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SPECTRA OF ALGEBRAS OF ENTIRE FUNCTIONS ON BANACH SPACES

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ABSTRACT. We obtain an explicit description of the spectrum (set of closed maximal ideals) of $H_b(X)$, algebra of analytic functions on a Banach space X which are bounded on bounded subsets. We show that the spectrum of $H_b(X)$ admits a natural linear structure. Some applications to the algebra of uniformly continuous and bounded analytic functions on the unit ball $B \subset X$ are indicated.

Let A be a complex commutative topological algebra. Let us denote by $M(A)$ the spectrum (set of closed maximal ideals = set of continuous characters = set of continuous complex-valued homomorphisms) of A. Recall that A is *semisimple* if the complex homomorphisms from $M(A)$ separate points of A. It is well known that every semisimple commutative Fréchet algebra A is isomorphic to some subalgebra of continuous functions on $M(A)$ endowed with a natural topology. More exactly, for every $a \in A$ there exists a function $\widehat{a} : M(A) \to \mathbb{C}$ defined by $\widehat{a}(\phi) := \phi(a)$. The weakest topology on $M(A)$ such that all functions $\widehat{a}$, $a \in A$, are continuous is called the *Gelfand topology.* The Gelfand topology coincides with the weak-star topology of the strong dual space A', restricted to $M(A)$. If A is a Banach algebra, $M(A)$ is a weak-star compact subset of the unit ball of A'.

If A is a uniform algebra of continuous functions on a metric space G, then for any $x \in G$ the *point evaluation functional* $\delta(x) : f \mapsto f(x)$ belongs to $M(A)$.

The purpose of this paper is to describe the spectrum of the Fréchet algebra $H_b(X)$ of entire analytic functions of bounded type on a Banach space X and to study some related questions of infinite-dimensional holomorphy.

The problem of description of the spectrum of $H_b(X)$ was first studied by Aron, Cole and Gamelin [3, 4]. Using the Aron-Berner extension operation [2, 10], they showed, in particular, that X'' belongs to the spectrum of $H_b(X)$. In [5] it is proved that this inclusion is proper if there exists a polynomial on X which is not weakly continuous on bounded sets. This approach was generalized for algebra-valued analytic functions by García et al. in [18]. Some analytic structure on the set of maximal ideals was considered in [5] (for generalization for algebra-valued functions see [17]). In [22] Mujica investigated ideals of analytic functions of bounded type on

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Tsirelson's space T and showed that each character on $H_b(T)$ is a point evaluation functional. Homomorphisms of H_b were studied by Carando, García and Maestre in [9]. In [1] Alencar et al. considered maximal ideals of algebras of symmetric analytic functions on ℓ_p.

In this paper we show that every element of the spectrum of $H_b(X)$ can be represented by a sequence of functionals $(u_k)_{k=1}^{\infty}$ such that each u_k belongs to a Banach space E_k, where $E_1 = X''$ and E_n coincides with a special subspace of linear functionals on n-homogeneous polynomials. It is also shown that the spectrum of $H_b(X)$ contains the linear space of all finite sequences $(u_1, \ldots, u_m, 0, 0, \ldots)$. Finally, some related examples are considered.

For background on analytic functions on infinite-dimensional spaces, we refer the reader to [13] or to [21]. For details on the Aron-Berner extension we refer to [8].

For a given complex Banach space X, $\mathcal{P}(^nX)$ (resp. $\mathcal{P}(^{\le n}X)$) denotes the Banach space of all continuous n-homogeneous complex-valued polynomials on X (resp. the Banach space of all continuous n-degree complex-valued polynomials on X). $\mathcal{P}_f(^nX)$ denotes the subspace of n-homogeneous polynomials of *finite type*, that is the subspace generated by all polynomials of the form $P(x) = (\gamma(x))^n$ with $\gamma \in X'$ and $\mathcal{P}_c(^nX)$ is the closure of $\mathcal{P}_f(^nX)$ with the topology of uniform convergence on bounded subsets of X. It is well known [6] that if X' has the approximation property, then $\mathcal{P}_c(^nX)$ coincides with $\mathcal{P}_{wu}(^nX)$, the space of n-homogeneous polynomials which are weakly uniformly continuous on bounded subsets of X.

Recall that for every polynomial $P \in \mathcal{P}(^nX)$ there exists a (necessarily unique) symmetric n-linear form A_P, *associated with* P such that $A_P(x, \ldots, x) = P(x)$. We will write $A_P(x_1^{k_1}, \ldots, x_n^{k_n})$ instead of $A_P(\underbrace{x_1, \ldots, x_1}_{k_1}, \ldots, \underbrace{x_n, \ldots, x_n}_{k_n})$. We will use the fact that $\mathcal{P}(^nX)$ is isomorphic to the dual space of the symmetric projective n-fold tensor product $\bigotimes_{s,\pi}^n X$ of X.

