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ON THE CROSSINGS NUMBER OF A HYPERPLANE BY A STABLE RANDOM PROCESS

The numbers of crossings of a hyperplane by discrete approximations for trajectories of an α-stable random process (with $1 < \alpha < 2$) and some processes related to it are investigated. We consider an α-stable process is killed with some intensity on the hyperplane and a pseudo-process that is formed from the α-stable process using its perturbation by a fractional derivative operator with a multiplier like a delta-function on the hyperplane. In each of these cases, the limit distribution of the crossing number of the hyperplane by some discret approximation of the process is related to the distribution of its local time on this hyperplane. Integral equations for characteristic functions of these distributions are constructed. Unique bounded solutions of these equations can be constructed by the method of successive approximations.

Key words and phrases: α-stable process, local time, pseudo-process.

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INTRODUCTION

Let $(x(t), \mathcal{M}_t, \mathbb{P}_x)$ denote a standard Markov process on $\mathbb{R}^d$ $(d \geq 1)$. Consider a fixed hyperplane $S = \{x \in \mathbb{R}^d : (x, \nu) = r\}$, in $\mathbb{R}^d$ and two open sets

$$D_- = \{x \in \mathbb{R}^d : (x, \nu) < r\}, \quad D_+ = \{x \in \mathbb{R}^d : (x, \nu) > r\},$$

where $\nu \in \mathbb{R}^d$ is a given unit vector and $r \in \mathbb{R}$ is a given constant.

Our goal is to describe a changes number of the sets D_- and D_+ before a fixed time $t > 0$ by the trajectories of the process $(x(t))_{t \geq 0}$ started at fixed point $x \in \mathbb{R}^d$.

Consider for $m, n \in \mathbb{N}$ the random variable

$$\xi_m^{(n)} = \sum_{k=1}^{m} v\left(x\left(\frac{k-1}{n}\right), x\left(\frac{k}{n}\right)\right),$$

where $v(x, y) = \mathbb{1}_{D_-}(x)\mathbb{1}_{D_+}(y) + \mathbb{1}_{D_+}(x)\mathbb{1}_{D_-}(y)$.

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The variable $\xi^{(n)}_{[nt]}$ equals to the number of crossings of the hyperplane S by the ordered set of points in $\mathbb{R}^d$: $x(0)$, $x(1/n)$,…,$x([nt]/n)$.

We are going to find out a sequence of normalizing multipliers $\{c_n : n \geq 1\}$ such that the limit distribution of the sequence $\{c_n \xi^{(n)}_{[nt]} : n \geq 1\}$ exists and to describe it. It is obvious that $c_n \to 0$, as $n \to \infty$.

The limit theorems of this type were initiated by I. I. Gikhman in connection with some problems of mathematical statistics. I. I. Gikhman considered sequences of one-dimensional Markov chains approaching a diffusion process with smooth local characteristics (see [1, 2]).

1 Some auxiliary results

We will use the following corollary of one A. V. Skorokhod's theorem (see [3, Th. 1]).

Lemma 1. *A limit distribution of the sequence of random variables* $c_n \xi^{(n)}_{[nt]}$ *exists if and only if a limit distribution exists for the variables* $c_n \eta^{(n)}_{[nt]}$, *where*

$$\eta^{(n)}_m = \sum_{k=1}^{m} v_n\left(x\left(\frac{k}{n}\right)\right), \quad v_n(x) = \mathbb{E}_x v\left(x(0), x\left(\frac{1}{n}\right)\right),$$

and these limit distributions coincide, if only they exist.

So, we will consider the random variables $c_n \eta^{(n)}_{[nt]}$.

For any fixed $t > 0$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$ we consider the characteristic function

$$u_n(t, x, \theta) = \mathbb{E}_x \exp\left\{i\theta c_n \eta^{(n)}_{[nt]}\right\}, \quad \theta \in \mathbb{R}$$

of the random variable $c_n \eta^{(n)}_{[nt]}$.

