On multinomial identities for Pell and Pell–Lucas polynomials

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Extending the classical Fibonacci and Lucas numbers, Horadam and Mahon [4] introduced Pell and Pell–Lucas polynomials. They are defined respectively by the recurrence relation: for $n \geq 2$,

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x), \qquad Q_n(x) = 2xQ_{n-1}(x) + Q_{n-2}(x)$$

with different initial conditions $P_0(x) = 0$, $P_1(x) = 2$, and $Q_0(x) = 2$, $Q_1(x) = 2x$.

We investigate some families of Toeplitz–Hessenberg determinants the entries of which are Pell and Pell–Lucas polynomials. This leads us to discover some identities for these polynomials. Our approach is similar in spirit to [1–3].

Denote $|s| = s_1 + \cdots + s_n$, $\sigma_n = s_1 + 2s_2 + \cdots + ns_n$, and $m_n(s) = \frac{|s|!}{s_1!\cdots s_n!}$ is the multinomial coefficient.

Proposition. *Let $n \geq 1$, except when noted otherwise. Then*

$$\sum_{\sigma_n=n} (-1)^{|s|} m_n(s) P_0^{s_1}(x) P_1^{s_2}(x) \cdots P_{n-1}^{s_n}(x) = -(2x)^{n-2}, \qquad n \geq 2, \tag{1}$$

$$\sum_{\sigma_n=n} m_n(s) P_0^{s_1}(x) P_1^{s_2}(x) \cdots P_{n-1}^{s_n}(x) = \frac{\left(x+\sqrt{x^2+2}\right)^{n-1} - \left(x-\sqrt{x^2+2}\right)^{n-1}}{2\sqrt{x^2+2}},$$

$$\sum_{\sigma_n=n} (-1)^{|s|} m_n(s) P_0^{s_1}(x) P_1^{s_2}(x) \cdots P_{n-1}^{s_n}(x) =$$
$$= \frac{\left(2x-1-\sqrt{4x^2-4x+5}\right)^n - \left(2x-1+\sqrt{4x^2-4x+5}\right)^n}{2^n \cdot \sqrt{4x^2-4x+5}},$$

$$\sum_{\sigma_n=n} m_n(s) P_0^{s_1}(x) P_1^{s_2}(x) \cdots P_{n-1}^{s_n}(x) =$$
$$= \frac{\left(1+2x+\sqrt{4x^2+4x+5}\right)^n - \left(1+2x-\sqrt{4x^2+4x+5}\right)^n}{2^n \cdot \sqrt{4x^2+4x+5}},$$

$$\sum_{\sigma_n=n} (-1)^{|s|} m_n(s) P_2^{s_1}(x) P_3^{s_2}(x) \cdots P_{n+1}^{s_n}(x) = 0, \qquad n \geq 3, \tag{2}$$

$$\sum_{\sigma_n=n} m_n(s) P_2^{s_1}(x) P_3^{s_2}(x) \cdots P_{n+1}^{s_n}(x) =$$
$$= \frac{\left(2x+\sqrt{4x^2+2}\right)^{n+1} - \left(2x-\sqrt{4x^2+2}\right)^{n+1}}{4\sqrt{4x^2+2}},$$

$$\sum_{\sigma_n=n} (-1)^{|s|} m_n(s) P_1^{s_1}(x) P_3^{s_2}(x) \cdots P_{2n-1}^{s_n}(x) = -4x^2(4x^2+1)^{n-2}, \qquad n \geq 2,$$

$$\sum_{\sigma_n=n} (-1)^{|s|} m_n(s) P_3^{s_1}(x) P_5^{s_2}(x) \cdots P_{2n+1}^{s_n}(x) = -4x^2, \tag{3}$$

where the summation is over integers $s_i \geq 0$ satisfying $\sigma_n = n$.

We establish similar formulas for Pell–Lucas polynomials.

Important special numerical cases are: $P_n(1) = P_n$ the n^{th} Pell number; $Q_n(1) = Q_n$ the n^{th} Pell–Lucas number; $P_n(1/2) = F_n$ the n^{th} Fibonacci number and $Q_n(1/2) = L_n$ the n^{th} Lucas number. Furthermore, $P_n(x/2) = F_n(x)$, the n^{th} Fibonacci polynomial, and $Q_n(x/2) = L_n(x)$, the n^{th} Lucas polynomial [5]. Also, there is a relationship between Pell and Pell–Lucas polynomials with Chebyshev polynomials of the first kind $T_n(x)$ and of the second kind $U_n(x)$ as follows

$$P_n(x) = (-i)^{n-1}U_{n-1}(ix), \quad n \geq 1, \qquad Q_n(x) = 2(-1)^n T_n(ix), \quad n \geq 0.$$

For example, from formulas (1)–(3) we obtain the following identities:

$$\sum_{\sigma_n=n} (-1)^{|s|} m_n(s) P_0^{s_1} P_1^{s_2} \cdots P_{n-1}^{s_n} = -2^{n-2}, \qquad n \geq 2,$$

$$\sum_{\sigma_n=n} (-1)^{|s|} m_n(s) F_0^{s_1} F_1^{s_2} \cdots F_{n-1}^{s_n} = -1 \qquad n \geq 2,$$

$$\sum_{\sigma_n=n} (-1)^{|s|} m_n(s) F_0^{s_1}(x) F_1^{s_2}(x) \cdots F_{n-1}^{s_n}(x) = -x^{n-2}, \qquad n \geq 2,$$

$$\sum_{\sigma_n=n} (-1)^{|s|} m_n(s) P_2^{s_1} P_3^{s_2} \cdots P_{n+1}^{s_n} = 0, \qquad n \geq 3,$$

$$\sum_{\sigma_n=n} (-1)^{|s|} m_n(s) F_2^{s_1} F_3^{s_2} \cdots F_{n+1}^{s_n} = 0, \qquad n \geq 3,$$

$$\sum_{\sigma_n=n} (-1)^{|s|} m_n(s) F_2^{s_1}(x) F_3^{s_2}(x) \cdots F_{n+1}^{s_n}(x) = 0, \qquad n \geq 3,$$

$$\sum_{\sigma_n=n} (-1)^{|s|} m_n(s) U_1^{s_1}(x) U_2^{s_2}(x) \cdots U_n^{s_n}(x) = 0, \qquad n \geq 3,$$

$$\sum_{\sigma_n=n} (-1)^{|s|} m_n(s) P_3^{s_1} P_5^{s_2} \cdots P_{2n+1}^{s_n} = -4, \qquad n \geq 2,$$

$$\sum_{\sigma_n=n} (-1)^{|s|} m_n(s) F_3^{s_1}(x) F_5^{s_2}(x) \cdots F_{2n+1}^{s_n}(x) = -1, \qquad n \geq 2.$$

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