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Representing measures and infinite-dimensional holomorphy ☆

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Abstract

We consider some applications of the Bishop–De Leeuw Theorem about representing measures for some algebras of analytic functions on unit balls of Banach spaces. In particular, we investigate Hardy spaces H^2 associated with corresponding algebras. Some examples are considered.

Keywords: Representing measure; Hardy spaces; Infinity-dimensional holomorphy; Spectra of algebras

0. Introduction

The classical Bishop–De Leeuw Theorem says that every complex homomorphism ϕ of a uniform Banach algebra A can be realized by a *representing measure* μ on the set of maximal ideals $\mathfrak{M}(A)$ of A. This result has many applications for algebras of analytic functions on open subsets of $\mathbb{C}^n$ (see e.g. [11]). Having μ we can consider the Hardy space $H^2(\mu)$, that is, the completion of A with respect to the "norm"

$$\|f\|_\mu = \sqrt{\int_{\mathfrak{M}(A)} |\hat{f}|^2 \, d\mu} =: \sqrt{\langle f \mid f \rangle},$$

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where $\hat{f}$ is the Gelfand transform of f considered as a continuous function on the compact set $\mathfrak{M}(A)$ endowed with the Gelfand topology.

Note that in general, $\|\cdot\|_\mu$ is a semi-norm on A. We say that a representing measure μ is *norming* for A if $\|\cdot\|_\mu$ is a norm. There are uniform algebras without norming representing measures. For example, if $A = C_c(T)$ is the algebra of all complex continuous functions on a Hausdorff space T, then the atomic measure concentrated at x_0, $\tilde{\delta}_{x_0}$ is a unique representing measure of a point-evaluation functional δ_{x_0}, $\delta_{x_0}(f) = f(x_0)$, $x_0 \in T$ and $\|\cdot\|_{\tilde{\delta}_{x_0}}$ is not a norm. Moreover, it is still true for any uniform Banach algebra A if x_0 is in the Choquet boundary of A (see [7] for details).

In [9] Cole and Gamelin developed the theory of Hardy spaces H^p on the infinite-dimensional polydisk using representing measures. The space H^2 on the infinite-dimensional polydisk was also independently constructed in [14] due to Neeb and Ørsted by infinite tensor products. In [12] the authors proposed an another approach to construction of analogues of Hardy spaces H^2 for infinite-dimensional domains using Hilbert symmetric tensor products and symmetric Fock spaces. Analytic functions which have integral representations with respect to Gaussian measure on abstract Wiener spaces were considered by Pinasco and Zaldendo in [16].

Section 1 of this paper is a brief reminder of the Bishop–De Leeuw Theorem for uniform Banach algebras and some related questions. In Section 2 we consider the Hardy space $H^2_{\mathrm{a}}(\mu)$ associated with a Banach algebra $A_{\mathrm{a}}(B)$ of analytic functions on the unit ball B of a Banach space which are generated by linear functionals for a norming measure μ that represents the zero-evaluation functional $\delta_0 \in A_{\mathrm{a}}(B)'$. In particular, we describe an orthogonal basis of $H^2_{\mathrm{a}}(\mu)$ for some special case of μ. In Section 3 we investigate under which conditions $H^2_{\mathrm{a}}(\mu)$ is a reproducing kernel Hilbert space and when the reproducing kernel is determined by an analytic map. In Section 4 we consider some connections between $H^2_{\mathrm{a}}(\mu)$ and a vector valued Hardy space and prove some analogue of boundary valued theorem for $H^2_{\mathrm{a}}(\mu)$. Some examples of representing measures μ and corresponding Hardy spaces $H^2(\mu)$ for c_0 and ℓ_p, $1 < p < \infty$, are considered in Section 5.

For background of analytic functions on infinite-dimensional Banach spaces, we refer the reader to [8,10] and [13].

1. Preliminaries on representing measures

Let T be a compact Hausdorff space. We denote by $C_r(T)$ and $C_c(T)$ the Banach algebras of real and complex valued functions f on T respectively with the uniform norm $\|f\| = \sup_{x\in T} |f(x)|$. Let $\mathcal{M}$ be the set of non-negative regular Borel measures on T which we think as a subset of the dual space $C_r(T)'$.

Let V be a subspace of $C_r(T)$ or $C_c(T)$ which contains the constant functions and $x \in T$. We define $\mathcal{M}_x(V)$ to be the subset of $\mathcal{M}$ consisting of all measures μ with

$$\int f\,d\mu = f(x)$$

for all $f \in V$. It is known that $\mathcal{M}_x(V)$ is a compact and convex subset of $C_r(T)'$ or $C_c(T)'$, respectively. $\mathcal{M}_x(V)$ is always non-empty since it must contain at least the point evaluation functional $f \mapsto f(x)$, $f \in C_c(T)$.

Let S be a subset of T and V a collection of scalar functions on T. We define

$$i_V(S) := \bigl\{y \in T\colon\ f(y) = f(x) \text{ for some } x \in S \text{ and all } f \in V\bigr\}.$$

Definition 1.1. A given subspace V of either $C_r(T)$ or $C_c(T)$, the *Choquet boundary* of V, denoted by $\partial_C(V)$ is defined to consist of those points x in T which are such that any μ in $\mathcal{M}_x(V)$ satisfies

$$\mu\big(i_V(x)\big) = 1.$$

Note the Choquet boundary is not necessarily closed (or even Borel) subset of T. From the definition it follows that if V separates points of T and $x \in \partial_C(V)$, then δ_x has a unique representing measure that is the atomic measure concentrated at x.

If V is a subspace of $C_c(T)$, we will denote by V_r the subspace of $C_r(T)$ consisting of real parts of the functions in V. It is known [7] that $\partial_C(V_r) = \partial_C(V)$ and for each $x \in T$, $\mathcal{M}_x(V_r) = \mathcal{M}_x(V)$.