Let us denote by $A_n(X)$ the closure of the algebra, generated by polynomials from $\mathcal{P}(^{\le n}X)$ with respect to the uniform topology on bounded subsets. It is clear that $A_1(X) \cap \mathcal{P}(^nX) = \mathcal{P}_c(^nX)$ and $A_n(X)$ is a Fréchet algebra of entire analytic functions on X for every n. The closure of the algebra of all polynomials $\mathcal{P}(X)$ with respect to the uniform topology on bounded subsets is denoted by $H_b(X)$ and is called the *algebra of entire functions of bounded type* on X. It is well known that $H_b(X)$ consists of all entire functions that are bounded on bounded subsets. The closure of the algebra of all polynomials with respect to the uniform topology on the unit ball B, $H^\infty_{uc}(B)$, is the algebra of all analytic functions on B which are uniformly continuous and bounded. We will use the short notation M_b and M_{uc} for the spectra $M(H_b(X))$ and $M(H^\infty_{uc}(B))$ respectively.

According to [3], every continuous functional $\phi \in H_b(X)'$ can be represented by $\phi = \sum_{k=0}^{\infty} \phi_k$, where $\phi_k = \pi_k(\phi)$ is the restriction of ϕ to $\mathcal{P}(^kX)$. The infimum of all $r > 0$, $R(\phi)$ such that ϕ is continuous with respect to the norm of uniform convergence on the ball rB is called the *radius function* of ϕ. It is known [3] that

$$R(\phi) = \limsup_{n\to\infty} \|\phi_n\|^{1/n}.$$

For every polynomial $P \in \mathcal{P}(^{mk}X)$ we denote by $P_{(m)}(u)$ the polynomial from $\mathcal{P}(^k\bigotimes_{s,\pi}^m X)$ such that $P_{(m)}(x^{\otimes m}) = P(x)$, where $x^{\otimes m} = \underbrace{x \otimes \cdots \otimes x}_{m\text{times}}$.

Lemma 1. *Let $\phi \in H_b(X)'$ such that $\phi(P) = 0$ for every $P \in \mathcal{P}(^m X) \cap A_{m-1}(X)$, where m is a fixed positive integer and $\phi_m \neq 0$. Then there is $\psi \in M_b$ such that $\psi_k = 0$ for $k < m$ and $\psi_m = \phi_m$. The radius function $R(\psi) = \|\phi_m\|^{1/m}$.*

Proof. Since $\phi_m \neq 0$, there is an element $w \in (\bigotimes_{s,\pi}^m X)''$, $w \neq 0$ such that for any m-homogeneous polynomial P, $\phi(P) = \phi_m(P) = \widetilde{P}_{(m)}(w)$, where $\widetilde{P}_{(m)}$ is the Aron-Berner extension of the linear functional $P_{(m)}$ from $\bigotimes_{s,\pi}^m X$ to $(\bigotimes_{s,\pi}^m X)''$ and $\|w\| = \|\phi_m\|$. For an arbitrary n-homogeneous polynomial Q we set

$$\psi(Q) = \begin{cases} \widetilde{Q}_{(m)}(w) & \text{if } n = mk \text{ for some } k \geq 0 \ , \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $\widetilde{Q}_{(m)}$ is the Aron-Berner extension of the k-homogeneous polynomial $Q_{(m)}$ from $\bigotimes_{s,\pi}^m X$ to $(\bigotimes_{s,\pi}^m X)''$.

Let (u_α) be a net from $\bigotimes_{s,\pi}^m X$ that converges to w in the weak-star topology of $(\bigotimes_{s,\pi}^m X)''$, where α belongs to an index set $\mathfrak{A}$. We can assume that u_α has a representation $u_\alpha = \sum_{j=1}^\infty x_{j,\alpha}^{\otimes m}$ for some $x_{j,\alpha} \in X$. Let us show that $\psi(PQ) = \psi(P)\psi(Q)$ for any homogeneous polynomials P and Q. Let us suppose first that $\deg(PQ) = mr + l$ for some integers $r \geq 0$ and $m > l > 0$. Then P or Q has degree equal to $mk + s$, $k \geq 0$, $m > s > 0$. Thus, by the definition, $\psi(PQ) = 0$ and $\psi(P)\psi(Q) = 0$. Suppose that $\deg(PQ) = mr$ for some integer $r \geq 0$. If $\deg P = mk$ and $\deg Q = mn$ for $k, n \geq 0$, then $\deg(PQ) = m(k+n)$ and $\psi(PQ) = (\widetilde{PQ})_{(m)}(w) = \widetilde{P}_{(m)}(w)\widetilde{Q}_{(m)}(w) = \psi(P)\psi(Q)$.

Now let $\deg P = mk + l$ and $\deg Q = mn + r$, $l, r > 0$, $l + r = m$. Write $\nu = 1/(\deg P + \deg Q)! = 1/(m(k+n+1))!$. Let A_{PQ} denote the symmetric multilinear map, associated with PQ. Then

$$\begin{aligned} &A_{PQ}(x_1, \ldots, x_{m(k+n+1)}) \\ &= \nu \sum_{\sigma \in S_{m(k+n+1)}} A_P(x_{\sigma(1)}, \ldots, x_{\sigma(mk+l)}) A_Q(x_{\sigma(mk+l+1)}, \ldots, x_{\sigma(m(k+n+1))}), \end{aligned}$$

