On polynomial orthogonality on Banach spaces

Andriy Zagorodnyuk
Vasyl Stefanyk Precarpathian National University

УДК 517.98

A. V. Zagorodnyuk*

ON POLYNOMIAL ORTHOGONALITY ON BANACH SPACES

A. V. Zagorodnyuk. *On polynomial orthogonality on Banach spaces*, Matematychni Studii, **14** (2000) 189–192.

In the paper the notion of orthogonality with respect to a homogeneous polynomial p of arbitrary degree on a Banach space is defined and some properties of p-orthogonal sequences are studied. Some application for divisibility of polynomials on Banach spaces are given.

А. В. Загороднюк. *О полиномиальной ортогональности в банаховых пространствах* // Математичні Студії. – 2000. – Т.14, №2. – С.189–192.

В статье определено понятие ортогональности в банаховом пространстве относительно однородного полинома p произвольной степени. Изучены некоторые свойства p-ортогональных последовательностей.

Let X be a Banach space over a field $\mathbb{K}$ of real or complex numbers. A function $p\colon X \to \mathbb{K}$ is an *n-homogeneous polynomial* if there is a symmetric n-linear mapping $\bar{p}\colon X \times \cdots \times X = X^n \to \mathbb{K}$ such that $p(x) = \bar{p}(x, \ldots, x)$ for all $x \in X$. A *polynomial* $p\colon X \to \mathbb{K}$ is just a finite sum of homogeneous polynomials.

It is well known that a function $p\colon X \to \mathbb{K}$ is a polynomial of degree n if and only if p is a polynomial of degree not greater than n on each affine line in X and on some affine line p is an n-degree polynomial ([2], p. 57).

Let p be an n-homogeneous polynomial on X, $n > 1$. We say that linearly independent vectors $x, y \in X$ are *p-orthogonal* ($x \perp_p y$) if $p(t_1x + t_2y) = t_1^n p(x) + t_2^n p(y)$ for all numbers t_1, t_2. It means that for every k, $1 < k < n$, $\bar{p}(\overbrace{x, \ldots, x}^{k}, y, \ldots, y) = 0$. A subspace X_1 is p-orthogonal to X_2 if each vector from X_1 is p-orthogonal to each vector of X_2. In [4] it is proved that if X is complex infinite-dimensional space, then there is an infinite-dimensional subspace in $p^{-1}(0)$. Moreover, from Lemma 4 of [4] (see also [7]) it follows

Theorem A. *For any finite numbers of homogeneous polynomials $p_1, \ldots, p_n$ on an infinite-dimensional complex Banach space X and any vector z from the common set of zeros of $p_1, \ldots, p_n$ there is an infinite-dimensional linear subspace Z in $\bigcap_{i=1}^k p_i^{-1}(0)$ such that $z \in Z$.*

The proof of Theorem A is based on a claim proved in [4] (see also [1]) that for any infinite-dimensional complex space X and homogeneous polynomial p there is an infinite sequence $(x_i)_{i=1}^{\infty}$ such that $p(t_1x_1 + \cdots + t_mx_m) = t_1^n p(x_1) + \cdots + t_m^n p(x_m)$ for every m. So, according to our definition of p-orthogonality we can write

2000 *Mathematics Subject Classification:* 46G20.

* The author was supported in part by NSF Grant P-1-2089

Theorem B. *For every homogeneous polynomial p on an infinite-dimensional complex Banach space X there is an infinite linearly independent sequence $(x_i)_{i=1}^{\infty} \subset X$ of p-orthogonal vectors.*

The purpose of this paper is to discuss the notion of p-orthogonality on Banach spaces and related questions.

Throughout this paper X is an infinite-dimensional Banach space and p is a homogeneous polynomial of degree $n > 1$ on X.