If V is a subspace of $C_r(T)$ or $C_c(T)$ and μ and ν are in $\mathcal{M}$, we write $\nu \prec \mu$ if

$$\int f\,d\nu = \int f\,d\mu \quad \forall f \in V_r,$$

and

$$\int f^2\,d\nu \leqslant \int f^2\,d\mu \quad \forall f \in V_r.$$

We say that ν is *maximal* if $\nu \prec \mu$ implies $\nu = \mu$. Let V be a subspace of $C_r(T)$ or $C_c(T)$. From Zorn's Lemma it follows that for each ν in $\mathcal{M}$ there exists a maximal measure $\mu \in \mathcal{M}$ such that $\nu \prec \mu$ [7].

Let A be an arbitrary complex commutative uniform Banach algebra with identity and let $\mathfrak{M}(A)$ be the set of its maximal ideals (nonzero complex homomorphisms) endowed with the Gelfand topology. Using the Gelfand transform $f \mapsto \hat{f}$, $f \in A$, we can consider A as some function algebra namely, as a closed subalgebra of the algebra of continuous functions on the compact Hausdorff space $\mathfrak{M}(A)$ and

$$\|f\|_A = \sup_{\phi \in \mathfrak{M}(A)} \big|\hat{f}(\phi)\big|,$$

where $\hat{f}(\phi) = \phi(f)$.

Theorem 1.2 *(Bishop and De Leeuw). For every $\phi \in \mathfrak{M}(A)$ there is a maximal probability measure μ (the so-called* representing measure for ϕ) *on Choquet boundary $\partial_C(A) \subset \mathfrak{M}(A)$ such that*

$$\phi(f) = \int \hat{f}\,d\mu.$$

Note, for example, that the disk algebra $A(D)$, where D is the closed unit disk of $\mathbb{C}$, admits a unique maximal measure which represents δ_0. This measure is the probability Lebesgue measure on the unit circle [11, p. 38].

Throughout in this paper we consider algebras of analytic function on unit balls of Banach spaces and corresponding Hardy spaces $H^2(\mu)$ for some representing measure μ.

There is a number of interesting algebras of analytic functions on the open unit ball B of a complex Banach space X. We use a stand notation $\mathcal{P}(^nX)$ for the space of all continuous n-homogeneous polynomials on X. Let $\mathcal{P}_f(^nX) := \operatorname{span}\{\varphi^n\colon\ \varphi \in X'\}$ be the space of finite type n-homogeneous polynomials and its closure in $\mathcal{P}(^nX)$, $\mathcal{P}_{\mathrm{a}}(^nX)$ the space of approximable polynomials. If X' has the approximation property, then $\mathcal{P}_{\mathrm{a}}(^nX)$ coincides with the space

$\mathcal{P}_{\mathrm{w}}({}^n X)$ of weakly (uniformly) continuous polynomials on bounded sets [6]. In the general case, $\mathcal{P}_{\mathrm{a}}({}^n X) \subset \mathcal{P}_{\mathrm{w}}({}^n X)$. In [5] Aron, Cole and Gamelin constructed an example of a separable Banach space X without the approximation property such that $\mathcal{P}_{\mathrm{a}}({}^n X) \neq \mathcal{P}_{\mathrm{w}}({}^n X)$.

Let us fix notations for some algebras of analytic functions on the unit ball B of X. $H^\infty_{\mathrm{uc}}(B)$ denotes the algebra of uniformly continuous analytic functions on B; $A_{\mathrm{w}}(B)$ is a subalgebra of $H^\infty_{\mathrm{uc}}(B)$, consisting of weakly uniformly continuous functions on B and $A_{\mathrm{a}}(B)$ is the subalgebra of $A_{\mathrm{w}}(B)$ generated by polynomials from $\mathcal{P}_{\mathrm{a}}({}^n X)$, $n = 0, 1, 2, \ldots$. If X' has the approximation property, then $A_{\mathrm{w}}(B) = A_{\mathrm{a}}(B)$. If $X = c_0$, then by the Littlewood–Bogdanowicz–Pełczyński Theorem [15], $A_{\mathrm{a}}(B) = H^\infty_{\mathrm{uc}}(B)$. An another space, for which these algebras coincide, is the Tsirelson space [4]. According to [5] $\mathfrak{M}(A_{\mathrm{a}}(B)) = \overline{B}_{X''}$. Some description of $\mathfrak{M}(H^\infty_{\mathrm{uc}}(B))$ for an arbitrary Banach space was obtained in [19].

Note that in [3] Aron et al. considered boundary points of $H^\infty_{\mathrm{uc}}(B)$ *which belong to* $\overline{B}$ and showed that the Choquet boundary of $H^\infty_{\mathrm{uc}}(B)$ has a dense intersection with the unit sphere of ℓ_p.

2. Hardy spaces for algebra $A_{\mathrm{a}}(B)$

Let μ be a norming measure on the Choquet boundary of $\mathfrak{M}(A_{\mathrm{a}}(B))$ which represents δ_0 and $H^2_{\mathrm{a}}(\mu)$ be the corresponding Hardy space. Since $\mathfrak{M}(A_{\mathrm{a}}(B)) = \overline{B}_{X''}$, the Choquet boundary of $A_{\mathrm{a}}(B)$ is a subset of $\overline{B}_{X''}$. Moreover, in [2] Arenson has shown in more general situation that the Choquet boundary of $A_{\mathrm{a}}(B)$ coincides with the set of complex extreme points of $\overline{B}_{X''}$.

