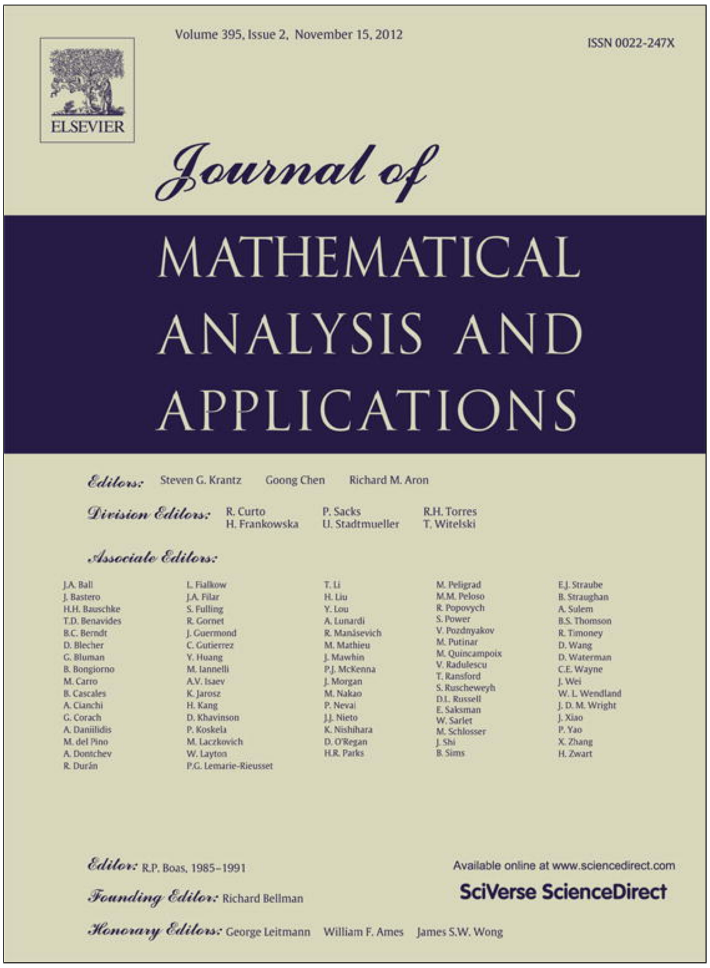
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# The convolution operation on the spectra of algebras of symmetric analytic functions

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ABSTRACT

We show that the spectrum of the algebra of bounded symmetric analytic functions on $\ell_p$, $1 \leq p < +\infty$ with the symmetric convolution operation is a commutative semigroup with the cancellation law for which we discuss the existence of inverses. For $p = 1$, a representation of the spectrum in terms of entire functions of exponential type is obtained which allows us to determine the invertible elements.



## 0. Introduction and preliminaries

The question of the description of the invariants of a linear transformations group on $\mathbb{C}^n$ which naturally acts on the algebra of polynomials is a typical problem of the classical Invariant Theory. Such invariants form algebras of symmetric polynomials with respect to given groups and have been investigated in the classical cases (see e.g. [1,2]). It is very important for these studies to describe the spectra of the algebras of invariants. The cases when a group (or even a semigroup) of symmetry acts on infinite-dimensional Banach spaces were considered in [3–6]. For the infinite-dimensional case we need to work with a natural completion of the algebra of continuous polynomials, that is, the algebra of analytic functions of bounded type. In this case, we can use some methods and ideas developed in [7,8].

Aron et al. introduced in [7] a convolution operation in the spectrum of the algebra $H_b(X)$ of analytic functions of bounded type defined on a complex Banach space $X$. This convolution is defined relying on translations on $X$. Later Aron et al. [8] discussed the commutativity of that convolution and proved that for $X = \ell_p$, it is not commutative.

By a *symmetric* function on $\ell_p$ we mean a function which is invariant under any reordering of the sequence in $\ell_p$. The algebra of symmetric analytic functions of bounded type with the topology of the uniform convergence on bounded sets will be denoted by $\mathcal{H}_{bs}(\ell_p)$. We denote by $\mathcal{M}_{bs}(\ell_p)$ its spectrum, that is the set of all continuous scalar valued homomorphisms.

When dealing with symmetric analytic functions the translation operators are not well defined anymore. This is why in [6] the authors introduced the so-called "intertwining" operators that lead them to define a "symmetric" convolution operation as is described in the next section. We prove that an endomorphism of $\mathcal{H}_{bs}(\ell_p)$ commutes with all intertwining operators if and only if it is a convolution operator. The results in this paper show that, contrary to the non-symmetric case, the symmetric convolution is indeed commutative. Also a representation of $\mathcal{M}_{bs}(\ell_1)$ in terms of entire functions

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of exponential type is obtained. Such representation allows us to determine the invertible elements in $\mathcal{M}_{bs}(\ell_1)$ for such symmetric convolution. Finally we present a description of the elements in the spectrum through certain points in $\ell_1^+$.

In [3] it is proved that, similarly to the classical finite dimensional case, the polynomials

$$F_k(x) = \sum_{i=1}^{\infty} x_i^k, \quad k = \lceil p \rceil, \lceil p \rceil + 1 \cdots \tag{0.1}$$

form an *algebraic basis* – named *the power series* basis – in the algebra of all symmetric polynomials on $\ell_p$ (here $\lceil p \rceil$ is the smallest integer that is greater than or equal to $p$). This means that for every symmetric polynomial $P$ of degree $\lceil p \rceil + n - 1, n \geq 1$ there is a polynomial $q$ on $\mathbb{C}^n$ such that $P(x) = q(F_{\lceil p \rceil}(x), \ldots, F_{\lceil p \rceil + n-1}(x))$. Actually, $q$ is unique as pointed out in [5].

For background on analytic functions on infinite-dimensional spaces, we refer the reader to [9] or to [10].