Proposition 1. *For every finite-dimensional subspace V of a complex Banach space X there is an infinite-dimensional subspace Z_0 such that $V \perp_p Z_0$.*

Proof. Let $\dim V = m$ and $e_1, \dots, e_m$ be a basis in V. Put

$$p_{i_1,\dots,i_m}(x) := \bar{p}(\overbrace{e_1,\dots,e_1}^{i_1}, \overbrace{e_2,\dots,e_2}^{i_2}, \dots, \overbrace{e_m,\dots,e_m}^{i_m}, x, \dots, x),$$

where $0 < i_1 + \dots + i_m < n$. Evidently,

$$\bigcap_{0<i_1+\dots+i_m<n} p^{-1}_{i_1,\dots,i_m}(0) \perp_p V.$$

From Theorem A it follows that

$$\bigcap_{0<i_1+\dots+i_m<n} p^{-1}_{i_1,\dots,i_m}(0)$$

contains an infinite-dimensional subspace Z_0. □

From Proposition 1 it follows that every finite sequence of p-orthogonal linearly independent vectors can be extended to some infinite sequence of p-orthogonal linearly independent vectors.

Recall that the set $\operatorname{ess\,ker} p := \{x_0 \in X : p(x + x_0) = p(x) \ \forall x \in X\}$ is said to be the *essential kernel* of a homogeneous polynomial p. In [4] it is shown that the essential kernel is always a closed linear subspace of X.

We say that a sequence $(x_i)_i \subset X$ is *p-orthonormal* if it is p-orthogonal and $p(x_i) = 1$. If $p(x_i) \neq 0$ for each i we will say that $(x_i)_i$ is *semi-p-orthonormal* sequence.

Proposition 2. *Let X be a separable Banach space and p be a continuous n-homogeneous polynomial and $\operatorname{ess\,ker} p = 0$. Let us suppose that there is a p-orthonormal sequence $(x_i)_{i=1}^{\infty}$ such that its linear span is dense in X. Then there is a norm $\|\cdot\|_n$ in X such that the completion $(X, \|\cdot\|_n)$ of X in the norm $\|\cdot\|_n$ is isomorphic to ℓ_n and for any finite sum $\sum a_i x_i$ we have $\|\sum a_i x_i\|_n = [p(\sum |a_i| x_i)]^{1/n}$.*

Proof. Let us consider the subspace $X_f \subset X$ of finite sums $\sum a_i x_i \subset X$. Evidently, $|||\sum a_i x_i|||_n := [p(\sum |a_i| x_i)]^{1/n} = (\sum |a_i|^n)^{1/n}$ is a norm on X_f and the completion $(X_f, |||\cdot|||_n)$ of X_f in the norm $|||\cdot|||_n$ is isomorphic to ℓ_n. On the other hand, since X_f is a dense subspace in X and the norm $|||\cdot|||_n$ is continuous in X we can extend it to a seminorm $\|\cdot\|_n$ on the whole space X by continuity. Let us show that $\|\cdot\|_n$ is a norm. Note first that $|p(x)| \le \|x\|_n^n$. This inequality is obvious for $x \in X_f$ and is true for every $x \in X$ by density of X_f. Let us suppose that $\|x_0\|_n = 0$ for some $x_0 \in X$. Then for every $x \in X$ and a number t

$$|p(x + tx_0)| \le \|x + tx_0\|_n^n \le (\|x\|_n + |t|\|x_0\|_n)^n = \|x\|_n^n.$$

Thus $p(x + tx_0) = p(x)$ (see [4] Corollary 10) and $x_0 \in \operatorname{ess\,ker} p$, hence $x_0 = 0$.

Thus $X \subset (X_f, \|\cdot\|_n)$ and therefore $(X, \|\cdot\|_n)$ is isomorphic to ℓ_n. □

Let us recall that a polynomial p is reducible if there are nonconstant polynomials p_1 and p_2 such that $p = p_1 p_2$. It is clear that if p is irreducible on some subspace then p is irreducible. In [3] it was announced that any irreducible polynomial on an infinite-dimensional space is irreducible on some finite-dimensional subspace. As far as we know [5], a proof of this result has not been published.