The next equation for the function $u_n(t, x, \theta)$

$$u_n(t, x, \theta) = 1 + n \int_0^{[nt]/n} d\tau \int_{\mathbb{R}^d} \left(1 - e^{-i\theta c_n v_n(y)}\right) u_n(\tau, y, \theta) g\left(\frac{[nt] - [n\tau]}{n}, x, y\right) dy \qquad (1)$$

follows from the identity $\exp\{\sum_{k=1}^m a_k\} = 1 + \sum_{k=1}^m (1 - e^{-a_k}) \exp\left\{\sum_{j=k}^m a_j\right\}$ holds true for each set of complex numbers a_1, a_2, …, a_m and each natural number m.

If the transition probability density of the process $(x(t))_{t \geq 0}$ is given by the equality

$$g(t, x, y) = (2\pi)^{-d} \int_{\mathbb{R}^d} \exp\{i(\lambda, y - x) - ct|\lambda|^\alpha\} \, d\lambda, \quad t > 0, \ x \in \mathbb{R}^d, \ y \in \mathbb{R}^d,$$

for fixed parameters $c > 0$ and $\alpha \in (1, 2]$, then the process $(x(t))_{t \geq 0}$ is called rotationally invariant α-stable random process. If $\alpha = 2$, this process is the Brownian motion. In this case, our problems have been addressed in many publications (see, for example, [4, 5] and others). Therefore, we will not consider this case. So, we will further assume that $1 < \alpha < 2$, although most of our results remain correct also for $\alpha = 2$.

Consider the function $f(t, x) = \int_0^t d\tau \int_S g(\tau, x, y) \, d\sigma_y$. It is a W-function for the process $(x(t))_{t \geq 0}$ satisfying the inequality $f(t, x) \leq N \frac{\alpha}{\alpha - 1} t^{1 - 1/\alpha}$. So, there exists a W-functional

$(l_t)_{t\geq 0}$ of the process $(x(t))_{t\geq 0}$ such that $\mathbb{E}_x l_t = f(t,x)$ (see [8, Th. 6.6]). This functional is called the local time on S for the process $(x(t))_{t\geq 0}$.

Using the following representation of the functional $(l_t)_{t\geq 0}$:

$$l_t = \lim_{h\to 0+} \int_0^t d\tau \int_S g(h, x(\tau), y)\, d\sigma_y \text{ in mean-square,}$$

and the Feynman-Kac formula, one can prove that the characteristic function of the random value l_t, that is $v(t,x,\theta) = \mathbb{E}_x \exp\{i\theta l_t\}$, satisfies the following equation

$$v(t,x,\theta) = 1 + i\theta \int_0^t d\tau \int_S g(t-\tau, x, y) v(\tau, y, \theta)\, d\sigma_y. \tag{2}$$

2 THE MAIN RESULTS

The first statement concerns to the rotationally invariant α-stable random process.

Theorem 1. *The limit distribution with respect to the measure $\mathbb{P}_x$ of the random variables sequence $n^{-1+1/\alpha}\xi^{(n)}_{[nt]}$ for fixed $t>0$ and $x\in\mathbb{R}^d$ has the characteristic function $(u(t,x,\theta))_{\theta\in\mathbb{R}}$, which is the unique bounded solution of the integral equation*

$$u(t,x,\theta) = 1 + i\varkappa\theta \int_0^t d\tau \int_S g(t-\tau, x, y) u(\tau, y, \theta)\, d\sigma_y,$$

where $\varkappa = \frac{2c^{1/\alpha}}{\pi}\Gamma(1-1/\alpha)$. This distribution coincides with the distribution of the multiplied by $\varkappa$ local time on the hyperplane S of the process $(x(t))_{t\geq 0}$.

