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# SOME WEAKER SUFFICIENT CONDITIONS OF $L$-INDEX BOUNDEDNESS IN DIRECTION FOR FUNCTIONS ANALYTIC IN THE UNIT BALL

We partially reinforce some criteria of $L$-index boundedness in direction for functions analytic in the unit ball. These results describe local behavior of directional derivatives on the circle, estimates of maximum modulus, minimum modulus of analytic function, distribution of its zeros and modulus of directional logarithmic derivative of analytic function outside some exceptional set. Replacement of universal quantifier on existential quantifier gives new weaker sufficient conditions of $L$-index boundedness in direction for functions analytic in the unit ball. The results are also new for analytic functions in the unit disc. The logarithmic criterion has applications in analytic theory of differential equations. This is convenient to investigate index boundedness for entire solutions of linear differential equations. It is also apllicable to infinite products.

Auxiliary class of positive continuous functions in the unit ball (so-denoted $Q_{\mathbf{b}}(\mathbb{B}^n)$) is also considered. There are proved some characterizing properties of these functions. The properties describe local behavior of these functions in the polydisc neighborhood of every point from the unit ball.

*Key words and phrases:* bounded $L$-index in direction, analytic function, unit ball, maximum modulus, directional derivative, distribution of zero.

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## INTRODUCTION

The paper is addendum to papers [4–6, 20]. There was introduced a concept of analytic functions in the unit ball of bounded $L$-index in a direction, where $L : \mathbb{B}^n \to \mathbb{R}_+$ is a continuous function, $R_+ = (0, +\infty)$, $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$. Besides, there were deduced necessary and sufficient conditions of belonging of analytic function in the unit ball to functions of bounded $L$-index in a direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, where $\mathbf{0} = (0, \ldots, 0)$. The conditions describe local behavior directional derivatives, maximum modulus and minimum modulus of the analytic function on the circle of arbitrary radii. There are also an estimate of logarithmic directional derivative outside some exceptional set by the function $L$ and an estimate of distribution of zeros for the analytic functions. Moreover, we established connection [4] betweeen analytic functions in the unit ball of bounded $L$-index in direction and analytic function in the unit ball of bounded value $L$-distribution.

Of course, there are two big classes of functions analytic in bounded domains from $\mathbb{C}^n$. These domains are unit ball and unit polydisc. The domains are not biholomorpic equivalent. Nevertheless, they are importance domains in function theory of several complex variables. Many methods are firstly developing for these domains. Particularly, there are papers [8–10]

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on the concept of bounded $\mathbf{L}$-index in joint variables for functions analytic in the unit polydisc or in the unit ball. It was demonstrated application [17] of the concept to study properties of analytic solutions of some systems of partial differential equations.

Recently, for entire functions of bounded $L$-index in direction new weaker sufficient conditions are obtained [2, 16]. They require validity of some conditions for one value of radius instead each positive value. Moreover, there was presented class [7] of entire functions of unbounded index in any direction. The proof of this fact checks validity of some conditions for some radius. It is simpler than for any radius. Also this idea [11] was applied to investigate $L$-index boundedness in direction of entire solutions of linear directional differential equations.

Here we will consider similar problems for analytic functions in the unit ball.

## 1 AUXILIARY CLASS OF POSITIVE CONTINUOUS FUNCTIONS IN THE UNIT BALL

This section is devoted to auxiliary class of of positive continuous functions in the unit ball. Note that positivity and continuity are still weak restrictions to construct a deep theory of bounded index. Thus, we suppose that the functions satisfy additional assumptions on local behavior.

Let $\mathbb{D} = \{t \in \mathbb{C} : |t| < 1\}$, $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$, $L : \mathbb{B}^n \to \mathbb{R}_+$ be a continuous function, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a fixed direction, where $\mathbf{0} = (0, \ldots, 0)$. For $z \in \mathbb{B}^n$ we denote $D_z = \{t \in \mathbb{C} : |t| \le \frac{1-|z|}{|\mathbf{b}|}\}$,

$$\lambda_{\mathbf{b}}(\eta) = \sup_{z\in\mathbb{B}^n} \sup_{t_1,t_2\in D_z} \left\{ \frac{L(z+t_1\mathbf{b})}{L(z+t_2\mathbf{b})} : |t_1 - t_2| \le \frac{\eta}{\min\{L(z+t_1\mathbf{b}), L(z+t_2\mathbf{b})\}} \right\}.$$

The notation $Q_{\mathbf{b}}(\mathbb{B}^n)$ stands for a class of positive continuous functions $L : \mathbb{B}^n \to \mathbb{R}_+$, satisfying

$$(\forall \eta \in [0, \beta]) : \ \lambda_{\mathbf{b}}(\eta) < +\infty \tag{1}$$

and

$$L(z) > \frac{\beta|\mathbf{b}|}{1 - |z|}, \tag{2}$$

where $\beta > 0$ is some constant. It is easy to check that class $Q_{\mathbf{b}}(\mathbb{B}^n)$ can be defined as follows. For $\eta \in [0, \beta]$, $z \in \mathbb{C}^n$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ and a positive continuous function $L : \ \mathbb{B}^n \to \mathbb{R}_+$, satisfying (2), we define

$$\lambda_1^{\mathbf{b}}(\eta) = \inf_{z\in\mathbb{B}^n} \inf \{L(z + t\mathbf{b})/L(z) : |t| \le \eta/L(z)\},$$

$$\lambda_2^{\mathbf{b}}(\eta) = \sup_{z\in\mathbb{B}^n} \sup \{L(z + t\mathbf{b})/L(z) : |t| \le \eta/L(z)\}.$$

Then the class $Q_{\mathbf{b}}(\mathbb{B}^n)$ consists from the functions $L$, providing inequality

$$(\forall \eta \in [0, \beta]) : \ 0 < \lambda_1^{\mathbf{b}}(\eta) \le \lambda_2^{\mathbf{b}}(\eta) < +\infty, \tag{3}$$

i.e., conditions (3) and (1) equivalent. Actually it is enough to require validity of any inequality in (3) for one value $\eta \in (0, \beta]$ (for $\eta = 0$ the inequality is trivial). If $n = 1$ then $Q(\mathbb{D}) \equiv Q_{\mathbf{1}}(\mathbb{B}^1)$.