## 1. The symmetric convolution

**Remark 1.1.** There is no $w \in \ell_p$, $w \neq 0$, such that $g(x) = f(x + w)$ is symmetric for every symmetric $f \in \mathcal{H}_{bs}(\ell_p)$.

**Proof.** There is $i_0 \in \mathbb{N}$, such that $|w_n| < \frac{1}{3}$ if $n \geq i_0$. Assume that $f(\cdot + w)$ belongs to $\mathcal{H}_{bs}(\ell_p)$ for every symmetric $f \in \mathcal{H}_{bs}(\ell_p)$. Then for every fixed permutation $\sigma$ and each element in the basis of $\ell_p$, $f(e_{\sigma(i)} + w) = g(e_{\sigma(i)}) = g(e_i) = f(e_i + w)$, $\forall f \in \mathcal{H}_{bs}(\ell_p)$. Thus $e_{\sigma(i)} + w$ is a permutation of $e_i + w$, that is, $1 + w_{\sigma(i)} = w_{j_i}$ for some index $j_i \in \mathbb{N}$.

Since $\sigma$ is a bijection, the set $\{\sigma(i) > i_0\}$ is infinite, so there are infinite terms $w_{j_i}$ with absolute value greater that $\frac{2}{3}$. Impossible. □

Next we recall some definitions.

**Definition 1.2** ([6]). Let $x, y \in \ell_p$, $x = (x_1, x_2, \ldots,)$ and $y = (y_1, y_2, \ldots,)$. We define the *intertwining* $x \bullet y \in \ell_p$ according to

$$x \bullet y = (x_1, y_1, x_2, y_2, \ldots,).$$

The mapping $f \mapsto T_y^s(f)$ where $T_y^s(f)(x) = f(x \bullet y)$ will be referred as to the *intertwining operator*. Observe that $T_x^s \circ T_y^s = T_{x \bullet y}^s = T_y^s \circ T_x^s$: Indeed, $[T_x^s \circ T_y^s](f)(z) = T_x^s[T_y^s(f)](z) = T_y^s(f)(z \bullet x) = f((z \bullet x) \bullet y) = f(z \bullet (x \bullet y))$, since $f$ is symmetric.

The above remark explains why we are led to use the intertwining operators to define the convolution in $\mathcal{M}_{bs}(\ell_p)$.

**Definition 1.3** ([6]). Given $f \in \mathcal{H}_{bs}(\ell_p)$ and $\theta \in \mathcal{H}_{bs}(\ell_p)'$, its symmetric convolution $\theta \star f$ is defined by $(\theta \star f)(x) = \theta[T_x^s(f)]$.

As pointed out in [6], it turns out that $\theta \star f \in \mathcal{H}_{bs}(\ell_p)$.

**Definition 1.4** ([6]). For any $\phi$ and $\theta$ in $\mathcal{H}_{bs}(\ell_p)'$, its *symmetric convolution* is defined according to

$$(\phi \star \theta)(f) = \phi(\theta \star f) = \phi(y \mapsto \theta(T_y^s f)).$$

**Corollary 1.5** ([6]). *If $\phi, \theta \in \mathcal{M}_{bs}(\ell_p)$, then $\phi \star \theta \in \mathcal{M}_{bs}(\ell_p)$.*

**Theorem 1.6.** (a) *For every $\varphi, \theta \in \mathcal{M}_{bs}(\ell_p)$ the following holds:*

$$(\varphi \star \theta)(F_k) = \varphi(F_k) + \theta(F_k). \tag{1.1}$$

(b) *The semigroup $(\mathcal{M}_{bs}(\ell_p), \star)$ is commutative, the evaluation at* 0, $\delta_0$, *is its identity and the cancellation law holds.*

**Proof.** Observe that for each element $F_k$ in the algebraic basis of polynomials, $\{F_k\}$, we have

$$(\theta \star F_k)(x) = \theta(T_x^s(F_k)) = \theta(F_k(x) + F_k) = F_k(x) + \theta(F_k).$$

Therefore,

$$(\varphi \star \theta)(F_k) = \varphi(F_k + \theta(F_k)) = \varphi(F_k) + \theta(F_k).$$

To check that the convolution is commutative, that is, $\phi \star \theta = \theta \star \phi$, it suffices to prove it for symmetric polynomials, hence for the basis $\{F_k\}$. Bearing in mind (1.1) and also by exchanging parameters $(\theta \star \varphi)(F_k) = \theta(F_k) + \varphi(F_k) = (\varphi \star \theta)(F_k)$ as we wanted.

It also follows from (1.1) that the cancellation rule is valid for this convolution: If $\varphi \star \theta = \psi \star \theta$, then $\varphi(F_k) + \theta(F_k) = \psi(F_k) + \theta(F_k)$, hence $\varphi(F_k) = \psi(F_k)$, and thus, $\varphi = \psi$. □

**Example 1.7.** There exist nontrivial invertible elements in the semigroup $(\mathcal{M}_{bs}(\ell_p), \star)$:

In [5, Example 3.1] it was constructed a continuous homomorphism $\varphi = \Psi_1$ on the uniform algebra $A_{us}(B_{\ell_p})$ such that $\varphi(F_p) = 1$ and $\varphi(F_i) = 0$ for all $i > p$. In a similar way, given $\lambda \in \mathbb{C}$ we can construct a continuous homomorphism $\Psi_\lambda$ on the uniform algebra $A_{us}(|\lambda|B_{\ell_p})$ such that $\Psi_\lambda(F_p) = \lambda$ and $\Psi_\lambda(F_i) = 0$ for all $i > p$: It suffices to consider for each $n \in \mathbb{N}$, the element $v_n = \left(\frac{\lambda}{n}\right)^{1/p}(e_1 + \cdots + e_n)$ for which $F_p(v_n) = \lambda$, and $\lim_n F_j(v_n) = 0$. Now, the sequence $\{\delta_{v_n}\}$ has an accumulation point $\Psi_\lambda$ in the spectrum of $A_{us}(|\lambda|B_{\ell_p})$. We use the notation $\psi_\lambda$ for the restriction of $\Psi_\lambda$ to the subalgebra $\mathcal{H}_{bs}(\ell_p)$ of $A_{us}(|\lambda|B_{\ell_p})$. It turns out that $\psi_\lambda \star \psi_{-\lambda} = \delta_0$ since for all elements $F_j$ in the algebraic basis, $(\psi_\lambda \star \psi_{-\lambda})(F_j) = \psi_\lambda(F_j) + \psi_{-\lambda}(F_j) = 0 = \delta_0(F_j)$.