Theorem 3. (Mazur and Orlicz) *Let p be an irreducible polynomial on an infinite-dimensional space X over the field $\mathbb{K}$. Then there exists a finite-dimensional subspace $W \subset X$ such that the restriction $p|_W$ of p on W is an irreducible polynomial.*

Proof. Let V be a finite-dimensional subspace. Let us denote by $l(V)$ the number of irreducible factors of $p|_V$. It is clear that if $V_2 \supset V_1$ then $l(V_2) \leq l(V_1)$. Let us denote by l the minimum of $l(V)$ over all finite-dimensional subspaces $V \subset X$. This number is well defined because there exists a minimal element in each subset of $\mathbb{N}$.

Let W be a finite-dimensional subspace such that $l(W) = l$. If $l = 1$ then W is the required subspace. Let us suppose that $l > 1$. Let x_0 be an arbitrary element of X. We denote by Z_{x_0} a subspace of X such that $Z_{x_0} = W + R_{x_0}$, where R_{x_0} is any finite-dimensional subspace, $x_0 \in R_{x_0}$. Z_{x_0} is a finite-dimensional subspace, so the polynomial $p|_{Z_{x_0}}$ can be decomposed into l nonconstant polynomials $r_1[Z_{x_0}](x)$, $r_2[Z_{x_0}](x)$,..., $r_l[Z_{x_0}](x)$, where the notation $r_k[Z_{x_0}](x)$ means that the polynomial $r_k[Z_{x_0}]$ is defined on Z_{x_0}. Let us write $r_k^0 = r_k[W] = r_k[Z_{x_0}]|_W$ for any x_0. So for every $x \in X$ the polynomial p can be decomposed into l nonconstant polynomials $r_1[Z_x], \dots, r_l[Z_x]$ on finite-dimensional subspace $Z_x = W + R_x$. Without loss of generality, we can assume that $r_k[Z_x] = r_k^0$ on W. So for every $x \in X$ there are defined functions

$$r_k(x) := r_k[Z_x](x), \ k = 1, \dots, l.$$

It is clear that the value of r_k at the point x is independent of the choice of R_x. Let us show that $r_k(x)$, $k = 1, \dots, l$ are polynomials on X. Indeed, let R_{x+th} be a finite-dimensional subspace which contains $x + th$ for some $x, h \in X$ and all $t \in \mathbb{K}$. Then $Z_{x+th} = W + R_{x+th}$ is a finite-dimensional subspace which contains the linear span of x and h. Since $r_k[Z_{x+th}]$ is a divisor of $p|_{Z_{x+th}}$ and $x, h \in Z_{x+th}$, we see that $r_k[Z_{x+th}](x + th)$ is a polynomial of variable t (for fixed x, h). Also, if $x_1 + t_1 h_1 = x_2 + t_2 h_2$ then $r_k(x_1 + t_1 h_1) = r_k(x_2 + t_2 h_2)$ because $r_k[Z_{x_1+t_1h_1}]$ and $r_k[Z_{x_2+t_2h_2}]$ coincide on the common domain. Thus, all $r_k(x)$ $k = 1, \dots, l$ are polynomials and $p(x) = r_1(x) \dots r_l(x)$. But this contradicts to the irreducibility of p. □

Proposition 4. *Let p be an n-homogeneous polynomial on a complex m-dimensional Banach space X and $n < m \leq \infty$. If there is a sequence $x_1, \dots, x_k$ of semi-p-orthonormal linear independent vectors in X, where $n < k \leq m$ then p is irreducible.*