where $S_{m(k+n+1)}$ is the group of permutations on $\{1, \ldots, m(k+n+1)\}$. Thus for $\alpha_1, \ldots, \alpha_{k+n+1} \in \mathfrak{A}$ we have

$$\begin{aligned} \psi(PQ) = (\widetilde{PQ})_{(m)}(w) &= \lim_{\alpha_1, \ldots, \alpha_{k+n+1}} \widetilde{A}_{PQ_{(m)}}(u_{\alpha_1}, \ldots, u_{\alpha_{k+n+1}}) \\ &= \lim_{\alpha_1, \ldots, \alpha_{k+n+1}} \widetilde{A}_{PQ_{(m)}} \left(\sum_{j=1}^\infty x_{j,\alpha_1}^{\otimes m}, \ldots, \sum_{j=1}^\infty x_{j,\alpha_{k+n+1}}^{\otimes m} \right) \\ &= \nu \sum_{\sigma \in S_{m(k+n+1)}} \lim_{\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k+n+1)}} \sum_{j_1, \ldots, j_{k+n+1}=1}^\infty A_P(x^m_{j_{\sigma(1)},\alpha_{\sigma(1)}}, \ldots, x^m_{j_{\sigma(k)},\alpha_{\sigma(k)}}, \\ &\quad x^l_{j_{\sigma(k+1)},\alpha_{\sigma(k+1)}}) A_Q(x^r_{j_{\sigma(k+1)},\alpha_{\sigma(k+1)}}, x^m_{j_{\sigma(k+2)},\alpha_{\sigma(k+2)}}, \ldots, x^m_{j_{\sigma(k+n+1)},\alpha_{\sigma(k+n+1)}}). \end{aligned}$$

Fix some $\sigma \in S_{m(k+n+1)}$ and fix all $x_{j_{\sigma(i)},\alpha_{\sigma(i)}}$, for $i \leq k$ and for $i > k+1$. Then

$$\begin{aligned} &\sum_{j_1, \ldots, j_{k+n+1}=1}^\infty \lim_{\alpha_{\sigma(k+1)}} A_P(x^m_{j_{\sigma(1)},\alpha_{\sigma(1)}}, \ldots, x^m_{j_{\sigma(k)},\alpha_{\sigma(k)}}, x^l_{j_{\sigma(k+1)},\alpha_{\sigma(k+1)}}) \\ &\times A_Q(x^r_{j_{\sigma(k+1)},\alpha_{\sigma(k+1)}}, x^m_{j_{\sigma(k+2)},\alpha_{\sigma(n+2)}}, \ldots, x^m_{j_{\sigma(k+n+1)},\alpha_{\sigma(k+n+1)}}) = 0 \end{aligned}$$

because for fixed $x_{k_{\sigma(i)},\alpha_{\sigma(i)}}$, $i \le k$,

$$P_\sigma(y) := \sum_{j_1,\dots,j_k,j_{k+2},\dots,j_{k+n+1}=1}^{\infty} A_P(x^m_{j_{\sigma(1)},\alpha_{\sigma(1)}},\dots,x^m_{j_{\sigma(k)},\alpha_{\sigma(k)}},y^l)$$

is an l-homogeneous polynomial and for fixed $x_{k_{\sigma(i)},\alpha_{\sigma(i)}}$, $i > k+1$,

$$Q_\sigma(y) := \sum_{j_1,\dots,j_k,j_{k+2},\dots,j_{k+n+1}=1}^{\infty} A_Q(y^r,x^m_{j_{\sigma(k+2)},\alpha_{\sigma(n+2)}},\dots,x^m_{j_{\sigma(k+n+1)},\alpha_{\sigma(k+n+1)}})$$

is an r-homogeneous polynomial. Thus $P_\sigma Q_\sigma \in A_{m-1}(X)$. Hence

$$\lim_\alpha (P_\sigma Q_\sigma)_{(m)}(u_\alpha) = \psi(P_\sigma Q_\sigma) = 0$$

for every fixed σ. Thus $\psi(PQ) = 0$. On the other hand, $\psi(P)\psi(Q) = 0$ by the definition of ψ. So $\psi(PQ) = \psi(P)\psi(Q)$.

Thus we have defined the multiplicative function ψ on homogeneous polynomials. We can extend it by linearity and distributivity to a linear multiplicative functional on the algebra of all continuous polynomials $\mathcal{P}(X)$. If ψ_n is the restriction of ψ to $\mathcal{P}(^nX)$, then $\|\psi_n\| = \|w\|^{n/m}$ if n/m is a positive integer and $\|\psi_n\| = 0$ otherwise. Hence $\psi = \sum_{n=0}^{\infty}\psi_n$ is a continuous linear multiplicative functional on $H_b(X)$ by [3, 2.4. Theorem] and the radius function of ψ can be computed by

$$R(\psi) = \limsup_{n\to\infty}\|\psi_n\|^{1/n} = \limsup_{n\to\infty}\|w\|^{n/mn} = \|w\|^{1/m} = \|\phi_m\|^{1/m}$$

as required. □

For each fixed $x \in X$, the translation operator T_x is defined on $H_b(X)$ by

$$(T_x f)(y) = f(y+x), \qquad f \in H_b(X).$$

It is not complicated to check that $T_x f \in H_b(X)$ and for fixed $\phi \in H_b(X)'$ the function $x \mapsto \phi(T_x f)$, $x \in X$, belongs to $H_b(X)$ (see [3]).