Next, let a continuous bounded function $(r(x))_{x\in S}$ with non-negative values be given. Consider the function $(G(t,x,y))_{t>0, x\in\mathbb{R}^d, y\in\mathbb{R}^d}$ which is a solution of to each one of the following equations

$$G(t,x,y) = g(t,x,y) - \int_0^t d\tau \int_S g(t-\tau, x, z) G(\tau, z, y) r(z)\, d\sigma_z,$$
$$G(t,x,y) = g(t,x,y) - \int_0^t d\tau \int_S G(t-\tau, x, z) g(\tau, z, y) r(z)\, d\sigma_z.$$

The function G is the transition probability density of the process $(x(t))_{t\geq 0}$ killed on the hyperplane S at some stopping time ζ (see [6]). The function $(r(x))_{x\in S}$ is the killing intensity of the process $(x(t))_{t\geq 0}$. It is clear that

$$\mathbb{P}_x(\{\zeta > t\}) = \int_{\mathbb{R}^d} G(t,x,y)\, dy = 1 - \int_0^t d\tau \int_S G(\tau, x, y) r(y)\, d\sigma_y.$$

Theorem 2. *The limit distribution with respect to the measure $\mathbb{P}_x$ of the random variables sequence $n^{-1+1/\alpha}\xi^{(n)}_{[nt]}$ for fixed $t>0$ and $x\in\mathbb{R}^d$ has the characteristic function $(u(t,x,\theta))_{\theta\in\mathbb{R}}$, which is the unique bounded solution of the integral equation*

$$u(t,x,\theta) = 1 + i\varkappa\theta \int_0^t d\tau \int_S G(t-\tau, x, y) u(\tau, y, \theta)\, d\sigma_y,$$

where $\varkappa = \frac{2c^{1/\alpha}}{\pi}\Gamma(1-1/\alpha)$. It is the distribution of the multiplied by $\varkappa$ local time on the hyperplane S for the process $(x(t))_{t\geq 0}$ killed at the stopping time ζ.

And the last, let a continuous bounded function $(q(x))_{x\in S}$ be given. Introduce an operator $\mathbf{B}_\nu$ determined by its symbol $(i|\xi|^{\alpha-2}(\xi,2c\nu))_{\xi\in\mathbb{R}^d}$. Define the function $(G(t,x,y))_{t>0,x\in\mathbb{R}^d,y\in\mathbb{R}^d}$ by the following formula

$$G(t,x,y)=g(t,x,y)+\int_0^t d\tau\int_S g(t-\tau,x,z)\mathbf{B}_\nu g(\tau,\cdot,y)(z)q(z)\,d\sigma_z.$$

This function is "a transition probability density" of some pseudo-process with a membrane on the hyperplane S (see [7]). The generator of this pseudo-process can be written in the following form: $\mathbf{A}+q(x)\delta_S(x)\mathbf{B}_\nu$, where $\mathbf{A}$ is the generator of the process $(x(t))_{t\geq 0}$ (that is a pseudo-differential operator whose symbol is given by the function $(-c|\xi|^\alpha)_{\xi\in\mathbb{R}^d}$).

Consider the function $(u(t,x,\theta))_{t\geq 0,x\in\mathbb{R}^d,\theta\in\mathbb{R}}$ defined by the equality

$$u(t,x,\theta)=\lim_{n\to\infty}\hat{\mathbb{E}}_x\exp\left\{i\theta n^{-1+1/\alpha}\eta^{(n)}_{[nt]}\right\}\overset{def}{=}$$
$$\lim_{n\to\infty}\int_{\mathbb{R}^d}\dots\int_{\mathbb{R}^d}\prod_{k=1}^{[nt]}\exp\left\{i\theta n^{-1+1/\alpha}\hat{v}_n(x_k)\right\}G\left(\frac{1}{n},x_{k-1},x_k\right)dx_k,$$

where $x_0=x$ and $\hat{v}_n(x)=\hat{\mathbb{E}}_x v\left(x(0),x\left(\frac{1}{n}\right)\right)\overset{def}{=}\int_{\mathbb{R}^d}v(x,y)G\left(\frac{1}{n},x,y\right)dy$. This function is "the characteristic function" of the the random variables sequence $n^{-1+1/\alpha}\xi^{(n)}_{[nt]}$ limit "distribution" for fixed $t>0$ and $x\in\mathbb{R}^d$.