Therefore, we obtain a complex line of invertible elements $\{\psi_\lambda : \lambda \in \mathbb{C}\}$.

As in the non-symmetric case [7, Theorem 5.5], the following holds:

**Proposition 1.8.** *Every $\varphi \in \mathcal{M}_{bs}(\ell_p)$ lies in a schlicht complex line through $\delta_0$.*

**Proof.** For every $z \in \mathbb{C}$, consider the composition operator $L_z : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ defined according to $L_z(f)((x_n)) := f((zx_n))$, and then, the restriction $L_z^*$ to $\mathcal{M}_{bs}(\ell_p)$ of its transpose map. Now put $\varphi^z := L_z^*(\varphi) = \varphi \circ L_z$. Observe that $\varphi^z(F_k) = \varphi \circ L_z(F_k) = \varphi((F_k(z\cdot))) = z^k\varphi(F_k)$. Also, $\varphi^0 = \delta_0$.

For each $f \in \mathcal{H}_{bs}(\ell_p)$ the self-map of $\mathbb{C}$ defined according to $z \rightsquigarrow \varphi^z(f)$ is entire by Aron et al. [7, Lemma 5.4(i)]. Therefore, the mapping $z \in \mathbb{C} \rightsquigarrow \varphi^z \in \mathcal{M}_{bs}(\ell_p)$ is analytic.

Since $\varphi \neq \delta_0$, the set $\Sigma := \{k \in \mathbb{N} : \varphi(F_k) \neq 0\}$ is non-empty. Let $m$ be the first element of $\Sigma$, so that $\varphi(F_m) \neq 0$. Then if $\varphi^z = \varphi^w$, one has $z^m\varphi(F_m) = w^m\varphi(F_m)$, hence $z^m = w^m$. Taking the principal branch of the $m$th root, the map $\xi \rightsquigarrow \varphi^{\sqrt[m]{\xi}}$ is one-to-one. $\square$

Recall that a linear operator $T : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ is said to be a *convolution operator* if there is $\theta \in \mathcal{M}_{bs}(\ell_p)$ such that $Tf = \theta \star f$. Let us denote $H_{conv}(\ell_p) := \{T \in L(\mathcal{H}_{bs}(\ell_p)) : T \text{ is a convolution operator}\}$.

**Proposition 1.9.** *A continuous homomorphism $T : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ is a convolution operator if and only if it commutes with all intertwining operators $T_y^s$, $y \in \ell_p$.*

**Proof.** Assume there is $\theta \in \mathcal{M}_{bs}(\ell_p)$ such that $Tf = \theta \star f$. Fix $y \in \ell_p$. Then $[T \circ T_y^s](f)(x) = [T(T_y^s(f))](x) = [\theta \star T_y^s(f)](x) = \theta[T_x^s(T_y^s(f))] = \theta[T_{x\bullet y}^s(f)]$. On the other hand, $[T_y^s \circ T](f)(x) = [T_y^s(Tf)](x) = Tf(x \bullet y) = (\theta \star f)(x \bullet y) = \theta[T_{x\bullet y}^s(f)]$.

Conversely, set $\theta = \delta_0 \circ T$. Clearly, $\theta \in \mathcal{M}_{bs}(\ell_p)$. Let us check that $Tf = \theta \star f$: Indeed, $(\theta \star f)(x) = \theta[T_x^s(f)] = [T(T_x^s(f))](0) = [T_x^s(T(f))](0) = Tf(0 \bullet x) = Tf(x)$. $\square$

Consider the mapping $\Lambda$ defined by $\Lambda(\theta)(f) = \theta \star f$, that is,

$$\Lambda : \mathcal{M}_{bs}(\ell_p) \to H_{conv}(\ell_p)$$
$$\theta \mapsto f \rightsquigarrow \theta \star f \equiv \Lambda(\theta)(f).$$

It is, clearly, bijective. Moreover we obtain a representation of the convolution semigroup

**Proposition 1.10.** *The mapping $\Lambda$ is an isomorphism from $(\mathcal{M}_{bs}(\ell_p), \star)$ into $(H_{conv}(\ell_p), \circ)$ where $\circ$ denotes the usual composition operation.*

**Proof.** First, notice that using the above proposition,

$$\begin{aligned}\Lambda(\varphi \star \theta)(f)(x) &= [(\varphi \star \theta) \star f](x) = (\varphi \star \theta)(T_x^s f) = \varphi(\theta \star T_x^s f)\\ &= \varphi[\Lambda(\theta)(T_x^s f)] = \varphi[(\Lambda(\theta) \circ T_x^s)(f)] = \varphi[(T_x^s \circ \Lambda(\theta))(f)].\end{aligned}$$

On the other hand,

$$[\Lambda(\varphi) \circ \Lambda(\theta)](f)(x) = \Lambda(\varphi)[\Lambda(\theta)(f)](x) = [\varphi \star \Lambda(\theta)(f)](x) = \varphi[T_x^s(\Lambda(\theta)(f))].$$

Thus the statement follows. $\square$

As a consequence, the homomorphism $\theta$ is invertible in $(\mathcal{M}_{bs}(\ell_p), \star)$, if and only if the convolution operator $\Lambda(\theta)$ is an algebraic isomorphism. Observe also that for $\psi \in \mathcal{M}_{bs}(\ell_p)$, one has

$$\psi \circ \Lambda(\theta) = \psi \star \theta,$$

because $[\psi \circ \Lambda(\theta)](f) = \psi[\Lambda(\theta)(f)] = \psi(\theta \star f) = (\psi \star \theta)(f)$.

