On new families of Fibonacci identities

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Fibonacci numbers are given by the recurrence $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with initial terms $F_0 = 0$ and $F_1 = 1$. Applications of these numbers include computer algorithms such as the Fibonacci search technique and the Fibonacci heap data structure, and graphs called Fibonacci cubes used for interconnecting parallel and distributed systems.

We investigate some families of Toeplitz-Hessenberg determinants and permanents (see, for example, [1, 2] and the bibliography given there) the entries of which are Fibonacci numbers. As result, we discover new Fibonacci identities with multinomial coefficients. For example, for $n \geq 2$, the following formulas hold:

$$\sum_{\sigma_n=n} (-1)^{|s|} p_n(s) \prod_{i=1}^{n} \left(\frac{F_{i-1}}{3}\right)^{s_i} = \frac{1}{\sqrt{33}} \left(\left(\frac{3-\sqrt{33}}{6}\right)^{n-1} - \left(\frac{3+\sqrt{33}}{6}\right)^{n-1} \right),$$

$$\sum_{\sigma_n=n} (-1)^{|s|} p_n(s) \prod_{i=1}^{n} \left(\frac{F_i}{3}\right)^{s_i} = \frac{1}{2\sqrt{10}} \left(\left(\frac{1-\sqrt{10}}{3}\right)^{n} - \left(\frac{1+\sqrt{10}}{3}\right)^{n} \right),$$

$$\sum_{\sigma_n=n} p_n(s) \prod_{i=1}^{n} \left(\frac{F_i}{3}\right)^{s_i} = \frac{1}{2\sqrt{13}} \left(\left(\frac{2+\sqrt{13}}{3}\right)^{n} - \left(\frac{2-\sqrt{13}}{3}\right)^{n} \right),$$

$$\sum_{\sigma_n=n} p_n(s) \prod_{i=1}^{n} \left(\frac{F_{i+1}}{3}\right)^{s_i} = \frac{3 \cdot 6^n + (-2)^n}{16 \cdot 3^n},$$

$$\sum_{\sigma_n=n} (-1)^{|s|} p_n(s) \prod_{i=1}^{n} \left(\frac{F_{i+4}}{3}\right)^{s_i} = \frac{(-2)^{n-2}}{3^n},$$

$$\sum_{\sigma_n=n} p_n(s) \prod_{i=1}^{n} \left(\frac{F_{2i}}{3}\right)^{s_i} = \frac{9^n - 1}{8 \cdot 3^n},$$

where $\sigma_n = s_1 + 2s_2 + \cdots + ns_n$, $|s| = s_1 + \cdots + s_n$, $p_n(s) = \frac{(s_1+\cdots+s_n)!}{s_1!\cdots s_n!}$, and the summation is over integers $s_i \geq 0$ satisfying $\sigma_n = n$.

References

1. *Goy T.* On new identities for Mersenne numbers // Appl. Math. E – Notes. 2018. Vol. 18. P. 100-105.
2. *Goy T.* Some families of identities for Padovan numbers // Proc. Jangjeon Math. Soc. 2018. Vol. 21, № 3 (in press).