The reasoning leads us to the proposition.

**Proposition 1.** *Let* $L:\mathbb{B}^n\to\mathbb{R}_+$ *be a positive continuous functions such that* $(\forall z\in\mathbb{B}^n):$ $L(z)>\frac{\beta|\mathbf{b}|}{1-|z|}$, *where* $\beta>1$. *Then the following statements are equivalent:*

1. $(\forall\eta\in[0,\beta]):\ \lambda_{\mathbf{b}}(\eta)<+\infty$;
2. $(\forall\eta\in[0,\beta]):\ 0<\lambda_1^{\mathbf{b}}(\eta)\le\lambda_2^{\mathbf{b}}(\eta)<+\infty$;
3. $(\exists\eta\in(0,\beta]):\ 0<\lambda_1^{\mathbf{b}}(\eta)\le\lambda_2^{\mathbf{b}}(\eta)<+\infty$.

The proof of this proposition is elementary and uses the definition of class $Q_{\mathbf{b}}(\mathbb{B}^n)$. Other propositions on class $Q_{\mathbf{b}}$ are in [1,14,20].

## 2 LOCAL BEHAVIOR OF DIRECTIONAL DERIVATIVE

Henceforth, we everywhere suppose that $\beta>1$.

Analytic function $F:\mathbb{B}^n\to\mathbb{C}$ is called a function of *bounded L-index [4–6,20] in a direction* $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$, if there exists $m_0\in\mathbb{Z}_+$ such that for every $m\in\mathbb{Z}_+$ and for each $z\in\mathbb{B}^n$

$$\frac{|\partial_{\mathbf{b}}^m F(z)|}{m!L^m(z)}\le\max_{0\le k\le m_0}\frac{|\partial_{\mathbf{b}}^k F(z)|}{k!L^k(z)},\tag{4}$$

where $\partial_{\mathbf{b}}^0F(z)=F(z),\partial_{\mathbf{b}}F(z)=\sum\limits_{j=1}^n\frac{\partial F(z)}{\partial z_j}b_j$, $\partial_{\mathbf{b}}^kF(z)=\partial_{\mathbf{b}}\left(\partial_{\mathbf{b}}^{k-1}F(z)\right)$, $k\ge2$. There is also papers about analytic functions in the unit ball of bounded $\mathbf{L}$-index in joint variables [19]. A connection between these classes is established in [17].

Theory of entire functions of bounded $L$-index in direction is deeply considered in [13].

We need the following criterion of $L$-index boundedness in direction.

**Theorem 1** ([5,6])**.** *Let* $L\in Q_{\mathbf{b}}(\mathbb{B}^n)$. *Analytic function* $F(z)$ *in* $\mathbb{B}^n$ *has bounded* $L$*-index in the direction* $\mathbf{b}\in\mathbb{C}^n$ *if and only if for every* $\eta$, $0<\eta\le\beta$, *there exist* $n_0=n_0(\eta)\in\mathbb{Z}_+$ *and* $P_1=P_1(\eta)\ge1$ *such that for each* $z\in\mathbb{B}^n$ *there exists* $k_0=k_0(z)\in\mathbb{Z}_+$, $0\le k_0\le n_0$, *and the following inequality*

$$\max\{|\partial_{\mathbf{b}}^{k_0}F(z+t\mathbf{b})|:|t|\le\eta/L(z)\}\le P_1|\partial_{\mathbf{b}}^{k_0}F(z)|$$

*holds.*

Let us formulate some auxiliary propositions.

**Lemma 1** ([5,6])**.** *Let* $L\in Q_{\mathbf{b}}(\mathbb{B}^n)$, $\frac{1}{\beta}<\theta_1\le\theta_2<+\infty$, $\theta_1L(z)\le L^*(z)\le\theta_2L(z)$. *Analytic function* $F(z)$ *in* $\mathbb{B}^n$ *has bounded* $L^*$*-index in the direction* $\mathbf{b}$ *if and only if the function* $F$ *has bounded* $L$*-index in the direction* $\mathbf{b}$.

**Lemma 2** ([5,6])**.** *Let* $L\in Q_{\mathbf{b}}(\mathbb{B}^n)$, $m\in\mathbb{C},m\neq0$. *Analytic function* $F(z)$ *in* $\mathbb{B}^n$ *is of bounded* $L$*-index in the direction* $\mathbf{b}\in\mathbb{C}^n$ *if and only if the function* $F(z)$ *is of bounded* $L$*-index in the direction* $m\mathbf{b}$.

**Theorem 2** ([5,6])**.** *Let* $\beta>1$, $L\in Q_{\mathbf{b},\beta}(\mathbb{B}^n)$. *Analytic function* $F(z)$ *in* $\mathbb{B}^n$ *has bounded* $L$*-index in the direction* $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ *if and only if for any* $r_1$ *and for any* $r_2$, $0<r_1<r_2\le\beta$, *there exists* $P_1=P_1(r_1,r_2)\ge1$ *such that for each* $z^0\in\mathbb{B}^n$

$$\max\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{r_2}{L(z^0)}\right\}\le P_1\max\left\{|F(z^0+t\mathbf{b})|:\ |t|=\frac{r_1}{L(z_0)}\right\}.\tag{5}$$

Theorem 2 is criterion of $L$-index boudnedness in direction providing maximum modulus estimate on the greater circle by maximum modulus estimate on the lesser circle. Also it is known some stronger proposition as sufficient conditions.