For fixed $\phi,\theta \in H_b(X)'$ the convolution product $\phi * \theta$ in $H_b(X)$ is defined by

$$(\phi * \theta)(f) = \phi(\theta(T_x f)), \qquad f \in H_b(X).$$

Let $\phi,\theta \in M_b$. According to [3, 4.7. Corollary], there exist nets $(x_\alpha),(y_\beta) \subset X$ such that

$$\phi(P) = \lim_\alpha P(x_\alpha), \qquad \theta(P) = \lim_\beta P(y_\beta) \tag{2}$$

for every polynomial P. We will write the condition (2) by $x_\alpha \xrightarrow{\mathcal{P}} \phi$ and $y_\beta \xrightarrow{\mathcal{P}} \theta$. Thus for every polynomial P we have: $(\phi * \theta)(P) = \lim_\beta \lim_\alpha P(x_\alpha + y_\beta)$. Note that M_b is a semigroup with respect to the convolution product and $\phi * \theta \ne \theta * \phi$ in general (see [5, Remark 3.5]). We denote $\phi_1 * \cdots * \phi_n$ briefly by $\mathop{\ast}\limits_{k=1}^{n}\phi_k$.

Let I_k be the minimal closed ideal in $H_b(X)$, generated by all m-homogeneous polynomials, $0 < m \le k$. Evidently, I_k is a proper ideal (contains no unit) so it is contained in a closed maximal ideal (see [21, p. 228]). Let

$$\Phi_k := \{\phi \in M_b \ : \ \ker\phi \supset I_k\}.$$

We set $\Phi_0 := M_b$. The functional $\delta(0)$, that is, point evaluation at zero, belongs to Φ_k for every $k > 0$.

Corollary 2. *If $A_m(X) \neq A_{m-1}(X)$ for some $m > 1$, then there exists $\psi \in \Phi_{m-1}$ such that $\psi \notin \Phi_m$.*

Proof. Let $P \in \mathcal{P}(^m X)$ and $P \notin A_{m-1}(X)$. Since $A_{m-1}(X)$ is a closed subspace of $H_b(X)$, by the Hahn-Banach Theorem there exists a linear functional $\phi \in H_b(X)'$ such that $\phi(Q) = 0$ for every $Q \in A_{m-1}(X)$ and $\phi(P) \neq 0$. So $\phi_k \equiv 0$ for $k < m$ and $\phi_m(P) \neq 0$. By Lemma 1 there exists $\psi \in M_b$ such that $\psi_k = \phi_k$ for $k = 1, \dots, m$. Thus $\psi \in \Phi_{m-1}$, but $\psi \notin \Phi_m$. □

Note that $A_1(c_0) = A_n(c_0)$ for every n, but $A_k(\ell_p) = A_m(\ell_p)$ for $k \neq m$ if and only if $k < p$ and $m < p$. Moreover, if X admits a polynomial which is not weakly sequentially continuous, then the chain of algebras $\{A_k(X)\}$ does not stabilize and if X contains ℓ_1, then $A_k(X) \neq A_m(X)$ for $k \neq m$ [19, 12].

Lemma 3. *If $\phi, \psi \in M_b$ and $\psi \in \Phi_{k-1}$, then $\phi * \psi(P) = \phi(P) + \psi(P)$ for every $P \in \mathcal{P}(^k X)$.*

Proof. Let (x_α) and (y_β) be nets in X such that $x_\alpha \xrightarrow{\mathcal{P}} \phi$ and $y_\beta \xrightarrow{\mathcal{P}} \psi$. For any fixed y_β and $0 < n < k$, $A_P(x^{k-n}, y_\beta^n)$ is a $(k-n)$-homogeneous polynomial. Thus

$$\phi(A_P(x^{k-n}, y_\beta^n)) = \lim_\alpha A_P(x_\alpha^{k-n}, y_\beta^n) = 0.$$

Therefore,

$$\begin{aligned}
\phi * \psi(P) &= \lim_{\beta,\alpha} P(x_\alpha + y_\beta) \\
&= \sum_{n+m=k} \lim_{\beta,\alpha} A_P(x_\alpha^n, y_\beta^m) = \sum_{n+m=k} \lim_\beta \Big(\lim_\alpha A_P(x_\alpha^n, y_\beta^m) \Big) \\
&= \lim_\beta \Big(\lim_\alpha A_P(x_\alpha, \dots, x_\alpha) + A_P(y_\beta, \dots, y_\beta) \Big) = \phi(P) + \psi(P).
\end{aligned}$$

□

Lemma 4. *If $P \in \mathcal{P}(^k X)$, $\phi_j \in \Phi_{j-1}$, then for every $m > k$, $\mathop{\circledast}\limits_{j=1}^{m} \phi_j(P) = \mathop{\circledast}\limits_{j=1}^{k} \phi_j(P)$.*

Proof. Since $\phi_j \in \Phi_{j-1}$, $\phi_j(P) = 0$ for every $j > k$. □

Given a sequence $(\phi_n)_{n=1}^\infty \subset M_b$, $\phi_n \in \Phi_{n-1}$, the infinite convolution $\mathop{\circledast}\limits_{n=1}^{\infty} \phi_n$ denotes a linear multiplicative functional on the algebra of all polynomials $\mathcal{P}(X)$ such that $\mathop{\circledast}\limits_{n=1}^{\infty} \phi_n(P) = \mathop{\circledast}\limits_{n=1}^{k}(P)$ if $P \in \mathcal{P}(^k X)$ for an arbitrary k. This multiplicative functional uniquely determines a functional in M_b (which we denote by the same symbol $\mathop{\circledast}\limits_{n=1}^{\infty} \phi_n$) if it is continuous.