Note that every bounded analytic function on B can be extended to a bounded analytic function $\hat{f}$ on $B_{X''}$ which is referred as the Aron–Berner extension of f (see [10] for details). Moreover, since B is weak-star dense in $B_{X''}$ and all functions in $A_{\mathrm{a}}(B)$ are continuous on $B \subset B_{X''}$ with respect to the weak-star topology on X'', the operator of Aron–Berner extension $f \mapsto \hat{f}$ coincides with the Gelfand transform on $A_{\mathrm{a}}(B)$ [5].

Proposition 2.1. *Let X be a separable Banach space. Then there exists a norming measure for $A_{\mathrm{a}}(B)$ which is defined on the Choquet boundary $\partial_C(A_{\mathrm{a}}(B))$ and represents δ_0.*

Proof. Let (y_n) be a dense sequence in the unit sphere S_X of X. Then (y_n) is weak-star dense in $B_{X''}$. For every y_n we consider a circle $e^{\mathrm{i}\vartheta} y_n$, $0 \leqslant \vartheta < 2\pi$. Put $\lambda_n = 2^{-n}\lambda$, where λ is the normalized Lebesgue measure on the unit circle. Let U be a Borel subset of $B_{X''}$. We set

$$\nu(U) = \sum_{n=1}^{\infty} \lambda_n\bigl(U \cap \bigl\{e^{\mathrm{i}\vartheta} y_n\colon 0 \leqslant \vartheta < 2\pi\bigr\}\bigr).$$

Let f be a nonzero function in $A_{\mathrm{a}}(B)$ and $\hat{f}$ be its Aron–Berner extension. Since $\hat{f}$ is weak-star continuous on $B_{X''}$, there is a weak-star open subset $V \in X''$ such that $|\hat{f}(x)|^2 > 0$ for every $x \in V \cap B_{X''}$. So

$$\int \bigl|\hat{f}(x)\bigr|^2\, d\nu \geqslant \int_{V \cap B_{X''}} \bigl|\hat{f}(x)\bigr|^2\, d\nu > 0.$$

Since

$$\int \hat{f}\, d\nu = \sum_{n=0}^{\infty} 2^{-n} f(0) = f(0),$$

ν is representing for δ_0. Let μ be a maximal representing measure for δ_0 such that $\nu \prec \mu$. Then μ is concentrated on the Choquet boundary $\partial_C(A_a(B))$ and is norming for $A_a(B)$ because $\int |\hat{f}(x)|^2\, d\mu \geqslant \int |\hat{f}(x)|^2\, d\nu$. $\square$

Proposition 2.2. *Let μ be a norming probability measure for $A_a(B)$. Then $\|f\|_\mu \leqslant \|f\|_{A_a(B)}$.*

Proof. Since μ is norming, the natural embedding of $A_a(B)$ into $H_a^2(\mu)$ is injective. Moreover,

$$\|f\|_\mu = \int \left|\hat{f}(x)\right|^2 d\mu \leqslant \sup_{x \in B_{X''}} \left|\hat{f}(x)\right| \mu(B_{X''}) = \|f\|_{A_a(B)}. \qquad \square$$

Now we suppose that X is a Banach space with the separable dual X' and μ is a δ_0-representing measure, norming for $A_a(B)$. Let E' be the completion of X' in $H_a^2(\mu)$. It is clear that E' is a Hilbert space. Let (e_i^*) be an orthonormal basis in E'. Since X' is a dense subspace of E', we can suppose that $e_i^* \in X'$ for every i.

By Proposition 2.2,

$$\left\|x^*\right\|_{E'} \leqslant \left\|x^*\right\|_{X'} \quad \text{or} \quad \left\|x^{**}\right\|_{E} \geqslant \left\|x^{**}\right\|_{X''},$$

where $x^* \in X'$ and $x^{**} \in X''$. It means, in particular that every open set of E with respect to the norm $\|\cdot\|_{X''}$ is open with respect to $\|\cdot\|_E$.

We will denote by (e_i) the orthonormal basis in E such that $e_i = \langle \cdot \mid e_i^* \rangle$.

Let (i) be a multi-index $(i) = (i_1, \ldots, i_n)$, $i_1 \leqslant \cdots \leqslant i_n$ for some n, $e_{(i)}^* = e_{i_1}^* \ldots e_{i_n}^*$ and if $x_{i_k} = e_{i_k}^*(x)$, then $x_{(i)} = x_{i_1} \ldots x_{i_n}$.

Proposition 2.3. *The Hilbert space $H_a^2(\mu)$ coincides with the closed linear span of $(e_{(i)}^*)_{|(i)| \geqslant 0}$, where e_0^* is the constant function, $e_0^*(x) = 1$.*

Proof. Since $e_i^* \in X'$ for every i, it follows that $e_{(i)}^* = e_{i_1}^* \ldots e_{i_n}^* \in A_a(B)$ and so $e_{(i)}^* \in H_a^2(\mu)$ for every multi-index (i).

Let H_0 be a closed linear span of polynomials $e_{(i)}^*$ in $H_a^2(\mu)$ and $f \in H_a^2(\mu)$. Then f can be approximated by functions from $A_a(B)$ in the topology of $H_a^2(\mu)$. Each function from $A_a(B)$ can be approximated by polynomials of finite type in the uniform topology, so in the topology of $H_a^2(\mu)$ as well. But polynomials of finite type belong to H_0. Indeed, let $l_1, \ldots, l_m$ be linear functionals in X'. Since $X' \subset E'$, each l_k can be represented by a convergent in E' series $l_k = a_1^k e_1^* + \cdots + a_n^k e_n^* + \cdots$. But each functional e_j^* belongs to X' so

$$l_1 l_2 \ldots l_m = \sum_{1 \leqslant i_1, \ldots, i_m < \infty} a_{i_1}^1 \ldots a_{i_m}^m e_{i_1}^* \ldots e_{i_m}^*.$$

It means that products of linear functionals in X' belong to H_0. Hence any polynomial of finite type that is a finite sum of finite products of linear functionals belong to H_0. Therefore $f \in H_0$ and so $H_0 = H_a^2(\mu)$. $\square$

Since the set of maximal ideals of $A_a(B)$ coincides with $\overline{B}_{X''}$ endowed with the weak-star topology, the space of all complex continuous functions on the set of maximal ideal coincides with the space $C_{w^*}(\overline{B}_{X''})$ of all weak-star continuous functions on $\overline{B}_{X''}$. By the Stone Theorem the set of finite sums $\sum \hat{f}_n \bar{\hat{g}}_m$ is dense in $C_{w^*}(\overline{B}_{X''})$, where f_n, and g_m are homogeneous polynomials in $A_a(B)$ and $\bar{\hat{g}}$ is the complex conjugate of $\hat{g}$.