**Theorem 3** ([5,6]). *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$. Analytic function $F(z)$ in $\mathbb{B}^n$ is of bounded $L$-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ if and only if there exist $r_1$ and $r_2$, $0 < r_1 < 1 < r_2 \le \beta$, and $P_1 \ge 1$ such that for every $z^0 \in \mathbb{B}^n$ inequality* (5) *is true.*

The theorems distinguish universal and existential quantifiers for $r_1$ and $r_2$ such that $0 < r_1 < 1 < r_2 < +\infty$.

This leads to a natural question: *Is it possible to replace quantifiers in other criteria of L-index boundedness in direction?*

Using Fricke's idea [21], we deduce a modification of Theorem 1.

**Theorem 4.** *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$. If there exist $\eta \in (0, \beta]$, $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \ge 1$ such that for any $z \in \mathbb{B}^n$ there exists $k_0 = k_0(z) \in \mathbb{Z}_+$, $0 \le k_0 \le n_0$, and*

$$\max\{|\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})| : |t| \le \eta / L(z)\} \le P_1 |\partial_{\mathbf{b}}^{k_0} F(z)|,$$

*then analytic function $F : \mathbb{B}^n \to \mathbb{C}$ has bounded $L$-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$.*

*Proof.* Besides mentioned paper of Fricke [21], our proof is similar to [3] (entire functions of bounded $L$-index in direction) and to [29] (entire functions of bounded $l$-index).

Assume that there exist $\eta \in (0, \beta]$, $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \ge 1$ such that for any $z \in \mathbb{B}^n$ there exists $k_0 = k_0(z) \in \mathbb{Z}_+$, $0 \le k_0 \le n_0$, and

$$\max\{|\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})| : |t| \le \frac{\eta}{L(z)}\} \le P_1 |\partial_{\mathbf{b}}^{k_0} F(z)|. \tag{6}$$

If $\eta \in (1, \beta]$, then we choose $j_0 \in \mathbb{N}$ such that $P_1 \le \eta^{j_0}$. And for $\eta \in (0; 1]$ we choose $j_0 \in \mathbb{N}$ such that $\frac{j_0! k_0!}{(j_0 + k_0)!} P_1 < 1$. The $j_0$ is well-defined because

$$\frac{j_0! k_0!}{(j_0 + k_0)!} P_1 = \frac{k_0!}{(j_0 + 1)(j_0 + 2) \cdot \ldots \cdot (j_0 + k_0)} P_1 \to 0, \ j_0 \to \infty.$$

Applying integral Cauchy's formula to the function $F(z + t\mathbf{b})$ as analytic function of one complex variable $t$ for $j \ge j_0$ we obtain that for every $z \in \mathbb{B}^n$ there exists $k_0 = k_0(z)$, $0 \le k_0 \le n_0$, and

$$\partial_{\mathbf{b}}^{k_0 + j} F(z) = \frac{j!}{2\pi i} \int\limits_{|t| = \frac{\eta}{L(z)}} \frac{\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})}{t^{j+1}} dt.$$

Taking into account (6), we deduce

$$\frac{|\partial_{\mathbf{b}}^{k_0 + j} F(z)|}{j!} \le \frac{L^j(z)}{\eta^j} \max \left\{ |\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})| : \ |t| = \frac{\eta}{L(z)} \right\} \le P_1 \frac{L^j(z)}{\eta^j} |\partial_{\mathbf{b}}^{k_0} F(z)|. \tag{7}$$

In view of choice $j_0$ with $\eta \in (1, \beta]$, for all $j \ge j_0$ one has

$$\frac{|\partial_{\mathbf{b}}^{k_0 + j} F(z)|}{(k_0 + j)! L^{k_0 + j}(z)} \le \frac{j! k_0!}{(j + k_0)!} \frac{P_1}{\eta^j} \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0! L^{k_0}(z + t_0 \mathbf{b})} \le \eta^{j_0 - j} \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0! L^{k_0}(z)} \le \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0! L^{k_0}(z)}.$$

Since $k_0 \le n_0$, the numbers $n_0 = n_0(\eta)$ and $j_0 = j_0(\eta)$ do not depend of $z$, and $z \in \mathbb{B}^n$ is arbitrary, the last inequality is equivalent to the assertion that $F$ has boudned $L$-index in the direction $\mathbf{b}$ and $N_{\mathbf{b}}(F, L) \le n_0 + j_0$.

If $\eta \in (0,1)$, then from (7) it follows that for all $j \ge j_0$

$$\frac{|\partial_{\mathbf{b}}^{k_0+j} F(z)|}{(k_0+j)! L^{k_0+j}(z)} \le \frac{j! k_0! P_1}{(j+k_0)!} \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{\eta^j k_0! L^{k_0}(z)} \le \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{\eta^j k_0! L^{k_0}(z)}$$

or in view of choice $j_0$

$$\frac{|\partial_{\mathbf{b}}^{k_0+j} F(z)|}{(k_0+j)!} \frac{\eta^{k_0+j}}{L^{k_0+j}(z)} \le \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0!} \frac{\eta^{k_0}}{L^{k_0}(z)}.$$

