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On the convergence of multidimensional regular C-fractions with independent variables

Досліджується збіжність багатовимірних регулярних C-дробів з нерівнозначними змінними, які є багатовимірним узагальненням регулярних C-дробів. Ці гіллясті ланцюгові дроби є ефективним інструментом для наближення функцій, заданих формальними кратними степеневими рядами. Показано, що перетин параболічної і кругової областей є областю збіжності багатовимірного регулярного C-дробу з нерівнозначними змінними, а параболічна область є областю збіжності гіллястого ланцюгового дробу, який є оберненим до багатовимірного регулярного C-дробу з нерівнозначними змінними.

Ключові слова: збіжність, рівномірна збіжність, багатовимірний регулярний C-дріб з нерівнозначними змінними.

Исследуется сходимость многомерных регулярных C-дробей с неравнозначными переменными, которые являются многомерным обобщением регулярных C-дробей. Эти ветвящиеся цепные дроби являются эффективным инструментом для приближения функций, заданных формальными кратными степенными рядами. Показано, что сечение параболической и круговой областей является областью сходимости многомерной регулярной C-дроби с неравнозначными переменными, а параболическая область является областью сходимости ветвящейся цепной дроби, которая является обратной к многомерной регулярной C-дроби с неравнозначными переменным.

Ключевые слова: сходимость, равномерная сходимость, многомерная регулярная C-дробь с неравнозначными переменными.

In this paper, we investigate the convergence of multidimensional regular C-fractions with independent variables, which are a multidimensional generalization of regular C-fractions. These branched continued fractions are an efficient tool for the approximation of multivariable functions, which are represented by formal multiple power series. We have shown that the intersection of the interior of the parabola and the open disk is the domain of convergence of a multidimensional regular C-fraction with independent variables. And, in addition, we have shown that the interior of the parabola is the domain of convergence of a branched continued fraction, which is reciprocal to the multidimensional regular C-fraction with independent variables.

Key words: convergence, uniform convergence, multidimensional regular C-fraction with independent variables.

Introduction

Let N be a fixed natural number and

$$\mathcal{I}_k = \big\{ i(k) : \ i(k) = (i_1, i_2, \ldots, i_k), \ 1 \le i_p \le i_{p-1}, \ 1 \le p \le k, \ i_0 = N \big\}, \quad k \ge 1,$$

be the sets of multiindices. Our research is devoted to the convergence of multidimensional regular C-fraction with independent variables

$$1 + \sum_{i_1=1}^{N} \frac{a_{i(1)} z_{i_1}}{1} + \sum_{i_2=1}^{i_1} \frac{a_{i(2)} z_{i_2}}{1} + \sum_{i_3=1}^{i_2} \frac{a_{i(3)} z_{i_3}}{1} + \cdots , \tag{1}$$

where the $a_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \ge 1$, are complex constants such that $a_{i(k)} \ne 0$, $i(k) \in \mathcal{I}_k$, $k \ge 1$, and where $\mathbf{z} = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N$, and the multidimensional regular C-fraction with independent variables which are reciprocal to it

$$\frac{1}{1} + \sum_{i_1=1}^{N} \frac{a_{i(1)} z_{i_1}}{1} + \sum_{i_2=1}^{i_1} \frac{a_{i(2)} z_{i_2}}{1} + \sum_{i_3=1}^{i_2} \frac{a_{i(3)} z_{i_3}}{1} + \cdots . \tag{2}$$

We note that these branched continued fractions with independent variables are the expansions of multiple power series [5, 7].

A convergence criteria have been given for multidimensional regular C-fractions with independent variables by T. M. Antonova and D. I. Bodnar [1], O. E. Baran [2], R. I. Dmytryshyn [6].

