**International Journal of Mathematical Analysis**
**Vol. 9, 2015, no. 2, 95 - 100**
**HIKARI Ltd, www.m-hikari.com**
**http://dx.doi.org/10.12988/ijma.2015.411351**

# On Cross-correlation of a Hyperfunction and a Real Analytic Function

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**Abstract**

We describe the cross-correlation operator over the space of real-analytic functions and generalize classic Schwartz's theorem on shift-invariant operators.

**Mathematics Subject Classification:** 46F05, 46H30, 47A60, 46F15

**Keywords:** hyperfunction, real-analytic function, shift-invariant operator

## 1 Introduction

By classic Schwartz's theorem for shift-invariant operators (see [14]) every continuous linear operator $L : \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n)$ commuting with the shift group $\tau_h : \varphi(\cdot) \longmapsto \varphi(\cdot - h)$, $h \in \mathbb{R}^n$, is the convolution operator with some distribution $f \in \mathcal{D}'(\mathbb{R}^n)$, i.e., $L\varphi = f * \varphi$. Hörmander in [4] performed a comprehensive analysis of the boundedness of translation-invariant operators on $L^p(\mathbb{R}^n)$. Such operators are of interest and have been considered by several authors in [6, 10]. The extension of the theory to Besov, Lorents and Hardy spaces was considered in [1], [2] and [15], respectively. The paper [9] is devoted to shift-invariant operators, commuting with contraction multi-parameter semigroups over a Banach space. For other results and references on the topic we refer the reader to [3, 5, 11].

The purpose of this paper is a generalization of classic Schwartz's theorem on shift-invariant operators. In Theorems 3.2 we describe shift-invariant operators for operator semigroups.

Let $\mathcal{B}_c(\mathbb{R}_+)$ be the space of hyperfunctions with a compact support in the positive semiaxis $\mathbb{R}_+$ and $\mathcal{A}(\mathbb{R}_+)$ be the space of germs of real-analytic functions. We consider the cross-correlation operator

$$C_f : \mathcal{A}(\mathbb{R}_+) \ni \varphi \longmapsto f \star \varphi, \qquad f \in \mathcal{B}_c(\mathbb{R}_+),$$

where $f \star \varphi$ is defined by the formula (3). In Theorem 3.2 we show that the algebra $\mathcal{B}_c(\mathbb{R}_+)$ can be represented by the isomorphism

$$\mathcal{B}_c(\mathbb{R}_+) \ni f \longmapsto C_f \in \mathscr{L}\big(\mathcal{A}(\mathbb{R}_+)\big)$$

onto the commutant $[T]^c$ of the shift semigroup $T$ (see formula (2)) in space of linear continuous operators on $\mathcal{A}(\mathbb{R}_+)$.

# 2 Preliminaries and notations

Let $\mathcal{A}(\mathbb{R}_+)$ be the space of germs of real-analytic functions on neighborhoods of the semiaxis $\mathbb{R}_+ := [0, \infty)$. A restriction of any element of $\mathcal{A}(\mathbb{R}_+)$ to $\mathbb{R}_+$ is uniquely defined function. In the sequel we will treat $\mathcal{A}(\mathbb{R}_+)$ as the space of such restrictions. It is known [7, Prop. 1.3.9], that a sequence $\{\varphi_n\}$ converges to $\varphi$ in $\mathcal{A}(\mathbb{R}_+)$ if and only if for any compact set $K \subset \mathbb{R}_+$ there exists a complex neighborhood $U$ of $K$, such that $\{\boldsymbol{\varphi}_n\}$ converges uniformly to $\boldsymbol{\varphi}$ in $U$ (here and subsequently bold symbol like $\boldsymbol{\varphi}$ denotes an analytic continuation of a corresponding function $\varphi$). Let $\mathcal{A}(\mathbb{R}_+)'$ be dual space of $\mathcal{A}(\mathbb{R}_+)$.