The point evaluation operator δ maps X into M_b by $x \mapsto \delta(x)$, $\delta(x)(f) = f(x)$. The operator $\widetilde{\delta}$ is the extension of δ onto X'', i.e. $\widetilde{\delta}(x'')(f) = \widetilde{f}(x'')$ for every $x'' \in X''$.

Theorem 5. *There exists a sequence of dual Banach spaces $(E_n)_{n=1}^\infty$ and a sequence of maps $\delta^{(n)} : E_n \to M_b$ such that $E_1 = X''$, $E_n = \mathcal{P}(^n X)' \cap I_{n-1}^\perp$, $\delta^{(1)} = \widetilde{\delta}$ and such that an arbitrary complex homomorphism $\phi \in M_b$ has a representation*

$$\phi = \mathop{\circledast}\limits_{n=1}^{\infty} \delta^{(n)}(u_n) \tag{3}$$

for some $u_n \in E_n$, $n = 1, 2, \dots$.

Proof. Put $E_1 = X''$. Then $\delta^{(1)}(x'') = \tilde{\delta}(x'') \in M_b$ for every $x'' \in X''$. Suppose that spaces E_k and maps $\delta^{(k)}$ are constructed for $k < n$. Denote by E_n the set $\{\pi_n(\phi) : \phi \in \Phi_{n-1}\}$, where $\pi_n(\phi) = \phi_n$ is the restriction of ϕ onto the subspace $\mathcal{P}(^nX)$. In other words, E_n consists of linear continuous functionals on $\mathcal{P}(^nX)$ that vanish on all polynomials in $\mathcal{P}(^nX) \cap A_{n-1}$. If $A_n = A_{n-1}$, then $E_n \equiv 0$. Otherwise, by Corollary 2, there are nonzero points in E_n.

By Lemma 3, for $P \in \mathcal{P}(^nX)$ and $\phi, \psi \in \Phi_{n-1} \subset M_b$, $\pi_n(\phi * \psi)(P) = \phi * \psi(P) = \phi(P) + \psi(P) = \pi_n\phi(P) + \pi_n\psi(P)$. Hence $\pi_n(\phi * \psi) = \pi_n(\phi) + \pi_n(\psi)$. For an arbitrary complex number a, $a\phi \in H_b(X)'$ and $\pi_k(a\phi) = a\pi_k(\phi)$. So $a\phi$ vanishes on all homogeneous polynomials of degree less than n. By Lemma 1 there exists $\psi \in M_b$ such that $\psi_k = a\phi_k$ for $1 \le k \le n$. Thus $\psi \in \Phi_{n-1}$ and $a\phi_n = \psi_n \in E_n$. Hence E_n is a linear space and polynomials from $\mathcal{P}(^nX)$ are acting on E_n as linear functionals. Put $W_n = \mathcal{P}(^nX)/(I_{n-1} \cap \mathcal{P}(^nX))$. Then W_n is a Banach space of linear functionals on E_n and the functionals from W_n separate points of E_n. Let us define a norm on E_n, $\|\cdot\|_n$ as the supremum of values of a vector from E_n on the unit ball of W_n. Therefore $W_n' = (\mathcal{P}(^nX)/(I_{n-1} \cap \mathcal{P}(^nX)))' = \mathcal{P}(^nX)' \cap I_{n-1}^{\perp} \supset E_n$. On the other hand, if $u \in \mathcal{P}(^nX)' \cap I_{n-1}^{\perp}$, then by Lemma 1, $u = \pi_n(\phi)$ for some $\phi \in M_b$ and so $u \in E_n$. Thus $E_n = W_n'$.

For given $w \in E_n$ let us define $\delta^{(n)}(w)(Q) = \psi(Q)$ on homogeneous polynomials Q by formula (1) and extend it to the unique complex homomorphism on $H_b(X)$ as in Lemma 1. So $\delta^{(n)}$ maps E_n into M_b. For any $\phi \in M_b$ put $u_1 := \phi_1 \in X'' = E_1$, $u_2 := \phi_2 - \pi_2(\delta^{(1)}(u_1))$. It is clear that $u_2 \in E_2$. Suppose that we have defined $u_k \in E_k$, $k < n$. Set

$$u_n := \phi_n - \pi_n\left(\mathop{\ast}_{k=1}^{n-1} \delta^{(k)}(u_k)\right). \tag{4}$$