In the present paper we derive some new convergence criteria for the mentioned above fractions. For establishing the convergence criteria, we use the convergence continuation theorem (Theorem 24.2 [9, pp. 108–109] (see also Theorem 2.17 [4, p. 66])) to extend the convergence, already known for a small region, to a larger region. The Theorems 1 and 2 give us the intersection of the interior of the parabola and the open disk for the domains of convergence of (1). In Theorems 3 and 4 the interior of the parabola are obtained for the domains of convergence of (2).

Convergence

We give two convergence criteria for multidimensional regular C-fraction with independent variables (1). For use in the following theorems we introduce the notation for the tails of (1):

$$F_{i(n)}^{(n)}(\mathbf{z}) = 1, \quad i(n) \in \mathcal{I}_n, \ n \ge 1, \tag{3}$$

$$F_{i(k)}^{(n)}(\mathbf{z}) = 1 + \sum_{i_{k+1}=1}^{i_k} \frac{a_{i(k+1)} z_{i_{k+1}}}{1} + \sum_{i_{k+2}=1}^{i_{k+1}} \frac{a_{i(k+2)} z_{i_{k+2}}}{1} + \cdots + \sum_{i_n=1}^{i_{n-1}} \frac{a_{i(n)} z_{i_n}}{1}, \tag{4}$$

where $i(k) \in \mathcal{I}_k$, $1 \le k \le n-1$, $n \ge 2$. Then it is clear that the following recurrence relations hold

$$F_{i(k)}^{(n)}(\mathbf{z}) = 1 + \sum_{i_{k+1}=1}^{i_k} \frac{a_{i(k+1)} z_{i_{k+1}}}{F_{i(k+1)}^{(n)}(\mathbf{z})}, \quad i(k) \in \mathcal{I}_k, \ 1 \le k \le n-1, \ n \ge 2. \quad (5)$$

Let

$$f_n(\mathbf{z}) = 1 + \sum_{i_1=1}^{N} \frac{a_{i(1)} z_{i_1}}{F_{i(1)}^{(n)}(\mathbf{z})}$$

be the nth approximant of (1), $n \ge 1$.

We shall prove the following result.

Theorem 1. *A multidimensional regular C-fraction with independent variables (1), where*

$$\sum_{i_{k+1}=1}^{i_k} \frac{l_{i_{k+1}} |a_{i(k+1)}|}{(1 - g_{i(k+1)})(1 + \cos(\arg(a_{i(k+1)})))} \le g_{i(k)}, \quad i(k) \in \mathcal{I}_k, \ k \ge 1, \quad (6)$$

$$\arg(a_{i(k)}) = \varphi, \quad -\pi < \varphi < \pi, \quad i(k) \in \mathcal{I}_k, \ k \ge 2, \quad (7)$$

$$l_k > 0, \quad 1 \le k \le N, \quad 0 < g_{i(k)} < 1, \quad i(k) \in \mathcal{I}_k, \ k \ge 1, \quad (8)$$

converges to a function holomorphic in the domain

$$\mathcal{D}_{l_1, l_2, \ldots, l_N, M} = \left\{ \mathbf{z} \in \mathbb{C}^N : \ |z_k| - \mathrm{Re}(z_k) < l_k, \quad |z_k| < M, \quad 1 \le k \le N \right\} \quad (9)$$

for every constant $M > 0$. The convergence is uniform on every compact subset of $\mathcal{D}_{l_1, l_2, \ldots, l_N, M}$.

Proof. We set

$$a_{i(k)} = r_{i(k)} e^{i\varphi}, \quad i(k) \in \mathcal{I}_k, \ k \ge 2. \quad (10)$$

Let n be an arbitrary natural number. By induction on k we show that, for arbitrary of multiindex $i(k) \in \mathcal{I}_k$, the following inequalities are valid

$$\mathrm{Re}(F_{i(k)}^{(n)}(\mathbf{z}) e^{-i\varphi/2}) > (1 - g_{i(k)}) \cos(\varphi/2) > 0, \quad (11)$$

where $1 \le k \le n$.