Next we address the question of solving the equation $\varphi = \psi \star \theta$ for given $\varphi, \theta \in \mathcal{M}_{bs}(\ell_p)$. We begin with a general lemma.

**Lemma 1.11.** *Let $A, B$ be Fréchet algebras and $T : A \to B$ an onto homomorphism. Then $T$ maps (closed) maximal ideals onto (closed) maximal ideals.*

**Proof.** Since $T$ is onto, it maps ideals in $A$ onto ideals in $B$. Let $\mathcal{J} \subset A$ be a maximal ideal. We prove that $T(\mathcal{J})$ is a maximal ideal in $B$: If $\mathcal{I}$ is another ideal with $T(\mathcal{J}) \subset \mathcal{I} \subset B$, it turns out that for the ideal $T^{-1}(\mathcal{I})$, $\mathcal{J} \subset T^{-1}(T(\mathcal{J})) \subset T^{-1}(\mathcal{I})$, hence either $\mathcal{J} = T^{-1}(\mathcal{I})$, or $A = T^{-1}(\mathcal{I})$. That is, either $T(\mathcal{J}) = \mathcal{I}$, or $B = \mathcal{I}$.

Let now $\varphi \in M(A)$ and $\mathcal{J} = Ker(\varphi)$, be a closed maximal ideal. Then $T(\mathcal{J})$ is a maximal ideal in $B$, so there is a character $\psi$ on $B$ such that $Ker(\psi) = T(\mathcal{J})$. Then $Ker(\varphi) \subset Ker(\psi \circ T)$, because if $\varphi(a) = 0$, that is, $a \in \mathcal{J}$, we have $T(a) \in Ker(\psi)$. By the maximality, either $\varphi = \psi \circ T$, or $\psi \circ T = 0$, hence $\psi = 0$. In the former case, $\psi$ is also continuous since being $T$ an open mapping, if $(b_n)$ is a null sequence in $B$, there is a null sequence $(a_n) \subset A$ such that $T(a_n) = b_n$; thus $\lim_n \psi(b_n) = \lim_n \psi \circ T(a_n) = \lim_n \varphi(a_n) = 0$. □

**Remark 1.12.** Let $A$, $B$ be Fréchet algebras and $T: A \to B$ be an onto homomorphism. If $T(Ker(\varphi))$ is a proper ideal, then there is a unique $\psi \in M(B)$ such that $\varphi = \psi \circ T$.

**Corollary 1.13.** *Let $\theta \in \mathcal{M}_{bs}(\ell_p)$. Assume that $\Lambda(\theta)$ is onto. If $\Lambda(\theta)(Ker\varphi)$ is a proper ideal, then the equation $\varphi = \psi \star \theta$ has a unique solution. In case $\Lambda(\theta)(Ker\varphi) = \mathcal{H}_{bs}(\ell_p)$, then the equation $\varphi = \psi \star \theta$ has no solution.*

**Proof.** The first statement is just an application of the remark, since $\psi \star \theta = \psi \circ \Lambda(\theta) = \varphi$. For the second statement, if some solution $\psi$ exists, then again $\psi \circ \Lambda(\theta) = \psi \star \theta = \varphi$, so $\psi(\mathcal{H}_{bs}(\ell_p)) = (\psi \circ \Lambda(\theta))((Ker\varphi)) = \varphi(Ker\varphi) = 0$. Therefore, then also $\varphi = 0$. □

## 2. A weak polynomial topology on $\mathcal{M}_{bs}(\ell_p)$

Let us denote by $w_p$ the topology in $\mathcal{M}_{bs}(\ell_p)$ generated by the following neighborhood basis:

$$U_{\varepsilon,k_1,\dots,k_n}(\psi) = \{\psi \star \varphi : \varphi \in \mathcal{M}_{bs}(\ell_p) \text{ and } |\varphi(F_{k_j})| < \varepsilon, j = 1, \dots, n\}.$$

It is easy to check that the convolution operation is continuous for the $w_p$ topology, since thanks to (1.1),

$$U_{\varepsilon/2,k_1,\dots,k_n}(\theta) \star U_{\varepsilon/2,k_1,\dots,k_n}(\psi) \subset U_{\varepsilon,k_1,\dots,k_n}(\theta \star \psi).$$

We say that a function $f \in \mathcal{H}_{bs}(\ell_p)$ is *finitely generated* if there are a finite number of the basis functions $\{F_k\}$ and an entire function $q$ such that $f = q(F_1, \dots, F_j)$.

**Theorem 2.1.** *A function $f \in \mathcal{H}_{bs}(\ell_p)$ is $w_p$-continuous if and only if it is finitely generated.*

**Proof.** Clearly, every finitely generated function is $w_p$-continuous. Let us denote by $V_n$ the finite dimensional subspace in $\ell_p$ spanned by the basis vectors $\{e_1, \dots, e_n\}$. First we observe that if there is a positive integer $m$ such that the restriction $f_{|V_n}$ of $f$ to $V_n$ is generated by the restrictions of $F_1, \dots, F_m$ to $V_n$ for every $n \ge m$, then $f$ is finitely generated. Indeed, for given $n \ge k \ge m$ we can write

$$f_{|V_k}(x) = q_1(F_1(x), \dots, F_m(x)) \quad \text{and} \quad f_{|V_n}(x) = q_2(F_1(x), \dots, F_m(x))$$

for some entire functions $q_1$ and $q_2$ on $\mathbb{C}^n$. Since

$$\{(F_1(x), \dots, F_m(x)) : x \in V_k\} = \mathbb{C}^m$$

(see e.g. [5]) and $f|_{V_n}$ is an extension of $f|_{V_k}$ we have $q_1(t_1, \dots, t_n) = q_2(t_1, \dots, t_n)$. Hence $f(x) = q_1(F_1(x), \dots, F_m(x))$ on $\ell_p$ because $f(x)$ coincides with $q_1(F_1(x), \dots, F_m(x))$ on the dense subset $\bigcup_n V_n$.