Denote $H(W)$ the vector space of all holomorphic functions on an open set $W \subset \mathbb{C}$. Let $\Omega$ be an open set in $\mathbb{R}$ and $V$ be an open set in $\mathbb{C}$ containing $\Omega$ as a relatively closed set. The vector space $\mathcal{B}(\Omega)$ of all hyperfunctions on $\Omega$ is defined to be the quotient space (see [8, 13])

$$\mathcal{B}(\Omega) = H(V \setminus \Omega)/H(V),$$

where $H(V)$ denotes the restriction of $H(V)$ to $V \setminus \Omega$. A hyperfunction represented by a holomorphic function $F \in H(V \setminus \Omega)$ is denoted as

$$f = [F] = F(t + i0) - F(t - i0) \quad \text{or} \quad f(t) = [F(z)]_{z=t}.$$

The representative $F$ is called a defining function of the hyperfunction $f$. For more details on the theory of hyperfunctions we refer the reader to [8, 13].

Let $\mathcal{B}_c(\mathbb{R}_+)$ denote the space of hyperfunctions with a compact support in $\mathbb{R}_+$. From Köte duality theorem [8] it follows that the isomorphism of vector spaces $\mathcal{B}_c(\mathbb{R}_+) \cong \mathcal{A}(\mathbb{R}_+)'$ is valid. Moreover, for a $\varphi \in \mathcal{A}(\mathbb{R}_+)$ and an

$f = [F] \in \mathcal{B}_c(\mathbb{R}_+)$ with $F \in H(V \setminus \operatorname{supp} f)$, the canonical bilinear functional is given by

$$\langle f, \varphi \rangle = -\oint_{\Gamma} F(z)\boldsymbol{\varphi}(z)dz, \tag{1}$$

where $\Gamma$ is a closed path in the intersection of the domains of $\boldsymbol{\varphi}$ and $F$, and surrounding $\operatorname{supp} f$ once in the positive orientation.

For any hyperfunctions $f = [F]$ and $g = [G]$ from $\mathcal{B}_c(\mathbb{R}_+)$ we define their convolution as a hyperfunction $f * g = [H]$, where

$$H(z) = -\oint_{\Gamma} F(w)G(z-w)\,dw,$$

and $\Gamma$ is a closed path in the intersection of the domains of analytic functions $w \longmapsto F(w)$ and $w \longmapsto G(z-w)$.

It is known [8], that the space $\mathcal{B}_c(\mathbb{R}_+)$ is an algebra with respect to the convolution with Dirac delta-function $\delta(x) = -\dfrac{1}{2\pi i}\left[\dfrac{1}{z}\right]_{z=x}$ as a unit element.

# 3 Main Result

Denote by $\mathscr{L}\big(\mathcal{A}(\mathbb{R}_+)\big)$ the space of all linear continuous operators on $\mathcal{A}(\mathbb{R}_+)$. We endow $\mathscr{L}\big(\mathcal{A}(\mathbb{R}_+)\big)$ with the locally convex topology of uniform convergence on bounded subsets of $\mathcal{A}(\mathbb{R}_+)$.

Given an $h \in \mathbb{R}_+$, consider the shift operator $T_h$ that is defined on the space $\mathcal{A}(\mathbb{R}_+)$ by the formula

$$T_h : \varphi(\cdot) \longmapsto \varphi(\cdot + h), \qquad \varphi \in \mathcal{A}(\mathbb{R}_+). \tag{2}$$

It is immediate that $T := \{T_h : h \in \mathbb{R}_+\}$ is an one-parameter $(C_0)$-semigroup.

Commutant of the semigroup $T$ is defined to be the set

$$[T]^c := \{A \in \mathscr{L}\big(\mathcal{A}(\mathbb{R}_+)\big) : T_h \circ A = A \circ T_h, \forall h \in \mathbb{R}_+\}.$$

The cross-correlation of a hyperfunction $f = [F] \in \mathcal{B}_c(\mathbb{R}_+)$ and a real-analytic function $\varphi \in \mathcal{A}(\mathbb{R}_+)$ is defined to be

$$(f \star \varphi)(t) := -\oint_{\Gamma} F(z)\boldsymbol{\varphi}(z+t)dz = \langle f, T_t\varphi \rangle, \qquad t \in \mathbb{R}_+, \tag{3}$$

where $\Gamma$ is a same path as in (1).