From (3) it is clear that for $k = n$ the inequalities (11) hold. By induction hypothesis that (11) hold for $k = p+1$, $p+1 \le n$, $i(p+1) \in \mathcal{I}_{p+1}$, we prove (11) for $k = p$, $i(p) \in \mathcal{I}_p$. Indeed, use of (5) and (10), for the arbitrary of multiindex $i(p) \in \mathcal{I}_p$, leads to

$$F_{i(p)}^{(n)}(\mathbf{z}) e^{-i\varphi/2} = e^{-i\varphi/2} + \sum_{i_{p+1}=1}^{i_p} \frac{r_{i(p+1)} z_{i_{p+1}}}{F_{i(p+1)}^{(n)}(\mathbf{z}) e^{-i\varphi/2}}.$$

In the proof of Lemma 4.41 [8] it is shown that if $x \geq c > 0$ and $v^2 \leq 4u + 4$,

$$\min_{-\infty < y < +\infty} \mathrm{Re}\frac{u + iv}{x + iy} = -\frac{\sqrt{u^2 + v^2} - u}{2x}. \tag{12}$$

We set $u = \mathrm{Re}(r_{i(p+1)}z_{i_{p+1}})$, $v = \mathrm{Im}(r_{i(p+1)}z_{i_{p+1}})$, $x = \mathrm{Re}(F_{i(p+1)}^{(n)}(\mathbf{z})e^{-i\varphi/2})$, $y = \mathrm{Im}(F_{i(p+1)}^{(n)}(\mathbf{z})e^{-i\varphi/2})$. Then for the arbitrary index i_{p+1}, $1 \leq i_{p+1} \leq i_p$, it follows from (6) that

$$|a_{i(p+1)}| \leq g_{i(p)}(1 - g_{i(p+1)})(1 + \cos(\varphi))/l_{i_{p+1}} < 2/l_{i_{p+1}}.$$

From this inequality it is easily shown that $v^2 \leq 4u + 4$.

Using (6)–(12) and induction hypothesis, we obtain

$$\mathrm{Re}(F_{i(p)}^{(n)}(\mathbf{z})e^{-i\varphi/2}) \geq \cos(\varphi/2) - \sum_{i_{p+1}=1}^{i_p} \frac{r_{i(p+1)}(|z_{i_{p+1}}| - \mathrm{Re}(z_{i_{p+1}}))}{2\mathrm{Re}(F_{i(p+1)}^{(n)}(\mathbf{z})e^{-i\varphi/2})} >$$

$$> \cos(\varphi/2) - \sum_{i_{p+1}=1}^{i_p} \frac{r_{i(p+1)}l_{i(p+1)}}{2(1 - g_{i(p+1)})\cos(\varphi/2)} \geq (1 - g_{i(p)})\cos(\varphi/2) > 0.$$

It follows from (11) that $F_{i(k)}^{(n)}(\mathbf{z}) \not\equiv 0$ for all indices. Thus, the approximants $f_n(\mathbf{z})$, $n \geq 1$, of (1) form a sequence of functions holomorphic in $\mathcal{D}_{l_1,l_2,\ldots,l_N,M}$.

Let

$$\mathcal{D}_{l_1,l_2,\ldots,l_N,M,\sigma} = \{\mathbf{z} \in \mathbb{C}^N : |z_k| - \mathrm{Re}(z_k) < \sigma l_k, \quad |z_k| < \sigma M, \quad 1 \leq k \leq N\}, \tag{13}$$

where $0 < \sigma < 1$. We set

$$a = \max_{1 \leq i_1 \leq N} |a_{i(1)}|. \tag{14}$$

Using (9)–(11), for the arbitrary $\mathbf{z} \in \mathcal{D}_{l_1,l_2,\ldots,l_N,M,\sigma}$, $\mathcal{D}_{l_1,l_2,\ldots,l_N,M,\sigma} \subset \mathcal{D}_{l_1,l_2,\ldots,l_N,M}$, we obtain for $n \geq 1$

