^{k-1} \binom{k}{i}(-2)^i M_{k-i} M_n^{k-i} M_{n-1}^i \cdot M_s \\
&= (-2)^k M_1 M_{n-1}^k M_{k+1} + \sum_{i=0}^{k-1} \binom{k}{i}(-2)^i (M_{s+1} M_{k-i+1} - 2M_s M_{k-i}) M_n^{k-i} M_{n-1}^i \\
&= (-2)^k M_{n-1}^k M_{k+1} + \sum_{i=0}^{k-1} \binom{k}{i}(-2)^i M_{s+k-i+1} M_n^{k-i} M_{n-1}^i \\
&= \sum_{i=0}^{k} \binom{k}{i}(-2)^i M_{k-i+s+1} M_n^{k-i} M_{n-1}^i.
\end{aligned}$$

So, by induction, the statement is true for all integers $n \geq 2$, $k \geq 0$, and $0 < r < k$.

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