Let us show that $u_n \in E_n$. It is enough to check that for every $P \in \mathcal{P}(^nX)$ such that $P = P_kP_m$, $\deg P_k = k \neq 0$, $\deg P_n = n \neq 0$ implies $u_n(P) = 0$. Note that for all n-homogeneous polynomials P_n,

$$\phi_n - \pi_n\left(\mathop{\ast}_{k=1}^{n-1} \delta^{(k)}(u_k)\right)(P_n) = \phi_n - \mathop{\ast}_{k=1}^{n-1} \delta^{(k)}(u_k)(P_n).$$

From the multiplicativity of ϕ and Lemma 4 it follows that

$$\begin{aligned}
u_n(P) &= \phi_n(P_kP_m) - \mathop{\ast}_{j=1}^{n-1} \delta^{(j)}(u_j)(P_kP_m) = \phi_k(P_k)\phi_m(P_m) \\
&- \left(\mathop{\ast}_{j=1}^{n-1} \delta^{(j)}(u_j)(P_k)\right)\left(\mathop{\ast}_{j=1}^{n-1} \delta^{(j)}(u_j)(P_m)\right) \\
&= \left(u_k(P_k) + \mathop{\ast}_{j=1}^{k-1} \delta^{(j)}(u_j)(P_k)\right)\left(u_m(P_m) + \mathop{\ast}_{j=1}^{m-1} \delta^{(j)}(u_j)(P_m)\right) \\
&- \left(\mathop{\ast}_{j=1}^{k} \delta^{(j)}(u_j)(P_k)\right)\left(\mathop{\ast}_{j=1}^{m} \delta^{(j)}(u_j)(P_m)\right) = 0.
\end{aligned}$$

The last equality holds because by the induction assumption, $u_k \in E_k$, $u_m \in E_m$ and hence, by Lemma 3,

$$u_k(P_k) + \mathop{\ast}_{j=1}^{k-1} \delta^{(j)}(u_j)(P_k) = \mathop{\ast}_{j=1}^{k} \delta^{(j)}(u_j)(P_k) \tag{5}$$

and

$$u_m(P_m) + \mathop{\circledast}_{j=1}^{m-1} \delta^{(j)}(u_j)(P_m) = \mathop{\circledast}_{j=1}^{m} \delta^{(j)}(u_j)(P_m).$$

Let us consider the functional $\mathop{\circledast}\limits_{j=1}^{\infty} \delta^{(j)}(u_j)$. Since $u_k \in E_k$, by Lemma 3,

$$\mathop{\circledast}_{j=1}^{\infty} \delta^{(j)}(u_j)(f) = f(0) + \sum_{n=1}^{\infty} \mathop{\circledast}_{j=1}^{n} \delta^{(j)}(u_j)(f_n),$$

where $f = \sum f_n$ is the Taylor series expansion of f. Hence $\mathop{\circledast}\limits_{j=1}^{\infty} \delta^{(j)}(u_j)$ is well defined on $\mathcal{P}(X)$. On the other hand, applying (4) and (5) we obtain

$$\begin{aligned} \left(\phi - \mathop{\circledast}_{j=1}^{\infty} \delta^{(j)}(u_j)\right)(P_n) &= \phi_n(P_n) - \mathop{\circledast}_{j=1}^{n} \delta^{(j)}(u_j)(P_n) \\ &= u_n(P) + \mathop{\circledast}_{j=1}^{n-1} \delta^{(j)}(u_j)(P_n) - \mathop{\circledast}_{j=1}^{n} \delta^{(j)}(u_j)(P_n) = 0 \end{aligned}$$

for arbitrary $P_n \in \mathcal{P}(^nX)$. Thus $\phi = \mathop{\circledast}\limits_{j=1}^{\infty} \delta^{(j)}(u_j)$ on $\mathcal{P}(X)$. Hence $\phi = \mathop{\circledast}\limits_{j=1}^{\infty} \delta^{(j)}(u_j)$ on $H_b(X)$. □

Let us denote by E^∞ the space of all finite sequences $(u_1, \ldots, u_m, 0, \ldots)$, $u_k \in E_k$. According to Theorem 5, every finite sequence $\mathfrak{u} = (u_1, \ldots, u_m, 0, \ldots)$ defines a character $\phi_{\mathfrak{u}} = \mathop{\circledast}\limits_{k=1}^{m} \delta^{(k)}(u_k) \in M_b$. Thus $E^\infty \subset M_b$ and for every $\mathfrak{u}, \mathfrak{v} \in E^\infty$, $\phi_{\mathfrak{u}+\mathfrak{v}} \in M_b$. Moreover, from the density of polynomials in $H_b(X)$ it follows that E^∞ is dense in M_b with respect to the Gelfand topology. So we have proved the following theorem.

Theorem 6. *M_b contains the dense linear subspace of all finite subsequences $(u_1, \ldots, u_m, 0, \ldots)$, $u_k \in E^k$.*

According to [3, 7], the operation of sum on X may be discontinuous with respect to the Gelfand topology, induced from M_b. Hence, in general, E^∞ is not a topological vector space. Thus, the density of E^∞ in M_b does not imply that M_b is a linear space.

We need to have some properties of the radius function, proved by Aron, Cole and Gamelin in [3].