Thus, the function $F$ is of bounded $\tilde{L}$-index in the direction $\mathbf{b}$, where $\tilde{L}(z) = \frac{L(z)}{\eta}$. Then by Lemma 1 the function $F$ has bounded $L$-index in the direction $\mathbf{b}$, if $\eta\beta > 1$. When $\eta \le \frac{1}{\beta}$, we choose arbitrary $\gamma > \frac{1}{\eta\beta}$. By Lemma 1 the function $F$ is of bounded $L_1$-index in the direction $\mathbf{b}$, where $L_1(z) = \eta\gamma\tilde{L}(z)$. Then be Lemma 2 the function $F$ has bounded $L_1$-index in the direction $\gamma\mathbf{b}$. Since $\partial_{\gamma\mathbf{b}}^k F = \gamma^k \partial_{\mathbf{b}}^k F$ and $L_1^k(z) = \gamma^k L^k(z)$, in inequality (4) with the definition of $L$-index boundedness in direction the corresponding multiplier $\gamma$ is reduced. Hence, the function $F$ is of bounded $L$-index in the direction $\mathbf{b}$. Theorem is proved. □

The following propostion is easy directly deduced from the definition of $L$-index boundedness in direction.

**Proposition 2.** *Let $L : \mathbb{B}^n \to \mathbb{C}$ be a positive continuous function. An analytic function $F : \mathbb{B}^n \to \mathbb{C}$ has bounded $L$-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ if and only if the function $G(z) = F(\mathbf{a}z + \mathbf{c})$ has bounded $L_*$-index in the direction $\frac{\mathbf{b}}{\mathbf{a}}$ for any $\mathbf{c} \in \mathbb{C}^n$ and $\mathbf{a} \in \mathbb{B}^n$ such that $|c| < 1 - |a|$, $a_j \ne 0$ $(\forall j)$, where $\mathbf{a}z + \mathbf{c} = (a_1 z_1 + c_1, \ldots, a_n z_n + c_n)$, $\frac{\mathbf{b}}{\mathbf{a}} = (\frac{b_1}{a_1}, \ldots, \frac{b_n}{a_n})$, $L_*(z) = L(\mathbf{a}z + \mathbf{c})$.*

The proof of the proposition is elementary and it is similar to proof in the case of entire functions (see [12]).

Analog of Proposition 2 for entire functions has generated the following still open problem.

**Problem 1** ([12]). *Does exist numbers $a_1$, $a_2$, $c_1$, $c_2 \in \mathbb{C}$ and an entire function $F(z_1, z_2)$ such that $F(z_1, z_2)$ is of bounded $L$-index in a direction $\mathbf{b} = (b_1, b_2)$, but $F(a_1 z_1 + c_1, a_2 z_2 + c_2)$ is of unbounded $L$-index in the same direction $\mathbf{b} = (b_1, b_2)$?*

## 3 Estimate maximum modulus by minimum modulus

Previously (see [5,6]) we proved few criteria of $L$-index boundedness in direction. They are analogs of one-dimensional criterion of $l$-index boundedness [29]. Moreover, we found that some assertions (Theorems 1 and 2) have modified stronger versions. In fact, their reinforcement is to replace universal quantifiers by existential quantifiers (see Theorems 3 and 4).

Also we can weaken sufficient conditions of Theorem 3, replacing the condition $0 < r_1 < 1 < r_2 < +\infty$ by $0 < r_1 < r_2 < +\infty$.

**Theorem 5.** *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, $F$ be a function analytic in $\mathbb{B}^n$. If there exist $r_1$ and $r_2$, $0 < r_1 < r_2 \le \beta$, and $P_1 \ge 1$ such that for all $z^0 \in \mathbb{B}^n$ inequality* (5) *is satisfied, then the function $F$ is of bounded $L$-index in the direction $\mathbf{b}$.*

*Proof.* Our proof is based on idea of A. D. Kuzyk and M. M. Sheremeta [24]. They proposed this method to investigate the $l$-index boundedness of entire solutions of linear differential equations. Later their idea was applied for entire functions of bounded $L$-index in the direction and in joint variables [2, 15].

Inequality (5) for $0 < r_1 < r_2 < \beta$ implies

$$\max\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{2r_2}{r_1+r_2}\frac{r_1+r_2}{2L(z^0)}\right\} \le P_1 \max\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{2r_1}{r_1+r_2}\frac{r_1+r_2}{2L(z_0)}\right\}.$$

Putting $L^*(z) = \frac{2L(z)}{r_1+r_2}$, we obtain

$$\begin{aligned}\max\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{2r_2}{(r_1+r_2)L^*(z^0)}\right\} \\ \le P_1 \max\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{2r_1}{(r_1+r_2)L^*(z^0)}\right\},\end{aligned} \tag{8}$$

where $0 < \frac{2r_1}{r_1+r_2} < 1 < \frac{2r_2}{r_1+r_2} < \frac{2\beta}{r_1+r_2}$. Clearly, $L^*(z) = \frac{2L(z)}{r_1+r_2} > \frac{2\beta|b|}{(r_1+r_2)(1-|z|)}$, i.e., $L^*$ satisfies (2) and belongs to the class $Q_{\mathbf{b}}(\mathbb{B}^n)$ with $\frac{2\beta}{r_1+r_2}$ instead $\beta$. From validity of inequality (8) we get that by Theorem 3 the function $F$ has bounded $L^*$-index in the direction $\mathbf{b}$. And by Lemma 1 the function $F$ has bounded $L$-index in the direction $\mathbf{b}$. □

**Theorem 6** ([5, 6])**.** *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$. An analytic function $F(z)$ in $\mathbb{B}^n$ has bounded $L$-index in the direction $\mathbf{b}$ if and only if for every $R$, $0 < R \le \beta$, there exist $P_2(R) \ge 1$ and $\eta(R) \in (0, R)$ such that for all $z^0 \in \mathbb{B}^n$ and some $r = r(z^0) \in [\eta(R), R]$ the inequality*

$$\max\left\{|F(z^0+t\mathbf{b})| : |t| = r/L(z^0)\right\} \le P_2 \min\left\{|F(z^0+t\mathbf{b})| : \ |t| = r/L(z^0)\right\} \tag{9}$$

*is true.*

Taking into account analogs of Theorems 4 and 5 for entire functions there was posed the following question in [12].

