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IDENTITIES INVOLVING SUMS OF PRODUCTS OF FIBONACCI POLYNOMIALS AND MULTINOMIAL COEFFICIENTS

МУЛЬТИНОМИАЛЬНЫЕ ТОЖДЕСТВА ДЛЯ СУММ ПРОИЗВЕДЕНИЙ ПОЛИНОМОВ ФИБОНАЧЧИ

We consider determinants for some families of Toeplitz-Hessenberg matrices the entries of which are Fibonacci polynomials introduced by E. C. Catalan. These determinant formulas may also be rewritten as identities involving sums of products of Fibonacci polynomials and multinomial coefficients.

Мы рассматриваем определители некоторых семейств матриц Теплица-Хессенберга, элементами которых являются полиномы Фибоначчи. Полученные формулы для указанных определителей записаны нами как тождества, включающие суммы произведений полиномов Фибоначчи и мультиномиальные коэффициенты.

Keywords: Fibonacci polynomial, Toeplitz-Hessenberg matrix, Trudi’s formula, multinomial coefficient.

Ключевые слова: полином Фибоначчи, матрица Теплица-Хессенберга, формула Труди, мультиномиальный коэффициент.

Fibonacci polynomials are polynomials that can be defined by Fibonacci-like recursion relations and they were studied in 1883 by E.C. Catalan and E. Jacobsthal. For example, Catalan studied the polynomials $F_n(x)$ defined by the recurrence [4]

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \qquad n \geq 2, \tag{1}$$

with $F_0(x) = 0$ and $F_1(x) = 1$.

The few next Fibonacci polynomials are $F_2(x)=x$, $F_3(x)=x^2+1$, $F_4(x)=x^3+2x$, $F_5(x)=x^4+3x^2+1$, $F_6(x)=x^5+4x^3+3x$, etc. A non-recursive expression for $F_n(x)$ is

$$F_n(x)=\sum_{i=0}^{\lfloor n/2\rfloor}\binom{n-i-1}{i}x^{n-2i-1},\qquad n\ge 0.$$

Note that $F_n(1)=F_n$, where F_n is the *n*-th Fibonacci number.

Fibonacci polynomials are special case of Chebyshev polynomials and have been studied by many mathematicians (see, [1, 4–6] for the complete bibliography). For example, Nalli and Haukkanen [5] introduced the *h*(*x*)-Fibonacci polynomials that generalized Fibonacci polynomials $F_n(x)$ and the *k*-Fibonacci numbers $F_{k;n}$. In [6], Tingting and Wenpeng studied some sums of powers of Fibonacci polynomials, and gave several interesting identities. Abd-Elhammed et al. established new connection formulas between Fibonacci polynomials and Chebyshev polynomials; these formulas are expressed in terms of certain values of hypergeometric functions [1].

Denote $m_n(s)=\dfrac{(s_1+\cdots+s_n)!}{s_1!\cdots s_n!}$, $\sigma_n=s_1+2s_2+...+ns_n$ and $|s|=s_1+s_2+...+s_n$.

Using Toeplitz-Hessenberg determinants with special entries $F_i(x)$ and Trudi's formula (see [2, 3] and the references given there), we derive the following Fibonacci identities with multinomial coefficients $m_n(s)$.

Theorem 1. *Let* $n\ge 2$, *except when noted otherwise. Then*

$$\sum_{\sigma_n=n}(-1)^{|s|}m_n(s)F_0^{s_1}(x)F_1^{s_2}(x)\cdots F_{n-1}^{s_n}(x)=-x^{n-2},$$

$$\sum_{\sigma_n=n}m_n(s)F_0^{s_1}(x)F_1^{s_2}(x)\cdots F_{n-1}^{s_n}(x)=\sum_{i=1}^{n-1}2^{i-1}\binom{n-1-i}{i-1}x^{n-2i},\qquad n\ge 1,$$

$$\sum_{\sigma_n=n}(-1)^{|s|}m_n(s)F_1^{s_1}(x)F_3^{s_2}(x)\cdots F_{2n-1}^{s_n}(x)=-x^2(x^2+1)^{n-2},$$

$$\sum_{\sigma_n=n}(-1)^{|s|}m_n(s)F_2^{s_1}(x)F_3^{s_2}(x)\cdots F_{n+1}^{s_n}(x)=0,\qquad n\ge 3,$$

$$\sum_{\sigma_n=n}(-1)^{|s|}m_n(s)F_3^{s_1}(x)F_5^{s_2}(x)\cdots F_{2n+1}^{s_n}(x)=-x^2,$$

$$\sum_{\sigma_n=n}(-1)^{|s|}m_n(s)\left(\frac{F_1(x)}{x}\right)^{s_1}\left(\frac{F_2(x)}{x}\right)^{s_2}\cdots\left(\frac{F_n(x)}{x}\right)^{s_n}=\sum_{i=0}^{n-1}(-1)^{n-i}x^{2i-n},\qquad n\ge 1,$$

$$\sum_{\sigma_n=n}(-1)^{|s|}m_n(s)\left(\frac{F_3(x)}{x}\right)^{s_1}\left(\frac{F_4(x)}{x}\right)^{s_2}\cdots\left(\frac{F_{n+2}(x)}{x}\right)^{s_n}=\frac{1}{(-x)^n},$$

$$\sum_{\sigma_n=n}m_n(s)\left(\frac{F_1(x)}{x}\right)^{s_1}\left(\frac{F_2(x)}{x}\right)^{s_2}\cdots\left(\frac{F_n(x)}{x}\right)^{s_n}=\sum_{i=1}^{n}\sum_{j=1}^{n}\binom{i-1}{j-1}\binom{n-j}{i-1}x^{n-2i},$$

$$\sum_{\sigma_n=n}(-1)^{|s|}m_n(s)\left(\frac{F_4(x)}{x}\right)^{s_1}\left(\frac{F_6(x)}{x}\right)^{s_2}\cdots\left(\frac{F_{2n+2}(x)}{x}\right)^{s_n}=0,\qquad n\ge 3,$$

$$\sum_{\sigma_n=n}m_n(s)\left(\frac{F_0(x)}{x}\right)^{s_1}\left(\frac{F_2(x)}{x}\right)^{s_2}\cdots\left(\frac{F_{2n-2}(x)}{x}\right)^{s_n}=(x^2+2)^{n-2},$$

where the summation is over all n-tuples $(s_1,\ldots,s_n)$ *of nonnegative integers* s_i *satisfying the Diophantine equation* $\sigma_n=n$.

Theorem 1 may be proved in the same way as Theorems 1 and 2 in [3].

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