$$|f_n(\mathbf{z})| \leq 1 + \sum_{i_1=1}^{N} \frac{|a_{i(1)}||z_{i_1}|}{\mathrm{Re}(F_{i(1)}^{(n)}(\mathbf{z})e^{-i\varphi/2})} < 1 + \sum_{i_1=1}^{N} \frac{a\sigma M}{\cos(\varphi/2)(1 - g_{i(1)})} = C(\mathcal{D}_{l_1,l_2,\ldots,l_N,M,\sigma}),$$

where the constant $C(\mathcal{D}_{l_1,l_2,\ldots,l_N,M,\sigma})$ depends only on the domain (13), i.e. the sequence $\{f_n(\mathbf{z})\}$ is uniformly bounded in $\mathcal{D}_{l_1,l_2,\ldots,l_N,M,\sigma}$.

Let $\mathcal{K}$ be an arbitrary compact subset of $\mathcal{D}_{l_1,l_2,\ldots,l_N,M}$. Let us cover $\mathcal{K}$ with domains of form (13). From this cover we choose the finite subcover

$$\mathcal{D}_{l_1,l_2,\ldots,l_N,M,\sigma^{(1)}}, \ \mathcal{D}_{l_1,l_2,\ldots,l_N,M,\sigma^{(2)}}, \ldots, \mathcal{D}_{l_1,l_2,\ldots,l_N,M,\sigma^{(k)}}.$$

We set

$$C(\mathcal{K}) = \max_{1 \leq r \leq k} C(\mathcal{D}_{l_1,l_2,\ldots,l_N,M,\sigma^{(r)}}).$$

Then for arbitrary $\mathbf{z} \in \mathcal{K}$ we obtain $|f_n(\mathbf{z})| \leq C(\mathcal{K})$, for $n \geq 1$, i.e. the sequence $\{f_n(\mathbf{z})\}$ is uniformly bounded on each compact subset of the domain (9).

Let $b = \min\{1/a, l_1, l_2, \ldots, l_N, 8MN\}$, where a is defined by (14), and let

$$\mathcal{L}_b = \left\{ \mathbf{z} \in \mathbb{R}^N : \ 0 < z_k < \frac{b}{8N}, \quad 1 \leq k \leq N \right\}.$$

Then for the arbitrary $\mathbf{z} \in \mathcal{L}_b$, $\mathcal{L}_b \subset \mathcal{D}_{l_1,l_2,\ldots,l_N,M}$, we obtain

$$|a_{i(1)}z_{i_1}| < \frac{ab}{8N} < \frac{1}{2N}, \quad |a_{i(k+1)}z_{i_{k+1}}| < \frac{b}{4Nl_{i_{k+1}}} \leq \frac{1}{4i_k}, \quad i(k) \in \mathcal{I}_k, \ k \geq 1.$$

It follows from Theorem 1 [3], with $g_{i(k)} = 1/2$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, that (1) converges in $\mathcal{L}_b$. Hence by Theorem 24.2 [9, pp. 108–109] (see also Theorem 2.17 [4, p. 66]), the multidimensional regular C-fraction with independent variables (1) converges uniformly on compact subsets of $\mathcal{D}_{l_1,l_2,\ldots,l_N,M}$ to a holomorphic function.

The following theorem can be proved in much the same way as Theorem 1 using Theorem 4 [3].

Theorem 2. *A multidimensional regular C-fraction with independent variables (1), where the $a_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 2$, satisfy the conditions*

$$|a_{i(k)}| \leq g_{i(k-1)}(1 - g_{i(k)})(1 + \cos(\arg(a_{i(k)})))/l_{i_k}, \quad i(k) \in \mathcal{I}_k, \ k \geq 2,$$

and the conditions (7) and (8), converges to a function holomorphic in the domain

$$\mathcal{O}_{l_1,l_2,\ldots,l_N,M} = \left\{ \mathbf{z} \in \mathbb{C}^N : \ \sum_{k=1}^{N} \frac{|z_k| - \text{Re}(z_k)}{l_k} < 1, \quad \sum_{k=1}^{N} |z_k| < M \right\}$$

for every constant $M > 0$. The convergence is uniform on every compact subset of $\mathcal{O}_{l_1,l_2,\ldots,l_N}$.