**Problem 2** ([12, Problem 6])**.** *Is the following Conjecture 1 true?*

**Conjecture 1** ([12, 1])**.** *Let $L \in Q^n_{\mathbf{b}}$. An entire function $F : \mathbb{C}^n \to \mathbb{C}$ has bounded $L$-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ if and only if there exist $R > 0$, $P_2(R) \ge 1$ and $\eta(R) \in (0, R)$ such that for all $z^0 \in \mathbb{C}^n$ and some $r = r(z^0) \in [\eta(R), R]$ inequality* (9) *is valid.*

The was fully proved for entire functions in [2, 16].

Now, we will try to deduce similar results for functions analytic in the unit ball.

**Theorem 7.** *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, $F : \mathbb{B}^n \to \mathbb{C}$ be an analytic function. If there exists $R \in (0, \beta/2)$ (or if there exists $R \in [\beta/2, \beta)$ and $(\forall z \in \mathbb{B}^n) : L(z) > \frac{2\beta|b|}{1-|z|}$) and there exist $P_2 \ge 1$, $\eta \in (0, R)$ such that for all $z^0 \in \mathbb{B}^n$ and some $r = r(z^0) \in [\eta, R]$ inequality* (9) *holds, thenthe function $F$ has bounded $L$-index in the direction $\mathbf{b}$.*

*Proof.* In view of Theorem 5 we need to show existence $P_1$ such that for all $z^0 \in \mathbb{B}^n$

$$\max\left\{|F(z^0+t\mathbf{b})|:|t|=(\beta-R)/L(z^0)\right\} \le P_1 \max\left\{|F(z^0+t\mathbf{b})|:|t|=R/L(z^0)\right\}. \qquad (10)$$

Assume that there exist $R \in (0,\beta/2)$, $P_2 \ge 1$ and $\eta \in \big(0,R\big)$ such that for every $z^0 \in \mathbb{B}^n$ and some $r = r(z^0) \in \big[\eta, R\big]$ we have

$$\max\left\{|F(z^0+t\mathbf{b})|:|t|=r/L(z^0)\right\} \le P_2 \min\left\{|F(z^0+t\mathbf{b})|:|t|=r/L(z^0)\right\}.$$

Denote $L^* = \max\left\{L(z^0+t\mathbf{b}):|t|\le \beta/L(z^0)\right\}$, $\rho_0 = R/L(z^0)$, $\rho_k=\rho_0+k\eta/L^*$, $k\in\mathbb{Z}_+$. We obtain

$$\frac{\eta}{L^*} < \frac{R}{L^*} \le \frac{R}{L(z^0)} = \frac{\beta}{L(z^0)} - \frac{\beta-R}{L(z^0)}.$$

Therefore, there exists $n^* \in \mathbb{N}$, independent of $z^0$ and such that

$$\rho_{p-1} < \frac{\beta-R}{L(z^0)} \le \rho_p \le \frac{\beta}{L(z^0)},$$

for some $p = p(z^0) \le n^*$. It is possible because $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$. Ar first, one has

$$\begin{aligned}\left(\frac{\beta}{L(z^0)}-\rho_0\right)\Big/\left(\frac{\eta}{L^*}\right) =&\frac{(\beta-R)L^*}{\eta L(z^0)}\\ =&\frac{\beta-R}{\eta}\max\left\{\frac{L(z^0+t\mathbf{b})}{L(z^0)}:|t|\le\frac{\beta}{L(z^0)}\right\}\le\frac{\beta-R}{\eta}\lambda_{\mathbf{b}}(\beta).\end{aligned}$$

Therefore, $n^* = \left[\frac{\beta-R}{\eta}\lambda_{\mathbf{b}}(\beta)\right]$, where $[a]$ is an entire part of number $a\in\mathbb{R}$. Let $|F(z^0+t_k^{**}\mathbf{b})| = \max\{|F(z^0+t\mathbf{b})|: t\in c_k\}$, $c_k=\{t\in\mathbb{C}:|t|=\rho_k\}$, and $t_k^*$ be the intersection point of the segment $[0,t_k^{**}]$ with the circle $c_{k-1}$. Hence, for every $r>\eta$ and for each $k\le n^*$ we get the inequality $|t_k^{**}-t_k^*| = \frac{\eta}{L^*} \le \frac{r}{L(z^0+t_k^*\mathbf{b})}$. Thus, for some $r=r(z^0+t_k^*\mathbf{b})\in[\eta,R]$ we deduce

$$\begin{aligned}|F(z^0+t_k^{**}\mathbf{b})| &\le \max\left\{|F(z^0+t\mathbf{b})|:|t-t_k^*|=r/L(z^0+t_k^*\mathbf{b})\right\}\\ &\le P_2\min\left\{|F(z^0+t\mathbf{b})|:|t-t_k^*|=r/L(z^0+t_k^*\mathbf{b})\right\}\\ &\le P_2\min\left\{|F(z^0+t\mathbf{b})|:|t-t_k^*|=r/L(z^0+t_k^*\mathbf{b}),|t-t_0|\le\rho_{k-1}\right\}\\ &\le P_2\max\{|F(z^0+t\mathbf{b})|:t\in c_{k-1}\}.\end{aligned}$$

Hence,

$$\begin{aligned}\max\left\{|F(z^0+t\mathbf{b})|:|t|=(\beta-R)/L(z^0)\right\} &\le \max\{|F(z^0+t\mathbf{b})|:t\in c_p\}\\ &\le P_2\max\{|F(z^0+t\mathbf{b})|:t\in c_{p-1}\}\\ &\le \ldots \le (P_2)^p\max\{|F(z^0+t\mathbf{b})|:t\in c_0\}\\ &\le (P_2)^{n^*}\max\left\{|F(z^0+t\mathbf{b})|:|t|=R/L(z^0)\right\}.\end{aligned}$$

We get (10) with $P_1=(P_2)^{n^*}$. Thus, for $R\in(0,\beta/2)$ Theorem 7 is proved.