Proposition 7. (1) *For each $r > 0$, the set of $\phi \in M_b$ satisfying $R(\phi) \leq r$ coincides with the spectrum of $H^\infty_{uc}(rB)$. In particular, $M_{uc} = \{\phi \in M_b : R(\phi) \leq 1\}$.*
(2) *For every $\phi, \psi \in H_b(x)'$, $R(\phi * \psi) \leq R(\phi) + R(\psi)$.*

Example 8. 1. Let X be c_0 or Tsirelson's space. Then $E_k = \{0\}$ for $k > 1$ [4, 22].

2. Let $X = \ell_1$ and $\phi \in H_b(\ell_1)'$, $\|\phi\| = 1$. According to [3], $\phi \in M_b(\ell_1)$ if and only if for every $m = 1, 2, \ldots$ there exists a symmetric measure on $\beta(\mathbb{N}^m)$, ν_m and a constant $c > 0$ such that $\|\nu_m\| \leq c^m$ and for each $P_m \in \mathcal{P}(^m\ell_1)$,

$$\phi(P_m) = \int_{\beta(\mathbb{N}^m)} \widehat{P}_m d\nu_m,$$

where $\widehat{P}_m$ is just P_m regarded as a vector from $\ell^\infty(\mathbb{N}^m)$. By Theorem 5, $\phi \in M_b(\ell_1)$ if and only if there is a sequence of symmetric measures (μ_m) which are orthogonal to $\beta(\mathbb{N}^j) \times \beta(\mathbb{N}^k) \subset \beta(\mathbb{N}^m)$, for $m > 1$, $k + j = m$, $k, j > 0$, and functionals

$$u_m(P_m) = \int_{\beta(\mathbb{N}^m)} \widehat{P}_m d\mu_m$$

determine ϕ by formula (3).

3. (Cf. [1, Example 3.1].) Let $X = \ell_p$ for some integer p, $1 < p < \infty$. For every n, put

$$v_n = \frac{1}{n^{1/p}}(e_1 + \cdots + e_n),$$

where (e_k) is the standard basis in ℓ_p. Since $\|v_n\| = 1$, $R(\delta(v_n)) = 1$ and so $\delta(v_n) \in M_{uc} \subset M_b$. By compactness of M_{uc} there is an accumulation point $\phi \in M_{uc}$ of the sequence $(\delta(v_n))$. If $0 < k < p$, then by Pitt's Theorem (see [16, Theorem 5.1]) every polynomial $P \in \mathcal{P}(^k\ell_p)$ is weakly continuous on bounded sets. Since v_n is weakly null in ℓ_p, $\phi(P) = 0$. On the other hand, $\phi(Q) = 1$ for the polynomial $Q(x) = \sum_{n=1}^\infty x_n^p$. Thus $\phi \in \Phi_{p-1}$ and $\phi \neq 0$. In other words, if $\phi = \mathop{\ast}\limits_{k=1}^{\infty} \delta^{(k)}(u_k)$ is the representation of ϕ by Theorem 5, then $u_k = 0$ for $k < p$ and $u_p \neq 0$.

Proposition 9. *Let $\phi \in M_b$ and let $\phi = \mathop{\ast}\limits_{k=1}^{\infty} \delta^{(k)}(u_k)$, $u_k \in E_k$, be its representation. Then*

$$\limsup_{k\to\infty} \|u_k\|_k^{1/k} \le R(\phi) \le \sum_{k=1}^\infty \|u_k\|_k^{1/k}.$$

Proof. The first inequality holds because $\|u_k\|_k \le \|\phi_k\|$ and by the definition of the radius function. The second inequality follows from Proposition 7 and the following calculation:

$$R(\delta^{(k)}(u_k)) = \limsup_{m\to\infty} \|\pi_{km}(\delta^{(k)}(u_k))\|^{1/km} = \|\delta^{(k)}(u_k)\|^{m/km} = \|u_k\|^{1/k}.$$

□

Let F be an analytic map from ℓ_1 to ℓ_1 defined by

$$F(x) = F\left(\sum_{n=1}^\infty x_n e_n\right) = \sum_{n=1}^\infty x_n^n e_n.$$

We denote by $F(\ell_1)$ the range of F and by $F(B_{\ell_1})$ the range of F restricted to the unit ball $B_{\ell_1} \subset \ell_1$.

Given a sequence of Banach spaces $(E_n, \|\cdot\|_n)_{n=1}^\infty$ and $0 < \rho \le \infty$ the Köthe sequence space $\lambda^1(K_\rho; (E_n))$ (where $K_\rho = \{(r^n)_{n=1}^\infty : 0 < r < \rho\}$) is the Fréchet space

$$\left\{(x_n)_{n=1}^\infty \in \prod_{n=1}^\infty E_n : p_r((x_n)_{n=1}^\infty) = \sum_{n=1}^\infty \|x_n\| r^n < \infty\ \forall r,\ 0 < r < \rho\right\},$$

endowed with the topology given by the seminorms $\{p_r\}_{0<r<\rho}$. By Cauchy-Hadamard's formula,

$$\lambda^1(K_\rho; (E_n)) = \left\{(x_n)_{n=1}^\infty \in \prod_{n=1}^\infty E_n :\ \limsup_{n\to\infty} \|x_n\|_n^{1/n} \le \frac{1}{\rho}\right\}.$$