Next, we give two convergence criteria for multidimensional regular C-fractions with independent variable (2). In addition to (3) and (4), for the tails of (2) we introduce the following notation

$$F_{i(0)}^{(0)}(\mathbf{z}) = 1, \quad F_{i(0)}^{(n)}(\mathbf{z}) = 1 + \sum_{i_1=1}^{N} \frac{a_{i(1)}z_{i_1}}{1} + \sum_{i_2=1}^{i_1} \frac{a_{i(2)}z_{i_2}}{1} + \cdots + \sum_{i_n=1}^{i_{n-1}} \frac{a_{i(n)}z_{i_n}}{1}, \quad n \geq 1.$$

And, thus, the nth approximant of (2) we may write as $g_n(\mathbf{z}) = 1/F_{i(0)}^{(n-1)}(\mathbf{z})$, $n \geq 1$.

Now we shall prove the following result.

Theorem 3. *A multidimensional regular C-fraction with independent variables (2), where the $a_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, satisfy the conditions*

$$\sum_{i_k=1}^{i_{k-1}} \frac{l_{i_k}|a_{i(k)}|}{(1 - g_{i(k)})(1 + \cos(\arg(a_{i(k)})))} \leq g_{i(k-1)}, \quad i(k) \in \mathcal{I}_k, \ k \geq 1, \tag{15}$$

$$\arg(a_{i(k)}) = \varphi, \quad -\pi < \varphi < \pi, \quad i(k) \in \mathcal{I}_k, \ k \geq 1, \tag{16}$$

$$l_k > 0, \quad 1 \leq k \leq N, \quad 0 < g_{i(k-1)} < 1, \quad i(k) \in \mathcal{I}_k, \ k \geq 1, \tag{17}$$

converges to a function holomorphic in the domain

$$\mathcal{D}_{l_1,l_2,\ldots,l_N} = \{\mathbf{z} \in \mathbb{C}^N : \ |z_k| - \mathrm{Re}(z_k) < l_k, \quad 1 \leq k \leq N\}. \tag{18}$$

The convergence is uniform on every compact subset of $\mathcal{D}_{l_1,l_2,\ldots,l_N}$.

Proof. By analogy (11) it is easy to prove the validity of the following inequalities

$$\mathrm{Re}(F_{i(k)}^{(n)}(\mathbf{z})e^{-i\varphi/2}) > (1 - g_{i(k)})\cos(\varphi/2) > 0, \tag{19}$$

where $n \geq 0$, $0 \leq k \leq n$, $i(k) \in \mathcal{I}_k$, if $k \geq 1$. It follows from (19) that $F_{i(k)}^{(n)}(\mathbf{z}) \not\equiv 0$ for all indices. It means that the approximants $g_n(\mathbf{z})$, $n \geq 1$, of (2) form a sequence of functions holomorphic in $\mathcal{D}_{l_1,l_2,\ldots,l_N}$. Let

$$\mathcal{D}_{l_1,l_2,\ldots,l_N,\sigma} = \{\mathbf{z} \in \mathbb{C}^N : \ |z_k| - \mathrm{Re}(z_k) < \sigma l_k, \quad 1 \leq k \leq N\}, \tag{20}$$

where $0 < \sigma < 1$. Using (10) and (19) for the arbitrary $\mathbf{z} \in \mathcal{D}_{l_1,l_2,\ldots,l_N,\sigma}$, $\mathcal{D}_{l_1,l_2,\ldots,l_N,\sigma} \subset \mathcal{D}_{l_1,l_2,\ldots,l_N}$, we obtain for $n \geq 1$