Now, suppose that $R \in [\beta/2, \beta)$ and $(\forall z \in \mathbb{B}^n) : L(z) > \frac{2\beta|b|}{1-|z|}$. Then inequality (9) can be rewritten as

$$\max\left\{|F(z^0 + \frac{t}{2} \cdot 2\mathbf{b})| : |t/2| = \frac{r/2}{L(z^0)}\right\} \le P_2 \min\left\{|F(z^0 + \frac{t}{2} \cdot 2\mathbf{b})| : |t/2| = \frac{r/2}{L(z^0)}\right\}.$$

Denoting $t' = t/2$, one has

$$\max\left\{|F(z^0 + t' \cdot 2\mathbf{b})| : |t'| = \frac{r/2}{L(z^0)}\right\} \le P_2 \min\left\{|F(z^0 + t' \cdot 2\mathbf{b})| : |t'| = \frac{r/2}{L(z^0)}\right\}.$$

Since $r \le R \in [\beta/2, \beta)$, we have $r/2 \le R \in [\beta/4, \beta/2) \subset (0, \beta/2)$. Therefore, as shown above the function $F$ has bounded $L$-index in the direction $2\mathbf{b}$, but by Lemma 2 the function is also of bounded $L$-index in the direction $\mathbf{b}$. □

## 4 ESTIMATE OF DIRECTIONAL LOGARITHMIC DERIVATIVE

Below we formulate another criterion of $L$-index boundedness in direction. It describes behavior of logarithmic derivative in direction and distribution of zeros. Firstly the criterion was obtained by Fricke [21,22] for entire function of bounded index.

We need additional notations.

Let $g_{z^0}(t) := F(z^0 + t\mathbf{b})$. If for given $z^0 \in \mathbb{B}^n$ $g_{z^0}(t) \neq 0$ for all $t \in D_{z^0}$, then $G_r^{\mathbf{b}}(F, z^0) := \varnothing$; if for given $z^0 \in \mathbb{B}^n$ $g_{z^0}(t) \equiv 0$, then $G_r^{\mathbf{b}}(F, z^0) := \{z^0 + t\mathbf{b} : t \in D_{z^0}\}$. And if for some $z^0 \in \mathbb{B}^n$ $g_{z^0}(t) \not\equiv 0$ and $a_k^0$ are zeros of the functions $g_{z^0}(t)$, i.e., $F(z^0 + a_k^0\mathbf{b}) = 0$, then

$$G_r^{\mathbf{b}}(F, z^0) := \bigcup_k \left\{z^0 + t\mathbf{b} : |t - a_k^0| \le \frac{r}{L(z^0 + a_k^0\mathbf{b})}\right\}, \quad r > 0.$$

Let

$$G_r^{\mathbf{b}}(F) = \bigcup_{z^0 \in \mathbb{B}^n} G_r^{\mathbf{b}}(F, z^0).$$

By $n(r, z^0, 1/F) = \sum_{|a_k^0| \le r} 1$ we denote counting functions of number of zeros $a_k^0$.

**Theorem 8** ([5,6]). *Let $F$ be an analytic function in $\mathbb{B}^n$, $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$ and $\mathbb{B}^n \setminus G_\beta^{\mathbf{b}}(F) \neq \varnothing$. The function $F(z)$ has bounded $L$-index in the direction $\mathbf{b}$ if and only if*

1) *for every $r \in (0, \beta]$ there exists $P = P(r) > 0$ such that for any $z \in \mathbb{B}^n \setminus G_r^{\mathbf{b}}(F)$*

$$\left|\frac{\partial_{\mathbf{b}} F(z)}{F(z)}\right| \le PL(z); \tag{11}$$

2) *for each $r \in (0, \beta]$ there exists $\widetilde{n}(r) \in \mathbb{Z}_+$ such that for all $z^0 \in \mathbb{B}^n$ with $F(z^0 + t\mathbf{b}) \not\equiv 0$ one has*

$$n\left(\frac{r}{L(z^0)}, z^0, \frac{1}{F}\right) \le \widetilde{n}(r). \tag{12}$$

We weak sufficient conditions in Theorem 8. The one-dimensional analog of Theorem 8 for entire functions revealed its efficiency in the investigation of boundedness of the $l$-index of infinite products in the one-dimensional case [27,28]. Recently, in [18], it has also used this criterion to establish the sufficient conditions of boundedness of the $L$-index in joint variables in terms of the restrictions imposed on the partial logarithmic derivatives and the distribution of zeros. There was posed the following problem.

**Problem 3** ([12, Problem 7]). *Is the following Conjecture 2 true?*

**Conjecture 2** ([12, 2]). *Let $F(z)$ be an entire function in $\mathbb{C}^n$, $L \in Q^n_{\mathbf{b}}$. The function $F$ has bounded $L$-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ if and only if*

1) *there exist $r > 0$, $P > 0$ such that for every $z \in \mathbb{C}^n \setminus G_r$ inequality (11) holds;*

2) *there exist $r > 0$, $\widetilde{n} \in \mathbb{Z}_+$ such that for every $z \in \mathbb{C}^n$ inequality (12) is true.*

By some additional restriction there was proved the conjecture in [2, 16].