A measure μ on $\overline{B}_{X''}$ is called *circular* or *scalar invariant* if it is invariant with respect to the scalar group $\overline{B}_{X''} \ni x \mapsto e^{\mathrm{i}\vartheta} x \in \overline{B}_{X''}$, $\vartheta \in [-\pi, \pi]$ that is

$$\int w\big(e^{\mathrm{i}\vartheta} x\big)\, d\mu(x) = \int w(x)\, d\mu(x)$$

for every $w \in C_{w^*}(\overline{B}_{X''})$.

We denote by E_n' the completion of $\mathcal{P}_{\mathrm{a}}(^n X)$ in $H_{\mathrm{a}}^2(\mu)$ and by E_n its predual.

Theorem 2.4. *A norming measure μ is circular if and only if E_m' is orthogonal to E_n' for $m \neq n$. Moreover, in this case the following decomposition formula holds*

$$\int w(x)\, d\mu(x) = \frac{1}{2\pi} \int d\mu(x) \int_{-\pi}^{\pi} w\big(e^{\mathrm{i}\vartheta} x\big)\, d\vartheta, \qquad w \in C_{w^*}(\overline{B}_{X''}), \tag{1}$$

and μ is necessary representing for zero evaluation complex homomorphism δ_0.

Proof. Suppose that E_m' is orthogonal to E_n' for $m \neq n$. Let w be a finite sums $w = \sum \hat{f}_n \bar{\hat{g}}_m$. By the orthogonality $\hat{f}_n \perp \hat{g}_m$ we have for every $\vartheta \in [-\pi, \pi]$,

$$\begin{aligned}
&\int w(x)\, d\mu(x) \\
&\quad = \sum_{n,m \in \mathbb{Z}_+} \int \hat{f}_n(x) \bar{\hat{g}}_m(x)\, d\mu(x) = \sum_{n \in \mathbb{Z}_+} \int \hat{f}_n(x) \bar{\hat{g}}_n(x)\, d\mu(x) \\
&\quad = \sum_{n \in \mathbb{Z}_+} \int \hat{f}_n(x) \bar{\hat{g}}_n(x) e^{\mathrm{i}(n-n)\vartheta}\, d\mu(x) \sum_{n \in \mathbb{Z}_+} \int \hat{f}_n\big(e^{\mathrm{i}\vartheta} x\big) \bar{\hat{g}}_n\big(e^{\mathrm{i}\vartheta} x\big)\, d\mu(x) \\
&\quad = \sum_{n,m \in \mathbb{Z}_+} \int \hat{f}_n\big(e^{\mathrm{i}\vartheta} x\big) \bar{\hat{g}}_m\big(e^{\mathrm{i}\vartheta} x\big)\, d\mu(x) = \int w\big(e^{\mathrm{i}\vartheta} x\big)\, d\mu(x).
\end{aligned}$$

In general, approaching $w \in C_{w^*}(\overline{B}_{X''})$ by the finite sums $\sum \hat{f}_n \bar{\hat{g}}_m$ and using the continuity of integration of functions in $C_{w^*}(\overline{B}_{X''})$ we obtain the equality

$$\int w(x)\, d\mu(x) = \int w\big(e^{\mathrm{i}\vartheta} x\big)\, d\mu(x)$$

for any $w \in C_{w^*}(\overline{B}_{X''})$. So the measure μ is circular. For every $w \in C_{w^*}(\overline{B}_{X''})$ the function $(\vartheta, x) \mapsto w(e^{\mathrm{i}\vartheta} x)$ is continuous on $[-\pi, \pi] \times \overline{B}_{X''}$. By the Fubini Theorem we have

$$\int d\mu(x) \int_{-\pi}^{\pi} w\big(e^{\mathrm{i}\vartheta} x\big)\, d\vartheta = \int_{-\pi}^{\pi} d\vartheta \int w\big(e^{\mathrm{i}\vartheta} x\big)\, d\mu(x).$$

However, the second integral of the right hand does not depend of ϑ and $w \circ e^{\mathrm{i}\vartheta} \in C_{w^*}(\overline{B}_{X''})$ for every $w \in C_{w^*}(\overline{B}_{X''})$. Taking into account the equality $\int_{-\pi}^{\pi} d\vartheta = 2\pi$ we get (1). If $f \in A_{\mathrm{a}}(B)$, then $t \to f(tx)$ is an analytic function on the unit disk $D \in \mathbb{C}$. Thus

$$\int \hat{f}(x)\, d\mu(x) = \frac{1}{2\pi} \int d\mu(x) \int_{-\pi}^{\pi} \hat{f}\big(e^{\mathrm{i}\vartheta} x\big)\, d\vartheta = f(0).$$

So μ is representing.