Here we use quotes with notions that apply to the pseudo-process, similar to the ordinary random process. These notions must be understood in some special way described above.

Theorem 3. *The function $(u(t,x,\theta))_{\theta\in\mathbb{R}}$ for fixed $t>0$ and $x\in\mathbb{R}^d$ is the unique bounded solution of the integral equation*

$$u(t,x,\theta)=1+i\varkappa\theta\int_0^t d\tau\int_S g(t-\tau,x,y)u(\tau,y,\theta)(1-q^2(y))\,d\sigma_y,$$

where $\varkappa=\frac{2c^{1/\alpha}}{\pi}\Gamma(1-1/\alpha)$.

3 PROOF OF THE MAIN RESULTS

The proofs of these results are executed according to the same scheme. Consider the first result (i.e. it is for the rotationally invariant α-stable random process).

First of all, one can prove two technical lemmas. The first one prompts us that we must choose $c_n=n^{-1+1/\alpha}$. And the second one allows to pass from equation (1) to some simpler one.

Lemma 2. *Let the real-valued function $(\varphi(x))_{x\in\mathbb{R}^d}$ be such that $\sup_{\rho\in\mathbb{R}}\int_{S_\rho}|\varphi(x)|\,d\sigma<\infty$, where $S_\rho=\{x\in\mathbb{R}^d:(x,\nu)=\rho\}$, and there exist the nontangentional limits $\varphi(x-)$ and $\varphi(x+)$ from the side of D_- and D_+ in each point $x\in S$.*

Then the following relation (with $\varkappa=\mathbb{E}_0|(x(1),\nu)|=\frac{2c^{1/\alpha}}{\pi}\Gamma(1-1/\alpha)$)

$$\lim_{n\to\infty}n^{1/\alpha}\int_{\mathbb{R}^d}v_n(x)\varphi(x)\,dx=\varkappa\int_S\frac{\varphi(y-)+\varphi(y+)}{2}\,d\sigma$$

is hold true. In addition, the inequality $\left|n^{1/\alpha}\int_{\mathbb{R}^d} v_n(x)\varphi(x)\,dx\right| \le \frac{\varkappa}{2}\sup\limits_{\rho\in\mathbb{R}}\int_{S_\rho}|\varphi(x)|\,d\sigma$ *is fulfilled.*

Let a measurable function $(\psi(t,x))_{t\ge 0, x\in\mathbb{R}^d}$ be such that $\sup\limits_{t\in[0,T],x\in\mathbb{R}^d}|\psi(t,x)| < \infty$ for any $T>0$. Consider its transformation Ψ_n for $n\in\mathbb{N}$ given by

$$\Psi_n(t,x) = n^{1/\alpha}\int_0^t d\tau\int_{\mathbb{R}^d} v_n(y)\psi(\tau,y)g(t-\tau,x,y)\,dy, \quad t>0,\ x\in\mathbb{R}^d.$$

Lemma 3. *For given numbers* $\varepsilon>0$, $L>0$, $T>0$, *there exists a number* $\delta>0$ *such that the inequality* $|\Psi_n(t',x')-\Psi_n(t,x)|<\varepsilon$ *is held for all* $t\in[0,T]$, $t'\in[0,T]$, $x\in\mathbb{R}^d$, $x'\in\mathbb{R}^d$, $n\in\mathbb{N}$ *and all measurable functions* ψ *with the property* $\sup\limits_{t\in[0,T],x\in\mathbb{R}^d}|\psi(t,x)|\le L$ *if only the inequality* $|t-t'|+|x-x'|<\delta$ *is fulfilled.*