Now we consider similar problem for analytic functions in the unit ball with $r \in (0, \beta]$ instead $r > 0$. Let us denote

$$G_r(F) := G^{\mathbf{b}}_r(F) = \bigcup_{z\colon F(z)=0} \{z + t\mathbf{b}\colon |t| < r/L(z)\},$$

$a^0_k$ are zeros of the function $F(z^0 + t\mathbf{b})$ for fixed $z^0 \in \mathbb{B}^n$. By $n_{z^0}(r, F) = n_{\mathbf{b}}(r, z^0, 1/F) := \sum_{|a^0_k| \le r} 1$ we denote the counting function of zeros $a^0_k$ for the slice function $F(z^0 + t\mathbf{b})$ in the disc $\{t \in \mathbb{C}\colon |t| \le r\}$. If for given $z^0 \in \mathbb{B}^n$ and for all $t \in D_z$ $F(z^0 + t\mathbf{b}) \equiv 0$, then we put $n_{z^0}(r) = -1$. Denote $n(r) = \sup_{z \in \mathbb{B}^n} n_z(r/L(z))$.

**Theorem 9.** *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, $\mathbb{B}^n \setminus G^{\mathbf{b}}_{\beta}(F) \ne \varnothing$, $F : \mathbb{B}^n \to \mathbb{C}$ be an analytic function. If the following conditions are satisfied*

1) *there exists $r_1 \in (0, \beta/2)$ (either there exists $r_1 \in [\beta/2, \beta)$ and $(\forall z \in \mathbb{B}^n) : L(z) > \frac{2\beta|b|}{1-|z|}$) such that $n(r_1) \in [-1; \infty)$;*

2) *there exist $r_2 \in (0, \beta)$, $P > 0$ such that $2r_2 \cdot n(r_1) < r_1/\lambda_{\mathbf{b}}(r_1)$ and for all $z \in \mathbb{B}^n \setminus G_{r_2}(F)$ inequality (11) is true;*

*then the function $F$ has bounded $L$-index in the direction $\mathbf{b}$.*

*Proof.* Suppose that conditions 1) and 2) are true.

At first, we consider the case $n(r_1) \in \{-1; 0\}$. Then in the best case the function $F$ can only identically equals zero on the complex line $z^* + t\mathbf{b}$ for some $z^* \in \mathbb{B}^n$, i.e., $F(z^* + t\mathbf{b}) \equiv 0$. For all points lying on such complex lines inequality (9) is obvious.

Let $z^0 \in \mathbb{B}^n \setminus G_{r_2}$. For any points $t_1$ and $t_2$ such that $|t_j| = \frac{r_2}{L(z_0)}$, $j \in \{1, 2\}$, one has

$$\ln\left|\frac{F(z^0 + t_2\mathbf{b})}{F(z^0 + t_1\mathbf{b})}\right| \le \int_{t_1}^{t_2} \left|\frac{\partial_{\mathbf{b}} F(z^0 + t\mathbf{b})}{F(z^0 + t\mathbf{b})}\right| |dt| \le P \int_{t_1}^{t_2} L(z^0 + t\mathbf{b})|dt| \le P\lambda_{\mathbf{b}}(r_2)\, L(z^0) \frac{\pi r_2}{L(z^0)} \le \pi r_2 P \lambda_{\mathbf{b}}(r_2)$$

(we also use that $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$). Hence,

$$\max\left\{|F(z^0 + t\mathbf{b})| : |t| = \frac{r_2}{L(z^0)}\right\} \le P_2 \min\left\{|F(z^0 + t\mathbf{b})| : |t| = \frac{r_1}{L(z^0)}\right\},$$

where $P_2 = \exp\{\pi r_2 P \lambda_2(r_2)\}$. Therefore, by Theorem 7 the function $F$ has bounded $L$-index in the direction $\mathbf{b}$.

Let $r_1 > 0$ be a such that $n(r_1) \in [1;\infty)$ and $2n(r_1)r_2 < r_1/\lambda_{\mathbf{b}}(r_1)$. Put $c = \frac{r_1}{2r_2\lambda_{\mathbf{b}}(r_1)} - n(r_1) > 0$. Clearly, $r_2 = r_1/(2(n(r_1)+c)\lambda_{\mathbf{b}}(r_1))$.

Under condition 1) each set $\overline{K} = \left\{z^0 + t\mathbf{b} : |t| \le \frac{r_1}{L(z^0)}\right\}$ has no more $n(r_1)$ zeros of the function $F$, where $F(z^0 + t\mathbf{b}) \not\equiv 0$.

Under condition 2) there exists $P > 0$ such that $|\frac{\partial_{\mathbf{b}}F(z)}{F(z)}| \le PL(z)$ for every $z \in \mathbb{B}^n \setminus G_{r_2}$, i.e., for all $z \in \overline{K}$, lying outside the sets $\left\{z^0 + t\mathbf{b} :\ |t - a_k^0| < \frac{r_2}{L(z^0 + a_k^0\mathbf{b})}\right\}$, where $a_k^0 \in \overline{K}$ are zeros of the slice function $F(z^0 + t\mathbf{b}) \not\equiv 0$. By definition $\lambda_{\mathbf{b}}$ we obtain $L(z^0)/\lambda_{\mathbf{b}}(r_1) \le L(z^0 + a_k^0\mathbf{b})$. Then $|\frac{\partial_{\mathbf{b}}F(z)}{F(z)}| \le PL(z)$ for every point $z \in \mathbb{B}^n$, lying outside union of the sets

$$c_k^0 = \left\{z^0 + t\mathbf{b} : |t - a_k^0| \le \frac{r_2\lambda_{\mathbf{b}}(r_1)}{L(z^0)} = \frac{r_1}{2(n(r_1) + c)L(z^0)}\right\}.$$