Conversely, let μ be circular. Suppose that $f_n \in \mathcal{P}(^n X)$ and $g_m \in \mathcal{P}(^m X)$. Then

$$\int \hat{f}_n(x)\bar{\hat{g}}_m(x)\,d\mu = \int \hat{f}_n\big(e^{\mathrm{i}\vartheta}x\big)\bar{\hat{g}}_m\big(e^{\mathrm{i}\vartheta}x\big)\,d\mu$$
$$= \frac{1}{2\pi}\int \hat{f}_n(x)\bar{\hat{g}}_m(x)\,d\mu(x)\int\limits_{-\pi}^{\pi} e^{\mathrm{i}(n-m)\vartheta}\,d\vartheta = 0$$

if $m \neq n$. So E'_m is orthogonal to E'_n. □

Let $l^* \in X' \subset E'$. Then $l := \langle \cdot \mid l^*\rangle \in E \subset X''$. Denote by $C^0_{w^*}(X'')$ the dense subspace of $C_{w^*}(\overline{B}_{X''})$ which consists of all finite sums $\sum \hat{f}_n\bar{\hat{g}}_m$, where $f_n \in \mathcal{P}_{\mathrm{a}}(^n X)$ and $g_m \in \mathcal{P}_{\mathrm{a}}(^m X)$. We say that a measure μ is *circular with respect to* $l^* \in X'$ if for every $w \in C^0_{w^*}(X'')$,

$$\int w\big(x - l^*(x)l + e^{\mathrm{i}\vartheta}l^*(x)l\big)\,d\mu = \int w(x)\,d\mu$$

for every $\vartheta \in [-\pi, \pi]$.

Repeating our arguments with Fubini's Theorem we have that if μ is circular with respect to l^*, then

$$\int w(x)\,d\mu(x) = \frac{1}{2\pi}\int d\mu(x)\int\limits_{-\pi}^{\pi} w\big(x - l^*(x)l + e^{\mathrm{i}\vartheta}l^*(x)l\big)\,d\vartheta.$$

Theorem 2.5. *The set of polynomials $\{e^*_{(i)}\}$ forms an orthogonal basis in $H^2_{\mathrm{a}}(\mu)$ if and only if μ is circular with respect to e^*_j for every j.*

Proof. Suppose that μ is circular with respect to e^*_j for every j. Due to Proposition 2.3 we need to show that polynomials $e^*_{(i)}$ are orthogonal. Suppose that $(i) \neq (j)$. Then there exists k such that

$$e^*_{(i)} = e^{*r_1}_{i_1}\ldots e^{*m}_{k}\ldots e^{*r_s}_{i_s} \quad\text{and}\quad e^*_{(j)} = e^{*q_1}_{j_1}\ldots e^{*n}_{k}\ldots e^{*q_l}_{j_l}$$

for some $m \neq n$ and $i_1 < \cdots < i_s$, $j_1 < \cdots < j_l$. Thus

$$e^*_{(i)}\big(x - e^*_k(x)e_k + e^{\mathrm{i}\vartheta}e^*_k(x)e_k\big) = x^{*r_1}_{i_1}\ldots e^{\mathrm{i}m\vartheta}x^{*m}_k\ldots x^{*r_s}_{i_s} = e^{\mathrm{i}m\vartheta}e^*_{(i)}(x)$$

and by the same reason

$$e^*_{(j)}\big(x - e^*_k(x)e_k + e^{\mathrm{i}\vartheta}e^*_k(x)e_k\big) = e^{\mathrm{i}n\vartheta}e^*_{(j)}(x).$$

Since $m \neq n$, we have

$$\int e^*_{(i)}(x)\overline{e^*}_{(j)}(x)\,d\mu = \frac{1}{2\pi}\int e^*_{(i)}(x)\overline{e^*}_{(j)}(x)\,d\mu(x)\int\limits_{-\pi}^{\pi} e^{\mathrm{i}(m-n)\vartheta}\,d\vartheta = 0.$$

Conversely, suppose that the set $\{e^*_{(i)}\}$ forms an orthogonal basis in $H^2_{\mathrm{a}}(\mu)$. For a given e^*_k and $e^*_{(i)}$ we denote by $n_{(i)}$ the multiplicity of e^*_k in $e^*_{(i)}$ that is the maximal number such that $(e^*_k)^{n_{(i)}}$ is a factor of $e^*_{(i)}$. Then for every function $w \in C^0_{w^*}(X'')$ and linear functional e^*_k,

$$\begin{aligned}
&\int w(x)\,d\mu(x)\\
&=\sum_{(i)(j)}\int a_{(i)(j)}e^*_{(i)}(x)\overline{e^*}_{(j)}(x)\,d\mu(x)=\sum_{(i)}\int a_{(i)(i)}e^*_{(i)}(x)\overline{e^*}_{(i)}(x)\,d\mu(x)\\
&=\sum_{(i)}\int a_{(i)(i)}e^*_{(i)}(x)\overline{e^*}_{(i)}(x)\exp\bigl\{\mathrm{i}(n_{(i)}-n_{(i)})\vartheta\bigr\}\,d\mu(x)\\
&=\sum_{(i)}\int a_{(i)(i)}e^*_{(i)}\bigl[x-e^*_k(x)e_k+e^{\mathrm{i}\vartheta}e^*_k(x)e_k\bigr]\overline{e^*}_{(i)}\bigl[x-e^*_k(x)e_k+e^{\mathrm{i}\vartheta}e^*_k(x)e_k\bigr]\,d\mu(x)\\
&=\sum_{(i)(j)}\int a_{(i)(j)}e^*_{(i)}\bigl[x-e^*_k(x)e_k+e^{\mathrm{i}\vartheta}e^*_k(x)e_k\bigr]\overline{e^*}_{(j)}\bigl[x-e^*_k(x)e_k+e^{\mathrm{i}\vartheta}e^*_k(x)e_k\bigr]\,d\mu(x)\\
&=\int w\bigl(x-e^*_k(x)e_k+e^{\mathrm{i}\vartheta}e^*_k(x)e_k\bigr)\,d\mu(x). \qquad \square
\end{aligned}$$

Note that some non-circular representing measures on the unit ball of $\mathbb{C}^2$ were discussed in [17].