The definition of cross-correlation and properties of an integral, depending on a parameter, imply that for any $f \in \mathcal{B}_c(\mathbb{R}_+)$ and $\varphi \in \mathcal{A}(\mathbb{R}_+)$ the cross-correlation $f \star \varphi$ is an infinite differentiable function, satisfying the equality

$$D^n(f \star \varphi) = f \star D^n(\varphi), \qquad n \in \mathbb{N},$$

where $D^n$ denotes the $n$-th derivative operator.

Moreover, in [12] it is proved the following assertion, which improves the above result.

**Proposition 3.1.** *For any $f \in \mathcal{B}_c(\mathbb{R}_+)$ and $\varphi \in \mathcal{A}(\mathbb{R}_+)$ the cross-correlation $f \star \varphi$ is a real-analytic function, belonging to $\mathcal{A}(\mathbb{R}_+)$.*

For any hyperfunction $f \in \mathcal{B}_c(\mathbb{R}_+)$, the cross-correlation operator over the space $\mathcal{A}(\mathbb{R}_+)$ is defined to be

$$C_f : \varphi \longmapsto f \star \varphi.$$

Let us show the correctness of the above definition, i.e. $C_f \in \mathscr{L}\big(\mathcal{A}(\mathbb{R}_+)\big)$. The linearity of $C_f$ is clear. Check its continuity. Let $\{\varphi_n\}$ be a sequence, converging to zero in the space $\mathcal{A}(\mathbb{R}_+)$. Denote $\psi_n := C_f\varphi_n$. Show that $\{\psi_n\}$ converges to zero in the topology of the space $\mathcal{A}(\mathbb{R}_+)$. For any compact set $K \subset \mathbb{R}_+$ and a complex neighborhood $U$ of $K$ we have

$$\begin{aligned} \sup_{z\in U} |\boldsymbol{\psi}_n(z)| = \sup_{z\in U} \left| -\oint_\Gamma F(s)\boldsymbol{\varphi}_n(s+z)\,ds \right| &\le \sup_{z\in U} \oint_\Gamma |F(s)|\,|\boldsymbol{\varphi}_n(s+z)|\,ds \\ &\le \sup_{z\in U}\sup_{s\in\Gamma} |\boldsymbol{\varphi}_n(s+z)| \cdot \sup_{s\in\Gamma} |F(s)| \cdot \mu(\Gamma), \end{aligned} \tag{4}$$

where $\mu(\Gamma)$ denotes the length of the path $\Gamma$. Since $K$ is compact, the path $\Gamma$ is finite, therefore $\sup_{s\in\Gamma} |F(s)| \cdot \mu(\Gamma) < \infty$.

Maximum modulus principle implies that there exists a point $s_0 \in \Gamma$, such that

$$\sup_{z\in U}\sup_{s\in\Gamma} |\boldsymbol{\varphi}_n(s+z)| = \sup_{z\in U} |\boldsymbol{\varphi}_n(s_0+z)|.$$

Since $\varphi_n \to 0$ as $n \to \infty$ in the space $\mathcal{A}(\mathbb{R}_+)$, we get $\sup_{z\in U} |\boldsymbol{\varphi}_n(s_0+z)| \to 0$. Hence, the inequality (4) implies $\sup_{z\in U} \big|C_f[\varphi_n](z)\big| \to 0$ as $n \to \infty$, so $C_f$ is continuous operator.

The next theorem is a generalization of classic Schwartz's theorem on shift-invariant operators.