Proof. Without loss of generality, we can assume that $p(x_i) = 1$. Then the restriction of p on a subspace V, that is on the linear span of $x_1, \dots, x_k$, is a symmetric polynomial with respect to permutations of $x_1, \dots, x_k$. Moreover, $p(\sum_{i=1}^k a_i x_i) = \sum_{i=1}^k a_i^n$. Let us suppose that p is reducible. Since $k > n$, each divisor of p is a symmetric polynomial [8]. On the other hand, every symmetric polynomial can be represented by an algebraic combination of polynomials q_r, where $q_r(\sum_{i=1}^k a_i x_i) = \sum_{i=1}^k a_i^r$, $r = 1, \dots, n-1$ ([6], p. 79). Since $p = q_n$, this contradicts to the algebraic independence of $q_1, \dots, q_n$. □

Thus from Proposition 4 it follows that if p is a reducible n-homogeneous polynomial then there are at most n linearly independent p-orthonormal vectors.

Theorem 5. *Let X be a complex infinite-dimensional linear space. Then for each polynomial $p\colon X \to \mathbb{C}$ there is an infinite-dimensional subspace $Z \subset X$ such that the restriction of p on Z is a product of one-degree polynomials.*

Proof. From Theorem A it follows that there exists an affine subspace Z_1 of infinite dimension such that $\ker p \supset Z_1$. We can suppose that Z_1 is not a proper subspace of any affine subspace in zero set of p. Let Z be some linear subspace of X, $Z \supset Z_1$ and Z_1 be a hyperplane in Z (i.e. Z has the codimension equal to 1 in Z). Then there is a polynomial $q\colon Z \to \mathbb{C}$, $\deg q = 1$, such that $\ker q = Z_1$. It is clear that q is a divisor of p in Z (see e.g. [3], [9]). A simple induction shows that we can choose an infinite-dimensional subspace Z such that there exist polynomials $q_1, \dots, q_n$, $\deg q_i = 1$, $n := \deg p$ and $p = q_1 \dots q_n$ on Z. □

Corollary 6. *Every continuous polynomial on a complex Banach space is weakly continuous polynomial of the same degree on some infinite-dimensional subspace.*

Proof. It is evident that every product of one-degree polynomials is weakly continuous. Thus, we can use Theorem 5. □

REFERENCES

1. Aron R. M., Rueda M. P., *A problem concerning zero-subspaces of homogeneous polynomials,* Linear Topological Spaces and Complex Analysis, **3** (1997), 20–23.
2. Dineen S., *Complex analysis in locally convex spaces,* North-Holland Mathematics Studies, **57** (1981).
3. Mazur S., Orlicz W., *Sur la divisibilité des polynômes abstraits,* C.R. Acad. Sci., Paris, **202** (1936), 621–623.
4. Plichko A., Zagorodnyuk A., *On automatic continuity and three problems of "The Scottish Book" concerning the boundedness of polynomial functions,* J. Math. Anal. Appl., **220** (1998), 477–494.
5. G. Köthe, *Stanislaw Mazur's contributions to functional analysis,* Math. Ann. **277** (1987), 489–452.
6. van der Waerden, B. L., *Modern Algebra*, Ungar (1964).
7. Zagorognyuk A. V., *The existence of a linear manifold in a kernel of a complex polynomial functional on a finite dimensional linear space,* Matematychni Studii, **8** (1997), no. 1, 115–118. (in Ukrainian)
8. Zagorognyuk A., *Groups of symmetry of zero-set of polynomial functionals on complex Banach spaces,* Mathematical Methods and Physicomechanical Fields, **41** (1998), no. 3, 20–23. (in Ukrainian)
9. Zagorodnyuk A., *On two statements of "The Scottish Book" concerning the ring of bounded polynomial functionals on Banach spaces,* Ukrain. Mat. J. **48** (1996), no. 10, 1507–1516.

Institute for Applied Problems of Mechanics and Mathematics,
3b Naukova str., L'viv, 290601 Ukraine
sirand@mebm.lviv.ua

Received 8.09.1999
Revised 10.02.2000