3. $H^2_{\mathrm{a}}(\mu)$ as a reproducing kernel space

Let us recall a definition of a reproducing kernel spaces (see [18] for details).

Definition 3.1. Given an abstract set Z and $\mathcal{H}$ a Hilbert space of complex valued functions f on Z equipped with inner product $\langle\cdot\mid\cdot\rangle_{\mathcal{H}}$ a function $K(x,z)$ defined on $Z\times Z$ is called *reproducing kernel* of a closed subspace $\mathcal{H}_K\subset\mathcal{H}$ if

(i) for any fixed $z\in Z$, $K(x,z)$ belongs to $\mathcal{H}_K$ as a function in $x\in Z$;
(ii) for any $f\in\mathcal{H}_K$ and for any $z\in Z$,

$$f(z)=\bigl\langle f(\cdot)\bigm| K(\cdot,z)\bigr\rangle_{\mathcal{H}}.$$

The Hilbert space $\mathcal{H}_K$ is called a *reproducing kernel Hilbert space.*

Now we consider question: *Does exist an open subset $\mathcal{U}$ in E such that every $f\in H^2_{\mathrm{a}}(\mu)$ can be expressed as an analytic function on $\mathcal{U}$?* We do not know the answer in the general case. Let us observe that each function in $A_{\mathrm{a}}(B)$ has the Aron–Berner extension to an analytic function on $B_{X''}$. On the other hand, the natural domain for linear functionals from $H^2_{\mathrm{a}}(\mu)$ is E. So if the open set $\mathcal{U}$ exists, it must be a subset of $B_{X''}\cap E$.

For every $x=\sum_{i=1}^\infty x_ie_i\in E$ we consider a formal power series

$$\eta^*(x)=\sum_{n=0}^\infty\sum_{i_1,\dots,i_n}c_{i_1\dots i_n}\bar{x}_{i_1}\dots\bar{x}_{i_n}e^*_{i_1}\dots e^*_{i_n}=\sum_{|(i)|\geqslant 0}c_{(i)}\bar{x}_{(i)}e^*_{(i)}, \tag{2}$$

where $c_{(i)}:=\|e^*_{(i)}\|_\mu^{-2}$ and $\bar{x}_{i_k}$ is the complex conjugate to x_{i_k}.

Theorem 3.2. *Let X be a complex Banach space with separable dual and μ be a norming representing measure for $A_{\mathrm{a}}(B)$ such that polynomials $e^*_{(i)}$ form an orthogonal basis in $H^2_{\mathrm{a}}(\mu)$ for some $\{e^*_j\}_{j=1}^{\infty}$ in X'. Then the following statements are equivalent.*

(i) *There exists an open subset $\mathcal{U} \subset E$ such that series* (2) *is convergent in $H^2_{\mathrm{a}}(\mu)$ for every $x \in \mathcal{U}$.*
(ii) *$H^2_{\mathrm{a}}(\mu)$ is a reproducing kernel space with the reproducing kernel $K(x,z) = \langle \eta^*(x) \mid \eta^*(z)\rangle$ which is defined on $\mathcal{U} \times \mathcal{U}$ for some open subset $\mathcal{U} \subset E$.*
(iii) *For every x in an open subset $\mathcal{U} \subset E$ the linear functional $x \mapsto f(x)$ is continuous on $H^2_{\mathrm{a}}(\mu)$ and each element $f \in H^2_{\mathrm{a}}(\mu)$ is an analytic function on $\mathcal{U}$.*

Proof. Suppose that (2) converges on an open subset $\mathcal{U}$ of E. Then

$$\langle e^*_{(i)} \mid \eta^*(x)\rangle = c_{(i)} x_{(i)} \|e^*_{(i)}\|^2_{\mu} = x_{(i)} = e^*_{(i)}(x).$$

So $\langle f \mid \eta^*(x)\rangle = f(x)$ if $f \in A_{\mathrm{a}}(B)$ and we can put $f(x) = \langle f \mid \eta^*(x)\rangle$ for every $f \in H^2_{\mathrm{a}}(\mu)$ and $x \in \mathcal{U}$. Since $\eta^*(x)$ is an element in $H^2_{\mathrm{a}}(\mu)$ for any fixed $x \in \mathcal{U}$, it follows that the linear functional $\langle \cdot \mid \eta^*(x)\rangle : f \mapsto f(x)$ is continuous on $H^2_{\mathrm{a}}(\mu)$ for every $x \in \mathcal{U}$. By [18, p. 34], $K(x,z) = \langle \eta^*(z) \mid \eta^*(x)\rangle$ is a reproducing kernel of $H^2_{\mathrm{a}}(\mu)$ which is defined on $\mathcal{U} \times \mathcal{U}$. So (i) implies (ii). According to [18, p. 40], the map $K(x,z)$ is continuous on $\mathcal{U} \times \mathcal{U}$. So $\eta^*(x)$ is continuous on $\mathcal{U}$ as well. On the other hand, from (2) we have that the linear functional on $H^2_{\mathrm{a}}(\mu)$, $\eta(x) := \langle \cdot \mid \eta^*(x)\rangle$ can be expressed by a convergent power series

$$\eta(x) = \sum_{n=0}^{\infty} \sum_{i_1,\dots,i_n} c_{i_1\dots i_n} x_{i_1} \dots x_{i_n} \langle \cdot \mid e^*_{i_1}\rangle \cdots \langle \cdot \mid e^*_{i_n}\rangle.$$

Hence η is G-analytic on $\mathcal{U}$ [8, p. 201]. Both G-analyticity and continuity of η imply that η is an analytic map on $\mathcal{U}$ (see [8, p. 198]). For every $f \in H^2_{\mathrm{a}}(\mu)$, $f(x)$ is the composition of analytic map $x \mapsto \eta(x)$ and linear functional $\eta(x)(f) = f(x)$. So it must be analytic.