**Theorem 3.2.** *The mapping $\mathcal{K} : \mathcal{B}_c(\mathbb{R}_+) \ni f \longmapsto C_f \in \mathscr{L}\big(\mathcal{A}(\mathbb{R}_+)\big)$ produces an algebraic isomorphism from the convolution algebra $\mathcal{B}_c(\mathbb{R}_+)$ onto the commutant $[T]^c$ of the shift semigroup $T$, i.e.*

$$C_{f*g} = C_f \circ C_g, \qquad f, g \in \mathcal{B}_c(\mathbb{R}_+). \tag{5}$$

*In particular, $C_\delta$ is the identity in $\mathscr{L}\big(\mathcal{A}(\mathbb{R}_+)\big)$.*

*Proof.* The following equalities

$$(C_f T_h\varphi)(t) = (C_f\varphi)(t+h) = T_h(C_f\varphi)(t) = (T_h C_f\varphi)(t)$$

hold for all $h \in \mathbb{R}_+$ and $\varphi \in \mathcal{A}(\mathbb{R}_+)$.

Let now $A \in \mathscr{L}\big(\mathcal{A}(\mathbb{R}_+)\big)$ be an arbitrary operator with the property

$$(AT_h)\varphi(t) = (T_h A)\varphi(t), \qquad h \in \mathbb{R}_+, \qquad \varphi \in \mathcal{A}(\mathbb{R}_+). \tag{6}$$

It is clear that the functional $\langle f_0, \varphi\rangle := (A\varphi)(0)$ belongs to $\mathcal{B}_c(\mathbb{R}_+)$. By definition of cross-correlation we get $(C_{f_0}\varphi)(0) = \langle f_0, \varphi\rangle$, i.e. $(A\varphi)(0) = (C_{f_0}\varphi)(0)$ for all $\varphi \in \mathcal{A}(\mathbb{R}_+)$. Substituting $T_h\varphi$ instead of $\varphi$ and using the property (6), we get that $A = C_{f_0}$ and hence that image of $\mathcal{K}$ coincides with the commutant $[T]^c$.

Check the equality (5). Let $f * g = [H]$. By definition of convolution of two hyperfunctions we have

$$(C_{f*g}\varphi)(t) = ((f*g)\star\varphi)(t) = -\oint\limits_{\Gamma_2}\Big(-\oint\limits_{\Gamma_1} F(z)G(\xi-z)dz\Big)\boldsymbol{\varphi}(\xi+t)d\xi, \quad t \in \mathbb{R}_+,$$

where $\Gamma_2$ is a closed path, surrounding $\operatorname{supp} f * g$ and belonging to intersection of domains of functions $\xi \longmapsto H(\xi)$ and $\xi \longmapsto \boldsymbol{\varphi}(\xi + t)$.

Applying Fubini's theorem and changing variable $w = \xi - z$ in the inner integral we get

$$\begin{aligned}(C_{f*g}\varphi)(t) &= -\oint\limits_{\Gamma_1} F(z)\Big(-\oint\limits_{\Gamma_2} G(\xi - z)\boldsymbol{\varphi}(\xi + t)d\xi\Big)dz\\ &= -\oint\limits_{\Gamma_1} F(z)\Big(-\oint\limits_{\Gamma_3} G(w)\boldsymbol{\varphi}(w + z + t)dw\Big)dz,\end{aligned}$$

where $\Gamma_3$ is a shift of a contour $\Gamma_2$. Let $\boldsymbol{\psi}$ be an analytic continuation of $\psi$, where $\psi(s) := (g \star \varphi)(s) = -\oint_{\Gamma_3} G(w)\,\boldsymbol{\varphi}(w + s)\,d\omega$, $s \in \mathbb{R}_+$. Then

$$(C_{f*g}\varphi)(t) = -\oint\limits_{\Gamma_1} F(z)\,\boldsymbol{\psi}(z + t)\,dz = (f \star (g \star \varphi))(t) = (C_f \circ C_g\varphi)(t).$$

In particular, $C_f \circ C_\delta = C_{f*\delta} = C_f = C_{\delta*f} = C_\delta \circ C_f$ for all $f \in \mathcal{B}_c(\mathbb{R}_+)$. So $C_\delta$ is the identity. □

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**Received: November 20, 2014; Published: January 9, 2015**