Let $f$ be a $w_p$-continuous function in $\mathcal{H}_{bs}(\ell_p)$. Then $f$ is bounded on a neighborhood $U_{\varepsilon,1,\dots,m} = \{x \in \ell_p : |F_1(x)| < \varepsilon, \dots, |F_m(x)| < \varepsilon\}$. For a given $n \ge m$ let

$$f|_{V_n}(x) = q(F_1(x), \dots, F_m(x))$$

be the representation of $f|_{V_n}(x)$ for some entire function $q$ on $\mathbb{C}^n$. Since $\{(F_1(x), \dots, F_m(x)) : x \in V_n\} = \mathbb{C}^m$, $q(t_1, \dots, t_n)$ must be bounded on the set $\{|t_1| < \varepsilon, \dots, |t_m| < \varepsilon\}$. The Liouville Theorem implies $q(t_1, \dots, t_n) = q(t_1, \dots, t_m, 0 \dots, 0)$, that is, $f|_{V_n}$ is generated by $F_1, \dots, F_m$. Since it is true for every $n$, $f$ is finitely generated. □

For example $f(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{n!}$ is not $w_p$-continuous.

**Proposition 2.2.** *The topology $w_p$ is Hausdorff.*

**Proof.** If $\varphi \ne \psi$, then there is a number $k$ such that

$$|\varphi(F_k) - \psi(F_k)| = \rho > 0.$$

Let $\varepsilon = \rho/3$. Then for every $\theta_1$ and $\theta_2$ in $U_{\varepsilon,k}(0)$,

$$|(\varphi \star \theta_1)(F_k) - (\varphi \star \theta_2)(F_k)| = |(\varphi(F_k) - \psi(F_k)) - (\theta_2(F_k)) - \theta_1(F_k)| \ge \rho/3. \quad \square$$

**Proposition 2.3.** *On bounded sets of $\mathcal{M}_{bs}(\ell_p)$ the topology $w_p$ is finer than the weak-star topology $w(\mathcal{M}_{bs}(\ell_p), \mathcal{H}_{bs}(\ell_p))$.*

**Proof.** Since $(\mathcal{M}_{bs}(\ell_p), w_p)$ is a first-countable space, it suffices to verify that for a bounded sequence $(\varphi_i)_i$ which is $w_p$ convergent to some $\psi$, we have $\lim_i \varphi_i(f) = \psi(f)$ for each $f \in \mathcal{H}_{bs}(\ell_p)$: Indeed, by the Banach–Steinhaus theorem, it is enough to see that $\lim_i \varphi_i(P) = \psi(P)$ for each symmetric polynomial $P$. Being $\{F_k\}$ an algebraic basis for the symmetric polynomials, this will follow once we check that $\lim_i \varphi_i(F_k) = \psi(F_k)$ for each $F_k$. To see this, notice that given $\varepsilon > 0$, $\varphi_i \in U_{\varepsilon,k}$ for $i$ large enough, that is, there is $\theta_i$ such that $\varphi_i = \psi \star \theta_i$ with $|\theta_i(F_k)| < \varepsilon$. Then, $|\varphi_i(F_k) - \psi(F_k)| = |\theta_i(F_K)| < \varepsilon$ for $i$ large enough. □

**Proposition 2.4.** *If $(\mathcal{M}_{bs}(\ell_p), \star)$ is a group, then $w_p$ coincides with the weakest topology on $\mathcal{M}_{bs}(\ell_p)$ such that for every polynomial $P \in \mathcal{H}_{bs}(\ell_p)$ the Gelfand extension $\widehat{P}$ is continuous on $\mathcal{M}_{bs}(\ell_p)$.*

**Proof.** The sets $F_k^{-1}(B(F_k(\psi), \varepsilon))$ generate the weakest topology such that all $\widehat{P}$ are continuous. Let $\theta \in \mathcal{M}_{bs}(\ell_p)$ be such that $|F_k(\theta) - F_k(\psi)| < \varepsilon$. Set $\varphi = \theta \star \psi^{-1}$. Then $|F_k(\varphi)| = |F_k(\theta) - F_k(\psi)| < \varepsilon$ and $\theta = \psi \star \varphi$. □

## 3. Representations of the convolution semigroup $(\mathcal{M}_{bs}(\ell_1), \star)$

In this section we consider the case $\mathcal{H}_{bs}(\ell_1)$. This algebra admits besides the power series basis $\{F_n\}$, another natural basis that is useful for us: It is given by the sequence $\{G_n\}$ defined by $G_0 = 1$, and

$$G_n(x) = \sum_{k_1<\cdots<k_n}^{\infty} x_{k_1} \cdots x_{k_n},$$

and we refer to it as the *basis of elementary* symmetric polynomials.

**Lemma 3.1.** *We have that $\|G_n\| = 1/n!$*

**Proof.** To calculate the norm, it is enough to deal with vectors in the unit ball of $\ell_1$ whose components are non-negative. And we may restrict ourselves to calculate it on $L_m$ the linear span of $\{e_1, \ldots, e_m\}$ for $m \geq n$. We do the calculation in an inductive way over $m$.

Since $G_{n|_{L_m}}$ is homogeneous, its norm is achieved at points of norm 1. If $m = n$, then $G_n$ is the product $x_1 \cdots x_n$. By using the Lagrange multipliers rule, we deduce that the maximum is attained at points with equal coordinates, that is at $\frac{1}{n}(e_1 + \cdots + e_n)$. Thus $|G_n(\frac{1}{n}, \overset{n}{\ldots}, \frac{1}{n}, 0, \ldots)| = 1/n^n \leq \frac{1}{n!}$.