Let now every $f \in H^2_{\mathrm{a}}(\mu)$ be an analytic function on an open set $\mathcal{U} \subset E$ and for every $x \in \mathcal{U}$ the x-evaluation linear functional $x \mapsto f(x)$, $f \in H^2_{\mathrm{a}}(\mu)$ be well defined and continuous. But this functional coincides with $\eta(x)$ on the basis polynomials $e^*_{(i)}$. Hence $f(x) = \langle f \mid \eta^*(x)\rangle$ and so (2) is convergent. □

Problem. Let μ be a norming circular representing measure for $A_{\mathrm{a}}(B_X)$ and X' is separable. Does necessary exist an orthonormal basis (e^*_j) in E' such that μ is circular with respect to e^*_j for every j?

4. Vector valued Hardy spaces

Let D be the complex unit open disk and T be the unit sphere in $\mathbb{C}$. For a given Hilbert space $\mathcal{H}$, $L^2(T,\mathcal{H})$ is the space of Bochner–Lebesgue 2-integrable functions and $H^2(D,\mathcal{H})$ denotes the Hardy space of $\mathcal{H}$-valued analytic functions F in D. Using the boundary values we will identify $H^2(D,\mathcal{H})$ with the space $H^2(T,\mathcal{H})$ of $\mathcal{H}$-valued functions from $L^2(T,\mathcal{H})$ which admit extensions to analytic functions in D. Note that

$$\|F\|^2_{H^2(D,\mathcal{H})} = \|F\|^2_{H^2(T,\mathcal{H})} = \frac{1}{2\pi} \sup_{r<1} \int_{-\pi}^{\pi} \|F(re^{\mathrm{i}\vartheta})\|^2 \, d\vartheta,$$

where the integral is in the Bochner–Lebesgue sense. If $F(z)=\sum_{n=0}^{\infty} z^n F_n$ is the Taylor series expansion of F, $z\in D$, $F_n\in\mathcal{H}$, then the norm of F can be computed by

$$\|F\|^2_{H^2(D,\mathcal{H})}=\sum_{n=0}^{\infty}\|F_n\|^2_{\mathcal{H}}.$$

Suppose that $H^2_{\mathrm{a}}(\mu)$ satisfies conditions of Theorem 3.2 and so the Hardy space $H^2_{\mathrm{a}}(\mu)$ consists of analytic functions on the open subset $\mathcal{U}$ of a Hilbert space E. Let $\overline{\mathcal{U}}$ be the closure of $\mathcal{U}$ in E. For every $f\in H^2_{\mathrm{a}}(\mu)$, we define the function $F: D\to H^2_{\mathrm{a}}(\mu)$ by

$$F\bigl(re^{\mathrm{i}\vartheta}\bigr)=f\bigl(re^{\mathrm{i}\vartheta}x\bigr),\quad x\in\overline{\mathcal{U}},\ r<1.$$

Denote by ξ the map $f\mapsto F$.

Theorem 4.1. *The map ξ is an isometric embedding of $H^2_{\mathrm{a}}(\mu)$ into the space $H^2(D,H^2_{\mathrm{a}}(\mu))$.*

Proof. For an arbitrary $f\in H^2_{\mathrm{a}}(\mu)$ with the Taylor series expansion $f(x)=\sum_{n=0}^{\infty} f_n(x)$,

$$\|F\|^2_{H^2(D,H^2_{\mathrm{a}}(\mu))}=\bigl\|\xi(f)\bigr\|^2_{H^2(D,H^2_{\mathrm{a}}(\mu))}=\sum_{n=0}^{\infty}\|F_n\|^2_{\mu}=\sum_{n=0}^{\infty}\|f_n\|^2_{\mu}=\|f\|^2_{\mu},$$

where $F_n(e^{\mathrm{i}\vartheta})=f_n(e^{\mathrm{i}\vartheta}x)$. $\square$

Note that the embedding ξ is not a surjection.

Corollary 4.2. *Let x belongs to the boundary of $\overline{\mathcal{U}}$ and $f\in H^2_{\mathrm{a}}(\mu)$. Then the following limit exists in $L^2(T,H^2_{\mathrm{a}}(\mu))$,*

$$\lim_{r\to 1^-} f\bigl(re^{\mathrm{i}\vartheta}x\bigr)=\lim_{r\to 1^-} F\bigl(re^{\mathrm{i}\vartheta}\bigr)(x)$$

for almost everyone $e^{\mathrm{i}\vartheta}\in T$.

5. Examples

Example 5.1. Let $X=c_0$ with the standard basis (e_i). Then $H_{\mathrm{uc}}(B_{c_0})=A_{\mathrm{a}}(B_{c_0})$. Denote by μ a measure on the unit ball B_{ℓ_∞} which is the infinity product of one-dimensional Lebesgue probability measures on intervals $[-1,1]$. It is easy to check that coordinate functionals e_i^*, $i=1,2,\ldots,$ form an orthogonal basis in E' and

$$\bigl\|e^*_{(i)}\bigr\|^2=\int\limits_{[-1,1]^n}\bigl|e^*_{i_1}\bigr|^2\ldots\bigl|e^*_{i_n}\bigr|^2\,dt_{i_1}\ldots dt_{i_n}=\int\limits_{[-1,1]^n} dt_{i_1}\ldots dt_{i_n}=1.$$

So $\eta^*(x)=\sum_{(i)} x_{(i)}e^*_{(i)}$. The space E coincides with the completion of the linear span of (e_i) in the ℓ_2-norm and the domain $\mathcal{U}$ of η^* is defined by

$$\mathcal{U}=\left\{x=\sum_{i=1}^{\infty}x_ie_i^*\in E':\ |x_i|<1\right\}=\ell_2\cap B_{\ell_\infty}.$$

The corresponding space $H^2_{\mathrm{a}}(\mu)$ consists of all analytic functions on $\mathcal{U}$ which can be expressed by

$$f(x) = \langle f \mid \eta^*(x) \rangle = \int f\overline{\eta^*(x)}\,d\mu.$$

Evidently that μ is circular and polynomials $e^*_{(i)}$ form an orthogonal basis.