The total sum of diameters of the sets $c_k^0$ does not exceed the value $\frac{r_1 n(r_1)}{(n(r_1)+c)L(z^0)} < \frac{r_1}{L(z^0)}$. Hence, there exists a set $\tilde{c}^0 = \left\{z^0 + t\mathbf{b} : |t| = \frac{r}{L(z^0)}\right\}$, where $\frac{r_1 \min\{1,c\}}{2(n(r_1)+c)} = \eta < r < r_1$, such that for all $z \in \tilde{c}^0$ we have $\left|\frac{\partial_{\mathbf{b}}F(z)}{F(z)}\right| \le PL(z) \le P\lambda_{\mathbf{b}}(r)L(z^0) \le P\lambda_{\mathbf{b}}(r_1)\, L(z^0)$. For any points $z_1 = z^0 + t_1\mathbf{b}$ and $z_2 = z^0 + t_2\mathbf{b}$ with $\tilde{c}^0$ one has

$$\ln\left|\frac{F(z^0 + t_2\mathbf{b})}{F(z^0 + t_1\mathbf{b})}\right| \le \int_{t_1}^{t_2}\left|\frac{\partial_{\mathbf{b}}F(z^0 + t\mathbf{b})}{F(z^0 + t\mathbf{b})}\right||dt| \le P\lambda_2(r_1)\, L(z^0)\frac{\pi r}{L(z^0)} \le \pi r_1\, P(r_2)\lambda_{\mathbf{b}}(r_1).$$

Therefore,

$$\max\left\{|F(z^0 + t\mathbf{b})| : |t| = \frac{r}{L(z^0)}\right\} \le P_2 \min\left\{|F(z^0 + t\mathbf{b})| :\ |t| = \frac{r}{L(z^0)}\right\}, \tag{13}$$

where $P_2 = \exp\{\pi r_1\, P(r_2)\lambda_{\mathbf{b}}(r_1)\}$. If $F(z^0 + t\mathbf{b}) \equiv 0$, then inequality (13) is obvious. By Theorem 7 the function $F(z)$ has bounded $L$-index in the direction $\mathbf{b}$. Theorem 9 is proved. □

**Remark 1.** We proved Hypothesis 2 for analytic function in the unit ball under the additional condition $2r_2 n(r_1) < r_1/\lambda_{\mathbf{b}}(r_1)$. The same condition was firstly appeared for entire functions in [16]. At present, we do not know whether this condition is essential (see Problem 3 in [16]).

Note that Theorems 4, 5, 7 and 9 are new even for analytic functions in the unit disc (cf. [23, 25, 26]). Particularly, for $n = 1$ and analytic functions of bounded $l$-index Theorem 9 implies the following corollary.

**Corollary 1.** *Let $l \in Q(\mathbb{D})$, $f : \mathbb{D} \to \mathbb{C}$ be an analytic function in the unit disc. If the function $f$ satisfies the condition:*

1) *there exists $r_1 \in (0, \beta/2)$ (either there exists $r_1 \in [\beta/2, \beta)$ and $(\forall t \in \mathbb{D}) : l(t) > \frac{2\beta}{1-|t|}$) such that $n(r_1) \in [0;\infty)$;*

2) *there exists $r_2 \in (0, \beta)$, $P > 0$ such that $2r_2 \cdot n(r_1) < r_1/\lambda_{\mathbf{b}}(r_1)$, $\mathbb{D}\setminus G_{r_2}(f) \neq \varnothing$ and for all $t \in \mathbb{D}\setminus G_{r_2}(f)$ $\frac{|f'(t)|}{|f(t)|} \le Pl(t)$;*

*then the function $f$ has bounded $l$-index.*

As we have written that similar criteria (estimate of maximum modulus, minimum modulus, logarithmic derivative and distribution of zeros for arbitrary radii) are also known for function analytic in the unit disc and in arbitrary domain on the complex plane [23,25,26]. But they contain the universal quantifier in their assumptions.

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Бандура А.І. *Деякі слабші достатні умови обмеженості L-індексу за напрямком для аналітичних в одиничній кулі функцій* // Карпатські матем. публ. — 2019. — Т.11, №1. — C. 14–25.

Частково посилюються деякі критерії обмеженості $L$-індексу за напрямком для аналітичних в одиничній кулі функцій. Ці результати описують локальне поводження похідних за напрямком на колі, оцінки максимуму модуля, мінімуму модуля аналітичної функції, розподілу її нулів та модуля логарифмічної похідної за напрямком від аналітичної функції зовні деякої виняткової множини. Заміна квантора універсальності на квантор загальності дає нові слабші достатні умови обмеженості $L$-індексу за напрямком для аналітичних в одиничній кулі функцій. Ці результати також є новими для функцій, аналітичних в одиничному крузі. Отриманий логарифмічний критерій має застосування в аналітичній теорії диференційних рівнянь. Він зручний у дослідженні обмеженості індексу цілих розв'язків лінійних диференційних рівнянь. Також він застосовний до нескінченних добутків.

Досліджено допоміжний клас додатних неперервних функцій в одиничній кулі (так званий $Q_{\mathbf{b}}(\mathbb{B}^n)$). Для функцій з цього класу доведено деякі характеристизаційні властивості. Ці властивості описують локальне поводження таких функцій в полікругових околах кожної точки з одиничної кулі.

*Ключові слова і фрази:* обмежений $L$-індекс за напрямком, аналітична функція, одинична куля, максимум модуля, похідна за напрямком, розподіл нулів.