Now for $m > n$, and $x \in L_m$, we have $G_n(x) = \sum_{k_1<\cdots<k_n\leq m} x_{k_1} \cdots x_{k_n}$. Again the Lagrange multipliers rule leads to either some of the coordinates vanish or they are all equal, hence they have the same value $\frac{1}{m}$. In the first case, we are led back to some the previous inductive steps, with $L_k$ with $k < m$, so the aimed inequality holds. While in the second one, we have $G_n(\frac{1}{m}, \overset{m}{\ldots}, \frac{1}{m}, 0, \ldots) = \binom{m}{n} \frac{1}{m^n} \leq \frac{1}{n!}$.

Moreover, $\|G_n\| \geq \lim_m \binom{m}{n} \frac{1}{m^n} = \frac{1}{n!}$. This completes the proof. □

Let $\mathbb{C}\{t\}$ be the space of all power series over $\mathbb{C}$. We denote by $\mathcal{F}$ and $\mathcal{G}$ the following maps from $\mathcal{M}_{bs}(\ell_1)$ into $\mathbb{C}\{t\}$

$$\mathcal{F}(\varphi) = \sum_{n=1}^{\infty} t^{n-1}\varphi(F_n) \quad \text{and} \quad \mathcal{G}(\varphi) = \sum_{n=0}^{\infty} t^n \varphi(G_n).$$

Let us recall that every element $\varphi \in \mathcal{M}_{bs}(\ell_1)$ has a radius-function

$$R(\varphi) = \limsup_{n\to\infty} \|\varphi_n\|^{\frac{1}{n}} < \infty,$$

where $\varphi_n$ is the restriction of $\varphi$ to the subspace of $n$-homogeneous polynomials [6].

**Proposition 3.2.** *The mapping $\varphi \in \mathcal{M}_{bs}(\ell_1) \xrightarrow{\mathcal{G}} \mathcal{G}(\varphi) \in \mathcal{H}(\mathbb{C})$ is one-to-one and ranges into the subspace of entire functions on $\mathbb{C}$ of exponential type. The type of $\mathcal{G}(\varphi)$ is less than or equal to $R(\varphi)$.*

**Proof.** Using Lemma 3.1,

$$\limsup_{n\to\infty} \sqrt[n]{n!|\varphi_n(G_n)|} \leq \limsup_{n\to\infty} \sqrt[n]{n!\|\varphi_n\| \, \|G_n\|} = \limsup_{n\to\infty} \sqrt[n]{\|\varphi_n\|} = R(\varphi) < \infty,$$

hence $\mathcal{G}(\varphi)$ is entire and of exponential type less than or equal to $R(\varphi)$. That $\mathcal{G}$ is one-to-one follows from the fact $\{G_n\}$ is a basis. □

**Theorem 3.3.** *The following identities hold:*

(1) $\mathcal{F}(\varphi\star\theta)=\mathcal{F}(\varphi)+\mathcal{F}(\theta)$.
(2) $\mathcal{G}(\varphi\star\theta)=\mathcal{G}(\varphi)\mathcal{G}(\theta)$.

**Proof.** The first statement is a trivial corollary of the properties of the convolution. To prove the second we observe that

$$G_n(x\bullet y)=\sum_{k=0}^{n}G_k(x)G_{n-k}(y).$$

Thus

$$(\theta\star G_n)(x)=\theta(T_x^s(G_n))=\theta\left(\sum_{k=0}^{n}G_k(x)G_{n-k}\right)=\sum_{k=0}^{n}G_k(x)\theta(G_{n-k}).$$

Therefore,

$$(\varphi\star\theta)(G_n)=\varphi\left(\sum_{k=0}^{n}G_k(x)\theta(G_{n-k})\right)=\sum_{k=0}^{n}\varphi(G_k)\theta(G_{n-k}).$$

Hence, being the series absolutely convergent,

$$\begin{aligned}\mathcal{G}(\varphi)\mathcal{G}(\theta)&=\sum_{k=0}^{\infty}t^k\varphi(G_k)\sum_{m=0}^{\infty}t^m\theta(G_m)=\sum_{n=0}^{\infty}\sum_{k+m=n}t^n\varphi(G_k)\theta(G_m)\\&=\sum_{n=0}^{\infty}t^n\sum_{k+m=n}\varphi(G_k)\theta(G_m)=\sum_{n=0}^{\infty}t^n(\varphi\star\theta)(G_n)=\mathcal{G}(\varphi\star\theta).\quad\square\end{aligned}$$

**Example 3.4.** Let $\psi_\lambda$ be as defined in Example 1.7. We know that $\mathcal{F}(\psi_\lambda)=\lambda$. To find $\mathcal{G}(\psi_\lambda)$ note that

$$G_k(v_n)=\left(\frac{\lambda}{n}\right)^k\binom{n}{k},\quad\text{hence }\varphi(G_k)=\lim_n G_k(v_n)=\frac{\lambda^k}{k!}$$

and so

$$\mathcal{G}(\psi_\lambda)(t)=\lim_{n\to\infty}\sum_{k=0}^{n}(\lambda t)^k\psi_\lambda(G_n)=\lim_{n\to\infty}\sum_{k=0}^{n}\frac{(\lambda t)^k}{k!}=e^{\lambda t}.$$

According to well-known Newton's formula we can write for $x\in\ell_1$,

$$nG_n(x)=F_1(x)G_{n-1}(x)-F_2(x)G_{n-2}(x)+\cdots+(-1)^{n+1}F_n(x).\tag{3.1}$$

Moreover, if $\xi$ is a complex homomorphism (not necessarily continuous) on the space of symmetric polynomials $\mathcal{P}_s(\ell_1)$, then

$$n\xi(G_n)=\xi(F_1)\xi(G_{n-1})-\xi(F_2)\xi(G_{n-2})+\cdots+(-1)^{n+1}\xi(F_n).\tag{3.2}$$

Next we point out the limitations of the construction's technique described in 1.7.