$$|g_n(\mathbf{z})| \leq \frac{1}{\mathrm{Re}(F_{i(0)}^{(n-1)}(\mathbf{z})e^{-i\varphi/2})} < \frac{1}{(1 - g_{i(1)})\cos(\varphi/2)} = C(\mathcal{D}_{l_1,l_2,\ldots,l_N,\sigma}),$$

where the constant $C(\mathcal{D}_{l_1,l_2,\ldots,l_N,\sigma})$ depends only on the domain (20), i.e. the sequence $\{g_n(\mathbf{z})\}$ is uniformly bounded in $\mathcal{D}_{l_1,l_2,\ldots,l_N,\sigma}$.

Let $\mathcal{K}$ be an arbitrary compact subset of $\mathcal{D}_{l_1,l_2,\ldots,l_N}$. Let us cover $\mathcal{K}$ with domains of form (20). From this cover we choose the finite subcover

$$\mathcal{D}_{l_1,l_2,\ldots,l_N,\sigma^{(1)}}, \ \mathcal{D}_{l_1,l_2,\ldots,l_N,\sigma^{(2)}}, \ldots, \mathcal{D}_{l_1,l_2,\ldots,l_N,\sigma^{(k)}}.$$

We set

$$C(\mathcal{K}) = \max_{1 \leq r \leq k} C(\mathcal{D}_{l_1,l_2,\ldots,l_N,\sigma^{(r)}}).$$

Then for arbitrary $\mathbf{z} \in \mathcal{K}$ we obtain $|g_n(\mathbf{z})| \leq C(\mathcal{K})$, for $n \geq 1$, i.e. the sequence $\{g_n(\mathbf{z})\}$ is uniformly bounded on each compact subset of the domain (18).

Let

$$\mathcal{L}_{l_1,l_2,\ldots,l_N} = \left\{\mathbf{z} \in \mathbb{R}^N : \ 0 < z_k < \frac{l_k}{8N}, \quad 1 \leq k \leq N\right\}.$$

Then from (15)-(17) for the arbitrary $\mathbf{z} \in \mathcal{L}_{l_1,l_2,\ldots,l_N}$, $\mathcal{L}_{l_1,l_2,\ldots,l_N} \subset \mathcal{D}_{l_1,l_2,\ldots,l_N}$, we obtain

$$|a_{i(k)}z_{i_k}| < \frac{g_{i(k-1)}(1 - g_{i(k)})(1 + \cos(\varphi))}{8N} < \frac{1}{4N} \leq \frac{1}{4i_{k-1}}, \quad i(k) \in \mathcal{I}_k,\ k \geq 1.$$

It follows from Theorem 2 [3], with $g_{i(k)} = 1/2$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, that (2) converges in $\mathcal{L}_{l_1,l_2,\ldots,l_N}$. Hence by Theorem 24.2 [9, pp. 108–109] (see also Theorem 2.17 [4, p. 66]), the multidimensional regular C-fraction with independent variables (2) converges uniformly on compact subsets of $\mathcal{D}_{l_1,l_2,\ldots,l_N}$ to a holomorphic function.

Finally, the following theorem can be proved in much the same way as Theorem 3 using Theorem 5 [3].

Theorem 4. *A multidimensional regular C-fraction with independent variables (2), where the $a_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, satisfy the conditions*

$$|a_{i(k)}| \leq g_{i(k-1)}(1 - g_{i(k)})(1 + \cos(\arg(a_{i(k)})))/l_{i_k}, \quad i(k) \in \mathcal{I}_k,\ k \geq 1,$$

and the conditions (16) and (17), converges to a function holomorphic in the domain

$$\mathcal{O}_{l_1,l_2,\ldots,l_N} = \left\{ \mathbf{z} \in \mathbb{C}^N : \sum_{k=1}^{N} \frac{|z_k| - \mathrm{Re}(z_k)}{l_k} < 1 \right\}.$$

The convergence is uniform on every compact subset of $\mathcal{O}_{l_1,l_2,\ldots,l_N}$.

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