Next, using Lemma 3 one can easily prove that solutions of equation (1) for the characteristic function $u_n(t,x,\theta)$ of $n^{-1+1/\alpha}\eta^{(n)}_{[nt]}$ and solutions of the following equation

$$u^*_n(t,x,\theta) = 1 + i\theta n^{1/\alpha}\int_0^t d\tau\int_{\mathbb{R}^d} v_n(y)u^*_n(\tau,y,\theta)g(t-\tau,x,y)\,dy$$

satisfy the relation $\lim_{n\to\infty}\sup\limits_{x\in\mathbb{R}^d}\sup\limits_{0<t\le T}\sup\limits_{\theta_1\le\theta\le\theta_2}|u_n(t,x,\theta)-u^*_n(t,x,\theta)| = 0$ for any $T>0$, $\theta_k\in\mathbb{R}$ $(k=1,2)$, $\theta_1<\theta_2$.

As the corollary of Lemma 2 one can say that the characteristic function $(u(t,x,\theta))_{\theta\in\mathbb{R}}$ (t and x are fixed) of the limit distribution with respect to the measure $\mathbb{P}_x$ for the sequence of the random variables $n^{-1+1/\alpha}\xi^{(n)}_{[nt]}$ (and $n^{-1+1/\alpha}\eta^{(n)}_{[nt]}$ also) satisfies the following equation

$$u(t,x,\theta) = 1 + i\theta\varkappa\int_0^t d\tau\int_S g(t-\tau,x,y)u(\tau,y,\theta)\,d\sigma_y \tag{3}$$

A solution of equation (3) can be constructed by the method of successive approximations, that is we have $u(t,x,\theta)=\sum_{k=0}^\infty u^{(k)}(t,x,\theta)(i\theta\varkappa)^k$, where $u^{(0)}(t,x,\theta)\equiv 1$, $u^{(k)}(t,x,\theta) = \int_0^t d\tau\int_S g(t-\tau,x,y)u^{(k-1)}(\tau,y,\theta)\,d\sigma_y$.

This follows from the estimation $|u^{(k)}(t,x,\theta)|\le C^k\frac{(\Gamma(\beta))^k}{\Gamma(1+k\beta)}t^{k\beta}$, getting by the induction, where $C>0$ is some constant, $\beta=1-1/\alpha$.

The solution of equation (3) is unique in the class of bounded functions, because the difference between each two solutions of equation (3) satisfies the following equation

$$w(t,x,\theta) = i\theta\varkappa\int_0^t d\tau\int_S g(t-\tau,x,y)w(\tau,y,\theta)\,d\sigma_y.$$

and we have inequalities $|w(t,x,\theta)|\le\frac{(C\theta\varkappa\Gamma(\beta))^k}{\Gamma(1+k\beta)}t^{k\beta}$ for each $k\in\mathbb{N}$.

Comparing equations (3) and (2) we get that the distribution of $\varkappa l_t$ and the limit distribution of $n^{-1+1/\alpha}\xi^{(n)}_{[nt]}$ (with respect to the measure $\mathbb{P}_x$) are equal.

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Осипчук М.М. *Про кількість перетинів гіперплощини стійким випадковим процесом* // Карпатські матем. публ. — 2018. — Т.10, №2. — С. 10–15.

Досліджено числа перетинів гіперплощини дискретними наближеннями траекторій α-стійкого випадкового процесу ($1 < \alpha < 2$) та деяких пов'язаних з ним процесів. Розглядаються α-стійкий випадковий процес з убиванням з даною інтенсивністю на гіперплощині та псевдопроцес, утворений з α-стійкого випадкового процесу збуренням його оператором дробової похідної з множником типу дельта-функції на гіперплощині. В кожному з цих випадків граничний розподіл кількості перетинів гіперплощини деякою дискретною апроксимацією процесу пов'язаний з розподілом його локального часу на цій гіперплощині. Побудовані інтегральні рівняння для характеристичних функцій цих розподілів. Єдині обмежені розв'язки цих рівнянь можна одержати методом послідовних наближень.