**Remark 3.5.** Let $\xi$ be a complex homomorphism on $\mathcal{P}_s(\ell_1)$ such that $\xi(F_m)=c\neq 0$ for some $m\geq 2$ and $\xi(F_n)=0$ for $n\neq m$. Then $\xi$ is not continuous.

**Proof.** Using formula (3.2) we can see that

$$\xi(G_{km})=(-1)^{m+1}\frac{\xi(F_m)\xi(G_{(k-1)m})}{km}$$

and $\xi(G_n)=0$ if $n\neq km$ for some $k\in\mathbb{N}$. By induction we have

$$\xi(G_{km})=\frac{\left((-1)^{m+1}c/m\right)^k}{k!}$$

and so

$$\mathcal{G}(\xi)(t)=1+\sum_{k=1}^{\infty}\frac{\left((-1)^{m+1}c/m\right)^k}{k!}t^{km}=1+\sum_{k=1}^{\infty}\frac{\left((-1)^{m+1}\frac{ct^m}{m}\right)^k}{k!}=e^{\left((-1)^{m+1}\frac{ct^m}{m}\right)}.$$

Hence $\mathcal{G}(\xi)(t) = e^{-\frac{(-ct)^m}{m}} = e^{-\frac{(-c)^m}{m}t^m}$. Since $m \geq 2$, $\mathcal{G}(\xi)$ is not of exponential type. So if $\xi$ were continuous, it could be extended to an element in $\mathcal{M}_{bs}(\ell_1)$, leading to a contradiction with Proposition 3.2. □

According to the Hadamard Factorization Theorem (see [11, p. 27]) the function of exponential type $\mathcal{G}(\varphi)(t)$ is of the form

$$\mathcal{G}(\varphi)(t) = e^{\lambda t}\prod_{k=1}^{\infty}\left(1-\frac{t}{a_k}\right)e^{t/a_k}, \tag{3.3}$$

where $\{a_k\}$ are the zeros of $\mathcal{G}(\varphi)(t)$. If $\sum |a_k|^{-1} < \infty$, then this representation can be reduced to

$$\mathcal{G}(\varphi)(t) = e^{\lambda t}\prod_{k=1}^{\infty}\left(1-\frac{t}{a_k}\right). \tag{3.4}$$

Recall how $\psi_\lambda$ was defined in Example 1.7.

**Proposition 3.6.** *If $\varphi \in (\mathcal{M}_{bs}(\ell_1), \star)$ is invertible, then $\varphi = \psi_\lambda$ for some $\lambda$. In particular, the semigroup $(\mathcal{M}_{bs}(\ell_1), \star)$ is not a group.*

**Proof.** If $\varphi$ is invertible then $\mathcal{G}(\varphi)(t)$ is an invertible entire function of exponential type and so has no zeros. By Hadamard's factorization (3.3) we have that $\mathcal{G}(\varphi)(t) = e^{\lambda t}$ for some complex number $\lambda$. Hence $\varphi = \psi_\lambda$ by Proposition 3.2.

The evaluation $\delta_{(1,0,\ldots,0,\ldots)}$ does not coincide with any $\psi_\lambda$ since, for instance, $\psi_\lambda(F_2) = 0 \neq 1 = \delta_{(1,0,\ldots,0,\ldots)}(F_2)$. □

Another consequence of our analysis is the following remark.

**Corollary 3.7.** *Let $\Phi$ be a homomorphism of $\mathcal{P}_s(\ell_1)$ to itself such that $\Phi(F_k) = -F_k$ for every $k$. Then $\Phi$ is discontinuous.*

**Proof.** If $\Phi$ is continuous it may be extended to a continuous homomorphism $\widetilde{\Phi}$ of $\mathcal{H}_{bs}(\ell_1)$. Then for $x = (1, 0, \ldots, 0, \ldots)$, we have $\delta_x \star (\delta_x \circ \widetilde{\Phi}) = \delta_0$. However, this is impossible since $\delta_x$ is not invertible. □

We close this section by analyzing further the relationship established by the mapping $\mathcal{G}$.

It is known from Combinatorics (see e.g. [12, pp. 3,4]) that

$$\mathcal{G}(\delta_x)(t) = \prod_{k=1}^{\infty}(1 + x_k t) \quad \text{and} \quad \mathcal{F}(\delta_x)(t) = \sum_{k=1}^{\infty}\frac{x_k}{1 - x_k t} \tag{3.5}$$

for every $x \in c_{00}$. Formula (3.5) for $\mathcal{G}(\delta_x)$ is true for every $x \in \ell_1$: Indeed, for fixed $t$, both the infinite product and $\mathcal{G}(\delta_x)(t)$ are analytic functions on $\ell_1$.

Taking into account formula (3.5) we can see that the zeros of $\mathcal{G}(\delta_x)(t)$ are $a_k = -1/x_k$ for $x_k \neq 0$. Conversely, if $f(t)$ is an entire function of exponential type which is equal to the right hand side of (3.4) with $\sum |a_k|^{-1} < \infty$, then for $\varphi \in \mathcal{M}_{bs}(\ell_1)$ given by $\varphi = \psi_\lambda \star \delta_x$, where $x \in \ell_1$, $x_k = -1/a_k$ and $\psi_\lambda$ as defined in Example 1.7, it turns out that $\mathcal{G}(\varphi)(t) = f(t)$. So we just have to examine entire functions of exponential type with Hadamard canonical product

$$f(t) = \prod_{k=1}^{\infty}\left(1-\frac{t}{a_k}\right)e^{t/a_k} \tag{3.6}$$

with $\sum |a_k|^{-1} = \infty$. Note first that the growth order of $f(t)$ is not greater than 1. According to Borel's theorem [11, p. 30] the series

$$\sum_{k=1}^{\infty}\frac{1}{|a_k|^{1+d}}$$

converges for every $d > 0$. Let

$$\Delta_f = \limsup_{n\to\infty}\frac{n}{|a_n|}, \qquad \eta_f = \limsup_{r\to\infty}\left|\sum_{|a_n|<r}\frac{1}{a_n}\right|$$

and $\gamma_f = \max(\Delta_f, \eta_f)$. Due to Lindelöf's theorem [11, p. 33] the type $\sigma_f$ of $f$ and $\gamma_f$ simultaneously are equal either to zero, or to infinity, or to positive numbers. Hence $f(t)$ of the form (3.6) is a function of exponential type if and only if $\sum |a_k|^{-1-d}$ converges for every $d > 0$ and $\gamma_f$ is finite.