Corollary 10. (1) *M_b contains every sequence $\mathfrak{u} = (u_k)_{k=1}^{\infty}$, $u_k \in E_k$, such that the sequence $(\|(u_k)\|)_{k=1}^{\infty}$ is in $F(\ell_1)$.*
(2) *M_{uc} contains every sequence $\mathfrak{u} = (u_k)_{k=1}^{\infty}$, $u_k \in E_k$, such that the sequence $(\|(u_k)\|)_{k=1}^{\infty}$ is in $F(B_{\ell_1})$.*
(3) *Every complex homomorphism $\phi \in M_b$ is contained in a Köthe sequence space $\lambda^1(K_\rho; (E_n))$ for $\rho = 1/R(\phi)$.*
(4) *M_{uc} is contained in $\lambda^1(K_1; (E_n))$.*

Proof. Since $F^{-1}((\|u_k\|)_{k=1}^{\infty}) \in \ell_1$, $\sum_{k=1}^{\infty} \|u_k\|^{1/k} \le \infty$ and by Proposition 9, $R(\phi_{\mathfrak{u}}) < \infty$. Thus $\phi_{\mathfrak{u}} \in M_b$. Moreover, if $F^{-1}((\|u_k\|)_{k=1}^{\infty}) \in F(B_{\ell_1})$, then $R(\phi_{\mathfrak{u}}) \le 1$ and $\phi_{\mathfrak{u}} \in M_{uc}$.

Suppose that $\phi_{\mathfrak{u}} \in M_b$ for some $\mathfrak{u} = (u_k)_{k=1}^{\infty}$. Then $R(\phi_{\mathfrak{u}}) < \infty$ and by Proposition 9, $\limsup\limits_{k\to\infty} \|u_k\|^{1/k} \le R(\phi_{\mathfrak{u}})$. Hence $\phi_{\mathfrak{u}} \in \lambda^1(K_{1/R(\phi_{\mathfrak{u}})}; (E_n))$. In particular, if $R(\phi_{\mathfrak{u}}) \le 1$, then $\phi_{\mathfrak{u}} \in \lambda^1(K_1; (E_n))$. □

Dixon [14] has given an example of an algebra of polynomials of infinitely many variables which admits discontinuous scalar-valued homomorphisms. In [15] a construction is given of a discontinuous scalar-valued homomorphism of an algebra of polynomials on an arbitrary infinite-dimensional Banach space. The next corollary shows that the restriction of a discontinuous complex homomorphism on $A_n(X) \cap \mathcal{P}(X)$ can be continuous for every n. Note that the problem of existence of discontinuous complex homomorphisms on $H_b(X)$ for an infinitele-dimensional Banach space X is still open and equivalent to the famous Michael Problem [20], [21, p. 240].

Corollary 11. *If the sequence of algebras $A_n(X)$ does not stabilize, then there is a discontinuous complex homomorphism ζ on $\mathcal{P}(X)$ such that the restriction of ζ on $A_n(X) \cap \mathcal{P}(X)$ is a continuous complex homomorphism for every n.*

Proof. By Corollary 2 and Theorem 5 there exists an infinite sequence $(u_k)_{k=1}^{\infty}$, $u_k \in E_k$, $u_k \ne 0$. Since each E_k is a linear space, we can choose u_k such that $\limsup\limits_{k\to\infty} \|u_k\|_k^{1/k} = \infty$. Put $\zeta = \mathop{\ast}\limits_{k=1}^{\infty} \delta^{(k)}(u_k)$. Evidently,

$$\zeta(f) = \mathop{\ast}\limits_{k=1}^{n} \delta^{(k)}(u_k)(f)$$

for every $f \in A_n(X)$. So ζ is well defined and continuous on $A_n(X) \cap \mathcal{P}(X)$. If ζ is continuous on $\mathcal{P}(X)$, then it can be extended to a continuous complex homomorphism on $H_b(X)$. But this contradicts Proposition 9. □

In [11] Deghoul, using Borsuk's theorem, shows that there is an "exceptional" character ϕ on $H_b(\ell_2)$ such that ϕ vanishes on odd degree homogeneous polynomials and is different from the evaluation at 0. The next proposition delivers the existence of exceptional characters on $H_b(X)$ for a large number of X.

Proposition 12. *Suppose that $A_m(X) \ne A_k(X)$ for some $m > 1$ and all $k < m$. Then there exists a nontrivial character $\psi_0 \in M_b$ such that $\psi_0(P) = 0$ for every homogeneous polynomial P, $\deg P \ne nm$, $n = 1, \dots, \infty$.*

Proof. By Corollary 2 there exists a nontrivial character $\psi \in M_b$ which vanishes on all k-homogeneous polynomials for $k < m$. From Theorem 5 it follows that E_m contains a nonzero vector u_m. Put $\psi_0 = \delta^{(m)}(u_m)$. Then ψ_0 vanishes on all homogeneous polynomials excepting nm-homogeneous polynomials, $n = 1, 2, \ldots$. □

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