Note that the space $H^2_{\mathrm{a}}(\mu)$ associated with $A_{\mathrm{a}}(B_{c_0})$ was constructed independently in [9] and [14]. This example may be generalized if we consider instead c_0 the c_0-sum of n-dimensional Hilbert spaces $\mathbb{C}^n$. In this case μ will be the infinite product of probability Lebesgue measures on the unit balls of $\mathbb{C}^n$.

Let $\phi \in \mathfrak{M}(H^\infty_{\mathrm{uc}}(B))$ and x_α be a net in X such that $\phi(P) = \lim_\alpha \phi(x_\alpha)$ for every polynomial P. Such net exists according to [4]. For a given $t \in \mathbb{C}$ let us consider the net tx_α. If $|t| \leqslant 1$, then the net tx_α determines an element in $\mathfrak{M}(H^\infty_{\mathrm{uc}}(B))$ which we denote by $t * \phi$, and

$$t * \phi(f) = \sum_{n=0}^{\infty} t^n \phi(f_n)$$

for every $f(x) = \sum f_n(x)$ in $H^\infty_{\mathrm{uc}}(B)$, where f_n are n-homogeneous polynomials. Let $t = e^{\mathrm{i}\vartheta}$, $-\pi \leqslant \vartheta < \pi$ and λ be the Lebesgue probability measure on the unit circle $S_1\phi = e^{\mathrm{i}\vartheta} * \phi \subset \mathfrak{M}(H^\infty_{\mathrm{uc}}(B))$. Denote by μ_ϕ the extension of λ to Borel subsets of $\mathfrak{M}(H^\infty_{\mathrm{uc}}(B))$, $\mu_\phi(U) = \lambda(U \cap S_1\phi)$. Then μ_ϕ represents δ_0 and

$$\int g\,d\mu_\phi = \int_{-\pi}^{\pi} g\big(e^{\mathrm{i}\vartheta} * \phi\big)\,d\lambda(\vartheta)$$

for every continuous function g on $\mathfrak{M}(H^\infty_{\mathrm{uc}}(B))$. However μ_ϕ is not norming.

The following example contains some nontrivial representing measure for $H^\infty_{\mathrm{uc}}(B_{\ell_p})$ which is trivial on the subalgebra $A_{\mathrm{a}}(B_{\ell_p})$. Note that we do not know if there exists a norming representing measure on the set of maximal ideals of $H^\infty_{\mathrm{uc}}(B_{\ell_p})$.

Example 5.2. (Cf. [1, Example 3.1].) Let $X = \ell_p$ for some positive integer $p > 1$. For every n, put

$$v_n = \frac{1}{n^{1/p}}(e_1 + \cdots + e_n).$$

Since $\|v_n\|_{\ell_p} = 1$, the functional $\delta_{v_n} : f \to f(v_n)$ belongs to the set of maximal ideals of $H^\infty_{\mathrm{uc}}(B_{\ell_p})$. By the compactness of $\mathfrak{M}(H^\infty_{\mathrm{uc}}(B_{\ell_p}))$, there is an accumulating point $\phi \in \mathfrak{M}(H^\infty_{\mathrm{uc}}(B_{\ell_p}))$ of the sequence (δ_{v_n}). On the other hand, (v_n) is a weak-zero sequence and so $\phi(f) = \lim_{n\to\infty} f(v_n) = f(0) = \delta_0(f)$ for every weakly continuous function and so for every $f \in A_{\mathrm{a}}(B_{\ell_p})$. Let μ_ϕ be the δ_0-representing measure associated with ϕ which is introduced above. The support of this measure coincides with the set $\{e^{\mathrm{i}\vartheta} * \phi\colon\ \vartheta \in [-\pi, \pi]\} = \{\psi \in \mathfrak{M}(H^\infty_{\mathrm{uc}}(B_{\ell_p}))\colon\ \psi(f) = \lim_{n\to\infty} f(e^{\mathrm{i}\vartheta} v_n),\ f \in H^\infty_{\mathrm{uc}}(B_{\ell_p}),\ \vartheta \in [-\pi, \pi]\}$. Since $(e^{\mathrm{i}\vartheta} v_n)$ is a weak-zero sequence for each ϑ, the Gelfand transform of every function $f \in A_{\mathrm{a}}(B)$, $\hat{f}$ vanishes on the support of μ_ϕ. So $\int |\hat{f}|^2\,d\mu_\phi = f(0)$ if $f \in A_{\mathrm{a}}(B)$ and $\mu_\phi = \delta_0$ on $A_{\mathrm{a}}(B)$.

But the situation is different if a function is not weakly continuous. For example, let

$$Q_p\left(\sum_{i=1}^{\infty} x_i e_i\right) := \sum_{i=1}^{\infty} x_i^p \in H^\infty_{\mathrm{uc}}(B_{\ell_p}).$$

Then $\phi(Q_p) = \lim_{n\to\infty} \delta_{v_n}(Q_p) = 1 \neq \delta_0(Q_p)$. Moreover,

$$\int |\widehat{Q}_p|^2 \, d\mu_\phi = \int_{-\pi}^{\pi} e^{\mathrm{i}p\vartheta} e^{-\mathrm{i}p\vartheta} \phi(\widehat{Q}_p) \overline{\phi(\widehat{Q}_p)} \, d\lambda(\vartheta) = 1.$$

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