**Corollary 3.8.** *If a sequence $(x_n) \notin \ell_p$ for some $p > 1$, then there is no $\varphi \in \mathcal{M}_{bs}(\ell_1)$ such that*

$$\varphi(F_k) = \sum_{n=1}^{\infty} x_n^k$$

*for all $k$.*

Let $x = (x_1, \dots, x_n, \dots)$ be a sequence of complex numbers such that $x \in \ell_{1+d}$ for every $d > 0$,

$$\limsup_{n\to\infty} n|x_n| < \infty, \qquad \limsup_{r\to 1} \left| \sum_{\frac{1}{|x_n|}<r} x_n \right| < \infty \tag{3.7}$$

and $\lambda \in \mathbb{C}$. Let us denote by $\delta_{(x,\lambda)}$ a homomorphism on the algebra of symmetric polynomials $\mathcal{P}_s(\ell_1)$ of the form

$$\delta_{(x,\lambda)}(F_1) = \lambda, \qquad \delta_{(x,\lambda)}(F_k) = \sum_{n=1}^{\infty} x_n^k, \quad k > 1.$$

**Proposition 3.9.** *Let $\varphi \in \mathcal{M}_{bs}(\ell_1)$. Then the restriction of $\varphi$ to $\mathcal{P}_s(\ell_1)$ coincides with $\delta_{(x,\lambda)}$ for some $\lambda \in \mathbb{C}$ and $x$ satisfying (3.7).*

**Proof.** Consider the exponential type function $\mathcal{G}(\varphi)$ given by (3.3) and the corresponding sequence $x = (\frac{-1}{a_n})$.

If $x \in \ell_1$, then according to (3.4), $\varphi = \psi_\lambda \star \delta_x$. If $x \notin \ell_1$, then $\mathcal{G}(\varphi)(t) = e^{\lambda t} \prod_{n=1}^{\infty} \Big(1 + tx_n\Big) e^{-tx_n}$ and, on the other hand, $\mathcal{G}(\varphi)(t) = \sum_{n=0}^{\infty} \varphi(G_n) t^n$.

We have

$$\begin{aligned}\left(e^{\lambda t} \prod_{n=1}^{\infty} (1 + tx_n)\, e^{-tx_n}\right)'_t &= \lambda e^{\lambda t} \prod_{n=1}^{\infty} (1 + tx_n)\, e^{-tx_n} \\ &\quad + e^{\lambda t} \left( -tx_1^2 e^{-tx_1} \prod_{n\neq 1} (1 + tx_n)\, e^{-tx_n} - tx_2^2 e^{-tx_2} \prod_{n\neq 2} (1 + tx_n)\, e^{-tx_n} - \cdots \right) \\ &= \lambda e^{\lambda t} \prod_{n=1}^{\infty} (1 + tx_n)\, e^{-tx_n} - te^{\lambda t} \sum_{k=1}^{\infty} x_k^2 e^{-tx_k} \prod_{n\neq k} (1 + tx_n)\, e^{-tx_n}\end{aligned}$$

and

$$\left(e^{\lambda t} \prod_{n=1}^{\infty} (1 + tx_n)\, e^{-tx_n}\right)' \Bigg|_{t=0} = \lambda.$$

So by the uniqueness of the Taylor coefficients, $\varphi(G_1) = \varphi(F_1) = \lambda$.

Now

$$\begin{aligned}\left(e^{\lambda t} \prod_{n=1}^{\infty} (1 + tx_n)\, e^{-tx_n}\right)''_t &= \left(\lambda e^{\lambda t} \prod_{n=1}^{\infty} (1 + tx_n)\, e^{-tx_n}\right)'_t - \left(te^{\lambda t} \sum_{k=1}^{\infty} x_k^2 e^{-tx_k} \prod_{n\neq k} (1 + tx_n)\, e^{-tx_n}\right)'_t \\ &= \lambda^2 e^{\lambda t} \prod_{n=1}^{\infty} (1 + tx_n)\, e^{-tx_n} - \lambda te^{\lambda t} \sum_{k=1}^{\infty} x_k^2 e^{-tx_k} \prod_{n\neq k} (1 + tx_n)\, e^{-tx_n} \\ &\quad - e^{\lambda t} \sum_{k=1}^{\infty} x_k^2 e^{-tx_k} \prod_{n\neq k} (1 + tx_n)\, e^{-tx_n} - t \left(e^{\lambda t} \sum_{k=1}^{\infty} x_k^2 e^{-tx_k} \prod_{n\neq k} (1 + tx_n)\, e^{-tx_n}\right)'_t\end{aligned}$$

and

$$\left(e^{\lambda t} \prod_{n=1}^{\infty} (1 + tx_n)\, e^{-tx_n}\right)'' \Bigg|_{t=0} = \lambda^2 - \sum_{k=1}^{\infty} x_k^2.$$

Then

$$\varphi(G_2) = \frac{\lambda^2 - F_2(x)}{2} = \frac{(\varphi(F_1))^2 - F_2(x)}{2}.$$

On the other hand,

$$\varphi(G_2) = \frac{\varphi(F_1^2) - \varphi(F_2)}{2}$$

and we have

$$\varphi(F_2) = F_2(x).$$

Now using induction we obtain the required result. $\square$

**Question 3.10.** Does the map $\mathcal{G}$ act onto the space of entire functions of exponential type?

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